

LOCAL OPERATORS AND DERIVATIONS ON C^* -ALGEBRAS

BY

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ABSTRACT. The variations on a theme of locality for a pair of operators (H, K) on a C^* -algebra \mathfrak{A} are expressed algebraically. If K is a $*$ -derivation generating an action of \mathbf{R} on \mathfrak{A} , and H is $*$ -linear and K -local, then, under certain restrictions, H is shown to be very closely related to K .

1. Introduction. A hierarchy of conditions on a pair of operators (δ, δ_0) on a C^* -algebra \mathfrak{A} has been introduced in [3, 4, 5], each representing a notion of locality of δ with respect to δ_0 . These conditions were applied to unbounded derivations and dissipations in order to identify circumstances in which C^* -dynamical systems are generated. The concept of locality was taken from [1], where $\mathfrak{A} = C(X)$ was assumed to be abelian, δ_0 was the generator of some flow on X , and δ was a derivation commuting with the flow. In this context, locality conditions ensured that δ acted along the orbits of the flow.

Both the premise and the object of [1] were somewhat different than those of [4] and a series of intermediate papers cited therein. In [1] locality conditions were regarded as an undesirable obstruction, and results were sought which depended on the weakest possible form of locality, even if this required closability or dissipativity as an extra condition. The results then described δ in terms of δ_0 . In [4], etc., stronger forms of locality were required, closability could sometimes be dropped from the assumptions, and the conclusion was that δ generates a C^* -dynamical system, which might be more or less explicitly described in terms of the action τ generated by δ_0 . Earlier it had been shown in [6] that, if τ is periodic, only weak forms of locality are needed to obtain similar conclusions.

Locality was defined by means of states. Although this is perfectly natural in many contexts, it may not give the deepest insight into the conditions. It will be shown in this paper that the locality conditions have elementary algebraic formulations. Thus, δ is completely strongly δ_0 -local if and only if there is a net x_α in \mathfrak{A} such that $\|x_\alpha \delta_0(a) - \delta(a)\| \rightarrow 0$ for all a in the joint domain \mathcal{D} of δ and δ_0 , and δ is (completely) weakly δ_0 -local if and only if, for each a in \mathfrak{A} , $\delta(a)$ is orthogonal to any ideal of \mathfrak{A} which is orthogonal to $\delta_0(a)$.

In [3, 5] a yet stronger form of locality, known as strict locality, was introduced, and it was shown that any strictly δ_0 -local operator δ on a suitable domain is of the

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form $L\delta_0$ for some central (unbounded) multiplier L on the Pedersen ideal of the ideal generated by the range of δ_0 , even if δ is not assumed to be either selfadjoint or a derivation. In the final part of this paper, some similar results are obtained for selfadjoint completely strongly δ_0 -local operators δ . The most explicit case is when τ is periodic and \mathfrak{A} is simple; then, either δ is a scalar multiple of δ_0 , or δ_0 is inner and implemented by a projection and δ is a linear combination of δ_0 and δ_0^2 .

2. Algebraic forms of locality. Let \mathfrak{A} be a C^* -algebra, $P(\mathfrak{A})$ the set of all pure states of \mathfrak{A} , \mathscr{D} a linear subspace of \mathfrak{A} , and H and K linear operators of \mathscr{D} into \mathfrak{A} . Then H is said to be *weakly K -local* if

$$a \in \mathscr{D}, U \text{ weak* open in } P(\mathfrak{A}), \omega(K(a)^*K(a)) = 0 \text{ for all } \omega \text{ in } U \Rightarrow \omega(H(a)^*H(a)) = 0 \text{ for all } \omega \text{ in } U.$$

H is said to be *strongly K -local* if

$$a \in \mathscr{D}, \omega \in P(\mathfrak{A}), \omega(K(a)^*K(a)) = 0 \Rightarrow \omega(H(a)^*H(a)) = 0.$$

H is said to be *strictly K -local* if

$$a \in \mathscr{D}, b \in \mathfrak{A}, \omega \in P(\mathfrak{A}), \omega(K(a)^*b^*bK(a)) = 0 \Rightarrow \omega(H(a)^*b^*bH(a)) = 0.$$

H is said to be *completely weakly (strongly, strictly) K -local* if, for each $n \geq 1$, H_n is weakly (strongly, strictly) K_n -local on the subspace $M_n(\mathscr{D})$ of the C^* -algebra $M_n(\mathfrak{A})$ of $n \times n$ matrices with coordinates in \mathfrak{A} , where

$$H_n[a_{ij}] = [H(a_{ij})] \quad (a_{ij} \in \mathscr{D}).$$

These properties were introduced in [4 and 3], where it was shown in particular that strict and complete strict locality are equivalent. It is shown below that weak and complete weak locality are equivalent. However strong locality does not imply complete strong locality, as is shown by the example $\mathfrak{A} = \mathscr{D} = M_2(\mathbb{C})$,

$$K[a_{ij}] = \begin{bmatrix} a_{11} & a_{21} + a_{12} \\ a_{21} + a_{12} & 0 \end{bmatrix}, \quad H[a_{ij}] = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

PROPOSITION 1. Let $H, K: \mathscr{D} \rightarrow \mathfrak{A}$ be linear maps. The following are equivalent:

- (i) H is weakly K -local.
- (ii) $a \in \mathscr{D}$, I (closed, two-sided) ideal in \mathfrak{A} , $K(a)I = (0) \Rightarrow H(a)I = (0)$.
- (iii) H is completely weakly K -local.

PROOF. (i) \Rightarrow (ii). Suppose (ii) is false, so there exist a in \mathscr{D} , an ideal I and b in I with $K(a)I = (0)$, $H(a)b \neq 0$. Let

$$U = P(I) = \{ \omega \in P(\mathfrak{A}) : \omega(I) \neq (0) \}.$$

Now U is an open subset of $P(\mathfrak{A})$, and, for ω in U and an approximate unit (u_λ) of I ,

$$\omega(K(a)^*K(a)) = \lim \omega(K(a)^*K(a)u_\lambda) = 0.$$

But there exists ω_0 in $P(I)$ with $\omega_0(b^*H(a)^*H(a)b) \neq 0$. The state $\omega_1 = \omega_0(b^*b)^{-1}\omega_0(b^* \cdot b)$ belongs to $P(I)$ and $\omega_1(H(a)^*H(a)) \neq 0$. Thus H is not weakly K -local.

(ii) \Rightarrow (i). Suppose H is not weakly K -local, so there exist a in \mathcal{D} , an open subset U of $P(\mathfrak{A})$, and ω_0 in U with

$$\omega(K(a)^*K(a)) = 0 \quad (\omega \in U), \quad \omega_0(H(a)^*H(a)) \neq 0.$$

Let $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ be the GNS Hilbert space, representation and cyclic vector associated with ω . For ω in U and η in \mathcal{H}_ω , the state

$$b \rightarrow \frac{\langle \pi_\omega(b)(\xi_\omega + \lambda\eta), \xi_\omega + \lambda\eta \rangle}{\|\xi_\omega + \lambda\eta\|^2}$$

belongs to U for sufficiently small $\lambda > 0$. Hence $\pi_\omega(K(a)) = 0$. Let

$$J = \{b \in \mathfrak{A} : \pi_\omega(b) = 0 \text{ for all } \omega \text{ in } U\},$$

$$I = \{b \in \mathfrak{A} : bJ = (0)\}.$$

Since U is open,

$$P(I) \supset U$$

[8, 3.4; 12, IV.4.15]. Now $K(a) \in J$, so $K(a)I = (0)$. But $\omega_0 \in P(I)$, $\omega_0(H(a)^*H(a)) \neq 0$, so $H(a)I \neq (0)$. Thus (ii) is false.

(ii) \Rightarrow (iii). Let $A = [a_{ij}] \in M_n(\mathcal{D})$, I be an ideal in $M_n(\mathfrak{A})$, and suppose $K_n(A)I = (0)$. There is an ideal J in \mathfrak{A} such that $I = M_n(J)$ and $K(a_{ij})J = (0)$ for each i and j . It follows from (ii) that $H(a_{ij})J = (0)$, so $H_n(A)I = (0)$. It now follows from the implication (ii) \Rightarrow (i) for (H_n, K_n) that H_n is weakly K_n -local.

(iii) \Rightarrow (i). This is clear.

COROLLARY 2. *If \mathfrak{A} is prime, then H is (completely) weakly K -local if and only if $a \in \mathcal{D}$, $K(a) = 0 \Rightarrow H(a) = 0$.*

PROPOSITION 3. *Let $H, K: \mathcal{D} \rightarrow \mathfrak{A}$ be linear maps.*

1. *H is strongly K -local if and only if, for each a in \mathcal{D} , $H(a)$ belongs to the closed left ideal of \mathfrak{A} generated by $K(a)$.*

2. *H is completely strongly K -local if and only if there is a net x_λ in \mathfrak{A} such that $\|x_\lambda K(a) - H(a)\| \rightarrow 0$ for all a in \mathcal{D} .*

PROOF. 1. By the general theory of C*-algebras [10, 3.10.7], the closed left ideal of \mathfrak{A} generated by $K(a)$ is

$$\{b \in \mathfrak{A} : \omega(b^*b) = 0 \text{ whenever } \omega \in P(\mathfrak{A}) \text{ and } \omega(K(a)^*K(a)) = 0\}.$$

Now the assertion follows immediately from the definition of strong locality.

2. Suppose first that H is completely strongly K -local, $a_1, \dots, a_n \in \mathcal{D}$, and $\varepsilon > 0$. Let $a_{1j} = a_j$, $a_{ij} = 0$ ($2 \leq i \leq n$; $1 \leq j \leq n$). By part 1, there exists $B = [x_{ij}] \in M_n(\mathfrak{A})$ such that $\|B[K(a_{ij})] - [H(a_{ij})]\| < \varepsilon$. Then $\|x_{11}K(a_j) - H(a_j)\| < \varepsilon$. This gives the required strong approximation.

Conversely, suppose that $\|x_\lambda K(a) - H(a)\| \rightarrow 0$ for all a in \mathcal{D} . Take $[a_{ij}] \in M_n(\mathcal{D})$ and $B_\lambda = x_\lambda \otimes I_n \in M_n(\mathfrak{A})$. Then $\|B_\lambda[K(a_{ij})] - [H(a_{ij})]\| \rightarrow 0$. It follows from part 1 that H_n is strongly K_n -local.

PROPOSITION 4. *Let $H, K: \mathcal{D} \rightarrow \mathfrak{A}$ be linear maps. The following are equivalent:*

- (i) *H is strictly K -local.*
- (ii) *For each a in \mathcal{D} and b in \mathfrak{A} , $bH(a)$ belongs to the closed left ideal of \mathfrak{A} generated by $bK(a)$.*
- (iii) *H is completely strictly K -local.*
- (iv) *There is a net x_λ in \mathfrak{A} such that $\|x_\lambda bK(a) - bH(a)\| \rightarrow 0$ for all a in \mathcal{D} and all b in \mathfrak{A} .*

PROOF. The proofs of (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (iv) are very similar to parts 1 and 2 of Proposition 3, respectively, while (i) \Leftrightarrow (iii) was established in [3].

Now suppose H is completely strongly K -local and π is a (nondegenerate) representation of \mathfrak{A} on \mathcal{H} . As in [4],

$$a_i \in \mathcal{D}, \xi_i \in \mathcal{H}, \sum_{i=1}^n \pi(K(a_i))\xi_i = 0 \Rightarrow \sum_{i=1}^n \pi(H(a_i))\xi_i = 0,$$

so there is a linear operator $L: \mathcal{D}(L) \rightarrow \mathcal{H}$ with domain $\mathcal{D}(L)$ spanned by $\pi(K(\mathcal{D}))\mathcal{H}$ such that

$$(*) \quad L(\pi(K(a))\xi) = \pi(H(a))\xi \quad (a \in \mathcal{D}, \xi \in \mathcal{H}).$$

Proposition 3 shows that

$$L\eta = \lim \pi(x_\lambda)\eta \quad (\eta \in \mathcal{D}(L)).$$

If e denotes the orthogonal projection of \mathcal{H} onto $\overline{\mathcal{D}(L)}$, then $e \in \pi(A)''$ and eL is affiliated with $e\pi(A)''e$.

Let f be the central support of e in $\pi(A)''$. If H is strictly K -local, then L extends to a densely-defined linear operator \tilde{L} on $f\mathcal{H}$, affiliated with the centre of $\pi(A)''f$, such that

$$(**) \quad \tilde{L}(\pi(bK(a))\xi) = \pi(bH(a))\xi \quad (a \in \mathcal{D}, b \in \mathfrak{A}, \xi \in \mathcal{H})$$

(see [3, 5]).

If π is the universal representation of \mathfrak{A} , and there is a linear operator L satisfying (*), then H is completely strongly K -local. If L has an extension affiliated with the centre of $\pi(\mathfrak{A})''f$, then H is strictly K -local.

These observations also indicate why $*$ -linear operators are rarely local with respect to each other. Suppose that H and K are both $*$ -linear and H is completely strongly K -local. For a in \mathcal{D} , ξ in $\mathcal{D}(L^*)$ and η in \mathcal{H} ,

$$\begin{aligned} \langle L\pi(K(a))\xi, \eta \rangle &= \langle \pi(H(a))\xi, \eta \rangle = \langle \xi, \pi(H(a^*))\eta \rangle \\ &= \langle \xi, L\pi(K(a^*))\eta \rangle = \langle \pi(K(a))L^*\xi, \eta \rangle. \end{aligned}$$

Thus

$$\pi(K(a))L^*\xi = L\pi(K(a))\xi.$$

This is a very strong condition if $K(\mathcal{D})$ and $\mathcal{D}(L^*)$ are both large.

Next suppose that \mathfrak{A} is unital, H is completely strongly K -local, and $K(a_0)$ is invertible for some a_0 in \mathcal{D} . In the notation of Proposition 3, $\|x_\lambda - x\| \rightarrow 0$, where $x = H(a_0)K(a_0)^{-1} \in \mathfrak{A}$. Hence $H(a) = xK(a)$ ($a \in \mathcal{D}$). In the notation used above, $e = 1$ and $L = \pi(x)$.

It is possible to replace the single operator K by a family \mathcal{K} of operators $K: \mathcal{D} \rightarrow \mathfrak{A}$ in the definitions and most of the results of this section. For example, define H to be *strongly \mathcal{K} -local* if

$$a \in \mathcal{D}, \omega \in P(\mathfrak{A}), \omega(K(a)^*K(a)) = 0 \text{ for all } K \text{ in } \mathcal{K} \Rightarrow \omega(H(a)^*H(a)) = 0,$$

and define H to be *completely strongly \mathcal{K} -local* if H_n is strongly \mathcal{K}_n -local for each $n \geq 1$, where $\mathcal{K}_n = \{K_n: K \in \mathcal{K}\}$. As in Proposition 3, one can show that H is completely strongly \mathcal{K} -local if and only if H is the strong limit of operators of the form $\sum_{i=1}^n x_i K_i$ ($x_i \in \mathfrak{A}$, $K_i \in \mathcal{K}$). The notion of strict \mathcal{K} -locality has been studied in [5] in the case when $\mathcal{K} = \{\delta, \delta^2\}$ for some derivation δ .

3. Local derivations. Suppose now that \mathcal{D} is a subalgebra of \mathfrak{A} and K is a derivation δ . If H is completely strongly δ -local, then the domain $\mathcal{D}(L)$ of the operator L of §2 is $\pi(\mathcal{D})$ -invariant (so $e = f$). Furthermore the following are equivalent:

- (i) H is strictly δ -local;
- (ii) H is a derivation;
- (iii) L is affiliated with the centre of $\pi(\mathfrak{A})''e$

(see [3, 5]). Also, if δ has an extension which generates an action τ of \mathbb{R} on \mathfrak{A} , and H is a $*$ -derivation commuting with τ , then H also extends to a generator [4, Theorem 2.1]. Our object here is to show that, under suitable conditions, any $*$ -linear completely strongly δ -local operator H must be very closely related to δ , even if H is not assumed to be a derivation or to commute with τ .

More precisely, $(\mathfrak{A}, \tau, \delta, \mathcal{D}, H)$ will be called a *completely strongly local system* if δ is the generator of an action τ of \mathbb{R} on \mathfrak{A} , \mathcal{D} is a selfadjoint core for δ , and $H: \mathcal{D} \rightarrow \mathfrak{A}$ is a completely strongly δ -local $*$ -linear operator. We shall consider an irreducible covariant representation (\mathcal{H}, π, u) of $(\mathfrak{A}, \mathbb{R}, \tau)$ and endeavour to show that either

(HM) there is a real number λ such that $\pi(H(a)) = \lambda\pi(\delta(a))$ ($a \in \mathcal{D}$), or

(HP) there is a projection p on \mathcal{H} and real numbers γ, λ_1 and λ_2 such that

$$\begin{aligned} \pi(\delta(a)) &= \gamma\delta_p(\pi(a)), \\ \pi(H(a)) &= \lambda_1\delta_p(\pi(a)) + \lambda_2\delta_p^2(\pi(a)) \quad (a \in \mathcal{D}), \end{aligned}$$

where

$$\delta_p(x) = i(px - xp).$$

A similar result has been obtained in [5, Theorem 3.1] under the assumptions that H is strictly $\{\delta, \delta^2\}$ -local, but not necessarily selfadjoint, and (\mathcal{H}, π) is irreducible, but not necessarily covariant.

The additional restrictions will be certain combinations of the following:

(CP) π is periodic, \mathcal{D} is the linear span of subspaces \mathcal{D}_n of the spectral spaces $\mathfrak{U}^\tau(n)$, $\mathcal{D}_0 = \mathfrak{U}^\tau$ and $\mathfrak{U}^\tau\mathcal{D}_n \subset \mathcal{D}_n$. (Here, \mathfrak{U}^τ is the fixed point algebra of τ , and

$$\mathfrak{U}^\tau(n) = \{a \in \mathfrak{A} : \tau_t(a) = \exp(2\pi int/T)a \ (t \in \mathbb{R})\},$$

where T is the period of τ .)

(CB) There exists x in the multiplier algebra $M(\mathfrak{A})$ of \mathfrak{A} such that $H(a) = x\delta(a)$ for all a in \mathcal{D} .

(CI) (\mathcal{H}, π) is irreducible.

(CC) \mathcal{D} is τ -invariant and $H\tau_t = \tau_t H$ for all t .

THEOREM 5. *Let $(\mathfrak{A}, \tau, \delta, \mathcal{D}, H)$ be a completely strongly local system, (\mathcal{H}, π, u) an irreducible covariant representation of $(\mathfrak{A}, \mathbf{R}, \tau)$, and suppose (CP) and (CI) are satisfied. Then either (HM) or (HP) holds.*

PROOF. The kernel of π is invariant under τ, δ and H (since H is strongly δ -local), and the induced system $(\pi(\mathfrak{A}), \tilde{\tau}, \tilde{\delta}, \pi(\mathcal{D}), \tilde{H})$ is completely strongly local and satisfies (CP). There is therefore no loss in identifying \mathfrak{A} with the C^* -algebra $\pi(\mathfrak{A})$. According to (CI), \mathfrak{A} is irreducible, but for the moment we do not assume this. There is also no loss in assuming that $T = 2\pi$ and $u_T = 1$. Let $L: \mathcal{D}(L) \rightarrow \mathcal{H}$ be the linear operator associated with H and δ , so that $\mathcal{D}(L)$ is spanned by $\{\delta(a)\xi: a \in \mathcal{D}, \xi \in \mathcal{H}\}$ and $L(\delta(a)\xi) = H(a)\xi$.

Let

$$\begin{aligned}\mathcal{H}_n &= \{ \xi \in \mathcal{H}: u_t \xi = e^{int} \xi \ (t \in \mathbf{R}) \}, \\ \mathcal{N} &= \{ n \in \mathbf{Z}: \mathcal{H}_n \neq (0) \},\end{aligned}$$

so that $\mathcal{H} = \bigoplus_{n \in \mathcal{N}} \mathcal{H}_n$.

If \mathcal{N} contains only one integer, then $\delta = 0$, and strong locality implies that $H = 0$. Thus, in the following, it will be assumed that \mathcal{N} contains at least two integers.

For each n in \mathcal{N} , choose a unit vector ξ_n in \mathcal{H}_n . Since (\mathcal{H}, π, u) is irreducible,

$$\mathcal{H} = [\mathfrak{A} u_{\mathbf{R}} \xi_n] = [\mathfrak{A} \xi_n].$$

Replacing a in \mathfrak{A} by its Fourier component in $\mathfrak{A}^\tau(k)$, it follows that

$$\mathcal{H}_{n+k} = [\mathfrak{A}^\tau(k) \xi_n] = [\mathcal{D}_k \xi_n].$$

In particular, $\mathcal{H}_n = [\mathfrak{A}^\tau \xi_n]$ for an arbitrary unit vector ξ_n in \mathcal{H}_n , so the restriction of \mathfrak{A}^τ to \mathcal{H}_n is irreducible.

Now let m and n be distinct integers in \mathcal{N} and $k = m - n$. It follows from the preceding paragraph that

$$\mathcal{H}_m = [\mathcal{D}_k \xi_n]$$

and, hence,

$$\mathcal{H}_m = \mathfrak{A}^\tau \mathcal{D}_k \xi_n \subset \mathcal{D}_k \xi_n$$

(using irreducibility of \mathfrak{A}^τ on \mathcal{H}_m , Kadison's Transitivity Theorem, and (CP)). Thus there exists a_1 in \mathcal{D}_k with $a_1 \xi_n = \xi_m$. Also by Kadison's Transitivity Theorem, there exists a_0 in \mathfrak{A}^τ with $a_0 \xi_n = \xi_n = a_0^* a_1^* \xi_m$. Let $a_2 = a_1 a_0 \in \mathcal{D}_k$. Then $a_2 \xi_n = \xi_m$, $a_2^* \xi_m = \xi_n$. In particular, $\xi_n = ik^{-1} \delta(a_2^*) \xi_m \in \mathcal{D}(L)$. Let $\mu_n = \langle L \xi_n, \xi_n \rangle$.

Now let ξ'_n be any unit vector in \mathcal{H}_n , and ξ'_r any vector in \mathcal{H}_r , where $r \neq n$, and assume (CI). By Kadison's Transitivity Theorem, there exists b in \mathfrak{A} with

$$b \xi_n = \xi'_n, \quad b^* \xi'_n = \xi_n, \quad b^* \xi'_r = 0.$$

Replacing b by its Fourier component in \mathfrak{U}^τ , it may be assumed that $b \in \mathfrak{U}^\tau$. Then $ba_2^* \in \mathcal{D}_{-k}$, and

$$\begin{aligned}\xi'_n &= ba_2^* \xi_m = ik^{-1} \delta(ba_2^*) \xi_m \in \mathcal{D}(L), \\ ik \langle \xi'_r, L\xi'_n \rangle &= \langle \xi'_r, H(ba_2^*) \xi_m \rangle = \langle H(a_2 b^*) \xi'_r, \xi_m \rangle = ik \langle La_2 b^* \xi'_r, \xi_m \rangle = 0.\end{aligned}$$

Thus $L\xi'_n \in \mathcal{H}_n$. Furthermore,

$$\begin{aligned}ik\bar{\mu}_n &= ik \langle \xi_n, L\xi'_n \rangle = \langle \xi_n, H(a_2^*) \xi_m \rangle = \langle H(a_2) \xi_n, \xi_m \rangle = ik \langle La_2 \xi_n, \xi_m \rangle \\ &= ik \langle L\xi_m, \xi_n \rangle = ik\mu_m = ik \langle La_2 b^* \xi'_n, \xi_m \rangle = \langle H(a_2 b^*) \xi'_n, \xi_m \rangle \\ &= \langle \xi'_n, H(ba_2^*) \xi_m \rangle = ik \langle \xi'_n, Lba_2^* \xi_m \rangle = ik \langle \xi'_n, L\xi'_n \rangle.\end{aligned}$$

This shows that

$$\langle L\xi'_n, \xi'_n \rangle = \mu_n = \bar{\mu}_n.$$

Since ξ'_n is an arbitrary unit vector in \mathcal{H}_n , $L\xi'_n = \mu_n \xi'_n$.

Since $\mu_n = \bar{\mu}_m$ for every distinct pair n, m in \mathcal{N} , there are only two possibilities:

(i) There is a real number λ such that $\mu_n = \lambda$ for all n . Then, for a in \mathcal{D}_k , ξ in \mathcal{H}_n , $H(a)\xi = L\delta(a)\xi = \lambda\delta(a)\xi$, so $H(a) = \lambda\delta(a)$.

(ii) There are integers $k \neq 0$ and n and real numbers λ_1 and λ_2 such that $\mathcal{N} = \{n, n+k\}$, $\mu_n = k^{-1}(\lambda_1 - i\lambda_2)$, $\mu_{n+k} = k^{-1}(\lambda_1 + i\lambda_2)$. Let p be the orthogonal projection of \mathcal{H} onto \mathcal{H}_{n+k} . Then, for a in \mathcal{D} ,

$$\delta(a) = ik(pa - ap) = k\delta_p(a), \quad \mathcal{D}(L) = \mathcal{H},$$

$$L = k^{-1}(\lambda_1 + i\lambda_2)p + k^{-1}(\lambda_1 - i\lambda_2)(1-p) = k^{-1}\lambda_1 + ik^{-1}\lambda_2(2p-1),$$

$$H(a) = L\delta(a) = \lambda_1\delta_p(a) - \lambda_2(2p-1)(pa - ap) = \lambda_1\delta_p(a) + \lambda_2\delta_p^2(a).$$

COROLLARY 6. *Let \mathfrak{A} be a simple C^* -algebra, δ the generator of a periodic action τ of \mathbf{R} on \mathfrak{A} , and $H: \mathcal{D} \rightarrow \mathfrak{A}$ a completely strongly δ -local $*$ -linear operator, where \mathcal{D} is as described in (CP). Then either*

- (i) *there is a real number λ such that $H = \lambda\delta|_{\mathcal{D}}$, or*
- (ii) *there exist a projection p in $M(\mathfrak{A})$ and real numbers γ , λ_1 and λ_2 such that*

$$\delta(a) = i\gamma(pa - ap) \quad (a \in \mathfrak{A}),$$

$$H(a) = \lambda_1\delta(a) + \lambda_2\delta^2(a) \quad (a \in \mathcal{D}).$$

PROOF. Suppose first that \mathfrak{A} is separable. Kishimoto [9, Theorem 2.1] has shown that \mathfrak{A} has a τ -invariant pure state. The corresponding GNS representation of \mathfrak{A} is irreducible and faithful, and the action τ is covariantly implemented. When \mathfrak{A} is considered in this representation, Theorem 5 shows that either (i) holds or (ii) holds for some projection p . But p implements a bounded, hence inner, derivation of \mathfrak{A} [11]. Since \mathfrak{A} is irreducible, $p \in M(\mathfrak{A})$.

If \mathfrak{A} is inseparable, any separable subspace of \mathfrak{A} is contained in a separable τ -invariant simple C^* -subalgebra \mathfrak{A}_0 (see [2]) which has a τ -invariant pure state ω_0 . Arguing by contradiction and using the GNS representation of a τ -invariant extension of ω_0 , the result may be recovered.

THEOREM 7. *Let $(\mathfrak{A}, \tau, \delta, \mathcal{D}, H)$ be a completely strongly local system, (\mathcal{H}, π, u) an irreducible covariant representation of $(\mathfrak{A}, \mathbf{R}, \tau)$, and suppose (CP) and (CC) are satisfied. Then either (HM) or (HP) holds.*

PROOF. The proof proceeds exactly as in Theorem 5, except that it is no longer possible to ensure that $b^*\xi'_r = 0$. However, this was only used to ensure that \mathcal{H}_n is L -invariant. Under condition (CC), $H(\mathcal{D}_{-k}) \subset \mathcal{D}_{-k}$, so $L\xi'_n = ik^{-1}H(ba_2^*)\xi_m \in \mathcal{H}_n$.

Note that if τ is periodic, $\mathcal{D}(H)$ is a dense τ -invariant subalgebra of \mathfrak{A} , and H is closed and commutes with τ , then one may take $\mathcal{D}_n = \mathfrak{A}^\tau(n) \cap \mathcal{D}(H)$, and thereby arrange that (CP), as well as (CC), is satisfied. Furthermore, \mathcal{D} is a core for δ and (by Theorem 7) for H .

Now consider the aperiodic case. Condition (CP) is replaced by (CB)—boundedness of the multiplier L . Although this appears to be a strong condition, it was observed in §2 that (CB) is automatically satisfied if $\delta(\mathcal{D})$ contains an invertible element.

THEOREM 8. *Let $(\mathfrak{A}, \tau, \delta, \mathcal{D}, H)$ be a completely strongly local system satisfying (CB) and (CC), and let (\mathcal{H}, π, u) be an irreducible covariant representation of $(\mathfrak{A}, \mathbf{R}, \tau)$. Then either (HP) or (HM) holds. If (HP) fails, then the projection p in (HM) belongs to $M(\pi(\mathfrak{A}))$.*

PROOF. As in Theorem 5, there is no loss in identifying \mathfrak{A} with $\pi(\mathfrak{A})$. The closed linear span \mathcal{H}_0 of $\delta(\mathcal{D})\mathcal{H}$ is invariant under both \mathfrak{A} and u . If $\mathcal{H}_0 = (0)$ then $\delta = 0$, and it follows from strong locality that $H = 0$. Otherwise, irreducibility implies that $\mathcal{H}_0 = \mathcal{H}$. Since H commutes with τ ,

$$u_t x \delta(a) = \tau_t(H(a))u_t = xu_t \delta(a) \quad (a \in \mathcal{D}),$$

so x commutes with u_t .

Since $x\delta(a) = \delta(a)x^*$ for all a in \mathcal{D} ,

$$x(\tau_t(a) - a) = \int_0^t x\delta(\tau_s(a)) ds = \int_0^t \delta(\tau_s(a))x^* ds = (\tau_t(a) - a)x^*.$$

Hence

$$\begin{aligned} u_t(xau_s - au_s x^*) &= (x\tau_t(a) - \tau_t(a)x^*)u_{s+t} \\ &= (xa - ax^*)u_{s+t} = (xau_s - au_s x^*)u_t. \end{aligned}$$

Thus $(xau_s - au_s x^*)$ commutes with u . But any bounded linear operator y on \mathcal{H} is a strong limit of linear combinations of operators of the form au_s , so $(xy - yx^*)$ commutes with u .

Let ξ and η be unit vectors in \mathcal{H} belonging to orthogonal spectral subspaces of u and take y to be the rank one operator $\xi \otimes \eta$. Then

$$0 = \langle (xy - yx^*)\xi, \eta \rangle = \langle x\eta, \eta \rangle - \langle \xi, x\xi \rangle.$$

As in Theorem 5, there are only two possibilities:

- (i) x is a scalar multiple λ of the identity operator. Then $H(a) = \lambda\delta(a)$ ($a \in \mathcal{D}$).
- (ii) The spectrum of u contains only two real numbers α and $\alpha + \gamma$ ($\gamma \neq 0$), and, if p is the orthogonal projection of \mathcal{H} onto the $(\alpha + \gamma)$ -spectral subspace, there are

real numbers λ_1 and λ_2 such that

$$x = \gamma^{-1}(\lambda_1 + i\lambda_2)p + \gamma^{-1}(\lambda_1 - i\lambda_2)(1 - p).$$

If $\lambda_2 = 0$, case (i) applies. Otherwise,

$$\begin{aligned} p &= i(2\lambda_2)^{-1}(\lambda_1 - \gamma x) \in M(\mathfrak{U}), \\ \delta(a) &= i\gamma(pa - ap) = \gamma\delta_p(a), \\ H(a) &= x\delta(a) = \lambda_1\delta_p(a) + \lambda_2\delta_p^2(a) \quad (a \in \mathscr{D}). \end{aligned}$$

THEOREM 9. *Let $(\mathfrak{U}, \tau, \delta, \mathscr{D}, H)$ be a completely strongly local system, (\mathscr{H}, π, u) a covariant representation of $(\mathfrak{U}, \mathbf{R}, \tau)$, and suppose (CB) and (CI) are satisfied. Then either (HM) or (HP) holds.*

PROOF. Since H is $*$ -linear and \mathscr{D} is a core for δ , $x\delta(a) = \delta(a)x^*$ for all a in \mathscr{D} and hence for all a in $\mathscr{D}(\delta)$. As in Theorem 8, $x(\tau_t(a) - a) = (\tau_t(a) - a)x^*$. So (identifying \mathfrak{U} with $\pi(\mathfrak{U})$),

$$x(u_t a u_t^* - a) = (u_t a u_t^* - a)x^*.$$

By irreducibility it follows that

$$x(u_t y u_t^* - y) = (u_t y u_t^* - y)x^*$$

for any bounded linear operator y .

Let E and F be disjoint compact subsets of \mathbf{R} , and let ξ and η be unit vectors in the spectral subspaces $\mathscr{H}^u(E)$ and $\mathscr{H}^u(F)$ respectively. Choose t so that $\{e^{iat} : \alpha \in E\} \cap \{e^{i\beta t} : \beta \in F\} = \emptyset$. Then $(1 - \langle u_t^* \eta, \eta \rangle u_t)|_{\mathscr{H}^u(E)}$ is invertible on $\mathscr{H}^u(E)$. Let ξ' be the vector in $\mathscr{H}^u(E)$ with $\xi' - \langle u_t^* \eta, \eta \rangle u_t \xi' = \xi$. Let y be the rank one operator $\eta \otimes \xi'$. Then

$$\begin{aligned} 0 &= \langle [x(y - u_t y u_t^*) + (u_t y u_t^* - y)x^*] \eta, \eta \rangle \\ &= \langle x(\xi' - \langle u_t^* \eta, \eta \rangle u_t \xi'), \eta \rangle = \langle x\xi, \eta \rangle. \end{aligned}$$

Thus x leaves invariant each spectral subspace $\mathscr{H}^u(E)$ and therefore commutes with u . The proof may now be completed as in Theorem 8.

It seems likely that a completely strongly local system will satisfy either (HM) or (HP) in any irreducible representation in which τ is covariantly implemented (condition (CI)), or in any irreducible covariant representation if H commutes with τ (condition (CC)), at least if some other weak conditions involving the nature of \mathscr{D} and/or closability of H are satisfied. Indeed, it is even conceivable that either (HM) or (HP) holds in each irreducible representation, irrespective of whether τ is covariantly implemented. If so, H is strictly $\{\delta, \delta^2\}$ -local, and it would follow from [5, Theorem 4.1] that $H = L\delta + M\delta^2$ for some (unbounded) central multipliers of the Pedersen ideal of the ideal of \mathfrak{U} generated by the range of δ . Here M would have to take on a very special form to ensure that H is completely strongly δ -local.

If \mathfrak{U} has a faithful irreducible representation in which τ is covariantly implemented, then Theorem 9 determines completely and explicitly the nature of completely strongly local systems satisfying (CB) (see Corollary 6). In general, it may not be possible to find such a representation, even if \mathfrak{U} is simple. However, there is a

representation of this type if \mathfrak{A} is \mathbf{R} -simple and $(\mathfrak{A}, \mathbf{R}, \tau)$ has a ground or ceiling state [7], or if \mathfrak{A} is simple and the strong Connes spectrum of τ is not \mathbf{R} [9].

The irreducible covariant representations of $(\mathfrak{A}, \mathbf{R}, \tau)$ correspond to the irreducible representations of the C^* -crossed product $\mathfrak{A} \times_{\tau, \mathbf{R}}$ [10, 7.6], and therefore their direct sum is faithful on \mathfrak{A} . Thus knowledge of the behaviour of a completely strongly local system in each irreducible covariant representation, as given by Theorems 7 and 8 for systems satisfying (CC) and either (CP) or (CB), may be regarded as determining the system completely.

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