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OF PLANAR EULERIAN GRAPHS AND PERMUTATIONS

BY

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ABSTRACT. Infinite planar Eulerian graphs are used to show that for $\nu > 0$ the covering number of the infinite simple group $H_\nu = S/S^\nu$ is two.

Here S denotes the group of all permutations of a set of cardinality $\aleph_\nu$, S^ν denotes its subgroup consisting of the permutations moving less than $\aleph_\nu$ elements, and the covering number of a (simple) group G is the smallest positive integer n satisfying $C^n = G$ for every nonunit conjugacy class C in G.

1. Introduction. The covering number of a group G is defined as the smallest positive integer n such that $C^n = G$ holds for every nonunit conjugacy class C in G. We write $\mathrm{cn}(G) = \omega$ when no such positive integer exists. See [**AHSt, ACM**] for further discussion and references

Obviously, if $\mathrm{cn}(G) < \omega$ then G is simple and nonabelian. If G is a finite nonabelian simple group, then $\mathrm{cn}(G) < \omega$ (see e.g. [**AHSt**]), but $\mathrm{cn}(G)$ rarely takes on the smallest conceivable value, namely two [**ACM**]. On the other hand, the alternating group A over any infinite set B is an infinite (nonabelian) simple group satisfying $\mathrm{cn}(A) = \omega$. A is also the first factor in the unique composition chain of the symmetric group $S = S_B$, consisting of all permutations of the set B. When $|B| = \aleph_\nu$, this chain is $1 \triangleleft A \triangleleft S^0 \triangleleft S^1 \triangleleft \cdots \triangleleft S^\nu \triangleleft S^{\nu+1} = S$, where S^τ denotes the group of all permutations of B moving less than $\aleph_\tau$ elements. Thus the composition factors are A, $\mathbf{Z}_2$, $H_0, \ldots, H_\nu$, where $\mathbf{Z}_2 = \{0, 1\}$ is the two-element group, and $H_\tau = S^{\tau+1}/S^\tau$. Rather surprisingly we have [**ACM**, Theorem 3.1].

THEOREM 1. *Let* $0 < \nu$. *Then* $\mathrm{cn}(H_\tau) = 2$ *for* $\tau = 0, \ldots, \nu$.

A proof of this theorem for $0 < \tau < \nu$ is presented in [**ACM**]. In this paper we shall establish it for the extreme cases, $\tau = 0$ and $\tau = \nu$. The argument provided in [**ACM**] for the case $0 < \tau < \nu$ combines Cayley's Representation Theorem with a theorem of G. A. Miller [**Mi**] (see §3). The proof we present here for the cases $\tau = 0$, $\tau = \nu$ involves a direct combinatorial argument, where the construction of suitable infinite planar Eulerian graphs serves to establish the necessary inclusion relations

$$(1) \qquad C_1 \subseteq C_2 \cdot C_3,$$

where C_1, C_2, C_3 are conjugacy classes of the symmetric group over a countably infinite set. Relations of the form (1) form the core of any argument for Theorem 1.

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The use of planar Eulerian graphs to establish such relations is both simple and extensive. In particular, it may replace the use of Miller's Theorem and Cayley's Representation Theorem, by providing a simple and purely elementary combinatorial argument for Theorem 1. The method is developed in §4, where the central connection between planar Eulerian graphs and relations (1) is stated as Theorem 3.

Most of the paper is devoted to a detailed and largely self-contained argument for Theorem 1 with $\tau = \nu$, namely

THEOREM 1′. *Let* $\nu > 0$. *Then* $\mathrm{cn}(H_\nu) = 2$.

In §2 an outline of the argument is given. §3 reduces the argument to the proof of two propositions, 3.8 and 3.9, which led us to the discovery of the method of planar Eulerian graphs treated in §4. In §5 this method is used to derive Propositions 3.8 and 3.9 from Proposition 5.1, which is established by making use of the existence of suitable planar Eulerian graphs.

§6 includes a proof of Theorem 1 for $\tau = 0$. In fact, we show there that the extended covering number of H_0 is 3 if $\nu > 0$ and 4 if $\nu = 0$. This number is the smallest integer k such that any product of k nonunit classes in H_0 equals H_0. These results rely on a recent paper of Droste [**D**], as well as on §5. Roughly speaking, Droste's work is useful whenever at least two of the classes in (1) have at least one infinite orbit, while Proposition 5.1 is used in the other case. Theorem 1 for $\tau = 0$ follows.

2. Outline of the argument. In this section we show how Theorem 1′ follows from a list of identities (1), where C_1, C_2, C_3 are conjugacy classes in S_0, the symmetric group over a countably infinite set.

We first develop some notation. $N = \{1, 2, \dots\}$ denotes the set of positive integers and $N^+ = N \cup \{\aleph_0\}$. A *type* is any cardinal-valued function t defined on N^+. n^* denotes the type defined by $n^*(n) = 1$, $n^*(m) = 0$ for $m \neq n$. For a type t and a cardinal number k a type $k \cdot t$ is defined by $(k \cdot t)(n) = k \cdot (t(n))$, $n \in N^+$. If $\{t_i$: $i \in I\}$ is an indexed family of types, then the *sum* $t = \sum_{i \in I} t_i$ is defined by the condition $t(n) = \sum_{i \in I} t_i(n)$, $n \in N^+$. Thus, for every type t we have $t = \sum_{n \in N^+} t(n) \cdot n^*$. The class T of all types forms a commutative semimodule over the class-semiring of cardinal numbers[2] with the semigroup operation $t_1 + t_2 = \sum_{i \in \{1,2\}} t_i$.

For any $t \in T$, let $|t|$ denote the cardinal number defined by

$$|t| = \sum_{n \in N^+} n \cdot t(n).$$

For any ordinal τ, let T_τ denote the set of all types T with $|t| < \aleph_\tau$. Then T_τ forms a subsemigroup of T. We let $\equiv_\tau$ denote the congruence relation in T modulo T_τ, that is,

$$t \equiv_\tau s \quad \text{iff } t = t_0 + r,\ s = s_0 + r \text{ for some } r \in T,\ t_0, s_0 \in T_\tau.$$

Let B be any set. Thus $|B|$ denotes its cardinality, S_B denotes the group of all permutations of B, and S_B^τ its subgroup consisting of all permutations moving less

[2] If the axiom of choice is not assumed, read "Alephs" instead of "cardinal numbers".

than $\aleph_\tau$ elements. The subscript B is omitted whenever the context allows. Permutations act on symbols from the left, and so $\eta\zeta$ means "first ζ, then η".

For $b \in B$, $\xi \in S_B$, let $(b)_\xi = \{\xi^m(b)\colon m = 0, \pm 1, \pm 2, \dots\}$ denote the ξ-orbit containing b. Thus $|(b)_\xi| \in N^+$ for every $b \in B$, $\xi \in S_B$. With every $\xi \in S_B$ associate a type $\bar{\xi} \in T$ by letting $\bar{\xi}(n)$ be the cardinality of the set of ξ-orbits of cardinality n, that is,

$$\bar{\xi}(n) = |\{(b)_\xi\colon b \in B, |(b)_\xi| = n\}|.$$

Notice that $|\bar{\xi}| = |B|$ holds for every $\xi \in S_B$. Conversely, whenever $t \in T$ and a set B satisfies $|B| = |t|$, there is a *t-permutation* ξ in S_B, i.e., a $\xi \in S_B$ satisfying $\bar{\xi} = t$.

Let $\{B_i\colon i \in I\}$ be a partition of B, and let $\varphi_i \in S_{B_i}$ for $i \in I$. Then $\varphi = \sum_{i \in I}\varphi_i$ denotes the permutation of B satisfying $\varphi | B_i = \varphi_i$ for all i (where $\varphi | B_i$ denotes the restriction of φ to the set B_i). Obviously, $\bar{\varphi} = \sum_{i \in I}\bar{\varphi}_i$. If, in addition, $\psi_i \in S_{B_i}$ for all $i \in I$ and $\psi = \sum_{i \in I}\psi_i$, then $\varphi\psi = \sum_{i \in I}\varphi_i\psi_i$. If $I = \{1, \dots, n\}$, then we write $\varphi_1 + \cdots + \varphi_n$ for $\sum_{i \in I}\varphi_i$.

Let $\sim_G$ denote the conjugacy relation in the group G, and let $[g]_G$ denote the conjugacy class of $g \in G$ in G; again, the subscript G is omitted whenever the context allows.

The following propositions are well known.

PROPOSITION 2.0 [**S**, 1.3.11]. *Let B be any set, $\xi, \eta \in S_B$. Then $\xi \sim \eta$ iff $\bar{\xi} = \bar{\eta}$.*

PROPOSITION 2.1 [**M**, THEOREM 4]. *Let $|B| = \aleph_\nu$, $0 < \tau \leqslant \nu$, ξ, $\eta \in S_B$. Then $\xi S^\tau \sim \eta S^\tau$ iff $\bar{\xi} \equiv_\tau \bar{\eta}$.*

Let t be a type, $|t| = \aleph_\nu$, and let $0 \leqslant \tau \leqslant \nu$. We say that t is *τ-proper* iff:

(1) $t(1) = 0$ or $t(1) = \aleph_\nu$,

(2) $t(n) = 0$ or $t(n) \geqslant \aleph_\tau$, $n > 1$.

Let $|B| = \aleph_\nu$, $\xi \in S_B$. We say that ξ *is τ-proper* iff $\bar{\xi}$ is τ-proper. Notice that τ-properness implies τ'-properness for $\tau' < \tau$.

Proposition 2.1 implies

PROPOSITION 2.2. *Let $|B| = \aleph_\nu$, $0 \leqslant \tau < \nu$, $\eta \in S_B$. Then there is a τ-proper $\xi \in S_B$ with $\xi S^\nu \sim \eta S^\nu$.*

PROOF. Let $\xi \in S_B$ satisfy $\bar{\xi}(n) = 0$ if $n = 1$ and $\bar{\eta}(1) < \aleph_\nu$, or $n > 1$ and $\bar{\eta}(n) < \aleph_\tau$. Let $\bar{\xi}(n) = \bar{\eta}(n)$ otherwise. Then ξ is τ-proper, $\bar{\xi} \equiv_\nu \bar{\eta}$ and so $\xi S^\nu \sim \eta S^\nu$. □

REMARK. If $\aleph_\nu$ is not the sum of countably many smaller cardinals, then in Proposition 2.2, ξ may be chosen to satisfy $\bar{\xi}(n) \in \{0, \aleph_\nu\}$ for all $n \in N^+$.

We are now in position to reduce Theorem 1′ to a theorem about the square of 0-proper classes of S. Indeed, if $|B| = \aleph_\nu$ then for all $\xi, \eta, \zeta \in S_B$, $[\xi S^\nu]_{H_\nu} \subseteq [\eta S^\nu]_{H_\nu} \cdot [\zeta S^\nu]_{H_\nu}$ follows from $[\xi] \subseteq [\eta] \cdot [\zeta]$. Thus, Theorem 1′ follows from

THEOREM 2. *Let $|B| = \aleph_\nu$, $\nu > 0$, and let $\xi, \eta \in S_B$ be 0-proper permutations, where η moves $\aleph_\nu$ elements. Then $\xi \in [\eta]^2$.*

Before presenting an argument for Theorem 2, it will be convenient to develop our notation, so that the relation (1) is modelled also in the realm of types.

If C_1, C_2, C_3 are conjugacy classes of a group G, let us write $P(C_1, C_2, C_3)$ if $C_1 \subseteq C_2 \cdot C_3$. Since $C_2 \cdot C_3 = C_3 \cdot C_2$, we have $P(C_1, C_2, C_3)$ iff $P(C_1, C_3, C_2)$. Since $C_1 \subseteq C_2 \cdot C_3$ iff $C_2 \subseteq C_1 \cdot C_3^{-1}$, we have $P(C_1, C_2, C_3)$ iff $P(C_2, C_1, C_3^{-1})$. It follows that if $C = C^{-1}$ holds for every class in G then P is a symmetric three-place relation on the set of G-classes. Since $C = C^{-1}$ holds for every class of S (see Proposition 2.0), $C = C^{-1}$ holds also for any class in any homomorphic image of S, and so P is a symmetric relation on the classes of S as well as on the classes of H_ν (and in fact, on those of H_τ, $0 \leqslant \tau \leqslant \nu$).

We now extend P to denote a three-place relation on any group G, setting $P(g_1, g_2, g_3)$ iff $P([g_1], [g_2], [g_3])$; and we extend P to denote a symmetric relation on types in the natural way; namely, $P(r, s, t)$ holds for $r, s, t \in T$ iff $P(\xi, \eta, \zeta)$ holds for some permutations ξ, η, ζ with $\bar{\xi} = r$, $\bar{\eta} = s$ and $\bar{\zeta} = t$. Thus $P(r, s, t)$ holds iff $\xi = \eta\zeta$ holds for some r-permutation ξ, s-permutation η and t-permutation ζ.

Theorem 2 is restated now as

THEOREM 2′. *Let $\nu > 0$. Let $r, s \in T$ be 0-proper types, $|r| = |s| = \sum_{n>1} n \cdot s(n) = \aleph_\nu$. Then $P(r, s, s)$ holds.*

Theorem 2′ is a consequence of the following lemmas, the first of which summarizes the most useful properties of P as a three-place relation on types.

LEMMA 1 (SYMMETRY, HOMOGENEITY AND SUPERADDITIVITY OF P). 1. *$P(t_1, t_2, t_3)$ iff $P(t_{\theta(1)}, t_{\theta(2)}, t_{\theta(3)})$ for any permutation θ of $\{1, 2, 3\}$.*

2. *$P(r, s, t)$ implies $p(k \cdot r, k \cdot s, k \cdot t)$ for any cardinal k.*

3. *$P(r_i, s_i, t_i)$ for all $i \in I$ implies $P(r, s, t)$, where $r = \sum_{i \in I} r_i$, $s = \sum_{i \in I} s_i$, $t = \sum_{i \in I} t_i$.*

PROOF. 1 was discussed above, and 2 follows from 3. To prove 3, let B be any set of cardinality $|r| = \sum_{n \in N^+} n \cdot r(n)$, and let $\xi \in S_B$ satisfy $\bar{\xi} = r$. By $r = \sum_{i \in I} r_i$, there is a partition $\{B_i\colon i \in I\}$ of B into ξ-invariant subsets such that $\xi_i = \xi | B_i \in S_{B_i}$ satisfies $\bar{\xi}_i = r_i$ for each $i \in I$. By $P(r_i, s_i, t_i)$, $|r_i| = |s_i| = |t_i| = |B_i|$ holds, and there are $\eta_i, \zeta_i \in S_{B_i}$ with $\bar{\eta}_i = s_i$, $\bar{\zeta}_i = t_i$, satisfying $\xi_i = \eta_i\zeta_i$. Let $\eta = \sum_{i \in I} \eta_i$, $\zeta = \sum_{i \in I} \zeta_i$. Then $\eta, \zeta \in S_B$, $\bar{\eta} = \sum_{i \in I} s_i = s$, $\bar{\zeta} = \sum_{i \in I} t_i = t$, and $\eta\zeta = \sum_{i \in I} \eta_i\zeta_i = \sum_{i \in I} \xi_i = \xi$. Hence, $P(r, s, t)$ holds. $\square$

LEMMA 2. *$P(k \cdot 1^*, s, s)$ holds for every cardinal number k and type s with $|s| = k$.*

PROOF. This is a restatement of the fact that every permutation is conjugate to its inverse. $\square$

The following two lemmas form the core of the proof of Theorem 1. Their proofs are discussed in §§3 and 5. Lemma 3 appears also in [**ACM**].

LEMMA 3. *$P(\aleph_0 \cdot n^*, \aleph_0 \cdot m^*, \aleph_0 \cdot l^*)$ holds for all $n, l, m \in N^+$, $1 < n, l, m$.*

LEMMA 4. *$P(\aleph_0 \cdot n^*, \aleph_0 \cdot (1^* + m^*), \aleph_0 \cdot (1^* + m^*))$ holds whenever $n, m \in N^+$, m greater than* 1.

PROOF OF THEOREM 2′. Let r, s be types satisfying $|r| = |s| = \Sigma_{1<n} s(n) = \aleph_\nu$, $r(n) = 0$ or $r(n)$ infinite, $s(n) = 0$ or $s(n)$ infinite for $n \in N^+$, and $s(1) = 0$ or $s(1) = \aleph_\nu$ (these assumptions hold in Theorem 2′). Let I be a set of cardinality $\aleph_\nu$. By our assumptions, we can define n_i, m_i for $i \in I$ so that $n_i, m_i \in N^+$, $1 < m_i$, and $|\{i: n_i = n\}| = nr(n) = r(n)$, $|\{i: m_i = m\}| = ms(m) = s(m)$ for $n, m \in N^+, m > 1$.

Define r_i, s_i for $i \in I$ by $r_i = \aleph_0 \cdot n_i^*$, and $s_i = \aleph_0 \cdot m_i^*$ if $s(1) = 0$, $s_i = \aleph_0 \cdot (1^* + m_i^*)$ if $s(1) = \aleph_\nu$. Then we have $r = \Sigma_{i\in I} r_i$, $s = \Sigma_{i \in I} s_i$. Also, by Lemmas 2–4 we have $P(r_i, s_i, s_i)$ for all $i \in I$. Hence, by Lemma 1, $P(r, s, s)$. □

3. Some preliminary observations. In this section we reduce the proof of Lemma 4 to that of Propositions 3.8 and 3.9 which are proved in §5. The method of proof involves repeated use of the properties of P listed in Lemma 1, in order to establish the P-relations in Lemmas 2 and 3 from simpler ones. Lemma 3 is proved in [**ACM**]. For the convenience of the reader we sketch its proof below.

PROPOSITION 3.1 (G. A. MILLER, 1900 [**Mi**]). *Let n, m, l be positive integers greater than one. Then there exists a finite group G with elements g_n, g_m, g_l in G of orders n, m, l, respectively, such that $g_n = g_m g_l$.*

Proposition 3.1 was rediscovered several times since 1900. (The author rediscovered it in 1981, in the course of establishing Theorem 1.) See e.g. [**F**].

Assume now that G mentioned in Proposition 3.1 satisfies $|G| = k$. Then $k = n' \cdot n = m' \cdot m = l' \cdot l$; $n', l', m' \in N$. Let $g_n^\circ, g_m^\circ, g_l^\circ$ be the permutations of G obtained from g_n, g_m, g_l in the proof of Cayley's Representation Theorem; that is, let $g_n^\circ(x) = g_n x$, $x \in G$. Then obviously $\bar{g}_n^\circ = n' \cdot n^*$, and similarly $\bar{g}_m^\circ = m' \cdot m^*$, $\bar{g}_l^\circ = l' \cdot l^*$. Also $g_n^\circ = g_m^\circ g_l^\circ$. Thus we obtain from Proposition 3.1

PROPOSITION 3.2. *Let $n, m, l \in N$, $1 < n, l, m$. Then there are $n', l', m' \in N$ such that $P(n' \cdot n^*, m' \cdot m^*, l' \cdot l^*)$ holds.*

It is not hard to extend Proposition 3.1 to the case where n, m, l are allowed to assume also the value $\aleph_0$ (see [**ACM**]). It follows that Proposition 3.2 still holds if we substitute N^+ for N. Now Lemma 3 follows, since $P(\aleph_0 \cdot n^*, \aleph_0 \cdot m^*, \aleph_0 \cdot l^*)$ for $n, m, l \in N^+$, $1 < n, l, m$, follows from 3.2 by the homogeneity of P.

Let us turn now to Lemma 4. It will be convenient to introduce the abbreviation:

$$Q(n, m) \text{ iff } P(\aleph_0 \cdot n^*, \aleph_0 \cdot (1^* + m^*), \aleph_0 \cdot (1^* + m^*)).$$

Thus, Lemma 4 is restated as

LEMMA 4′. *$Q(n, m)$ holds whenever $n, m \in N^+$, $m > 1$.*

We establish first several special cases.

PROPOSITION 3.3. $Q(1, m)$, $1 \leqslant m \leqslant \aleph_0$.

This follows from Lemma 2.

PROPOSITION 3.4. $Q(2, m)$, $m = 2, 3$.

PROOF. (a) $Q(2,2)$: indeed, with $B = N$ we have

$$(1,2)(3,4)(5,6)(7,8)\cdots = [(1,2)(5,6)\cdots][(3,4)(7,8)\cdots].$$

(b) $Q(2,3)$: indeed, with $B = \{1,2,3,4\}$ we have $(1,2)(3,4) = (2,4,3)(2,1,3)$ whence $Q(2,3)$ follows by Lemma 1. □

PROPOSITION 3.5.[3] $Q(n,2)$, $1 \leqslant n \leqslant \aleph_0$.

PROOF. The following are well known [**S**, 10.1.17]:

$$P((2m+1)^*, 1^* + m \cdot 2^*, 1^* + m \cdot 2^*), \qquad 1 \leqslant m < \aleph_0,$$
$$P((2m)^*, 2 \cdot 1^* + (m-1) \cdot 2^*, m \cdot 2^*), \qquad 1 \leqslant m < \aleph_0,$$
$$P(\aleph_0^*, 1^* + \aleph_0 \cdot 2^*, \aleph_0 \cdot 2^*).$$

$Q(n,2)$ for $1 \leqslant n \leqslant \aleph_0$ follows by Lemma 1. □

PROPOSITION 3.6.[3,4] $Q(n,\aleph_0)$, $1 < n \leqslant \aleph_0$.

PROOF. Assume first $n < \aleph_0$. Let $Z = \{0,1,-1,2,-2,\ldots\}$ denote the set of integers, and let $B = Z \times \{0,1,\ldots,n-1\}$. Let $\dot{+}$ denote addition modulo n on $\{0,1,\ldots,n-1\}$.

Define $\xi, \eta, \zeta \in S_B$ by setting

$$\begin{aligned}
&\xi(j,i) = (j, i \dot{+} 1), && j \in \mathbf{Z}, i \in \{0,\ldots,n-1\},\\
&\eta(j,0) = (j-1,0), && j \in \mathbf{Z},\\
&\eta(j,i) = (j,i), && j \in \mathbf{Z}, i \in \{1,\ldots,n-1\},\\
&\zeta(j,i) = (j, i \dot{+} 1), && j \in \mathbf{Z}, i \in \{0,\ldots,n-2\},\\
&\zeta(j,n-1) = (j+1,0) && j \in \mathbf{Z}.
\end{aligned}$$

Then $\xi = \eta\zeta$, $\bar{\xi} = \aleph_0 \cdot n^*$, $\bar{\eta} = \aleph_0 \cdot 1^* + \aleph_0^*$ and $\bar{\zeta} = \aleph_0^*$. Thus, we have $P(\aleph_0 \cdot n^*, \aleph_0 \cdot 1^* + \aleph_0^*, \aleph_0^*)$ and by Lemma 1, we have $Q(n,\aleph_0)$, $1 < n < \aleph_0$.

$Q(\aleph_0,\aleph_0)$ is a consequence of the obvious relation

$P(2 \cdot \aleph_0^*, \aleph_0 \cdot 1^* + \aleph_0^*, \aleph_0 \cdot 1^* + \aleph_0^*)$. □

PROPOSITION 3.7.[3] $Q(\aleph_0, m)$, $2 < m < \aleph_0$.

PROOF. Let $B = \mathbf{Z} \times \{0,1,\ldots,m-1\}$. Let $\dot{+}$ denote addition modulo m on $\{0,\ldots,m-1\}$. Define $\xi, \eta, \zeta \in S_B$ by setting for $j \in \mathbf{Z}$, $i \in \{0,\ldots,m-1\}$,

$$\begin{aligned}
&\xi(j,i) = (j,i+1) \qquad (i < m-1),\\
&\xi(j,m-1) = (j+1,0),\\
&\eta(2j,i) = (2j, i \dot{+} 1),\\
&\eta(2j+1,i) = (2j+1,i),\\
&\zeta(2j+1,i) = (2j+1, i \dot{+} 1),\\
&\zeta(2j,i) = (2j,i).
\end{aligned}$$

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[3] See Figure 1 for a graphical proof.

[4] This proposition was obtained by A. B. Gray in 1960 [**G**]. We are grateful to the referee for calling our attention to this remark.

Then

$$\xi = \eta\zeta, \quad \bar{\xi} = \aleph_0^*, \quad \bar{\eta} = \bar{\zeta} = \aleph_0(1^* + m^*),$$

and so $P(\aleph_0^*, \aleph_0 \cdot (1^* + m^*), \aleph_0 \cdot (1^* + m^*))$ holds. $Q(\aleph_0, m)$ follows by homogeneity of P. □

In §5 we shall prove

PROPOSITION 3.8. $Q(n, m)$, $2 < n, m < \aleph_0$.

PROPOSITION 3.9. $Q(2, m)$, $3 < m < \aleph_0$.

It follows from Propositions 3.3–3.9 that $Q(n, m)$ holds for all $1 \leqslant n \leqslant \aleph_0$ and $1 < m \leqslant \aleph_0$. Thus Lemma 4′ is proved, and the proof of Theorem 1′ is complete.

4. Planar Eulerian graphs and permutations. Recall that for types r, s, t, $P(r, s, t)$ means that an r-permutation is a product of an s-permutation and a t-permutation (see §2 for definitions). In this section a method to establish $P(r, s, t)$ by constructing suitable planar Eulerian graphs is developed. Briefly, the method is this:

Any planar Eulerian graph G with a Black-White coloring of its plane regions defines three types b_G, w_G, d_G by the edge length of boundaries of its black regions, of its white regions and the halves of its vertex degrees. We show that $P(b_G, d_G, w_G)$ holds. The details follow.

By a *planar graph* G we shall mean, as usual, a pair $G = (V_G, E_G)$ where $V = V_G$ is a set of points in the plane—the set of *vertices*—and $E = E_G$ is the set of *edges*. Each edge e is a subset of the plane satisfying $e = f([0,1])$, where f is a continuous mapping of the closed unit interval $[0,1]$ into the plane such that $f|(0,1)$ is a homeomorphism. $f(0)$, $f(1)$ are called the endpoints of e. The endpoints of $e \in E$ must belong to V but need not be distinct (*loops* are allowed) and two distinct edges may have the same endpoints (*multiplicity of edges* (including loops) is allowed). Any edge meets V by the set of its endpoints; and the intersection of two distinct edges is a set of vertices (so any two edges meet at most at common endpoints). In addition to those usual conditions, we require two conditions of local-finiteness.

(a) Every bounded set meets V in a finite set.

(b) Every bounded set meets only finitely many edges.

The *degree* $d(v)$ of $v \in V$ is defined as the number of times v occurs as an endpoint to an edge (so a loop contributes 2 to this count). It follows from (b) that every vertex v has a finite degree.

We let $G_* = V \cup (\bigcup E)$ denote the set of points in V or points in one of the edges (so $V \subseteq G_*$ and $e \subseteq G_*$ for every $e \in E$). We let G_*^c denote the complement of G_* in the plane. It follows from (a) and (b) that G_*^c is an open set. We refer to its connected components as *the regions* of the graph G, or the *G-regions*.

By a planar Eulerian graph (PEG) we mean a planar graph G satisfying the following two extra conditions.

(c) G_* is connected.

(d) $d(v)$ is even for every $v \in V$.

Let G be a PEG and let $x \in e \in E_G$. Then any closed simple curve meeting G^* only at x encloses an open region U disjoint from G^*. (*Negate*. Then by (a), (b), U must witness a G-vertex of odd degree, contradicting (d).) Thus we have

PROPOSITION 4.0 *Let G be a PEG. Then every edge e lies between two distinct G-regions.*

[In other words, every region U of a PEG is a regular open set: $U = \operatorname{int} clU$.]

Let C be a set of edges of a planar graph G. Then we say that C is a *circuit* if C has an enumeration without repetition $C = \{e_i\colon i \in A\}$, such that one of the following holds.

1. $A = \{0\}$ and e_0 is a loop.

2. Either (i) $A = \{0,\dots,n-1\}$ for $n > 1$, or (ii) $A = Z$; and for each $i \in A$, the set of endpoints of e_i is $(e_{i-1} \cap e_i) \cup (e_i \cap e_{i+1})$ (where $e_{-1} = e_{n-1}$, $e_n = e_0$ in case (i)).

For any subset U of the plane, let $C(U)$ denote the set of edges included in the closure of U. If G is an arbitrary plane graph and U is a G-region, then little can be said of $C(U)$. If, however, G is assumed to be Eulerian, $C(U)$ must be a circuit. Indeed, this is an immediate consequence of our definitions and Proposition 4.0. We state it as

PROPOSITION 4.1. *Let G be a PEG, and let U be a G-region. Then $C(U)$ is a circuit, and $\bigcup_{e \in C(U)} e$ is the boundary of U.*

We let $c(U) = |C(U)|$. Thus, $c(U)$ is the number of G-edges included in the closure of the G-region U (in short, touching U). Obviously, $c(U) \in N^+$.

Let G be a PEG. By a *bicoloring of G* or *a B-W-coloring of G* we mean a Black-White coloring of its regions; (that is, a mapping f of the set of G-regions into the set of two colors $\{B = \text{Black}, W = \text{White}\}$) so that every edge e lies between a black region and a white region. Obviously, the existence of a bicoloring of G implies (d). We leave to the reader the easy proof of

PROPOSITION 4.2. *Every PEG has precisely two bicolorings obtained from each other by interchanging Black and White.*

A *bicolored planar Eulerian graph* (BPEG) is a triple $G = (V, E, f)$ where f is a bicoloring of the PEG (V, E).

With a BPEG G we associate three types b_G, d_G, w_G defined as follows:

$$b_G(n) = |\{U\colon U \text{ is a black } G\text{-region and } c(U) = n\}|,$$

$$d_G(m) = |\{v\colon v \text{ is a } G\text{-vertex and } d(v) = 2m\}|,$$

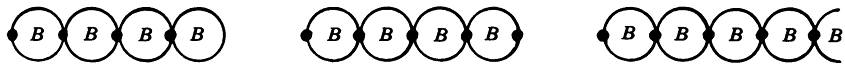
$$w_G(l) = |\{U\colon U \text{ is a white } G\text{-region and } c(U) = l\}|.$$

The following theorem provides the connection between PEGs and the present context.

THEOREM 3. *Let G be a BPEG. Then* $P(b_G, d_G, w_G)$.

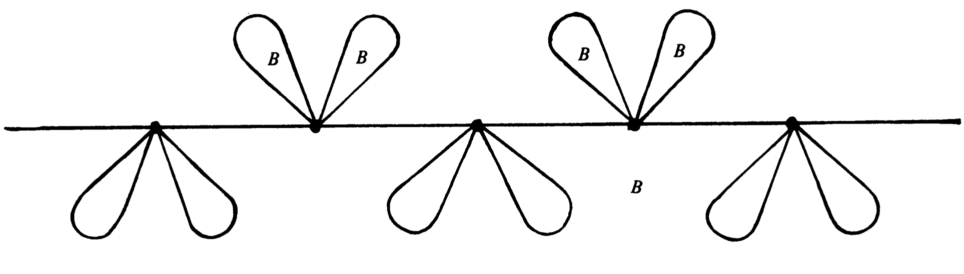
The following illustrated examples deomonstrate a few applications of Theorem 3.

EXAMPLE 1. The bicolored planar Eulerian graphs listed in Figure 1 constitute, by Theorem 3, proofs of Propositions 3.5–3.7.



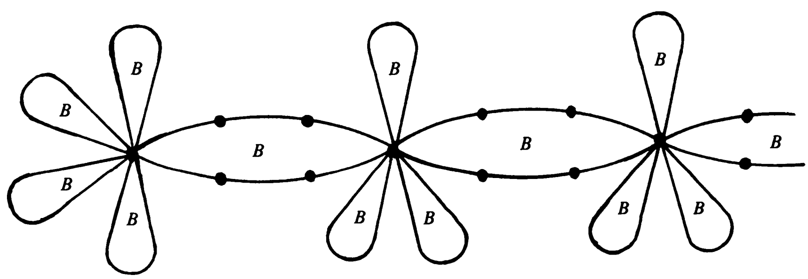
(a1) (a2) (a3)

By (a1), $P(1^* + m \cdot 2^*, 1^* + m \cdot 2^*, (2m+1)^*)$ $(m = 3)$.
By (a2), $P(m \cdot 2^*, 2 \cdot 1^* + (m-1) \cdot 2^*, (2m)^*)$ $(m = 4)$.
By (a3), $P(\aleph_0 \cdot 2^*, 1^* + \aleph_0 \cdot 2^*, \aleph_0^*)$.
Proposition 3.5 follows.



(b)

$P(\aleph_0 \cdot 1^* + \aleph_0^*, \aleph_0 \cdot m^*, \aleph_0 \cdot 1^* + \aleph_0^*)$ $(m = 3)$.
Proposition 3.6 follows.



(c)

$P(\aleph_0 \cdot (1^* + n^*), \aleph_0 \cdot (1^* + m^*), \aleph_0^*)$ $(n = 6, m = 5)$.
Proposition 3.7 follows.

FIGURE 1

EXAMPLE 2. By Figure 2, $P(\aleph_0 \cdot 3^*, 1^* + \aleph_0 \cdot 3^*, \aleph_0 \cdot 3^*)$ holds. Hence, by homogeneity, $P(\aleph_0 \cdot 3^*, \aleph_0 \cdot 1^* + \aleph_0 \cdot 3^*, \aleph_0 \cdot 3^*)$ and, by additivity, $P(\aleph_0 \cdot 3^*, \aleph_0 \cdot (1^* + 3^*), \aleph_0 \cdot (1^* + 3^*))$; i.e., $Q(3,3)$, holds a special case of Proposition 3.8.[5]

[5] In fact, $Q(n, n)$ is a trivial relation for all $n \in N^+$; but this derivation of $Q(3,3)$ models the derivation of $Q(n, m)$ for arbitrary n, m specified in Propositions 3.8 and 3.9.

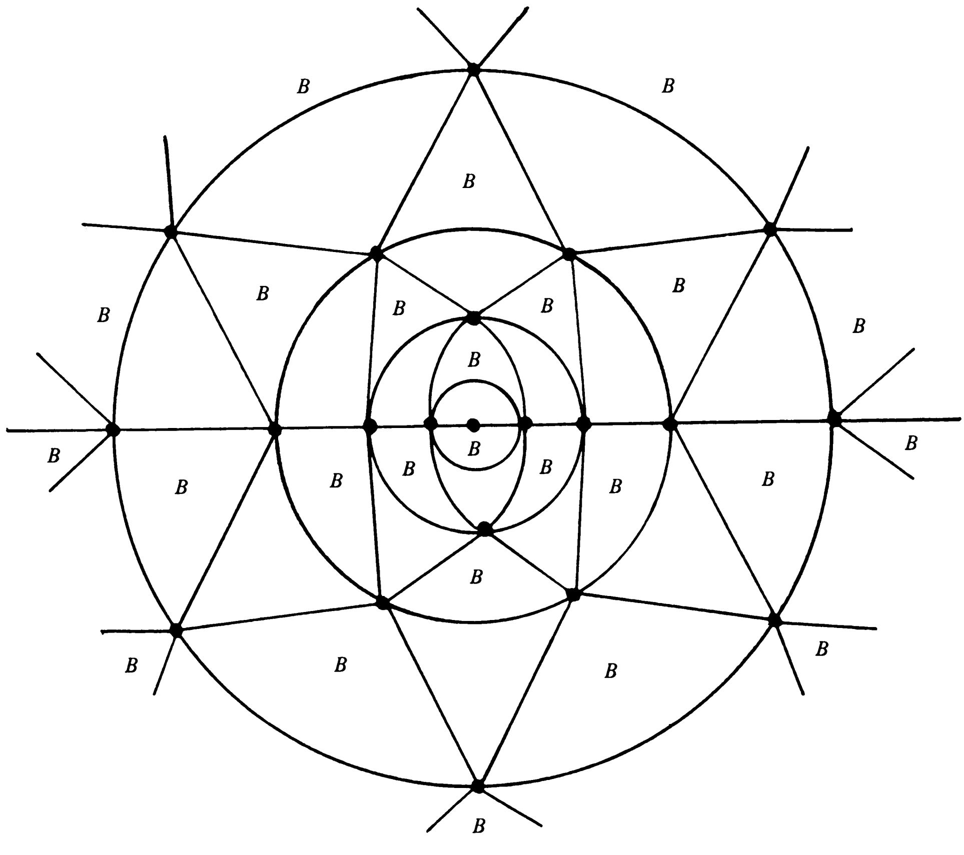


FIGURE 2

EXAMPLE 3. By Figure 3, $P(\aleph_0 \cdot 4^*, \aleph_0 \cdot 2^*, 1^* + \aleph_0 \cdot 4^*)$ holds. Hence (see Example 2), $Q(2,4)$ holds, a special case of Proposition 3.9.

PROOF OF THEOREM 3. Let G be a bicolored PEG. We shall derive from G a directed graph D whose edges are colored η, ζ such that:

1. Every vertex v of D is the tail of just one η-edge and just one ζ-edge, and it is the head of just one η-edge and just one ζ-edge.

By 1, D defines two permutations η and ζ of the set of its vertices. Let $\xi = \eta\zeta$.

2. $\bar{\xi} = b_G$, $\bar{\eta} = d_G$, $\bar{\zeta} = w_G$.

Thus, the existence of D establishes $P(b_G, d_G, w_G)$.

First step. Transform G into another bicolored PEG $\tilde{G}$, as follows:

(1) Around each vertex $v \in V$ draw a small circle C_v so that the circles are disjoint.

(2) Choose $\frac{1}{2}d(v)$ new points on C_v—*the new v-points*—one in each white region that C_v meets. $V_{\tilde{G}}$ is the set of all new v-points for every $v \in V_G$.

(3) Let $e \in E_G$, let U be the white region touching e, and let $v, v' \in V_G$ be the (not necessarily distinct) endpoints of e. Let u and u' be the new v-point on $C_v \cap U$ and

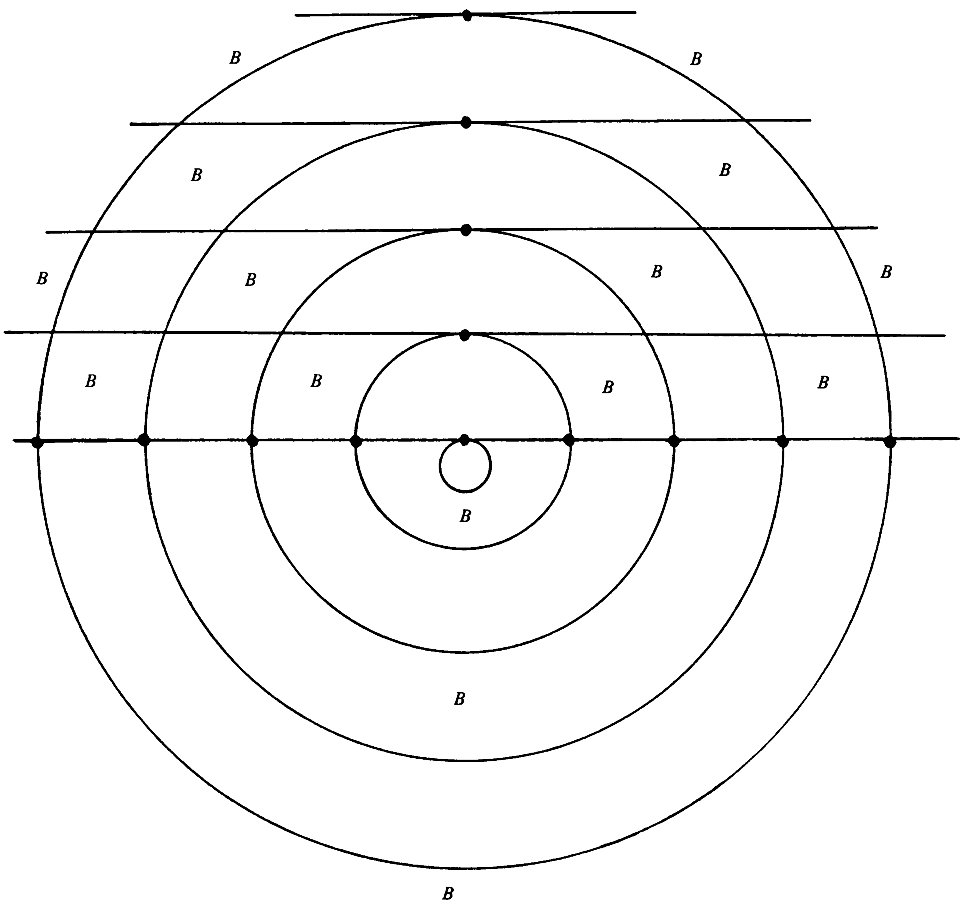


FIGURE 3

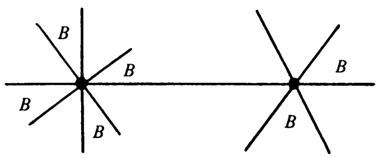
the new v'-point on $C_{v'} \cap U$. (Notice that $u = u'$ if and only if $v = v'$ and e encloses a white G-region.) Connect u and u' by a curve e' lying in U, intersecting C_v in u and $C_{v'}$ in u'. $E_{\tilde{G}}$ consists of all edges e' so obtained, together with all C_v-arcs determined by the new v-points for each $v \in V$.

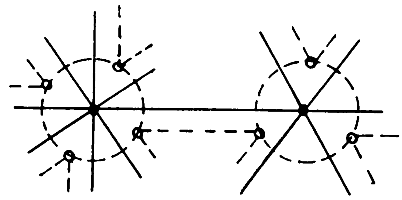
(4) Color white the $\tilde{G}$-regions included in white regions of G and the interiors of the circles C_v. Color black the other $\tilde{G}$-regions.

Second step. Color the $\tilde{G}$-edges by η and ζ as follows: Color the C_v-arcs in $E_{\tilde{G}}(v \in V)$ by η, and all other $\tilde{G}$-edges by ζ.

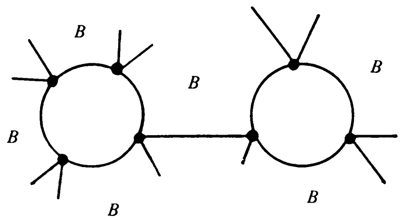
Third step. Obtain D from $\tilde{G}$ by orienting its edges as follows: Let $\tilde{e}$ be any $\tilde{G}$-edge. Then $\tilde{e}$ is so oriented, that the white region is on the right (so that the circumferences of the bounded white regions are oriented clockwise). By Proposition 2.0 this definition makes sense.

We leave it to the reader to check that 1 and 2 hold. $\square$



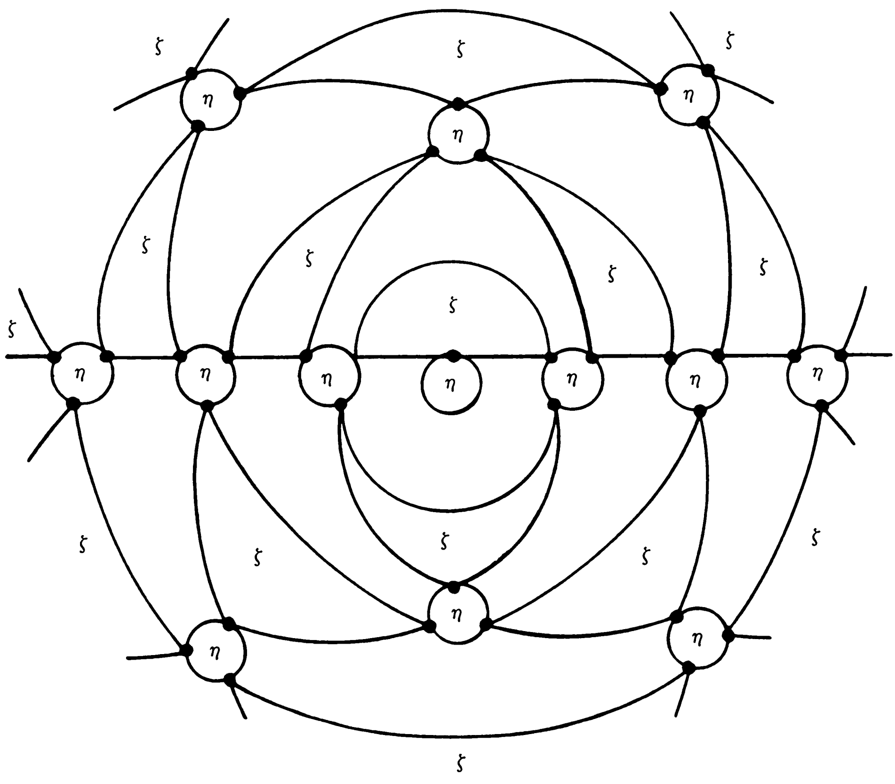


(0) Configuration in G (1) Introducing the circles C_v and the new v-points,



(2) The arising configuration in $\tilde{G}$

FIGURE 4



The digraph D derived by the proof of Theorem 3 from the BPEG G in Figure 2 is given in Figure 5. The edges are so colored and oriented that an edge touching a region marked $\zeta(\eta)$ is marked $\zeta(\eta)$ and the circuits bounding the marked regions are oriented clockwise.

FIGURE 5

5. The planar graphs establishing Theorem 1. In this section we use Theorem 3 to prove Propositions 3.8 and 3.9, thereby completing the proof of Theorem 1. These propositions follow by Lemma 1 from

PROPOSITION 5.0. 1. *$P(\aleph_0 \cdot n^*, 1^* + \aleph_0 \cdot m^*, \aleph_0 \cdot l^*)$ holds for all $n, m, l \in N$, $2 < n, m, l$.*

2. *$P(\aleph_0 \cdot m^*, \aleph_0 \cdot 2^*, 1^* + \aleph_0 \cdot l^*)$ holds for all $m, l \in N$, $3 < m, l$.*

We shall establish this proposition by showing that suitable bicolored planar Eulerian graphs (BPEGs) exist. At this point, the reader may wish to consult again Figures 2 and 3, which establish assertions 1 and 2 with $n = m = l = 3$ and $m = l = 4$, respectively.

Our method of proof is also indicated by the above figures. We obtain the required BPEGs as an increasing union of finite bicolored disc-maps, taking care in transitions from one such map to the next that the new black regions, new degrees defined for internal vertices, and new white regions are as required by the triple of types b_G, d_G, w_G in question. A somewhat more general proposition will be proved, namely,

PROPOSITION 5.1. *Let $(n_i)_{i\in N}$, $(m_j)_{j\in N}$, $(l_k)_{k\in N}$ be sequences of integers satisfying at least one of the following two conditions.*

1. $m_1 = 1$; $2 < n_i, m_j, l_k$, $\quad i, j, k \in N, 1 < j$,

2. $l_1 = 1$; $3 < n_i, l_k, 1 < m_j$, $\quad i, j, k \in N, 1 < k$.

Then there is a bicolored planar Eulerian graph G and enumerations $(B_i)_{i\in N}$, $(v_j)_{j\in N}$, $(W_k)_{k\in N}$ of the black G-regions, the set of G-vertices, and the white G-regions, respectively, so that $c(B_i) = n_i$, $d(v_j) = 2m_j$, $c(W_k) = l_k$; $i, j, k \in N$.

(Recall that $c(B_i)$, $d(v_j)$, $c(W_k)$ are the number of edges of the circuit enclosing B_i, the degree of v_j and the number of the edges of the circuit enclosing W_k, respectively.)

The proof of Proposition 5.1 requires some more notation. For brevity, take the plane to be the set of complex numbers. For a positive integer n, let $C_n = \{z: |z| = n\}$, $D_n = \{z: |z| \leqslant n\}$. By an *$n$-disc graph* ($n$-DG) we shall mean a triple $G = (V_G, E_G, f_G)$, satisfying the following conditions.

(A) (V_G, E_G) is a planar graph such that

(A0) $G_* = V_G \cup \bigcup_{e\in E_G} e$ is connected;

(A1) $C_n \subseteq G_* \subseteq D_n$;

(A2) for $v \in V_G \cap C_n$, $d(v) \leqslant 4$.

(B) f_G is a B-W coloring of the bounded G-regions; that is, every edge $e \in E_G$ not on C_n touches a black and a white region.

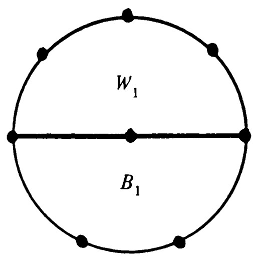
Notice that by (a), (b) and (A1), V_G and E_G are finite, and $d_G(v) \geqslant 2$ for $v \in V_G \cap C_n$; and by (A0) and (B), $d_G(v) > 0$ is even for every $v \in V_G \setminus C_n$ (such a vertex v will be called an inner vertex of G).

We say that the $(n+1)$-DG $G' = (V', E', f')$ *extends* the n-DG $G = (V, E, f)$ if $V = V' \cap D_n$, $E = \{e \in E': e \subseteq D_n\}$ and $f'(U) = f(U)$ for every bounded G-region U.

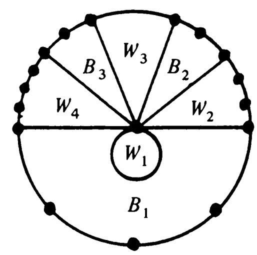
Our proof of Proposition 5.1 consists in showing that a bicolored planar graph G with the required properties can be constructed as the union $G = \bigcup_{n=1}^{\infty} G_n$, where for each n, $G_n = (V_n, E_n, f_n)$ is an n-DG and G_{n+1} extends G_n. f_n will be specified by a pair of sequences $(B_i)_i$ and $(W_k)_k$ enumerating the black and white G_n-regions.

We shall refer to the assumption that the condition 1 (2) of Proposition 5.1 holds as "Case 1" ("Case 2").

1. *Base of the inductive construction*. Let $G_1 = (V_1, E_1, f_1)$ be the 1-DG shown in Figure 6. Formally, $V_1 = \{-1, 0, 1\} \cup V_1^+ \cup V_1^-$, where $V_1^+(V_1^-)$ is the set of vertices on the upper (lower) half of the circle C_1.



Case 1. With $n_1 = 5$, $l_1 = 6$.



Case 2. With $(n_1, n_2, n_3) = (7, 4, 4)$, $m_1 = 4$, $(l_2, l_3, l_4) = (5, 3, 6)$.

FIGURE 6

Case 1. $V_1^+(V_1^-)$ is chosen so that $|V_1^+| = l_1 - 3(|V_1^-| = n_1 - 3)$. E_1 consists of the closed arcs of C_1 determined by $V_1 \cap C_1$, and the radial segments connecting 0 with 1, -1, i.e. the closed intervals $[0, 1]$, $[-1, 0]$.

Case 2. *Step* 1. We specify a subset E' of E_1. Let E' consist of $[-1, 0]$, $[0, 1]$ and a loop e with endpoint 0, all of whose nonzero points lie in the interior of the lower half of D_1.

Step 2. We specify a subset V' of V^+. Choose arbitrarily $2(m_1 - 2)$ points in $C_1 \cap \{z\colon \operatorname{Im}(z) > 0\}$.

Step 3. We specify another subset E'' of E_1. Let E'' consist of all radial segments connecting 0 with vertices in V'.

Step 4. We assign colors to the regions into which D_1 was divided. Let W_1 be the region bounded inside the loop e, let B_1 be the other region in the lower half of D_1, and let $W_2, B_2, \ldots, W_{m_1-1}, B_{m_1-1}, W_{m_1}$ be enumeration of the triangular regions into which the upper half of D_1 was divided in Step 3, in cyclical (say, positive) order.

Step 5. We specify another subset V'' of $V_1^+ \cup V_1^-$ by choosing arbitrarily extra points on C_1 so as to satisfy the equalities $C(W_1) = l_1, \ldots, C(W_{m_1}) = l_{m_1}$, $C(B_1) = n_1, \ldots, C(B_{m_1-1}) = n_{m_1-1}$. Let E''' be the set of arcs into which C_1 is divided by vertices chosen on C_1.

Step 6. Let $V_1 = \{-1, 0, 1\} \cup V' \cup V''$; $E_1 = E' \cup E'' \cup E'''$, and f_1 as determined in Step 4.

2. *Induction step*. Assume that n-DG $G_n = (V_n, E_n, f_n)$ is defined. Let $B_1, \ldots, B_p$; $v_1, \ldots, v_q$; $W_1, \ldots, W_r$ enumerate the black G_n-regions, the inner G_n-vertices and the

white G_n-regions. Let $u_1 = v_{q+1},\dots,u_s = v_{q+s}$ enumerate $V_n \cap C_n$ in positive cyclic order, let $\lfloor u_i, u_{i+1} \rfloor \in E_n$ denote the arc on C_n from u_i to u_{i+1} ($u_{s+1} = u_1$) and let $X_i \in \{B, W\}$ be the color of the bounded G-region touching $\lfloor u_i, u_{i+1} \rfloor$. We now extend G_n to G_{n+1} as follows.

Case 1. *Step* 1. For $t = 1,\dots,s$ let y_t denote the intersection of the ray from 0 through u_t by C_{n+1}. Choose $y_t^+ \in C_{n+1}$ between y_t and y_{t+1} (where $y_{s+1} = y_1$) ($y_t^+ = y_{t+1}$ is possible, if so required in Step 3). Let $y_t, y_t^+ \in V_{n+1}$. Let $[u_t, y_t]$, $[u_t, y_t^+]$ belong to E_{n+1} (where $[x, y]$ denotes the straight line segment from x to y).

Step 2. Color the regions bounded by the segments $[u_t, y_t^+]$, $[u_{t+1}, y_{t+1}]$ and the arcs $\lfloor u_t, u_{t+1} \rfloor$, $\lfloor y_t^+, y_{t+1} \rfloor$ properly; i.e., B if $X_t = W$ and W otherwise. Let $B_{p+1},\dots,B_{p'}$; $W_{r+1},\dots,W_{r'}$ denote the black and white new regions so obtained.

Step 3. Let $i \in \{p + 1,\dots,p'\}$ ($k \in \{r + 1,\dots,r'\}$). Let B_i (W_k) touch the arc $\lfloor u_t, u_{t+1} \rfloor \subseteq C_n$. Let $y_t^+ = y_{t+1}$ if $n_i = 3$ ($l_k = 3$). Otherwise, let $\lfloor y_t^+, y_{t+1} \rfloor$ be a nontrivial arc on C_{n+1} containing $n_i - 4$ ($l_k - 4$) new vertices. (This ensures $c(B_i) = n_i$, $c(W_k) = l_k$, $i = p + 1,\dots,p'$; $k = r + 1,\dots,r'$.)

Step 4. Let $4 \leqslant m_t' \leqslant 6$ be the degree of u_t in the graph drawn so far, $t = 1,\dots,s$. Choose $2m_{q+t} - m_t'$ points on the arc $\lfloor y_t, y_t^+ \rfloor$. These points are vertices in $V_{n+1} \cap C_{n+1}$. Every such point is then connected to u_t by a straight line segment. All these segments belong to E_{n+1}. (This ensures $d(v_j) = 2m_j$, $j = q + 1,\dots,q + s$.) In this step the triangular regions with vertices u_t, y_t, y_t^+ are subdivided into triangular regions with common vertex u_t and the two other vertices on C_{n+1}.

Step 5. Color properly the triangular regions obtained in Step 4 Black and White. Denote by $B_{p'+1},\dots,B_{p''}$, $W_{r'+1},\dots,W_{r''}$ the new triangular colored regions.

Step 6. Add extra vertices in $V_{n+1} \cap C_{n+1}$ as needed to ensure $c(B_i) = n_i$, $c(W_k) = l_k$ for $i = p' + 1,\dots,p''$, $k = r' + 1,\dots,r''$. Let E_{n+1} contain all arcs determined on C_{n+1} by $V_{n+1} \cap C_{n+1}$.

G_{n+1} is now defined.

Case 2. Here our induction hypothesis on $u_1,\dots,u_s$ includes the extra assumption:

$$(*) \qquad \text{if } t \neq t' \text{ and } d(u_t) = d(u_{t'}) = 4, \text{ then } |t - t'| > 1.$$

Step 0. Declare the degrees of $u_1,\dots,u_s$ to be $2m_{q+1},\dots,2m_{q+s}$, respectively.

Proceed through Steps 1–6 of Case 1, modified as required by Step 0. In particular, in Step 1, no $y_t, y_t^+ \in C_{n+1}$, are defined if $m_{q+t} = 2$ and the degree of u_t is 4. (($*$) and $n_t, l_t > 3$ are used in Step 3. In Step 6, ($*$) is carried over to G_{n+1} by $n_t, l_t > 3$.)

It is easily checked that the $(n + 1)$-DG $G_{n+1} = (V_{n+1}, E_{n+1}, f_{n+1})$ so defined extends (G_n, f_n), and that the union graph is as required by Proposition 5.1.

The proof of Theorem 1′ is complete.

6. The covering number of H_0. In this section we briefly indicate the proof that $\mathrm{cn}(H_0) = 2$ if $\nu > 0$. In **[ACM]** an analogous but stronger result was obtained for H_τ with $0 < \tau < \nu$. For a group G let *the extended covering number of* G, $\mathrm{ecn}(G)$, **[AHS]** denote the smallest integer $k + 1$ such that whenever $C_1,\dots,C_{k+1}$ are nonunit classes of G, we have $G = C_1 \cdots C_{k+1}$. It is not hard to show that $k + 1 \geqslant \mathrm{ecn}(G)$ iff, whenever $C_1,\dots,C_k$ are nonunit classes of G, we have $C_1 \cdots C_k \supseteq G \setminus \mathbf{1}$, where

1 denotes the unit-class. Thus, if $k + 1 = \mathrm{ecn}(G)$ and $\mathbf{1} \subseteq C^k$ for every class $C \neq \mathbf{1}$ then $\mathrm{cn}(G) \leqslant k$.

In [**ACM**] it is actually shown that if $\nu > 0$, $0 < \tau < \nu$, then $\mathrm{ecn}(H_\tau) = 3$ (whence $\mathrm{cn}(H_\tau) = 2$ follows). This result is true also when $\tau = 0$. We state it as

THEOREM 4. *Let* $\nu > 0$. *Then* $\mathrm{ecn}(H_0) = 3$.

COROLLARY. $\mathrm{cn}(H_0) = 2$.

For types t, s let $t \equiv s$ iff $t = t' + r$, $s = s' + r$ for some $t', s', r \in T$, with $|t'| = |s'| < \aleph_0$.

PROPOSITION 6.0 [**M**, Theorem 4]. *Let* $\nu \geqslant 0$, *let* $G = S/S^0$ *and let* $\xi, \eta \in S$. *Then* $[\xi S^0]_G = [\eta S^0]_G$ *iff* $\bar{\xi} \equiv \bar{\eta}$.

Define two sets of countable types U_0, U_1 by $t \in U_0 \Leftrightarrow t = \aleph_0 \cdot 1^* + t'$, where $t'(1) = 0$ and $n > 1 \Rightarrow t(n) \in \{0, \aleph_0\}$, $s \in U_1 \Leftrightarrow s = \aleph_0 \cdot 1^* + s'$, where $s'(1) = s'(2) = s'(3) = 0$, and $n > 1 \Rightarrow s'(n) < \aleph_0$, and $\Sigma_{n>3} ns(n) \in \{0, \aleph_0\}$. Let $U = U_0 + U_1$. By a U-type (U_i-type) we shall mean a type lying in U (U_i, $i = 0, 1$). From Proposition 6.0 follows

PROPOSITION 6.1. *Let C be a class in H_0. Then $C = [\xi S^0]$ where $\xi \in S$ satisfies $\bar{\xi} = \aleph_\nu \cdot 1^* + r$, r a U-type.*

Theorem 4 follows by Lemma 1 from

PROPOSITION 6.2. *Let r, s, t be nonunit U-types. Then $P(r, s, t)$.*

Define sets V_i' of countable types, $i = 1, 2, 3$ by $v' \in V_1'$ iff $v' = \aleph_0 \cdot n^*$, $1 < n < \aleph_0$; $v' \in V_2'$ iff $v' = m \cdot \aleph_0^*$, $m \in N$; $v' \in V_3'$ iff $v' = \Sigma_{n \in A} n^*$, where $A \subseteq N \setminus \{1, 2, 3\}$, $|A| = \aleph_0$. Let $V' = V_1' \cup V_2' \cup V_3'$ and define V_i, V by $v \in V_i$ iff $v = \aleph_0 \cdot 1^* + v'$, $v' \in V_i'$, $i = 1, 2, 3$; $v \in V$ iff $v = \aleph_0 \cdot 1^* + v'$, $v' \in V'$.

Obviously, $V \subseteq U$ and every $u \in U$ is a sum of members of V. Moreover, if $u \notin V_2$, i.e., u is not $\aleph_0 \cdot 1^* + m \cdot \aleph_0^*$ for some $1 \leqslant m \leqslant \aleph_0$, then u is a sum of countably many members of V. To establish Proposition 6.2 we shall need the following recent result of Droste.

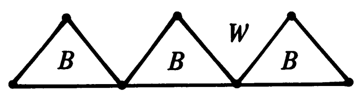
PROPOSITION 6.3 [**D**, Theorem A]. *$P(r, s, t)$ holds whenever $|r| = |s| = |t| = \Sigma_{n>1} nt(n) = \aleph_0$ and $r(\aleph_0), s(\aleph_0) > 0$.*

PROPOSITION 6.4. *$P(v_1, v_2, v_3)$ holds for every $v_1, v_2, v_3 \in V$.*

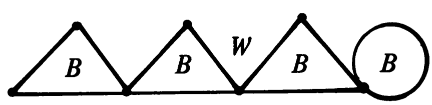
PROOF. We leave it to the reader to fill in the details of the proofs of the following inclusions:

$V_1 \times V_1 \times V_1 \subseteq P$ (by Lemma 3),
$V_1 \times V_1 \times V_2 \subseteq P$ (modify proof of Propostion 3.7 (Figure 1)),
$V_1 \times V_1 \times V_3 \subseteq P$ (use 5.1, 3.5 and Figure 7),

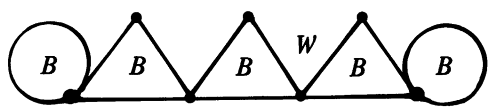
$V_1 \times V_2 \times V_2 \subseteq P$ (modify proof of Proposition 3.6 (Figure 1)),
$V_1 \times V_2 \times V_3 \subseteq P$ (modify proof of Proposition 3.6 (Figure 1)),
$V_1 \times V_3 \times V_3 \subseteq P$ (use Proposition 5.1),
$V_2 \times V_2 \times V_2 \subseteq P$ (use Proposition 6.3),
$V_2 \times V_2 \times V_3 \subseteq P$ (modify proof of Proposition 3.6 (Figure 1)),
$V_2 \times V_3 \times V_3 \subseteq P$ (modify proof of Proposition 3.7 (Figure 1)),
$V_3 \times V_3 \times V_3 \subseteq P$ (use Proposition 5.1). $\square$



(0) $P(k \cdot 3^*, (k+2) \cdot 1^* + (k-1) \cdot 2^*, (3k)^*)$



(1) $P(1^* + k \cdot 3^*; (k+1) \cdot 1^* + k \cdot 2^*, (3k+1)^*)$



(2) $P(2^* + k \cdot 3^*, k \cdot 1^* + (k+1) \cdot 2^*, (3k+2)^*)$

The relations (0), (1), (2) are established by the indicated bicolored Eulerian graphs (for $k = 3$). (0), (1), (2) imply $P(\aleph_0 \cdot (1^* + 3^*), \aleph_0 \cdot (1^* + 2^*), t)$ for any type t with $t(2) = t(\aleph_0) = 0$, $|t| = t(1) = \Sigma_{n>1} t(n) = \aleph_0$.

FIGURE 7

PROOF OF PROPOSITION 6.2. Let $r, s, t \in U$. We distinguish three cases.

Case 1. $r, s, t \notin V_2$. Then we can put $r = \Sigma_{n \in N} r_n$, $s = \Sigma_{n \in N} s_n$, $t = \Sigma_{n \in N} t_n$ with $r_n, s_n, t_n \in V$ for all $n \in N$. $P(r, s, t)$ follows by Proposition 6.4 and Lemma 1.

Case 2. $r \in V_2$, $s, t \notin V_2$.

2.1. $r(\aleph_0) = s(\aleph_0) = 0$: modify the proof of Proposition 3.7.

2.2. $s(\aleph_0) > 0$: use 6.3.

Case 3. $r, s \in V_2$: use 6.3. $\square$

The proof of Theorem 4 is complete.

We conclude by summarizing the situation when $\nu = 0$.

THEOREM 5. *Let* $\nu = 0$. *Then* $\mathrm{ecn}(H_0) = 4$.

PROOF. $\mathrm{ecn}(H_0) \geqslant 4$ since $P(\aleph_0 \cdot 2^*, \aleph_0 \cdot 2^*, \Sigma_{n \in N} n^*)$ does *not* hold (see [**ACM**]). $\mathrm{ecn}(H_0) \leqslant 4$ follows from a recent result of M. Droste, [**D**, Corollary 4.10] which asserts that $[x] \subseteq [\xi] \cdot [\eta] \cdot [\zeta]$ whenever x, ξ, η, ζ are permutations of a countable set, each moving infinitely many elements. □

PROPOSITION 6.5. Let $\nu = 0$ and let C be a class in H_0. Then $C \subseteq C^2$, and so by $C = C^{-1}$, $\mathbf{1} \subseteq C^3$.

PROOF. Let $C \neq \mathbf{1}$ be a class in H_0. Then by Proposition 6.0, $C = \xi S^0$, where $\bar{\xi} \in U$ or else $\bar{\xi}(1) < \aleph_0$ and $\Sigma_{1<n} n\bar{\xi}(n) = \aleph_0$. In the first case $C^2 \subseteq C$, as $P(\bar{\xi}, \bar{\xi}, \bar{\xi})$ holds by Proposition 6.2. In the second case we distinguish two cases.

Case 1. $\bar{\xi}(\aleph_0) > 0$. Then $P(\bar{\xi}, \bar{\xi}, \bar{\xi})$ holds by Proposition 6.3, and so again $C^2 \subseteq C$.

Case 2. $\bar{\xi}(\aleph_0) = 0$. We may assume $\bar{\xi}(2) \in \{0, \aleph_0\}$ (if $0 < \bar{\xi}(2) = m < \aleph_0$ replace ξ by ξ' with $\bar{\xi}'(1) = \bar{\xi}(1) + 2m$, $\bar{\xi}'(2) = 0$, $\bar{\xi}'(n) = \bar{\xi}(n)$ for $n > 2$). Then $P(\bar{\xi}, \bar{\xi}, \bar{\xi})$ follows from $P(2 \cdot 2^*, 2 \cdot 2^*, 2 \cdot 2^*)$ and Proposition 5.1. □

From Theorem 5 and Proposition 6.5 follows

THEOREM 6 [**ACM**]. *Let* $\nu = 0$. *Then* $\mathrm{cn}(H_0) = 3$.

The proof of Theorem 6 outlined in [**ACM**] reduces it to the following proposition, which is of independent interest.

PROPOSITION 6.6. *Let ξ be any permutation of a countable set moving infinitely many symbols. Then* $[\xi] \subseteq [\xi]^2$.

This proposition follows from the argument for Proposition 6.5 and the relation $P(\bar{\xi}, \bar{\xi}, \bar{\xi})$, where $\bar{\xi} = 2^* + (2m)^*$, proved in [**Hsu**].

In view of Theorems 1, 4, 5, a natural guess concerning the extended covering number of H_ν, $\nu > 0$, is that $\mathrm{ecn}(H_\nu) = 3$. In fact, a general conjecture of J. Stavi supports this guess. It states

Conjecture (*Stavi*). Let G be a finite simple group. Then $\mathrm{ecn}(G) = \mathrm{cn}(G) + 1$.

Unexpectedly, it turns out that this guess is false (and so the word "finite" may not be omitted in Stavi's conjecture). We leave to the reader the proof that if $\nu > 0$, $|B| = \aleph_\nu$, and if $\xi, \eta, \zeta \in S$ satisfy $\bar{\xi} = \aleph_\nu \cdot 3^*$, $\bar{\eta} = \aleph_\nu \cdot (1^* + 2^*)$, $\bar{\zeta} = \aleph_\nu \cdot 2^*$ then $[\xi S^\nu] \cap ([\eta S^\nu] \cdot [\zeta S^\nu]) = \varnothing$. A full discussion of the relation P on the classes of H_ν will appear elsewhere [**M1**].

ADDED IN PROOF. Recently S. Karni [**K**] found that $\mathrm{cn}(C_3) = 3$, $\mathrm{ecn}(C_3) = 5$ where C_3 is the sporadic finite group Conway 3.

REFERENCES

[**ACM**] Z. Arad, D. Chilag and G. Moran, *Groups with a small covering number*, Products of Conjugacy Classes in Groups (Z. Arad and M. Herzog, Eds.), Lecture Notes in Math., Springer, Berlin and New York (to appear).

[**AHSt**] Z. Arad, M. Herzog and J. Stavi, *Powers and products of conjugacy classes in groups*, Products of Conjugacy Classes in Groups (Z. Arad and M. Herzog, Eds.), Lecture Notes in Math., Springer, Berlin and New York (to appear).

[**D**] M. Droste, *Products of conjugacy classes of the infinite symmetric group*, Discrete Math. **47** (1983), 35–48.

[F] R. H. Fox, *On Fenchel's conjecture about F-groups*, Mat. Tidsskr. **B** (1952), 61–65.

[G] A. B. Gray, *Infinite symmetric and monomial groups*, Ph. D. Thesis, New Mexico State University, Las Cruces, 1960.

[Hu] C. L. Hsu, *The commutators of the alternating group*, Sci. Sinica Ser. **14** (1965), 339–342.

[K] S. Karni, *Covering numbers of small order and sporadic groups*, Products of Conjugacy Classes in Groups (Z. Arad and M. Herzog, Eds.), Lecture Notes in Math., Springer, Berlin and New York (to appear).

[Mi] G. A. Miller, *On the products of two substitutions*, Amer. J. Math. **22** (1900), 185–190.

[M] G. Moran, *Parity features for classes of the infinite symmetric group*, J. Combin. Theory Ser. A **33** (1982), 82–98.

[M1] ______, *Products of conjugacy classes in some infinite simple groups* (to appear).

[S] W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

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