GRADE SCHEMES AND GRADE FUNCTIONS

BY

STEPHEN McADAM

ABSTRACT. In recent years, two concepts similar to R-sequences have appeared, essential sequences and asymptotic sequences. This work explores the general nature of such sequences.

Introduction. In the last three years, two concepts similar to that of R-sequences have appeared, asymptotic sequences and essential sequences. Motivated by these examples, this paper will study the general nature of such sequences, which we will refer to as grade sequences. We will generally restrict our attention to a fixed Noetherian ring R and all of its localizations. For any ideal I_S in any localization R_S, we will assign to I_S a finite subset of $A(I_S)$ of spec R_S, the function A having certain nice properties to be specified in the next section. Foremost among these nice properties is the following. Suppose $x_1, \ldots, x_n$ are in I_S and $x_1 \notin A(OR_S)$, and for $i = 1, \ldots, n-1$, $x_{i+1} \notin \bigcup \{ P \in A((x_1, \ldots, x_i)R_S)$, and also suppose there is a prime $P_S \in A((x_1, \ldots, x_n)R_S)$ with $I_S \subseteq P_S$. Now suppose $y_1, \ldots, y_m$ in I_S behaves similarly. We will then insist that $n = m$. If so, A will be called a grade scheme on R, and to A we will associate a so-called grade function f defined by $f(I_S) = n$, n as above. The most well-known instance of such an A and f is, of course, $A(I_S) = \mathrm{Ass}(R_S/I_S)$, and $f(I_S)$ is the grade of I_S. As we will use the term 'grade' more generally, this famous special case will herein be called classical grade. In the next section, we will also give essential grade and asymptotic grade as examples of what we are studying. As mentioned earlier, we will usually just be looking at a grade scheme or grade function on a fixed ring R, and we will often custom build a grade function to exist on a given ring R. Certain grade functions, primarily classical, essential, and asymptotic grade, as well as the height function, automatically exist on all Noetherian rings, and we will refer to these as natural grade functions. The two-fold purpose of this paper is to study grade functions in the abstract, as well as to investigate which properties of the known natural grade functions are abstract properties, as opposed to being specific to the example at hand.

In §1, we make precise our definitions, and give the various natural examples. An important point made in our examples is that two different grade schemes can give rise to the same grade function. We give some useful technical lemmas concerning

that fact. §2 is the heart of the paper. It contains a characterization of grade functions which is used repeatedly in the rest of the paper, both for proving results and giving concrete examples. Starting with a grade function f on a Noetherian ring R, we also give a method of constructing a grade scheme A_f for f, called the canonical grade scheme for f. This canonical grade scheme is easy to work with, and plays a central role in what follows. In §3 we see how a grade function on R induces grade functions on certain homomorphic images of R, and conversely how grade functions on certain collections of homomorphic images of R induce a grade function on R. §4 studies passing between R and certain extension rings of R, notably faithfully flat extensions. In §5, we use the results of §§3 and 4 to attempt to find a new natural grade function. We develop a concept which gives a grade function on any local ring, and we suggest how it might be extended to all Noetherian rings. Next, let f be any of classical, essential, or asymptotic grade on a local ring (R, M). It is known that if I is an ideal and if $a \in M$, then $f(I, a) \leqslant f(I) + 1$. In §6 we study arbitrary grade functions on (R, M) which have this property, and show that if f is one such and $f(M) = $ height M, then $f(I) = $ height I for all ideals I. We also study the abstract version of the unmixedness theorem, which incorporates the known cases for classical, essential, and asymptotic grade. In §7, we consider grade sequences over ideal, and given an easy proof of a strong generalization of an inequality known to hold for essential and asymptotic cograde.

1. Definitions and examples. Throughout this paper, R will be a Noetherian ring and S will be an arbitrarily multiplicatively closed subset of R with $0 \notin S$. Thus R_S will represent an arbitrary localization of R. I_S will denote a (proper) ideal of R_S (here I is an ideal of R with $I \cap S = \varnothing$). We will be interested in a function $A(I_S)$ with $A(I_S) \subseteq$ spec R_S. (More formally, we should say A is a function defined on the set $\{I_S | I_S$ is some ideal in some localization R_S of $R\}$. However, for the most part we will eschew such formality.)

DEFINITION. The function A mentioned above is called a *proto-grade scheme on R* if (i) $A(I_S)$ is a finite nonempty subset of spec R_S, (ii) $P_S \in A(I_S)$ implies $I_S \subseteq P_S$ and (iii) if $P \in$ spec R and $P \cap S = \varnothing$, then $P \in A(I)$ if and only if $P_S \in A(I_S)$.

DEFINITION. Let A be a proto-grade scheme. The sequence of elements $x_1, \ldots, x_n$ in R_S is called an *avoiding sequence for A* if (i) $(x_1, \ldots, x_n)R_S \neq R_S$ and (ii) for $i = 1, \ldots, n$, $x_i \notin \bigcup\{P \in A((x_1, \ldots, x_{i-1})R_S)\}$.

REMARK. We will often give proofs which induct on the length of some sequence of elements $x_1, \ldots, x_n$. For these proofs, it will be convenient to agree that when $n = 0$, the (empty) sequence is automatically an avoiding sequence for A, and the ideal it generates is the zero ideal. In particular, in the previous definition, when $i = 1$ we have $\bigcup\{P \in A(0)\}$.

LEMMA 1.1. *Let A be a proto-grade scheme on R.*
 (i) *If P_S is a prime minimal over I_S, then $P_S \in A(I_S)$.*
 (ii) *If $x_1, \ldots, x_n$ is an avoiding sequence for A from R_S, then* height$(x_1, \ldots, x_n)R_S = n$.

PROOF. (i) As I_P is a proper ideal of R_P, by definition, $A(I_P)$ is nonempty. As every prime in $A(I_P)$ must contain I_P, we see that $A(I_P) = \{P_P\}$. Thus $P \in A(I)$, and so $P_S \in A(I_S)$.

(ii) This is easy by (i) and the principal ideal theorem.

Lemma 1.1 shows that if one builds an avoiding sequence for A by successively adding elements on to the end, the process must stop. This motivates our next definition.

DEFINITION. Let A be a proto-grade scheme on R, and let the elements $x_1, \ldots, x_n$ in I_S be an avoiding sequence for A. Suppose that for every $x \in I_S$, $x_1, \ldots, x_n$, x is not an avoiding sequence for A. Then $x_1, \ldots, x_n$ will be called *a maximal avoiding sequence for A in I_S.*

DEFINITION. Let A be a proto-grade scheme on R. Suppose for any I_S, all maximal avoiding sequences for A in I_S have the same length. Then A will be called a *grade scheme on R*, and avoiding sequences for A will be called *A-sequences*.

DEFINITION. Let A be a grade scheme on R. Let $f(I_S)$ be the length of a maximal A-sequence in I_S. The function f will be called *the grade function of A*. (It is only defined on proper ideals.)

DEFINITION. A function f defined on the set of ideals I_S will be called a *grade function on R*, if it is the grade function of some grade scheme on R.

LEMMA 1.2. *Let A be a grade scheme on R with grade function f.*

(i) *If $I_S \subseteq J_S$ are ideals of R_S, $f(I_S) \leqslant f(J_S)$.*

(ii) *If $x_1, \ldots, x_n$ is an A-sequence in R, and if $(x_1, \ldots, x_n)R_S \neq R_S$, then $x_1, \ldots, x_n$ is an A-sequence in R_S. (Here, we should refer to the images of $x_1, \ldots, x_n$ in R_S, but this is another formality we will ignore.)*

(iii) *If $I_S \neq R_S$, then $f(I) \leqslant f(I_S)$.*

(iv) *Let $x_1, \ldots, x_n$ be an A-sequence in R, and let $(x_1, \ldots, x_n)R \subseteq P \in \operatorname{spec} R$. Then $P \in A((x_1, \ldots, x_n)R)$ if and only if $f(P_P) = n$.*

(v) $f(I) = \min\{f(P_P) | I \subseteq P \in \operatorname{spec} R\}$.

(vi) $f(I_S) \leqslant \operatorname{height} I_S$. *In particular, for $P \in \operatorname{spec} R$, $f(P_P) \leqslant \operatorname{height} P$.*

(vii) $f(OR_S) = 0$.

PROOF. (i), (ii), and (iii) are straightforward. For (iv), if $P \in A((x_1, \ldots, x_n)R)$, then $P_P \in A((x_1, \ldots, x_n)R_P)$ so that $f(P_P) = n$, using (ii). Conversely, if $f(P_P) = n$, then by (ii) we see that $x_1, \ldots, x_n$ is a maximal A-sequence in P_P. Thus $P_P \in A((x_1, \ldots, x_n)R_P)$, so that $P \in A((x_1, \ldots, x_n)R)$. For (v), if $I \subseteq P \in \operatorname{spec} R$, then by (i) and (iii), $f(I) \leqslant f(P) \leqslant f(P_P)$. Now let $x_1, \ldots, x_n$ be a maximal A-sequence in I. By the prime avoidance lemma, we must have some $P \in A((x_1, \ldots, x_n)R)$ with $I \subseteq P$. By (iv), $f(I) = n = f(P_P)$. Now (vi) follows from Lemma 1.1(ii), and implies (vii).

We now give numerous examples. We will not prove every statement we make about these examples, but will give enough references that the interested reader may pursue matters. Here we define $A(I)$, the definition extending to $A(I_S)$ in the natural way.

EXAMPLE 1. $A_1(I) = \text{Ass}(R/I)$. The A_1-sequences are R-sequences and the grade function is classical grade.

Recall that for I an ideal, the sequence $\text{Ass}(R/I^n)$, $n = 1, 2, 3, \ldots$, stabilizes for large n. We denote that stable set by $A^*(I)$ (see [5, Chapter 1]).

EXAMPLE 2. $A_2(I) = A^*(I)$. Now in general, neither $\text{Ass}(R/I) \subseteq A^*(I)$ nor $A^*(I) \subseteq \text{Ass}(R/I)$. That is $A_1(I) \not\subseteq A_1(I)$ and $A_2(I) \not\subseteq A_1(I)$. However if I is generated by an R-sequence, then $A_1(I) = A_2(I)$ [1, Exercise 13, p. 103]. From this, one easily sees that the A_2-sequences and the grade function of A_2 are identical to those of A_1.

Ideals I and J are called projectively equivalent, denoted $I \sim J$, if for some positive integers n and m we have $\overline{I^n} = \overline{J^m}$, where the bar means take the integral closure of the ideal.

EXAMPLE 3. $A_3(I) = \bigcap A^*(J)$ over all $J \sim I$. This example and the next are discussed in [2 and 6]. The A_3-sequences are called essential sequences, the grade function is called essential grade.

EXAMPLE 4. $A_4(I) = \{P \in \text{spec } R | I \subseteq P,$ there is a $q \in \text{Ass}(R_P)^*$ with $(P_P)^*$ minimal over $I(R_P)^* + q\}$. Here, $(R_P)^*$ is the completion of R_P. In general, $A_4(I) \subseteq A_3(I)$ (this following easily from [6, Corollary 2]), but $A_3(I) \not\subseteq A_4(I)$. However, for I an ideal generated by an essential sequence, $A_3(I) = A_4(I)$. Thus the A_4-sequences and the grade function for A_4 are the same as for A_3.

For I an ideal, and $\overline{I}$ its integral closure, the sequence $\text{Ass}(R/\overline{I^n})$, $n = 1, 2, 3, \ldots$, stabilizes for large n to a set denoted $\overline{A}^*(I)$ (see [5, Chapter 3]). It is known that $\overline{A}^*(I) \subseteq A^*(I)$, the inclusion often being proper.

EXAMPLE 5. $A_5(I) = \overline{A}^*(I)$. The A_5-sequences are called asymptotic sequences. The grade function is called asymptotic grade (see [5, Chapter 5]).

EXAMPLE 6. $A_6(I) = \{P \in \text{spec } R | I \subseteq P,$ there is a minimal prime q of $(R_P)^*$ with $(P_P)^*$ minimal over $I(R_P)^* + q\}$. In general $A_6(I) \subseteq A_5(I)$ [5, Proposition 3.18] but $A_5(I) \not\subseteq A_6(I)$. The grade function is again asymptotic grade.

EXAMPLE 7. $A_7(I) = \{P \in \text{spec } R | P \text{ is minimal over } I\}$. The grade function is easily seen to be $f(I) = \text{height } I$.

REMARKS. (a) By Lemma 1.1(i), $A_7(I) \subseteq A_6(I)$. Clearly $A_6(I) \subseteq A_4(I)$, and [6, Proposition 3.3.1] shows $A_3(I) \subseteq A_2(I)$. Thus if f_n is the grade function of example n, then $f_1 = f_2 \leq f_3 = f_4 \leq f_5 = f_6 \leq f_7$. In general, none of the inequalities are equalities.

(b) Lemma 1.2(vi) shows $f(I) \leq \text{height } I$. If f is a natural grade function (i.e. existing on all Noetherian rings) we can ask for which rings will $f(I) = \text{height } I$ for all ideals I. For $f_1 = f_2$ the answer, of course, is Cohen-Macaulay rings. For $f_3 = f_4$ it is locally unmixed rings (the local ring (R, M) is unmixed if depth $q = \dim R$ for all $q \in \text{Ass } R^*$). For $f_5 = f_6$ the answer is locally quasi-unmixed rings $((R, M)$ is quasi-unmixed if depth $q = \dim R$ for all minimal primes q of R^*). This is discussed further in §6.

We saw in the examples that two different grade schemes can have the same grade function. As this will be important to us, we will explore it in the next two lemmas.

First, however, we will make a remark which will often save us the trouble of worrying about localizations of R.

REMARK. If A is a grade scheme on R, its domain is the set of all ideals in all localizations of R. If R_S is a localization of R, then A can be restricted to all ideals in all localizations of R_S, and the result is easily seen to be a grade scheme on R_S (called A *restricted to* R_S). Similar statments hold for proto-grade schemes and grade functions. In many instances we will need to prove results involving arbitrary R_S, but since we will often have hypotheses which restrict to R_S, we will just deal with R itself. This is a notational convenience which we will use frequently (usually tacitly). The proof of (i) $\Rightarrow$ (ii) of the next lemma and the proof of Lemma 1.4 are our first applications of this remark.

LEMMA 1.3. *Let A and B be two grade schemes on R. The following are equivalent.*

(i) *The grade functions of A and B are equal.*

(ii) $x_1, \ldots, x_n$ *is an A-sequence if and only if it is a B-sequence.*

PROOF. (ii) $\Rightarrow$ (i) is obvious. Assume now that (i) holds but (ii) fails, letting $x_1, \ldots, x_n$ be a minimal counterexample. Without loss, assume this is an A-sequence but not a B-sequence. Since $x_1, \ldots, x_{n-1}$ is an A-sequence, by minimality it is also a B-sequence. Therefore we must have some $P \in B(x_1, \ldots, x_{n-1})$ with $x_n \in P$. By Lemma 1.2(iv), if f is the common grade function of A and B, then we have both $f(P_P) = n - 1$ and $P \in A(x_1, \ldots, x_{n-1})$. This contradicts that $x_1, \ldots, x_n$ is an A-sequence.

REMARK. The reader will note that in the preceding proof we introduced the convention that if $x_1, \ldots, x_n$ is a sequence in R, $A(x_1, \ldots, x_n)$ will be used to denote $A((x_1, \ldots, x_n)R)$. When doubt exists as to which ring $x_1, \ldots, x_n$ comes from, we will use the more formal notation.

According to Lemma 1.3, the next definition is unambiguous.

DEFINITION. Let f be a grade function on R. Then $x_1, \ldots, x_n$ is an *f-sequence* if it is an A-sequence for any grade scheme A on R whose grade function is f.

LEMMA 1.4. *Let A be a grade scheme on R whose grade function is f. Let B be a proto-grade scheme on R. Then the following are equivalent.*

(i) *B is a grade scheme whose grade function is f.*

(ii) *If $x_1, \ldots, x_n$ is an f-sequence, then $A(x_1, \ldots, x_n) = B(x_1, \ldots, x_n)$.*

PROOF. (i) $\Rightarrow$ (ii). Assume (i) holds, and let $x_1, \ldots, x_n$ be an f-sequence. Thus it is both an A-sequence and a B-sequence. If P is a prime containing $(x_1, \ldots, x_n)$, then Lemma 1.2(iv) shows $P \in A(x_1, \ldots, x_n)$ if and only if $P \in B(x_1, \ldots, x_n)$ (if and only if $f(P_P) = n$).

(ii) $\Rightarrow$ (i). Assuming (ii), we will show that $x_1, \ldots, x_n$ is an avoiding sequence for B if and only if it is an A-sequence. The truth of (i) is then trivial. Since $A(0) = B(0)$ (the case $n = 0$ of (ii)), our statement is true for x_1. Assume inductively that it is true for sequences of length $n - 1$. Let $x_1, \ldots, x_n$ be either an avoiding sequence for B or an A-sequence. The same can be said for $x_1, \ldots, x_{n-1}$, which by induction is therefore an A-sequence. By (ii), $A(x_1, \ldots, x_{n-1}) = B(x_1, \ldots, x_{n-1})$. By the nature of

$x_1, \ldots, x_n$, we see that x_n is not in any prime contained in $A(x_1, \ldots, x_{n-1}) = B(x_1, \ldots, x_{n-1})$, and so $x_1, \ldots, x_n$ is both an avoiding sequence for B and an A-sequence. This completes the proof.

DEFINITION. If A and B are grade schemes on R, write $A \subseteq B$ if $A(I_S) \subseteq B(I_S)$ for every ideal I_S in every localization R_S. (Of course it is enough to have $A(I) \subseteq B(I)$ for each ideal I of R.)

We next show that any grade function has a minimal grade scheme associated with it.

DEFINITION. By $v(I)$ we will mean the smallest number of elements in a generating set for the ideal I. (Note that $f(I) \leqslant$ height $I \leqslant v(I)$.)

LEMMA 1.5. *Let f be a grade function on R. If $f(I) = v(I)$, then I can be generated by an f-sequence (necessarily of length $f(I)$).*

PROOF. This is similar to the proof of [**1**, Theorem 125].

THEOREM 1.6. *Let f be a grade function on R. Let $A_f^m(I) = \{ P \in \operatorname{spec} R \mid I \subseteq P$ and either P is minimal over I or $f(P_P) = f(I_P) = v(I_P) \}$. Define $A_f^m(I_S)$ analogously. Then A_f^m is a grade scheme for f, and if A is any grade scheme for f, then $A_f^m \subseteq A$.*

PROOF. Let A be any grade scheme for f. Suppose $P \in A_f^m(I)$. If P is minimal over I, then $P \in A(I)$. Otherwise, $f(P_P) = f(I_P) = v(I_P)$, and by Lemma 1.5, there is an f-sequence $x_1, \ldots, x_n$ in R_P with $I_P = (x_1, \ldots, x_n)R_P$. Since $f(P_P) = f(I_P) = n$, $P_P \in A((x_1, \ldots, x_n)R_P) = A(I_P)$, and so $P \in A(I)$. Therefore $A_f^m \subseteq A$.

We now see that $A_f^m(I)$ is finite, since it is contained in $A(I)$. We easily see that A_f^m is a proto-grade scheme, and we will apply Lemma 1.4 to it and A. Letting $y_1, \ldots, y_m$ be an f-sequence, we already have $A_f^m(y_1, \ldots, y_m) \subseteq A(y_1, \ldots, y_m)$. Conversely, if $P \in A(y_1, \ldots, y_m)$, then

$$f((y_1, \ldots, y_m)R_P) = f(P_P) = m \geqslant v((y_1, \ldots, y_m)R_P) \geqslant f((y_1, \ldots, y_m)R_P).$$

Equality holds throughout, showing $P \in A_f^m(y_1, \ldots, y_m)$, and completing our proof.

LEMMA 1.7. *Let f be a grade function on R, and let I and J be ideals. Then $f(I \cap J) = f(IJ) = \min\{ f(I), f(J) \}$.*

PROOF. $f(I \cap J) = f(IJ)$ by Lemma 1.2(v), since $I \cap J$ and IJ have the same radical. Now let $IJ \subseteq P \in \operatorname{spec} R$ with $f(IJ) = f(P_P)$. Without loss, assume $I \subseteq P$. Then $f(I) \leqslant f(P_P) = f(IJ) \leqslant f(I)$, the last step since $IJ \subseteq I$. Thus $f(I) = f(IJ)$. As $f(IJ) \leqslant f(J)$, we are done.

We close this section by noting that if $A(I) = \operatorname{Ass}(R/\bar{I})$, then A is not a grade scheme, in general. [**5**, pp. 40–41] contains an example of a local ring in which two different maximal avoiding sequences for A have different lengths.

2. Characterizing grade functions.

We now turn to a more subtle fact about grade functions. It will be a key property in our characterization of grade functions, for it captures the requirement in a grade scheme, $A(I)$ is finite.

DEFINITION. Let $Q \in \operatorname{spec} R$, and let U be an infinite subset of $\operatorname{spec} R$ such that every $P \in U$ properly contains Q. If for any infinite subset U' of U we have $\bigcap \{ P \in U' \} = Q$, then (Q, U) is called a conforming pair.

LEMMA 2.1. *Let I be an ideal of R, and let U_0 be an infinite subset of* spec R *such that $I \subseteq P$ for all $P \in U_0$. Then there is a conforming pair (Q, U) with $I \subseteq Q$ and $U \subseteq U_0$.*

PROOF. Expand I to an ideal Q maximal with respect to being properly contained in infinitely many primes of U_0. It is straightforward to see that Q is prime. If $U = \{P \in U_0 | Q \subsetneq P\}$, it is straightforward to see that (Q, U) is a conforming pair.

THEOREM 2.2. *Let f be a grade function on R, and let (Q, U) be a conforming pair in R. Suppose $f(P_P) \leqslant n$ for all $P \in U$. Then $f(Q_Q) \leqslant n - 1$.*

PROOF. If not, let $x_1, \ldots, x_n$ be elements in R whose images in R_Q form an f-sequence in Q_Q. Let A be a grade scheme for f, and let

$$W = \bigcup \{ p \in A(x_1, \ldots, x_{i-1}) | x_i \in p \},$$

the union over $i = 1, \ldots, n$. If $p \in W$, we must have $p \not\subseteq Q$ since otherwise $x_1, \ldots, x_n$ would not be an f-sequence in Q_Q. As (Q, U) is a conforming pair, if $p \in W$, then at most finitely many $P \in U$ can contain p. Since W is finite, by deleting finitely many P from U we may assume that $p \not\subseteq P$ whenever $p \in W$ and $P \in U$. We now see that the images of $x_1, \ldots, x_n$ form an f-sequence in R_P for all $P \in U$. Since $f(P_P) \leqslant n$, we have $P_P \in A((x_1, \ldots, x_n)R_P)$, so that $P \in A(x_1, \ldots, x_n)$ for all $P \in U$. This is impossible, since U is infinite.

REMARK. Using [7, Example 2, pp. 202–205] it is not hard to produce a Noetherian ring R with a conforming pair (Q, U) such that little height $Q =$ little height $P = 2$ for all $P \in U$. (Little height P is the length of the shortest saturated chain of primes from P down to a minimal prime.) Thus Theorem 2.2 shows that, in general, little height is not a grade function.

The next corollary combines three results which have appeared in three different places.

COROLLARY 2.3. *Let f be a grade function on R and let $Q \in$ spec R with $f(Q_Q) =$ height Q. Let $W = \{P \in$ spec $R | Q \subset P$ and height $P/Q = 1\}$. Then for all but finitely many $P \in W$, $f(P_P) =$ height P. In particular if R_Q is Cohen-Macaulay (respectively, unmixed) (respectively, quasi-unmixed) then the same is true of R_P for all but finitely many $P \in W$.*

PROOF. If false, then W contains an infinite subset U with $f(P_P) <$ height P for all $P \in U$. By [4, Theorem 1] height $P =$ height $Q + 1$ for all but finitely many $P \in U$, and deleting those finitely many does no harm. Clearly (Q, U) is now a conforming pair with $f(P_P) <$ height $P =$ height $Q + 1 = f(Q_Q) + 1$ for all $P \in U$. By Theorem 2.2, $f(Q_Q) < f(Q_Q)$, a contradiction. The three special cases follow by letting f be, respectively, classical, essential, and asymptotic grade.

We now come to our main result.

THEOREM 2.4. *Let f be a nonnegative integer valued function defined on the set $\{I_S | I_S$ is a proper ideal in some localization R_S of $R\}$. Then f is a grade function if and only if it satisfies the following three conditions.*

(i) $f(I_S) = \min\{f(P_P) | I_S \subseteq P_S \in$ spec $R_S\}$.

(ii) $f(P_P) \leqslant$ height P *for all* $P \in$ spec R.

(iii) *If* (Q, U) *is a conforming pair in* R, *and if* $f(P_P) \leqslant n$ *for all* $P \in U$, *then* $f(Q_Q) \leqslant n - 1$.

PROOF. One direction is by Lemma 1.2(v) and (vi) and Theorem 2.2. Thus assume f satisfies the given conditions. We seek a grade scheme A_f whose grade function is f. We define $A_f(I_S)$ to be $\{ P_S \in$ spec $R_S | I_S \subseteq P_S$ and $f(P_P) = f(I_P)\}$. Before proceeding, we note that condition (i) implies that if $I_S \subseteq J_S$ are ideals, then $f(I_S) \leqslant f(J_S)$ and $f(I) \leqslant f(I_S)$. We will use these inequalities repeatedly.

By condition (i), find a $P \in$ spec R with $I \subseteq P$ such that $f(I) = f(P_P)$. Then $f(I) \leqslant f(I_P) \leqslant f(P_P) = f(I)$, so that $f(I_P) = f(P_P)$. Thus $P \in A_f(I)$ which is therefore nonempty. We also claim $A_f(I)$ is finite, since if not, by Lemma 2.1 we could find a conforming pair (Q, U) with $I \subseteq Q$ and $U \subseteq A_f(I)$. For all $P \in U$, we would have $f(P_P) = f(I_P)$ By condition (i), $f(I_P) \leqslant f(Q_Q)$. Thus $f(P_P) \leqslant f(Q_Q)$ for all $P \in U$, and condition (iii) gives $f(Q_Q) \leqslant f(Q_Q) - 1$, a contradiction. Therefore $A_f(I)$ is a finite nonempty set, and by the remark preceding Lemma 1.3, so is $A_f(I_S)$. Also, it is clear from the definition that $P_S \in A_f(I_S)$ implies $I_S \subseteq P_S$, and if $P \in$ spec R with $P \cap S = \varnothing$, then $P \in A_f(I)$ if and only if $P_S \in A_f(I_S)$. Therefore, we have that A_f is a proto-grade scheme on R.

In order to complete our proof, we will prove the following two statements.

(a) If $x_1, \ldots, x_n$ is an avoiding sequence for A_f, and if $P \in A_f(x_1, \ldots, x_n)$, then $f(P_P) = n$.

(b) If $f(I) = n$, then any maximal avoiding sequence for A_f in I has length n.

Obviously (b) implies that A_f is a grade scheme for f, as desired. To prove (b), we will simultaneously treat (a) and (b) by inducting on n. For $n = 0$, suppose $P \in A_f(0)$. Then $f(P_P) = f(OR_P)$. However, applying conditions (i) and (ii) to any minimal prime of R_P shows that $f(OR_P) = 0$, so (a) holds when $n = 0$. As for (b), suppose $f(I) = 0$. By condition (i), there is an $I \subseteq P \in$ spec R with $f(P_P) = 0 = f(OR_P)$, so that $P \in A_f(O)$, clearly showing (b) holds.

Inductively assume both (a) and (b) hold for arbitrary $0 \leqslant m < n$. We will show they hold for n. For (a), let $x_1, \ldots, x_n$ be an avoiding sequence for A_f, and let $P \in A_f(x_1, \ldots, x_n)$. If $f(P_P) < n$, then $f(P) \leqslant f(P_P) < n$, and (b) is inductively violated. Thus $f(P_P) \geqslant n$. As $P \in A_f(x_1, \ldots, x_n)$, $f(P_P) = f((x_1, \ldots, x_n)R_P)$. Let P_P be a prime minimal over $(x_1, \ldots, x_n)R_P$. Then $f(P_P) = f((x_1, \ldots, x_n)R_P) \leqslant f(p_P) \leqslant f(p_p) \leqslant$ height $p \leqslant n$, using condition (ii) and the principal ideal theorem. Thus $f(P_P) = n$ and (a) is inductively proved.

For (b), let $f(I) = n$ and let $x_1, \ldots, x_k$ be an avoiding sequence for A_f in I. If $k < n$, then for any $P \in A_f(x_1, \ldots, x_k)$, statement (a) shows $f(P_P) = k < n = f(I)$, and so condition (i) shows $I \not\subseteq P$. Since $A_f(x_1, \ldots, x_k)$ is finite, we can find an $x_{k+1} \in I$ with $x_1, \ldots, x_k, x_{k+1}$ an avoiding sequence for A_f.

Now let $y_1, \ldots, y_l$ be a maximal avoiding sequence for A_f in I. We want $l = n$, and by the preceding paragraph have $l \geqslant n$. By condition (i), there is an $I \subseteq P \in$ spec R with $f(P_P) = f(I) = n$. Since $l \geqslant n$ we may consider $y_1, \ldots, y_n$, which is an avoiding sequence for A_f. To prevent statement (b) from being inductively violated, we must

571 GRADE SCHEMES AND GRADE FUNCTIONS

have $f(y_1,\ldots,y_n) \geqslant n$. Therefore $n \leqslant f((y_1,\ldots,y_n)R_P) \leqslant f(P_P) = n$. This shows that $f(P_P) = f((y_1,\ldots,y_n)R_P)$, so that $P \in A_f(y_1,\ldots,y_n)$. Since $I \subseteq P$, we now see that $y_1,\ldots,y_n$ is a maximal avoiding sequence for A_f in I. Thus $l = n$ as desired. This proves (b), and the theorem.

We examine the grade scheme constructed in Theorem 2.4 a bit more closely, first naming it, and then showing some of its nice properties.

DEFINITION. Let f be a grade function on R. For I_S an ideal in some localization R_S of R, let $A_f(I_S) = \{P_S \in \operatorname{spec} R_S | I_S \subseteq P_S \text{ and } f(I_P) = f(P_P)\}$. A_f is called *the canonical grade scheme of f.*

REMARK. It is easily seen that $A_f(I) = \{P \in \operatorname{spec} R | \text{there is an ideal } J \text{ with } J \subseteq I \subseteq P \in A_f^m(J)\}$.

LEMMA 2.5. *Let A_f be the canonical grade scheme of the grade function f on R. Let $I \subseteq J \subseteq P \subseteq Q$ and K be ideals of R with P and Q prime.*

(a) *If $P \in A_f(I)$, then $P \in A_f(J)$.*

(b) *If $f(I) = f(P_P)$, then $Q \in A_f(I)$ if and only if $Q \in A_f(J)$, in which case $f(I) = f(Q_Q)$.*

(c) *If $f(I) = f(P_P)$, then P is maximal in $A_f(I)$ if and only if P is maximal in $A_f(J)$.*

(d) *The hypothesis $f(I) = f(P_P)$ in (b) and (c) hold if I is generated by an f-sequence and $P \in A_f(I)$.*

(e) *$P \in A_f(IK)$ if and only if either (i) $K \nsubseteq P$ and $P \in A_f(I)$ or (ii) $K \subseteq P$ and $P \in A_f(I) \cap A_f(K)$.*

PROOF. (a) $f(P_P) \geqslant f(J_P) \geqslant f(I_P) = f(P_P)$, the equality since $P \in A_f(I)$. Thus equality holds throughout, and $P \in A_f(J)$.

(b) If $Q \in A_f(I)$, then $Q \in A_f(J)$ by (a). If $Q \in A_f(J)$, then $f(Q_Q) = f(J_Q) \leqslant f(J_P) \leqslant f(P_P) = f(I) \leqslant f(I_Q) \leqslant f(Q_Q)$, which shows $Q \in A_f(I)$ and also $f(I) = f(Q_Q)$.

(c) This follows easily from (a) and (b) since $f(I) = f(P_P)$ gives $P \in A_f(I)$.

(d) This is easy.

(e) If (i) holds, then $(IK)_P = I_P$, and since $P \in A_f(I)$ we have $P_P \in A_f(I_P) = A_f((IK)_P)$. Thus $P \in A_f(IK)$. If (ii) holds, then $f(I_P) = f(P_P) = f(K_P)$. Lemma 1.6 now shows $P \in A_f(IK)$. Conversely, suppose $P \in A_f(IK)$. If $K \nsubseteq P$, then $P_P \in A_f((IK)_P) = A_f(I_P)$, so $P \in A_f(I)$. If $K \subseteq P$, then since $IK \subseteq K \subseteq P$, part (a) shows $P \in A_f(K)$. Similarly $P \in A_f(I)$.

We will now identify the canonical grade scheme A_f when f is asymptotic grade. We will use the following definition.

DEFINITION. For P prime in a Noetherian ring R,

$$z(P) = \min\{\operatorname{depth} q | q \text{ is a minimal prime in } (R_P)^*\},$$

with $(R_P)^*$ the completion of R_P.

THEOREM 2.6. *Let f be asymptotic grade on R. Let $I \subseteq P \in \operatorname{spec} R$. Then $P \in A_f(I)$ if and only if for each minimal prime q of $(R_P)^*$, $\operatorname{height}(I(R_P)^* + q/q) \geqslant z(P)$. In particular, $A_f \subseteq A_6$.*

PROOF. By [5, Proposition 6.10] we have both $f(P_P) = z(P)$ and $f(I_P) = \min\{\text{height}(I(R_P)^* + q/q)|q$ is a minimal prime in $(R_P)^*\}$. Noting that we always have $f(I_P) \le f(P_P)$ and that equality holds exactly when $P \in A_f(I)$, the first part of the result is clear. Now suppose $P \in A_f(I)$, and let q be a minimal prime in $(R_P)^*$ such that depth $q = z(P)$. By the preceding, height$(I(R_P)^* + q/q) \ge$ depth q. As the reverse inequality is obvious, we have equality. Thus $(P_P)^*$ is minimal over $I(R_P)^* + q$. By definition, this gives $P \in A_6$. Therefore $A_f \subseteq A_6$.

In constructing Example A_6, we used all minimal primes of $(R_P)^*$. We now show that a single minimal prime of depth $z(P)$ would suffice.

THEOREM 2.7. *For each prime P of R, let $q(P)$ be a minimal prime in $(R_P)^*$, with* depth $q(P) = z(P)$. *Define* $A(I_S) = \{P_S \in \text{spec } R_S | I_S \subseteq P_S$ *and* $(P_P)^*$ *is minimal over* $I(R_P)^* + q(P)\}$. *Then A is a grade scheme for asymptotic grade on R. Also A contains the canonical grade scheme for asymptotic grade.*

PROOF. Let f be asymptotic grade on R. If $P_S \in A_f(I_S)$, then $P_S \in A(I_S)$ by the argument used in proving Theorem 2.6, with $q = q(P)$. Now A is easily seen to be a proto-grade scheme ($A(I_S)$ being finite since it is contained in $A_6(I_S)$). Let $x_1, \ldots, x_n$ be an asymptotic sequence. We have $A_f(x_1, \ldots, x_n) \subseteq A(x_1, \ldots, x_n) \subseteq A_6(x_1, \ldots, x_n) = A_f(x_1, \ldots, x_n)$, the equality by Lemma 1.4. As equality holds throughout, Lemma 1.4 shows that A is a grade scheme for f.

EXAMPLE. We wish to show that with f being asymptotic grade, in general $A_6 \nsubseteq A_f$. In fact we will show $A \nsubseteq A_f$ with A as constructed in Theorem 2.7. This is very easy. Let (R, M) be a complete local ring with exactly two minimal primes q_1 and q_2, both of depth 1. Let $q(M) = q_1$. Since M is minimal over $q_2 + q(M)$, $M \in A(q_2)$. However, height$(q_2 + q_2/q_2) = 0 < 1 = z(M)$, and so Theorem 2.6 shows $M \notin A_f(q_2)$.

REMARK. The preceding two theorems and example concern asymptotic grade and minimal primes of $(R_P)^*$. Analogous results hold for essential grade and prime divisors of $(R_P)^*$, using [6, Corollary 5.4].

EXAMPLE. We now let f be classical grade and give examples showing that in general A_f is not contained in A_1 or A_2, and A_1 and A_2 are not contained in A_f.

Let (R, M) be a 2-dimensional local domain which is not Cohen-Macaulay, and let P be a height 1 prime. Since $f(M_M) = 1 = f(P_M)$, $M \in A_f(P)$. Clearly $M \notin A_1(P)$. Thus $A_f \nsubseteq A_1$.

Let R be a Cohen-Macaulay domain and let P be a prime such that for some n, P^n is not P-primary. Then for some $P \subsetneq Q \in \text{spec } R$, $Q \in A_1(P^n)$. However $f(Q_Q) = \text{height } Q > \text{height } P = f(P_Q^n)$, so $Q \notin A_f(P^N)$. Thus $A_1 \nsubseteq A_f$.

Let (R, M) be a 2-dimensional local domain which is not Cohen-Macaulay, but which is unmixed (i.e. every $q \in \text{Ass } R^*$ has depth $q = 2$). For any ideal $I \ne 0$, $f(I_M) = 1 = f(M_M)$ so that $M \in A_f(I)$ for all $I \ne 0$. However since R is unmixed, [5, Proposition 10.11] shows for some $I \ne 0$, $M \notin A_2(I)$. Thus $A_f \nsubseteq A_2$.

Let (R, M) be a 2-dimensional normal domain, and let P be a nonprincipal height 1 prime. By [5, Corollary 4.7], $M \in A_2(P)$. However, being 2-dimensional and

normal, R is Cohen-Macaulay. Therefore, $f(M) = 2 > 1 = f(P_M)$, and so $M \notin A_f(P)$. Thus $A_2 \not\subseteq A_f$.

We now show that the canonical grade scheme of a grade function can often be enlarged in a natural way.

DEFINITION. Let f be a grade function on R. If q is an ideal of R, we will say that q is *f-modest* if whenever $q \subseteq P \in \operatorname{spec} R$, we have $f(P_P) \leqslant \operatorname{height}(P/q)$.

REMARKS. (i) If q is an ideal in the nilradical of R, then q is easily seen to be f-modest for any grade function f.

(ii) If f is asymptotic grade and q is a minimal prime of R, then q is f-modest using [**5**, Proposition 6.10].

(iii) If f is essential grade and $q \in \operatorname{Ass} R$, then q is f-modest using [**6**, Corollary 5.4].

(iv) If f is classical grade and $q \in \operatorname{Ass} R$, then q is f-modest by (iii) and the fact that classical grade does not exceed essential grade.

(v) Let q be f-modest and let Q be a prime minimal over q. Then $f(Q_Q) \leqslant \operatorname{height} Q/q = 0$. So $f(Q_Q) = 0$ and $Q \in A_f(0)$. (This does not characterize f-modesty. In fact Q may not be f-modest.)

(vi) If every prime minimal over Q is f-modest, then q is f-modest (but not conversely), as is easily seen.

LEMMA 2.8. *Let f be a grade function on R, and let q be an f-modest ideal of R. Let $x_1, \dots, x_n$ be an f-sequence in R with $(x_1, \dots, x_n) + q \neq R$. Then*

(a) $f((x_1, \dots, x_n) + q) = n$,

(b) $A_f((x_1, \dots, x_n) + q) = \{P \in A_f(x_1, \dots, x_n) | q \subseteq P\}$.

PROOF. (a) Let P be minimal over $(x_1, \dots, x_n) + q$. Then $n \leqslant f((x_1, \dots, x_n) + q) \leqslant f(P_P) \leqslant \operatorname{height} P/q \leqslant n$, using f-modesty and the principal ideal theorem.

(b) One containment is by Lemma 2.5(a). Conversely, if $P \in A_f((x_1, \dots, x_n) + q)$, then $f(P_P) = f((x_1, \dots, x_n)_P + q_P) = n = f((x_1, \dots, x_n)_P)$ showing $P \in A_f(x_1, \dots, x_n)$. Here we used part (a) applied in R_P to get the second equality.

THEOREM 2.9. *Let f be a grade function on R. Let V be a finite set of f-modest ideals of R. Define $A_{fV}(I_S) = A_f(I_S) \cup (\bigcup A_f(I_S + q_S))$ over all $q \in V$ with $I_S + q_S \neq R_S$. Then A_{fV} is a grade scheme for f.*

PROOF. It is easily seen that A_{fV} is a proto-grade scheme. We will use Lemma 1.4. Let $x_1, \dots, x_n$ be an f-sequence. We want $A_f(x_1, \dots, x_n) = A_{fV}(x_1, \dots, x_n)$. One inclusion is by definition of A_{fV}, and the other is by Lemma 2.8(b).

Let f be asymptotic grade. We previously saw $A_f \subseteq A_6 \subseteq A_5$. If V is the set of minimal primes in R, we now give an example in which $A_{fV} \not\subseteq A_5$.

EXAMPLE. Let R be $K[[X, Y, Z, W]]$ modulo $(X) \cap (Y, Z)$. Let x, y, z, w be the images of X, Y, Z, W, and let M be the maximal ideal of R. Let $V = \{(x), (y, z)\}$. Now let $I = (y, w)$. We claim that $M \in A_{fV}(I)$. It will suffice to show that $M \in A_f(I + (x))$. Note that

$$\operatorname{height}(I + (x)/(x)) = \operatorname{height}(I + (x) + (y, z)/(y, z)) = 2 = z(M).$$

574   STEPHEN McADAM

By Theorem 2.6, $M \in A_f(I + (x))$, so that $M \in A_{fV}(I)$. We now claim that $M \notin A_5(I) = \bar{A}^*(I)$. If it were, then by [5, Proposition 3.18] we would have either $M/(x) \in \bar{A}^*(I + (x)/(x))$ or $M/(y, z) \in \bar{A}^*(I + (y, z)/(y, z))$. However, since $R/(x)$ and $R/(y, z)$, being complete local domains, satisfy the altitude formula, [5, Proposition 4.1] would give either $3 = $ height $M/(x) = l(I + (x)/(x)) \leqslant 2$ (since the analytic spread of an ideal does not exceed its minimal number of generators) or $2 = $ height $M/(y, z) = l(I + (y, z)/(y, z)) = 1$. Thus $M \notin A_5(I)$, and so $A_{fV} \not\subseteq A_5$.

EXAMPLE. Let B be an arbitrary grade scheme for f and let V be a finite set of f-modest ideals. In analogy to A_{fV}, we can try to define

$$B_V(I_S) = B(I_S) \cup (\cup B(I_S + q_S))$$

over all $q \in V$ with $I_S + q_S \neq R_S$. However, in general the result will not be a grade scheme for f. In fact, it may not be a grade scheme at all, as we now show, using an example in which B differs from A_f by very little. Let (R, M) be $K[X, Y, Z]$ localized at (X, Y, Z). Let $q = (X)$ and $p = (X, Y)$. Theorem 2.4 makes it clear that in order to define a grade function on R, it suffices to specify its values on P_P for all $P \in \operatorname{spec} R$, in such a way that conditions (ii) and (iii) of Theorem 2.4 are satisfied. Thus we let $f(P_P) = $ height P, with the following exceptions: $f(q_q) = 0$, $f(M) = 2$, and $f(P_P) = 1$ whenever P is a height 2 prime containing q. We see that q is f-modest, and will let $V = \{0, q\}$. We now let $B(I_S) = A_f(I_S)$ in all cases except $B(p)$ which we take to be $A_f(p) \cup \{M\} = \{p, M\}$. (Note that by condition (i) of Theorem 2.4, $f(p) = f(p_p) = 1$, and since $f(M) = 2$, $M \notin A_f(p)$.) Since $f(p) = 1 < $ height p, p is not generated by an f-sequence. Since B and A_f agree everywhere but at p, Lemma 1.4 shows that B is a grade scheme for f. We now claim that B_V, defined as above, is not a grade scheme. (It clearly is a proto-grade scheme.) We first note that $B_V(0) = B(0) \cup B(q) = A_f(0) \cup A_f(q) = \{0, q\}$. Thus Y and Z are both avoiding sequences for B_V. We will show that Y is maximal in M, but Z is not, showing that B_V is not a grade scheme. Since $B_V(Y) = B(Y) \cup B((Y) + q) = B(Y) \cup B(p)$, and since $M \in B(p)$, Y is a maximal avoiding sequence for B_V. On the other hand $B_V(Z) = B(Z) \cup B((Z) + q) = A_f(Z) \cup A_f((Z, X))$, which does not contain M, since $f(M) = 2$ while $f(Z) = 1 = f((Z, X))$.

3. Factor rings. We mentioned that to define a grade function f, it suffices to specify $f(P_P)$ for all $P \in \operatorname{spec} R$ in such a way that conditions (ii) and (iii) of Theorem 2.4 hold. We use this (on R/q) in our next theorem.

THEOREM 3.1. *Let f be a grade function on R and let q be an f-modest ideal. For $P/q \in \operatorname{spec}(R/q)$, let $h(P_P/q_P) = f(P_P)$. Then h defines a grade function on R/q. If J/q is an ideal of R/q, then $h(J/q) = f(J)$. Also $P \in A_f(J)$ if and only if $P/q \in A_h(J/q)$.*

PROOF. Since q is f-modest, $h(P_P/q_P) = f(P_P) \leqslant $ height(P/q). Thus condition (ii) of Theorem 2.4 holds for h. As for condition (iii), let $(Q/q, U)$ be a conforming pair in R/q and suppose $h(P_P/q_P) \leqslant n$ for all $P/q \in U$. We want $h(Q_Q/q_Q) \leqslant n - 1$. Obviously $(Q, \{P|P/q \in U\})$ is a conforming pair in R, and for these P, $f(P_P) = h(P_P/q_P) \leqslant n$. Therefore $f(Q_Q) \leqslant n - 1$, and so $h(Q_Q/q_Q) \leqslant n - 1$ as desired.

Now h is seen to give a grade function on R/q defined via condition (i) of Theorem 2.4. If J/q is an ideal in R/q, then

$$h(J/q) = \min\left\{ h(P_P/q_p) \mid J/q \subseteq P/q \in \mathrm{spec}(R/q) \right\}$$

$$= \min\left\{ f(P_P) \mid J \subseteq P \in \mathrm{spec}\, R \right\} = f(J).$$

Finally, $P \in A_f(J)$ if and only if $f(P_P) = f(J_P)$ if and only if $h(P_P/q_P) = h(J_P/q_P)$ (by applying the preceding sentence in R_P and R_{P}/q_P) if and only if $P/q \in A_h(J/q)$.

REMARK. Let B be a grade scheme for f, and for J/q in R/q let $B_q(J/q) = \{ P/q \mid P \in B(J) \}$. If $B = A_f$, Theorem 3.1 shows that $B_q = A_h$. However in general, B_q will not be a grade scheme at all, as the example at the end of §2 works here as well.

COROLLARY 3.2. *Let f, q, and h be as in Theorem 3.2. Let $x_1, \ldots, x_n$ be an f-sequence with $(x_1, \ldots, x_n) + q \neq R$. Then*

(a) $x_1 + q, \ldots, x_n + q$ *is an h-sequence.*

(b) $A_h(x_1 + q, \ldots, x_n + q) = \{ P/q \mid q \subseteq P \in A_f(x_1, \ldots, x_n) \}$.

PROOF. We induct on n simultaneously for (a) and (b). When $n = 0$, (a) is trivial. For (b), suppose $q \subseteq P$. Then $P \in A_f(0)$ if and only if $f(P_P) = 0$ if and only if $h(P_P/q_P) = 0$ if and only if $P/q \in A_h(0)$.

Now suppose (a) and (b) both hold for $n - 1$. We will show they both hold for n. Suppose this fails for (a). Then $x_1 + q, \ldots, x_n + q$ is not an h-sequence, but by induction, $x_1 + q, \ldots, x_{n-1} + q$ is. Thus for some $P/q \in A_h(x_1 + q, \ldots, x_{n-1} + q)$, we must have $x_n + q \in P/q$. By (b), inductively, $P \in A_f(x_1, \ldots, x_{n-1})$. As $x_n \in P$, we have contradicted that $x_1, \ldots, x_n$ is an f-sequence.

As for (b), since by (a) we already have that $x_1 + q, \ldots, x_n + q$ is an h-sequence, if $P/q \in A_h(x_1 + q, \ldots, x_n + q)$, then $h(P_P/q_P) = n$. Thus $f(P_P) = n$. As $x_1, \ldots, x_n$ is an f-sequence in P_P, we see that $P \in A_f(x_1, \ldots, x_n)$. The reverse is similar.

We already noted that if q is in the nilradical of R, then q is f-modest for any grade function f. We will show that for such q, Theorem 3.1 gives a bijection between all grade functions on R and all grade functions on R/q.

LEMMA 3.3. *Let q be an ideal in the nilradical of R and let f be a grade function on R. For any ideal I of R, $f(I) = f(I + q)$ and $A_f(I) = A_f(I + q)$.*

PROOF. For some $I \subseteq P \in \mathrm{spec}\, R$, $f(I) = f(P_P)$. As $q \subseteq P$, $f(I) \leqslant f(I + q) \leqslant f(P_P) = f(I)$, and equality holds throughout. Thus $f(I) = f(I + q)$. Also, $P \in A_f(I)$ if and only if $f(P_P) = f(I_P)$ if and only if $f(P_P) = f(I_P + q_P)$ (since q_P is in the nilradical of R_P) if and only if $P \in A_f(I + q)$.

THEOREM 3.4. *Let q be an ideal contained in the nilradical of R.*

(a) *The map $f \to h$ (as in Theorem 3.1) is a bijection between the set of grade functions on R and the set of grade functions on R/q.*

(b) *If $f \to h$ is as above, then $P \in A_f(I)$ if and only if $P/q \in A_h(I + q/q)$.*

(c) *If $f \to h$ is as above, $x_1, \ldots, x_n$ is an f-sequence if and only if $x_1 + q, \ldots, x_n + q$ is an h-sequence.*

(d) *The bijection in (a) takes asymptotic grade on R to asymptotic grade on R/q.*

PROOF. (a) We will construct an inverse map for the map $f \to h$. Suppose that k is a grade function on R/q, and define l as follows. For P any prime in R, we define $l(P_P)$ to equal $k(P_P/q_P)$. We claim that l is a grade function on R. This is proved using Theorem 2.4 and is similar to the proof of Theorem 3.1. Now start with k and define l as above. Then let $f = l$ and define h as in Theorem 3.1. We have $h(J/q) = f(J) = l(J) = k(J + q/q) = k(J/q)$ (since $q \subseteq J$). That is, $h = k$. Conversely, start with f and h as in Theorem 3.1, let $k = h$ and define l as above. We see that $l(I) = k(I + q/q) = h(I + q/q) = f(I + q) = f(I)$ by Lemma 3.3. That is, $l = f$ and (a) is proved.

(b) Immediate from Lemma 3.3 and the final sentence of Theorem 3.1.

(c) One direction is by Corollary 3.2(a). Thus suppose that $x_1 + q, \ldots, x_n + q$ is an h-sequence, but that $x_1, \ldots, x_n$ is not an f-sequence. Then for some $i = 1, 2, \ldots, n$ there is a $P \in A_f(x_1, \ldots, x_{i-1})$ with $x_i \in P$. By (b), $P/q \in A_h(x_1 + q, \ldots, x_{i-1} + q)$. As $x_i + q \in P/q$, we have a contradiction.

(d) In view of (c), we need only show that $x_1, \ldots, x_n$ is an asymptotic sequence in R if and only if $x_1 + q, \ldots, x_n + q$ is an asymptotic sequence in R/q. For this, note that $P \in \bar{A}^*(I)$ if and only if there is a minimal prime $Q \subseteq P$ with $P/Q \in \bar{A}^*(I + Q/Q)$ [5, Proposition 3.18], from which it easily follows that $P \in \bar{A}^*(I)$ if and only if $P/q \in \bar{A}^*(I + q/q)$. Using this, our statement about asymptotic sequences is straightforward.

We will now show that grade functions on certain collections of factor rings of R induce a grade function on R. Then we will combine this with Theorem 3.1.

THEOREM 3.5. *Let V be a finite set of ideals of R, and suppose for any $P \in$ spec R there is a $q \in V$ with $q \subseteq P$. For each $q \in V$, let f_q be a grade function on R/q. Define f' in the following way. For I_S an ideal in some localization R_S of R, $f'(I_S) = \min\{f_q(I_S + q_S/q_S) | q \in V, I_S + q_S \neq R_S\}$. Then f' is a grade function on R.*

PROOF. Note that the hypothesis on V assures that for any I_S, there is a $q \in V$ with $I_S + q_S \neq R_S$, so that $f'(I_S)$ is defined. We will show that f' satisfies Theorem 2.4. For condition (i), we need only to consider an ideal I in R itself. Say $I \subseteq P \in$ spec R. Then

$$f'(P_P) = \min\{f_q(P_P/q_P) | q \in V, q \subseteq P\}$$

$$\geq \min\{f_q(I + q/q) | q \in V, q \subseteq P\} \geq f'(I).$$

We must now show $f'(P_P) = f'(I)$ for some P. For some $q \in V$, we have $f'(I) = f_q(I + q/q)$, and for some $I + q/q \subseteq P/q \in$ spec R/q, $f_q(I + q/q) = f_q(P_P/q_P)$. As $f_q(P_P/q_P) \geq f'(P_P) \geq f'(I)$, combining the above shows this is the P we seek.

Condition (ii) is easy, since $f'(P_P) = \min\{f_q(P_P/q_P) | q \in V, q \subseteq P\} \leq \min\{\text{height } P/q | q \in V, q \subseteq P\} \leq \text{height } P$.

As for condition (iii), let (Q, U) be a conforming pair in R, with $f'(P_P) \leq n$ for all $P \in U$. Thus for each $P \in U$, there is a $q \in V, q \subseteq P$, with $f_q(P_P/q_P) = f'(P_P) \leq n$. As U is infinite while V is finite, we find that for some fixed $q \in V$ there are infinitely many $P \in U$ for which $q \subseteq P$ and $f_q(P_P/q_P) \leq n$. Deleting the other P's

from U does no harm, as we still have a conforming pair. We now have $q \subseteq P$ for all $P \in U$, and so $q \subseteq Q$. It is clear that $(Q/q, \{P/q | P \in U\})$ is a conforming pair in R/q, and since $f_q(P_P/q_P) \leqslant n$ for all such P/q, we have $f_q(Q_Q/q_Q) \leqslant n - 1$. Thus $f'(Q_Q) \leqslant f_q(Q_Q/q_Q) \leqslant n - 1$, as desired.

COROLLARY 3.6. *Let* $R, V, f_q,$ *and* f' *be as in Theorem* 3.5.
(a) *If* $P \in A_{f'}(I)$, *then for some* $q \in V$ *with* $q \subseteq P$, $P/q \in A_{f_q}(I + q/q)$.
(b) *If for all* $q \in V$ *with* $q \subseteq P$ *we have* $P/q \in A_{f_q}(I + q/q)$, *then* $P \in A_{f'}(I)$.
(c) *If* I *is generated by an* f'-*sequence, then* $P \in A_{f'}(I)$ *if and only if for some* $q \in V$ *with* $q \subseteq P$, $P/q \in A_{f_q}(I + q/q)$.
(d) *If* $x_1, \ldots, x_n$ *is an* f'-*sequence, and if* $q \in V$ *with* $(x_1, \ldots, x_n) + q \neq R$, *then* $x_1 + q, \ldots, x_n + q$ *is an* f_q-*sequence.*
(e) *If* R *is a local ring, then* $x_1, \ldots, x_n$ *is an* f'-*sequence if and only if* $x_1 + q, \ldots, x_n + q$ *is an* f_q-*sequence for each* $q \in V$.

PROOF. For (a), let $P \in A_{f'}(I)$, so that $f'(P_P) = f'(I_P)$. By definition of f', $f'(I_P) = f'(P_P) = f_q(P_P/q_P) \geqslant f_q(I_P + q_P/q_P) \geqslant f'(I_P)$ for some $q \in V$ with $q \subseteq P$. Therefore equality holds throughout, and (a) is true. For (b), it is easily seen that we may assume R is local at P. Now the hypothesis shows $f_q(P/q) = f_q(I + q/q)$ for all $q \in V$. The definition of f' now shows that $f'(P) = f'(I)$ so that $P \in A_{f'}(I)$. To prove (c) and (d), we argue similarly to the proof of Corollary 3.2. As for (e), one direction is by (d), while the other is any easy induction using (a).

REMARK. The converse of Corollary 3.6(a) fails even when R is local. Let (R, M) be 1-dimensional with $V = \{q, Q\}$ being exactly the minimal primes of R. Let f_q and f_Q be classical grade. Let $I = q$. Now $f_Q(M/Q) = 1 = f_Q(I + Q/Q)$ so that $M/Q \in A_Q(I + Q/Q)$. However $f'(M) = \min\{f_q(M/q), f_Q(M/Q)\} = 1$ while $f'(I) = \min\{f_q(q/q), f_Q(q + Q/Q)\} = 0$. Thus $M \notin A_{f'}(I)$. The next corollary corrects this situation, at the cost of possibly going to a larger grade scheme. That is, if the B_q in Corollary 3.7 all equal the canonical grade scheme A_q of f_q, then the B produced wil have $A_{f'} \subseteq B$.

COROLLARY 3.7. *Let* R, V, f_q *and* f' *be as in Theorem* 3.5, *and suppose that for each* $q \in V$, B_q *is a grade scheme for* f_q. *Define* $B(I_S) = \{P_S \in \operatorname{spec} R_S | I_S \subseteq P_S,$ *and for some* $q \in V$ *with* $q \subseteq P$, $P_S/q_S \in B_q(I_S + q_S/q_S)\}$. *Then* B *is a grade scheme for* f'.

PROOF. It is straightforward to verify that B is a proto-grade scheme. We will invoke Lemma 1.4 applied to $A_{f'}$ and B. Thus let $x_1, \ldots, x_n$ be an f'-sequence. We must show that $A_{f'}(x_1, \ldots, x_n) = B(x_1, \ldots, x_n)$. Suppose $P \in A_{f'}(x_1, \ldots, x_n)$. By Corollary 3.6(a), for some $q \in V$ with $q \subseteq P$, $P/q \in A_{f_q}(x_1 + q, \ldots, x_n + q)$. However by (d) of that corollary, $x_1 + q, \ldots, x_n + q$ is an f_q-sequence, so that $P/q \in B_q(x_1 + q, \ldots, x_n + q)$ by Lemma 1.4. By definition $P \in B(x_1, \ldots, x_n)$, so that $A_{f'}(x_1, \ldots, x_n) \subseteq B(x_1, \ldots, x_n)$. The reverse containment is proved similarly, using Corollary 3.6(c).

THEOREM 3.8. *Let* V *be a finite set of ideals of* R *such that every prime of* R *contains some* $q \in V$.

(a) *Let f be a grade function on R such that every $q \in V$ is f-modest. For each $q \in V$ define h_q on R/q as in Theorem 3.1. Now letting $f_q = h_q$, define f' as in Theorem 3.5. Then $f' = f$.*

(b) *For each $q \in V$ let f_q be a grade function on R/q, and define f' as in Theorem 3.5. Then every $q \in V$ is f'-modest. Now letting $f = f'$, define h_q on R/q as in Theorem 3.1. Then $f_q \geqslant h_q$.*

PROOF. (a) For any ideal I of R, $f'(I) = \min\{f_q(I + q/q) | q \in V, I + q \neq R\}$. Since $f_q = h_q$, the characterization of h_q in Theorem 3.1 gives

$$f'(I) = \min\{f(I + q) | q \in V, I + q \neq R\}.$$

We will show that $f(I)$ equals the same minimum. Clearly it is equal to or less than that minimum. Now there is a prime P containing I with $f(I) = f(P_P)$. Since P contains some $q \in V$, $f(I) \leqslant f(I + q) \leqslant f(P_P) = f(I)$, and equality holds throughout.

(b) If $q \in V$ and $q \subseteq P \in \operatorname{spec} R$, then $f'(P_P) \leqslant f_q(P_P/q_P) \leqslant$ height P/q since f_q is a grade function. Thus q is f'-modest. Now for an ideal J/q in R/q, $h_q(J/q) = f(J) = f'(J) = \min\{f_Q(J + Q/Q) | Q \in V, J + Q \neq R\}$. Letting $Q = q$ shows $h_q(J/q) \leqslant f_q(J/q)$.

4. Extensions. Recall that rings $R \subseteq T$ satisfy *lying over* if for every $P \in \operatorname{spec} R$, there is a $p \in \operatorname{spec} T$ with $p \cap R = P$.

LEMMA 4.1. *Let the Noetherian rings $R \subseteq T$ satisfy lying over. Then for every $P \in \operatorname{spec} R$, there is a $p \in \operatorname{spec} T$ with $p \cap R = P$ and height $p \leqslant$ height P.*

PROOF. Let $S = R - P$, and let $a_1, \ldots, a_n$ be a system of parameters in $R_P = R_S$ (so that height $P = n$). By lying over, we have $(a_1, \ldots, a_n)T_S \neq T_s$. If $p_S \in \operatorname{spec} T_S$ with p_S minimal over $(a_1, \ldots, a_n)T_S$, clearly $p \cap R = P$ and height $p \leqslant n$ by the principal ideal theorem.

THEOREM 4.2. *Let $R \subseteq T$ be Noetherian rings which satisfy lying over. Let g be a grade function on T. For I_S any ideal in any localization R_S of R, let $f(I_S)$ be defined to be $g(IT_S)$. Then*

(a) *f is a grade function on R.*

(b) *If $P \in A_f(I)$, then there is a $p \in A_g(IT)$ with $p \cap R = P$.*

(c) *If I is generated by an f-sequence, then the converse of (b) also holds.*

(d) *Elements $x_1, \ldots, x_n$ in R are an f-sequence if and only if they are a g-sequence.*

(e) *Let B be any grade scheme for g. For I_S any ideal in any localization R_S of R, let $C(I_S) = \{P_S \in \operatorname{spec} R_S | there is a $p_s \in B(IT_S)$ with $p_S \cap R_S = P_S\}$. Then C is a grade scheme for f.*

(f) *If T is a faithfully flat extension of R and if g is any of classical, essential, or asymptotic grade, f is the same.*

(g) *If T is an integral extension of R such that every minimal prime of T contracts to a minimal prime of R, and if g is asymptotic grade, so is f.*

(h) *If T is a finite module extension of R such that every $p \in \operatorname{Ass} T$ has $p \cap R \in$ $\operatorname{Ass} R$, and if g is essential grade, so is f.*

PROOF. (a) First note that if I is an ideal of R and S is a multiplicatively closed subset of R disjoint from I, our hypothesis easily shows that $IT \cap S = \varnothing$. Thus $f(I_S) = g(IT_S)$ is defined. Furthermore, it is easily seen that for $P \in \operatorname{spec} R$, $f(P_P) = \min\{ g(p_p) | p \in \operatorname{spec} T, p \cap R = P\}$. Using this, it is not hard to see that f satisfies condition (i) of Theorem 2.4. It also satisfies condition (ii) of that theorem, using Lemma 4.1. For condition (iii), let (Q, U) be a conforming pair in R, with $f(P_P) \leqslant n$ for all $P \in U$. We want $f(Q_Q) \leqslant n - 1$. For each $P \in U$, the preceding shows there is a $p \in \operatorname{spec} T$ with $p \cap T = P$ and $g(p_p) = f(P_P) \leqslant n$. For each $P \in U$, consider one such $p \in \operatorname{spec} T$, and let W denote the set of those p. Clearly $QT \subseteq p$ for all $p \in W$, and so by Lemma 2.1 there is a conforming pair (q, W') of T with $QT \subseteq q$ and $W' \subseteq W$. As $g(p_p) \leqslant n$ for all $p \in W'$, we have that $g(q_q) \leqslant n - 1$. By the preceding, in order to show that $f(Q_Q) \leqslant n - 1$, it will suffice to show that $q \cap R = Q$. Clearly $Q \subseteq q \cap R$. Conversely, $q \cap R \subseteq \bigcap\{ p \cap R | p \in W'\}$. By construction, $\{ p \cap R | p \in W'\}$ is an infinite subset of U, and so $q \cap R \subseteq Q$ as desired.

(b) Let $P \in A_f(I)$. Then $f(P_P) = f(I_P)$. As noted in the proof of (a), there is a $p \in \operatorname{spec} T$, $p \cap R = P$, with $g(p_p) = f(P_P)$. If $S = R - P$, then $f(I_P) = g(IT_S) \leqslant g(IT_p) \leqslant g(p_p) = f(P_P) = f(I_P)$. Equality holds throughout, showing $p \in A_g(IT)$.

(c) and (d) We prove these simultaneously inducting on n. For $n = 0$, (d) is trivial. For (c), say $p \in A_g(0)$ and $p \cap R = P$. Then $g(p_p) = 0$, and so $f(P_P) = 0$, showing $P \in A_f(0)$. Now suppose (c) and (d) both hold for $n - 1$. Then the case n of (d) follows easily from (b) and the cases $n - 1$ of (c) and (d). As for the case n of (c), say $p \in A_g(x_1, \ldots, x_n)$, with $p \cap R = P$ and $x_1, \ldots, x_n$ an f-sequence. By the case n of (d), these are also a g-sequence. Thus $g(p_p) = n$, so that $f(P_P) \leqslant n$. As $x_1, \ldots, x_n$ are in P_P, $f(P_P) = n$ and $P \in A_f(x_1, \ldots, x_n)$.

(e) Clearly C is a proto-grade scheme. Let $x_1, \ldots, x_n$ be an f-sequence, so that by part (d), it is also a g-sequence. $P \in C(x_1, \ldots, x_n)$ if and only if there is a $p \in B(x_1, \ldots, x_n)$ with $p \cap R = P$ if and only if there is a $p \in A_g(x_1, \ldots, x_n)$ with $p \cap R = P$ (using Lemma 1.4 applied to B and A_g) if and only if $P \in A_f(x_1, \ldots, x_n)$ (by part (c)). Thus Lemma 1.4 applied to C and A_f shows that C is a grade scheme for f.

(f) With T a faithfully flat extension of R and I an ideal of R, it is well known that classical grade I = classical grade IT. Also essential grade I = essential IT by [6, Proposition 5.6.1]. That the same is true of asymptotic grade is shown in [10, Corollary 5.2].

(g) This follows easily from [5, Proposition 3.22].

(h) This is given by [6, Proposition 5.6.2].

Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings. Let $p \in \operatorname{spec} T$ and let $P = p \cap R$. It is easily seen that T_p is an extension of R_P, and furthermore it follows from [7, 19.2(2)] that $R_P \subseteq T_p$ satisfies lying over. In the proof of the next corollary we shall start with h being one of classical, essential, or asymptotic grade on T (hence also on T_p) and we will apply Theorem 4.2 to $R_P \subseteq T_p$ and $g = h$ on T_p.

A key step in that proof will be to show that the resulting grade function f on R_P also equals h. We do that now as a lemma.

LEMMA 4.3. *Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings. Let $p \in \operatorname{spec} T$ and let $P = p \cap R$. Let h be any one of classical, essential or asymptotic grade. Let f be the grade function on R_P obtained by applying Theorem 4.2 to $R_P \subseteq T_p$ starting with $g = h$ on T_p. Then $f = h$ on R_P.*

PROOF. Let $S = R - P$ and let $q \in \operatorname{spec} T$ with $q \subseteq p$ and q minimal over PT. Let $T_1 = T_S$, $T_2 = T_p$ and $T_3 = T_q$. Note that T_{i+1} is a localization of T_i, $i = 1, 2$. Let f_i be the grade function on R_P obtained by applying Theorem 4.2 to $R_P \subseteq T_i$, starting with $g = h$ on T_i, $i = 1, 2, 3$. Now $R_P \subseteq T_1$ is a faithfully flat extension, and so Theorem 4.2(f) shows that $f_1 = h$. Also [7, 19.2(2)] shows that $R_P \subseteq T_3$ is a faithfully flat extension, and so we also have $f_3 = h$. To prove our lemma, we will show $f_1 \leqslant f_2 \leqslant f_3$, giving $f_2 = h$ as desired. Let $x_1, \ldots, x_n$ be an f_1-sequence in R_P. Theorem 4.2(d) applied to $R_P \subseteq T_1$ shows that $x_1, \ldots, x_n$ is an h-sequence in T_1. As T_2 is a localization of T_1, $x_1, \ldots, x_n$ is also an h-sequence in T_2. Theorem 4.2(d) applied to $R_P \subseteq T_2$ now shows that $x_1, \ldots, x_n$ is an f_2-sequence in R_P. Thus any f_1-sequence is also an f_2-sequence, and so clearly $f_1 \leqslant f_2$. The proof that $f_2 \leqslant f_3$ is identical.

COROLLARY 4.4. *Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings. Let h be any of classical, essential, or asymptotic grade. Then if I is an ideal of R and $I \subseteq P \in \operatorname{spec} R$, the following are equivalent.*

(a) $P \in A_h(I)$.

(b) *There is a $p \in A_h(IT)$ with $p \cap R = P$.*

(c) *If $p \in \operatorname{spec} T$ is minimal over PT, then $p \in A_h(IT)$.*

PROOF. (a) $\Rightarrow$ (c). Let the prime p be minimal over PT. By [7, 19.2(2)], T_p is a faithfully flat extension of R_P. If in Theorem 4.2 applied to $R_P \subseteq T_p$, we start with $g = h$ on T_p, then the resulting f on R_P is also h as is shown by part (f) of that theorem. Now if $P \in A_h(I)$, then $P_P \in A_h(I_P) = A_f(I_P)$, and so Theorem 4.2(a) shows that some prime in $A_g(IT_P) = A_h(IT_P)$ lies over P_P. However p_p is the only prime of T_p lying over P_P, and so we must have $p \in A_h(IT)$. Thus (a) $\Rightarrow$ (c).

(c) $\Rightarrow$ (b) is trivial from [6, 18.11].

(b) $\Rightarrow$ (a) Let $p \in A_h(IT)$ with $p \cap R = P$. We can apply Theorem 4.2 to $R_P \subseteq T_p$, starting with $g = h$ on T_p. If f is the resulting grade function on R_P, then $f = h$ by Lemma 4.3. We now have $h(I_P) \leqslant h(P_P) = f(P_P) = g(P_P T_P) = h(PT_P) \leqslant h(p_p) = h(IT_P)$ (since $p \in A_h(IT)$) $= g(I_P T_P) = f(I_P) = h(I_P)$. As equality must hold throughout, $P \in A_h(I)$.

REMARK. Corollary 4.4 remains true if A_h is replaced by A_1, A_4, or A_5. For A_1, use [7, 18.11]. For A_4 use [6, Proposition 3.7]. For A_5 use [9, Theorems 8.5 and 8.8]. (It may well also hold for A_2, A_3, and A_6, but we have not checked.)

EXAMPLE. Corollary 4.4 shows that the converse of Theorem 4.2(b) holds if $R \subseteq T$ is faithfully flat and g is one of classical, essential, or asymptotic grade. We now show that the converse fails for arbitrary g, even if $R \subseteq T$ is faithfully flat. Let

(R, M) be a 2-dimensional local domain with P a height 1 prime. Let $T = R[X]$, X an indeterminate. Define g by specifying $g(p_p) = $ height p for all $p \in$ spec T except, for $p = MT$, let $g(p_p) = 1$ and, for $p = (P, X)$, let $g(p_p) = 0$. With f as in Theorem 4.2, it is easily seen that $MT \in A_g(PT)$ but $M \notin A_f(P)$.

We now look at integral extensions and will show that A_4 and A_5 are better behaved than the canonical grade schemes for essential and asymptotic grade. We first note that if I is an ideal in R and if the Noetherian ring T is an integral extension of R such that every minimal prime of T contracts to a minimal prime of R, then $P \in A_5(I)$ if and only if there is a $p \in A_5(IT)$ with $p \cap R = P_S$ [**5**, Proposition 3.22]. Similarly, if T is a finite module extension of R and every prime in Ass T contracts to a prime in Ass R, then $P \in A_4(I)$ if and only if there is a $p \in A_4(IT)$ with $p \cap R = T$ [**6**, Proposition 3.9]. We will now give an example showing that neither of these holds for A_f, with f asymptotic grade and essential grade, respectively. (Of course one direction does hold, by Theorem 4.2(b).)

EXAMPLE. Let T be $K[X, Y]$ localized at the complement of $(X, Y) \cup (X, Y + 1)$. Thus T has exactly two maximal ideals N_1 and N_2. Now $R = K + (N_1 \cap N_2)$ is a local ring, its maximal ideal being $M = N_1 \cap N_2$. Also T is a finite R-module. Furthermore on R and T, essential and asymptotic grades are equal, both being just height. We will call this f. Let p be a height 1 prime of T not in N_1. Let $I = p \cap N_1$, an ideal of both T and R. Since $IT_{N_1} = N_1$, $N_1 \in A_f(IT)$. However $M = N_1 \cap R \notin A_f(I)$, since $f(M) = 2$ while $f(I) = 1$.

THEOREM 4.5. *Let f be a grade function on R, and let X be an indeterminate. If P is a prime of $R[X]$ with $P \cap R = p$, then define $g(P_P)$ as follows. If $P = pR[X]$, let $g(P_P) = f(p_p)$. If $P \neq pR[X]$, let $g(P_P) = f(p_p) + 1$. Then*

(a) *g determines a grade function on $R[X]$ (via condition (i) of Theorem 2.4).*

(b) *If I is an ideal in R, then $P \in A_g(IR[X])$ if and only if $P = (P \cap R)R[X]$ and $P \cap R \in A_f(I)$.*

(c) *If $a_1, \dots, a_n$ are elements in R, they are an f-sequence if and only if they are a g-sequence. Also if I is an ideal of R, then $f(I) = g(IR[X])$.*

(d) *If f is either classical or asymptotic grade on R, then g is the same on $R[X]$.*

PROOF. (a) We must show that g satisfies conditions (ii) and (iii) of Theorem 2.4. Condition (ii) is easy, since it holds for f and since with P and p as in the statement of this theorem, height $P = $ height p if $P = pR[X]$ while height $P = $ height $p + 1$ if $P \neq pR[X]$. For condition (iii), we let (Q, U) be a conforming pair in $R[X]$, and assume $g(P_P) \leqslant n$ for all $P \in U$. We need $g(Q_Q) \leqslant n - 1$, and will proceed in cases. *Case* 1. There is some $p \in$ spec R such that infinitely many $P \in U$ satisfy $P \cap R = p$. Then we easily see that $Q = pR[X]$. Choose $P \in U$ with $P \cap R = p$ and $P \neq pR[X]$. Then $n \geqslant g(P_P) = f(p_p) + 1$. As $g(Q_Q) = f(p_p) \leqslant n - 1$, we are done. *Case* 2. If Case 1 fails, then it does no harm to delete from U the finitely many P satisfying $P \cap R = Q \cap R$. We now partition U into $U_1 = \{P \in U | P = (P \cap R)R[X]\}$ and $U_2 = \{P \in U | P \neq (P \cap R)R[X]\}$. One of U_1 or U_2 must be infinite. *Subcase* 2a. U_1 is infinite. Recalling our deletions from U, we easily see that $(Q \cap R, \{P \cap R | P \in U_1\})$ is a conforming pair in R. Furthermore, if $q = Q \cap R$, we see that $Q = qR[X]$, since any polynomial in Q has all of its coefficients in $(P \cap R)R[X]$ for all $P \in U_1$.

With $P \in U_1$, $n \geqslant g(P_P) = f(p_p)$ (with $p = P \cap R$), and our conforming pair shows that $f(q_q) \leqslant n - 1$. Thus $g(Q_Q) = f(q_q) \leqslant n - 1$. *Subcase* 2b. U_2 is infinite. Now we see that $(q, \{P \cap R | P \in U_2\})$ is a conforming pair. If $P \in U_2$ and $p = P \cap R$, then $g(P_P) \leqslant n$ gives $f(p_p) = g(P_P) - 1 \leqslant n - 1$, and so we get $f(q_q) \leqslant n - 2$. Since $g(Q_Q) \leqslant f(q_q) + 1$, we again have $g(Q_Q) \leqslant n - 1$. This completes the proof of part (a).

(b) Let $P \in A_g(IR[X])$ and let $p = P \cap R$. We have $g(P_P) = g(IR[X]_P) = \min\{g(Q_Q) | IR[X]_P \subseteq Q_P \in \operatorname{spec} R[X]_P\}$. Taking the particular case $Q = pR[X]$, we get $g(P_P) \leqslant g(Q_Q) = f(p_p)$. Since $p = P \cap R$, the definition of g now shows that $P = pR[X]$. It remains to show that $p \in A_f(I)$. We have $g(IR[X]_P) = g(P_P) = f(p_p)$; call this number n. Letting $f(I_p) = m$, we want $n = m$, and clearly have $n \geqslant m$ since $I_p \subseteq p_p$. Now there is some $I_p \subseteq q_p \in \operatorname{spec} R_p$ with $f(q_q) = f(I_p) = m$. With $Q = qR[X]$, we have $IR[X]_P \subseteq Q_P$, and so $n = g(IR[X]_P) \leqslant g(Q_Q) = f(q_q) = m$. Thus $n = m$. This proves half of (b). The other half is similar to the last part of the above.

(c) This is easy using (b).

(d) We will assume that f is asymptotic grade, the case of classical grade being similar. By [5, Proposition 3.21], if I is an ideal of R then $\bar{A}^*(IR[X]) = \{pR[X] | p \in \bar{A}^*(I)\}$. Thus if $a_1, \ldots, a_n$ are elements of R, they are an asymptotic sequence in R if and only if they are an asymptotic sequence in $R[X]$. We want $g = f$ on $R[X]$. It is enough to show $f(P_P) = g(P_P)$ for any $P \in \operatorname{spec} R[X]$. If $p = P \cap R$, clearly we may assume that R is local at p. Let $a_1, \ldots, a_n$ be a maximal asymptotic sequence in p. Thus $f(p) = n$ and so $g(P_P) = n$ if $P = pR[X]$, while $g(P_P) = n + 1$ if $P \neq pR[X]$. We must show the same dichotomy applies to $f(P_P)$. If $P = pR[X]$, we have already noted that since $p \in \bar{A}^*(a_1, \ldots, a_n)$, $P \in \bar{A}^*((a_1, \ldots, a_n)R[X])$ and so $f(P_P) = n$. Now suppose $P \neq pR[X]$. Then there is a monic polynomial $h(X)$ with $P = (pR[X], h(X))$. Our previous statements show that $a_1, \ldots, a_n, h(X)$ is an asymptotic sequence in P, so that $f(P_P) \geqslant n + 1$. Now letting $P' = pR[X]$, since $P' \in \bar{A}^*((a_1, \ldots, a_n)R[X])$, $f(P'_P) = n$. By [5, Proposition 5.17] applied to $R[X]_P$ and P'_P, and since P_P is the only prime larger than P'_P, we see that $f(P_P) = f(P'_P) + 1 = n + 1$.

COROLLARY 4.6. *Let X be an indeterminate over R, and let f be either classical or asymptotic grade. Let I be an ideal of R. Then $A_f(IR[X]) = \{pR[X] | p \in A_f(I)\}$.*

PROOF. This follows immediately from parts (b) and (d) of Theorem 4.5.

REMARK. Theorem 4.5(d) and Corollary 4.6 probably hold for essential grade as well. To mimic the above proofs, we would need for I an ideal of R and $A = A_3$ (alternatively $A = A_4$) that $A(IR[X]) = \{pR[X] | p \in A(I)\}$. This is probably true for one of (both of?) A_3 or A_4, but we have not checked.

QUESTION. Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings and let f be a grade function on R. For $P \in \operatorname{spec} T$ with $p = P \cap R$, define $g(P_P) = f(p_p) + \operatorname{height}(P/pT)$. Does g determine a grade function on T?

REMARK. Let $R \subseteq T$ be an integral extension of Noetherian rings, and let f be a grade function on R. For $P \in \operatorname{spec} T$ with $P \cap R = p$, let $g(P_P) = f(p_p)$. It is easily seen that g satisfies condition (iii) of Theorem 2.4. However it may not satisfy

condition (ii) (since possibly height P < height p), and therefore g may not determine a grade function on T. If every minimal prime of T contracts to a minimal prime of R, and if f is asymptotic grade on R, then g will be a grade function [5, Proposition 5.14], but need not be asymptotic grade. If

$$h(P_P) = \min\{ f(p_p), \text{height } p \},$$

then h always defines a grade function on T.

5. Another grade function. We start with an easy consequence of Theorems 3.5 and 4.2, mentioning two known special cases. We then introduce a third special case which we feel deserves attention, but about which we know very little.

PROPOSITION 5.1. *Let* (R, M) *be a local ring with completion* R^*. *Let* V *be a finite set of ideals of* R^* *such that every prime ideal of* R^* *contains some* $q \in V$. *For each* $q \in V$, *let* f_q *be a grade function on* R^*/q. *For* I_S *any ideal in any localization* R_S *of* R, *define* $f(I_S)$ *to be* $\min\{ f_q(IR_S^* + q_S/q_S) | q \in V, I_S + q_S \neq R_S^* \}$. *Then*
  (a) *f is a grade function on R.*
  (b) *If* $V = \{ q \in \text{spec } R^* | q \text{ is minimal} \}$ *and if each* f_q *is the height function, then* f *is asymptotic grade.*
  (c) *If* $V = \text{Ass } R^*$ *and if each* f_q *is the height function, then* f *is essential grade.*

PROOF. To prove (a), combine Theorems 3.5 and 4.2. For (b) and (c), use [5, Proposition 5.10 and **6**, Corollary 5.4] respectively.

Let (R, M) be a local ring with completion R^*. Let $V = \text{Ass } R^*$, and for every $q \in V$, let f_q be the classical grade function on R^*/q. Define f as in Proposition 5.1(a). Thus f is a grade function on R. This section reveals our ignorance, and is devoted to questions concerning f.

QUESTION. Is there a natural grade function, defined on any Noetherian ring R, which coincides with f whenever R is local?

A possible candidate for a grade function answering the previous question is given by the next question.

QUESTION. For I_S any ideal in any localization R_S of a Noetherian ring R, define $g(I_S)$ to be $\min\{\text{classical grade}(P_P)^*/q | I_S \subseteq P_S \in \text{spec } R_S, \text{ and } q \in \text{Ass}(R_P)^* \}$. Is g a grade function? If R is local, will g equal the above f?

The preceding question is motivated by a comparison of [5, Proposition 5.6] to [5, Proposition 6.10], as well as by comparing [**6**, Proposition 4.3] to [**6**, Corollary 5.4].

Recall now that A_1 and A_2 were defined in our earlier examples.

QUESTION. For I_S any ideal in any localization R_S of R, let $B(I_S) = \{ P_S \in \text{spec } R_S | I_S \subseteq P_S$, and there is a $q \in \text{Ass}(R_P)^*$ with

$$(P_P)^*/q \in A_1\left(\left(I(R_P)^* + q\right)/q\right)\right\}.$$

If R is local, is B a grade scheme for our f? Is it a grade scheme at all? What if R is arbitary? What if in the definition of B we replace A_1 by A_2?

REMARK. We do not know if the B of the preceding question is a grade scheme. However combining Corollary 3.7 and Theorem 4.2(e) shows that for a local ring (R, M) with completion R^*, we can construct a grade scheme C for the above f in

the following way. Let $C(I) = \{ P \in \operatorname{spec} R \,|\, \text{there is a } p \in \operatorname{spec} R^* \text{ with } p \cap R = P,$ and a $q \in \operatorname{Ass} R^*$ with $q \subseteq p$, such that $p/q \in A_1(IR^* + q/q)\}$ and let $C(I_S)$ be defined appropriately. (Of course A_2 would do as well as A_1.)

6. Modulation.

DEFINITION. Let f be a grade function on a local ring (R, M). If for every ideal I and element $a \in M$, $f(I, a) \leqslant f(I) + 1$, then we say f is *modulated*.

DEFINITION. The grade function f on R is *locally modulated* if f restricted to R_P is modulated for every $P \in \operatorname{spec} R$.

REMARKS. (a) It is easy to produce examples in which f restricted to R_M is modulated for all maximal ideals M, but for which f is not locally modulated.

(b) Let f be any of classical, essential, or asymptotic grade. Then f is locally modulated, [1, Theorem 127; 6, Proposition 5.8 and 5, Proposition 5.16].

THEOREM 6.1. *Let f be a grade function on R. The following are equivalent.*
(a) *f is locally modulated.*
(b) *If $P \subset Q$ are primes of R and* $\operatorname{height}(Q/P) = 1$, *then* $f(Q_Q) \leqslant f(P_P) + 1$.

PROOF. (a) $\Rightarrow$ (b) Suppose (a) holds, and $P \subset Q$ are as in (b). Pick $a \in Q_Q - P_Q$. As Q_Q is the only prime of R_Q containing (P_Q, a), we have that $f(P_Q, a) = f(Q_Q)$. Now (a) implies $f(Q_Q) \leqslant f(P_Q) + 1$. As the only primes containing P_Q are P_Q and Q_Q, we have $f(P_Q) = \min\{ f(P_P), f(Q_Q)\}$. Whichever is the minimum, we see that (b) holds.

(b) $\Rightarrow$ (a) Suppose (b) holds. To prove (a), we may assume that R is local. Let I be an ideal and a be a nonunit. For some $I \subseteq p \in \operatorname{spec} R$, $f(I) = f(p_p)$. If $a \in p$, then $f(I, a) \leqslant f(p_p) = f(I)$ and we are done. If $a \notin p$, let q be a prime minimal over (p, a). Then $\operatorname{height} q/p = 1$, and by (b), $f(q_q) \leqslant f(p_p) + 1$. As $(I, a) \subseteq q$, $f(I, a) \leqslant f(q_q) \leqslant f(p_p) + 1 = f(I) + 1$, and we are done.

COROLLARY 6.2. *Let f be a locally modulated grade function on R, and let* $P_0 \subset P_1 \subset \cdots \subset P_n$ *be a saturated chain of primes in R with P_0 a minimal prime. Then there is a saturated chain of primes $P_0 = p_0 \subset \cdots \subset p_{n-1} \subset P_n$ with $f(p_{i_{p_i}}) = i$* $= \operatorname{height} p_i$ *for* $i = 1, \ldots, n - 1$.

PROOF. Since P_0 is minimal, $f(P_{0_{P_0}}) = 0$. If $n = 1$, we are done. If $n > 1$, then let $W = \{ p \in \operatorname{spec} R \,|\, P_0 \subset p \subset P_2 \text{ is saturated}\}$. By [8, Proposition 2.2], W is infinite. As f is locally modulated, $f(p_p) \leqslant 1$ for all $p \in W$. Let $W_0 = \{ p \in W \,|\, f(p_p) = 0\}$. If W_0 is infinite, then (P_0, W_0) is a conforming pair, forcing $f(P_{0_{P_0}}) < 0$. This cannot be. Thus all but finitely many $p \in W$ have $f(p_p) = 1$. By [4, Theorem 1] all but finitely many $p \in W$ also have $\operatorname{height} p = 1$. We thus replace P_1 by some $p_1 \in W$ with $f(p_{1_{p_1}}) = 1 = \operatorname{height} p_1$. The other p_i, $i = 1, 2, \ldots, n - 1$, are similarly found.

Let f be a grade function on R. In Theorem 4.5 we saw a way of extending f to a grade function g on $R[X]$. If g is locally modulated on $R[X]$, it is trivial to see that f is locally modulated on R. For nice rings, the converse also holds, as we will see in Theorem 6.3. (For instance any locally quasi-unmixed ring satisfies the hypothesis of Theorem 6.3, and so fits our present use of 'nice'. Most Noetherian rings found in

classical situations are locally quasi-unmixed.) However, for nonnice rings the converse may fail.

EXAMPLE. By [7, Example 2, pp. 203–205] there is a 2-dimensional local domain (R, M) such that $R[X]$ has primes $0 \subset p \subset q$ with $p \cap R = 0$ but $p \neq 0$ and $q \cap R = M$ but $q \neq MR[X]$, and with height$(q/p) = 1$. Let f be the height function on R which is locally modulated. Now g is the height function on $R[X]$, but is not locally modulated since height $q = 3 >$ height $p + 1$. Our next result shows that this is the only sort of way that g can escape being locally modulated if f is.

THEOREM 6.3. *Let X be an indeterminate over R. Suppose whenever $p \subset q$ are primes in $R[X]$ with $p \neq (p \cap R)R[X]$ and $q \neq (q \cap R)R[X]$, if height $q/p = 1$ then* height $q \cap R/p \cap R = 1$. *Furthermore, let f be a locally modulated grade function on R, and let g be the grade function on $R[X]$ defined in Theorem 4.5. Then g is locally modulated on $R[X]$.*

PROOF. Let $p \subset q$ be primes of $R[X]$ with height$(q/p) = 1$. By Theorem 6.1 we need $f(q_q) \leqslant f(p_p) + 1$. We proceed in various ways, using $P = p \cap R$ and $Q = q \cap R$.

Case 1: $p \neq PR[X]$ and $q \neq QR[X]$. Then by hypothesis, height$(Q/P) = 1$, and since f is modulated, $f(Q_Q) \leqslant f(P_P) + 1$. By definition of g, $g(q_q) = f(Q_Q) + 1 \leqslant f(P_P) + 2 = g(p_p) + 1$.

Case 2: $p = PR[X]$ and $q = QR[X]$. Clearly height$(Q/P) = 1$, and $g(q_q) = f(Q_Q) \leqslant f(P_P) + 1 = g(q_q) + 1$.

Case 3: $p \neq PR[X]$ and $q = QR[X]$. By [4, Theorem 6] there is a prime P' of R with $P \subset P' \subset Q$ saturated. Since f is modulated, we have $f(Q_Q) \leqslant f(P_P) + 2$. Thus $g(q_q) = f(Q_Q) \leqslant f(P_P) + 2 = g(p_p) + 1$. The only remaining case is easy.

REMARK. The converse of Theorem 6.3 fails. That is, g being locally modulated does not imply that $R[X]$ has the stated property. We again take the ring mentioned in the example prior to Theorem 6.3, but this time we define f by specifying $f(P_P) =$ height P for all $P \in$ spec $R - \{M\}$, while $f(M) = 0$. The g produced on $R[X]$ is easily seen to be locally modulated. However we already have primes $p \subset q$ in $R[X]$ with $p \neq (p \cap R)R[X]$, $q \neq (q \cap R)R[X]$, height$(q/p) = 1$, but with height$((q \cap R)/(p \cap R)) > 1$.

LEMMA 6.4. *Let f be a grade function on R, and let I be an ideal of R. There is a maximal ideal M of R such that $I \subseteq M$ and $f(I) = f(I_M)$.*

PROOF. By Lemma 1.2(v), there is an $I \subseteq P \in$ spec R with $f(I) = f(P_P)$. Let M be a maximal ideal containing P. Now $f(I_M) \geqslant f(I) = f(P_P) \geqslant f(P_M) \geqslant f(I_M)$, and so equally holds throughout.

Recall that a Noetherian ring R satisfies the *first chain condition* if every maximal chain of primes (i.e. every saturated chain of primes between a minimal prime and a maximal prime) has length equal to dim R.

THEOREM 6.5. *Let f be a grade function on R. The following are equivalent.*

(a) *f restricted to R_M is modulated and $f(M) =$ height M for all maximal ideals M.*

(b) *$f(I) =$ height I for all ideals I and R_M satisfies the first chain condition for all maximal ideals M.*

PROOF. (a) $\Rightarrow$ (b) Let (a) hold, and suppose for some I, $f(I) \ne$ height I. We will derive a contradiction. We have $f(I) <$ height I, and by Lemma 6.4, for some maximal M containing I, $f(I_M) = f(I) <$ height $I \le$ height I_M. We may assume that R is local at M. We may also assume that I is maximal with respect to having $f(I) <$ height I. Let $x_1, \ldots, x_n$ be a maximal f-sequence in I. Then there is some $I \subseteq P \in A_f(x_1, \ldots, x_n)$. Now height $P \ge$ height $I > f(I) = n = f(P)$, and by the maximality of I, $I = P$ is prime. Since, by assumption, $f(M) =$ height M, $I \ne M$. Pick $a \in M - I$. As f is modulated on R, the maximality of I gives height$(I, a) = f(I, a) \le f(I) + 1 <$ height $I + 1$. Thus height$(I, a) =$ height I, and since I is prime, $a \in I$. This is our contradiction.

We must show that (a) implies R_M satisfies the first chain condition for each maximal ideal M. Again we may assume R is local at M. Let $p_0 \subset p_1 \subset \cdots \subset p_{m-1} \subset M$ be a maximal chain of primes. We need height $M = m$. By [4, Theorem 5] we may assume that height $p_{m-1} = m - 1$. Let $P = p_{m-1}$. Now $P \subset M$ is saturated and $m - 1 =$ height $P = f(P)$ (by the first part of this proof). Pick $b \in M - P$. As f is modulated, $f(P, b) \le f(P) + 1 = m$. Since $P \subset M$ is saturated, M is the only prime containing (P, b), and so $m \ge f(P, b) = f(M) =$ height M. Since our original chain of primes shows height $M \ge m$, we are done.

(b) $\Rightarrow$ (a) Assuming (b), we must show that for any maximal ideal M, f restricted to R_M is modulated. If $P \in$ spec R, (b) implies height $P = f(P) \le f(P_P) \le$ height P. That is, $f(P_P) =$ height P. From this it easily follows that f restricted to R_M is just the height function on R_M. Therefore we may assume that R is local at M. Let I be an ideal and $a \in M$. Since R satisfies the first chain condition, $f(I, a) =$ height$(I, a) \le$ height $I + 1 = f(I) + 1$, so that f is modulated.

REMARKS. (i) It is easily seen that condition (b) in Theorem 6.5 implies that f is locally modulated, since condition (b) localizes.

(ii) We previously noted that if f is any of classical, essential, or asymptotic grade, then f is locally modulated. Thus if on some ring R, $f(M) =$ height M for all maximal ideals M, it follows that $f(I) =$ height I for all ideals I.

We now look at unmixedness. The classical unmixedness theorem states that if R is a Cohen-Macaulay ring and $I = (x_1, \ldots, x_m)$ with $m =$ height I, and $n \ge 1$ is an integer, then $P \in \text{Ass}(R/I^n) = A_1(I^n)$ implies height $P =$ height I. We will show that if the grade function f on R satisfies the equivalent conditions of Theorem 6.5, then with I as above and A any grade scheme for f, $P \in A(I)$ implies height $P =$ height I. As for powers of I, we show that $P \in A_f(I^n)$ implies height $P =$ height I, and that this also works for our natural grade schemes A_1 through A_6.

THEOREM 6.6 *Let f be a grade function on R and suppose for each maximal ideal M of R, $f(M) =$ height M and f is modulated on R_M. Let A be any grade scheme for f, and let $I = (x_1, \ldots, x_m)$ with $m =$ height I. Also let $n \ge 1$ be an integer.*

(a) *If $P \in A(I)$, then* height $P =$ height I.

(b) *If $A \subseteq A_f$ and if $P \in A(I^n)$, then* height $P =$ height I.

(c) *If f is classical grade and either $P \in A_1(I^n)$ or $P \in A_2(I^n)$, then* height $P =$ height I.

(d) *If f is essential grade and either $\overset{\spadesuit}{P} \in A_3(I^n)$ or $P \in A_4(I^n)$, then* height $P =$ height I.

(e) *If f is asymptotic grade and either $P \in A_5(I^n)$ or $P \in A_6(I^n)$, then* height $P =$ height I.

PROOF. (a) Since condition (a) of Theorem 6.5 holds, so does condition (b). Thus f is just the height function (on all localizations of R by the previous remark). We consider the case $A = A_f$. If $P \in A_f(I)$ then height $P = f(P_P) = f(I_P) =$ height I_P $=$ height I, the last equality following from the hypothesis on I. Thus (a) holds when $A = A_f$. For arbitrary A, if $P \in A(I)$ then since the hypotheses localize, we may assume that R is local at P and satisfies the first chain condition (Theorem 6.5(b)). Since $I = (x_1,\ldots,x_m)$ with $m =$ height I, it is not hard to see that height$(x_1,\ldots,x_i)$ $= i$ for all $i = 1,\ldots,m$, and so by the first part of this proof, $A_f(x_1,\ldots,x_i)$ consists of the primes minimal over $(x_1,\ldots,x_i)$. Therefore we see that $x_1,\ldots,x_m$ is an f-sequence. Using Lemma 1.4, $P \in A(I) = A_f(I)$ and so height $P =$ height I by the preceding.

(b) Since f is just height, $A_f(J)$ consists of the primes minimal over J for any ideal J. Thus $A \subseteq A_f$ implies $A = A_f$, and also $A_f(I^n) = A_f(I)$. Therefore $P \in A(I^n)$ implies $P \in A_f(I)$ and so height $P =$ height I.

(c) In part (a) we saw that $x_1,\ldots,x_m$ is an f-sequence. By [1, Exercise 13, p. 103], $A_1(I^n) = A_1(I)$. Thus $P \in A_1(I^n)$ implies height $P =$ height I by part (a). Now for any ideal J, it is easily seen that $A_2(J^n) = A_2(J)$, and so the same argument works for A_2.

(d) For any ideal J it is easily seen that $A_3(J^n) = A_3(J)$ and $A_4(J^n) = A_4(J)$. Thus argue as in (c).

(e) For any ideal J, $A_5(J^n) = A_5(J)$ and $A_6(J^n) = A_6(J)$. The latter is easy. The former uses [5, Propositions 3.18 and 4.1]. Now argue as in (c).

Finally, a difficult question.

QUESTION. Let R and R' be two Noetherian rings such that spec R and spec R' are isomorphic as partially ordered sets under inclusion. Then a grade function f on R can be transferred to R' by letting $f'(P'_{P'}) = f(P_P)$, where $P' \in$ spec R' is the image of $P \in$ spec R under the isomorphism. Suppose that f is a locally modulated grade function on R. Is there an R' with spec $R' \approx$ spec R such that the transferred grade function f' on R' equals classical grade (or essential or asymptotic grade) on R'?

7. Sequences over ideals.

DEFINITION. Let A be a grade scheme on R, and let I be an ideal of R. Define an *A-sequence over I* to be a sequence $x_1,\ldots,x_n$ of elements of R such that $(I, x_1,\ldots,x_n) \neq R$ and for each $i = 1,\ldots,n$, $x_i \notin A(I, x_1,\ldots,x_{i-1})$.

In [5, Chapter VI] this concept is considered for A_5, while [3 and 11] look at it for A_3 and A_4. (To the best of our knowledge it has not been studied for A_1, A_2 or A_6.) We note that since $A_4 \subseteq A_3$ an A_3-sequence over I will be an A_4-sequence over I, but the converse fails despite the fact that A_3 and A_4 are both grade schemes for essential

grade. Thus we must specify the grade scheme, not just the grade function. Also it is not hard to find a grade scheme on a local ring and an ideal I having two maximal A-sequences over I of different length. (A_3, A_4 and A_5 escape this flaw.)

In this section we will look at A-sequences over I when A is a grade scheme for f and $A_f \subseteq A$. We give a very easy proof of a generalization of the following known facts. If f is essential (respectively, asymptotic) grade, A is A_3 (respectively, A_5) and I is an ideal in a local ring with $x_1, \ldots, x_n$ and A-sequence over I, then $n \leqslant f(M) - f(I)$ [3, Theorem 6.5] (respectively, [5, Proposition 6.9]). We then consider A_f-sequences over I when f is locally modulated.

LEMMA 7.1. *Let f be a grade function on R. Let A be any grade scheme for f with $A_f \subseteq A$. Let I be an ideal and let $x_1, \ldots, x_n$ be an A-sequence over I. Then $f(I, x_1, \ldots, x_n) \geqslant f(I) + n$.*

PROOF. Since $x_2, \ldots, x_n$ is an A-sequence over (I, x_1), it will suffice to let $n = 1$. Clearly $f(I, x_1) \geqslant f(I)$. If equality holds, then for some $(I, x_1) \subseteq P \in \operatorname{spec} R$, $f(P_P) = f(I, x_1) = f(I)$. However, this shows that $P \in A_f(I) \subseteq A(I)$, contradicting that x_1 is an A-sequence over I.

Since $A_f \subseteq A_4 \subseteq A_3$ for $f = $ essential grade, and $A_f \subseteq A_6 \subseteq A_5$ for $f = $ asymptotic grade, our next theorem incorporates the known results mentioned above.

THEOREM 7.2. *Let f be a grade function on a local ring (R, M) and let A be any grade scheme for f with $A_f \subseteq A$. If $x_1, \ldots, x_n$ is an A-sequence over I, then $n \leqslant f(M) - f(I)$.*

PROOF. $f(M) \geqslant f(I, x_1, \ldots, x_n) \geqslant f(I) + n$ by Lemma 7.1.

LEMMA 7.3. *Let f be a modulated grade function on a local ring (R, M), and let A be any grade scheme for f with $A_f \subseteq A$. Let I be an ideal and let $x_1, \ldots, x_n$ be an A-sequence over I. Then $f(I, x_1, \ldots, x_n) = f(I) + n$.*

PROOF. Lemma 7.1 gives one inequality and modulation gives the other.
REMARK. Lemma 7.3 requires that R be local, even if f is locally modulated.

THEOREM 7.4. *Let f be modulated on a local ring (R, M). Let I be an ideal, and let $x_1, \ldots, x_n$ be a maximal A_f-sequence over I. Then $n = f(M) - f(I)$.*

PROOF. By maximality, $M \in A_f(I, x_1, \ldots, x_n)$. Thus $f(M) = f(I, x_1, \ldots, x_n) = f(I) + n$, using Lemma 7.3.
The next corollary is similar to [5, Proposition 6.14 and 3, Theorem 6.2].

COROLLARY 7.5. *Let f be a locally modulated grade function on a local ring (R, M). Let A be any grade scheme for f with $A_f \subseteq A$. Let I be an ideal and let $x_1, \ldots, x_n$ be an A-sequence over I. Then $n \leqslant \min\{$little depth $P | I \subseteq P \in \operatorname{spec} R$ and $f(I) = f(P_P)\}$.*

PROOF. We have $n \leqslant f(M) - f(I)$ by Theorem 7.2. If P is as above, then $n \leqslant f(M) - f(P_P)$. Suppose little depth $P = m$. Then Theorem 6.1 easily shows $f(M) \leqslant f(P_P) + m$, and so $n \leqslant m$.

COROLLARY 7.6. *Let f be a locally modulated grade function on R. Let I be an ideal and let $x_1, \ldots, x_n$ be an A_f-sequence over I. Let $I \subseteq P \in \operatorname{spec} R$ with $f(I) = f(P_P)$. If $(P, x_1, \ldots, x_n) \neq R$, then $x_1, \ldots, x_n$ is an A_f-sequence over P.*

PROOF. We induct on n, the case $n = 0$ being trivial. Thus suppose we already have that $x_1, \ldots, x_{n-1}$ is an A_f-sequence over P. If our result fails, then for some $Q \in A_f(P, x_1, \ldots, x_{n-1})$ we have $x_n \in Q$. It is clear that in R_Q, $x_1, \ldots, x_n$ and $x_1, \ldots, x_{n-1}$ are A_f-sequences over I_Q and P_Q respectively, the latter being maximal. By Theorem 7.4, $f(Q_Q) - f(P_Q) = n - 1$ and $f(Q_Q) - f(I_Q) \geqslant n$. Therefore $f(P_Q) > f(I_Q)$. However, by hypothesis, $f(P_P) = f(I) \leqslant f(I_Q) < f(P_Q) \leqslant f(P_P)$, an obvious contradiction.

EXAMPLE. The preceding fails for arbitrary $P \in A_f(I)$. Let (R, M) be a 3-dimensional local domain. Let Q and P be height 2 primes and let $I = P \cap Q$. Define f by specifying $f(p_p) = \text{height } p$ for all $p \in \operatorname{spec} R - \{Q, M\}$, while $f(Q_Q) = 1$ and $f(M) = 2$. Thus f is locally modulated. Clearly $A_f(I) = \{Q, P\}$. Pick $x \in M - (Q \cup P)$ so that x is an A_f-sequence over I. However x is not an A_f-sequence over P since $f(M) = f(P)$ implies $M \in A_f(P)$.

PROPOSITION 7.7. *Let f be a locally modulated grade function on R. Let I be an ideal and let $x_1, \ldots, x_n$ be an A_f-sequence over I. Suppose Q is a prime containing $(I, x_1, \ldots, x_n)$. The following are equivalent.*

(i) $Q \in A_f(I, x_1, \ldots, x_n)$.
(ii) $f(Q_Q) = f(I_Q) + n$.
(iii) *There is a $P \in A_f(I)$ with $P \subseteq Q$, $f(P_P) = f(I_Q)$ and $Q \in A_f(P, x_1, \ldots, x_n)$.*

PROOF. (i) $\Rightarrow$ (ii). If (i) holds, then in R_Q, $x_1, \ldots, x_n$ is a maximal A_f-sequence over I_Q, and so Theorem 7.4 gives (ii).

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. There is some $I_Q \subseteq P_Q \in \operatorname{spec} R_Q$ with $f(I_Q) = f(P_P)$. Clearly $P \in A_f(I)$. By Corollary 7.6 applied in R_Q, $x_1, \ldots, x_n$ is an A_f-sequence over P_Q. By Lemma 7.3, $f((P, x_1, \ldots, x_n)R_Q) = f(P_Q) + n$. Since $f(P_P) = f(I_Q) \leqslant f(P_Q) \leqslant f(P_P)$, we have $f(P_Q) = f(I_Q)$. Thus $f((P, x_1, \ldots, x_n)R_Q) = f(I_Q) + n = f(Q_Q)$ by (ii). This shows $Q \in A_f(P, x_1, \ldots, x_n)$.

(iii) $\Rightarrow$ (i). If (iii) holds, then

$$f(Q_Q) = f((P, x_1, \ldots, x_n)R_Q) \leqslant f(P_Q) + n \quad \text{(by local modulation)}$$

$$\leqslant f(P_P) + n = f(I_Q) + n \leqslant f((I, x_1, \ldots, x_n)R_Q) \quad \text{(by Lemma 7.1)}$$

$$\leqslant f(Q_Q).$$

Equality holds throughout, and so $Q \in A_f(I, x_1, \ldots, x_n)$.

COROLLARY 7.8. *Let f be a locally modulated grade function on R. Let I be an ideal. Suppose $x_1, \ldots, x_n$ is an A_f-sequence over P for every $P \in A_f(I)$. Then $x_1, \ldots, x_n$ is an A_f-sequence over I.*

PROOF. We induct on n. Suppose we already know $x_1, \ldots, x_{n-1}$ is an A_f-sequence over I. If our result fails, then for some $Q \in A_f(I, x_1, \ldots, x_{n-1})$ we have $x_n \in Q$. By Proposition 7.7, for some $P \in A_f(I)$, $Q \in A_f(P, x_1, \ldots, x_{n-1})$. Since $x_n \in Q$ we have contradicted that $x_1, \ldots, x_n$ is an A_f-sequence over P.

REMARK. Comparing Corollary 7.6 (and the example following it) to Corollary 7.8 shows that it might be of interest to study the set of ideals I for which $A_f(I) = \{P \in \operatorname{spec} R \mid I \subseteq P$ and $f(I) = f(P_P)\}$. We will resist the temptation to do so here, except to point out that I has this property if and only if $f(P_P) = f(I)$ for every minimal prime divisor P of I, as is easily seen.

BIBLIOGRAPHY

1. I. Kaplansky, *Commutative rings*, Univ. of Chicago Press, Chicago, Ill., 1974.
2. D. Katz, *Prime divisors, asymptotic R-sequences, and unmixed local rings* (manuscript).
3. D. Katz and L. J. Ratliff, Jr., *Essential sequences over an ideal and essential cograde*. II (manuscript).
4. S. McAdam, *Saturated chains in Noetherian rings*, Indiana Univ. Math. J. **23** (1974), 719–728.
5. _____, *Asymptotic prime divisors*, Lecture Notes in Math., vol. 1023, Springer-Verlag, Berlin and New York, 1983.
6. S. McAdam and L. J. Ratliff, Jr., *Essential sequences* (manuscript).
7. M. Nagata, *Local rings*, Interscience, New York, 1962.
8. L. J. Ratliff, Jr., *Characterizations of catenary rings*, Amer. J. Math. **93** (1971), 1070–1108.
9. _____, *On asymptotic prime divisors* (manuscript).
10. _____, *Asymptotic sequences*, J. Algebra **85** (1983), 337–360.
11. _____, *Essential sequences over ideals and essential cograde* (manuscript).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712