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FREE LATTICE-ORDERED GROUPS REPRESENTED AS o-2 TRANSITIVE l-PERMUTATION GROUPS[1]

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ABSTRACT. It is easy to pose questions about the free lattice-ordered group F_η of rank $\eta > 1$ whose answers[2] are "obvious", but difficult to verify. For example:

1. What is the center of F_η?
2. Is F_η directly indecomposable?
3. Does F_η have a basic element?
4. Is F_η completely distributive?

Question 1 was answered recently by Medvedev, and both 1 and 2 by Arora and McCleary, using Conrad's representation of F_η via right orderings of the free group G_η. Here we answer all four questions by using a completely different tool: The (faithful) representation of F_η as an o-2-transitive l-permutation group which is pathological (has no nonidentity element of bounded support). This representation was established by Glass for most infinite η, and is here extended to all $\eta > 1$. Curiously, the existence of a transitive representation for F_η implies (by a result of Kopytov) that in the Conrad representation there is some right ordering of G_η which suffices all by itself to give a faithful representation of F_η. For finite η, we find that every transitive representation of F_η can be made from a pathologically o-2-transitive representation by blowing up the points to o-blocks; and every pathologically o-2-transitive representation of F_η can be extended to a pathologically o-2-transitive representation of F_{ω_0}.

Whether F_η has a pathologically o-2-transitive representation when η is finite was described by Glass as a "basic unsolved problem" [7, p. 138]. Appropriately, establishing the existence of such a representation will make the four introductory questions exceedingly easy to answer.

For background, see [**7** or **8**]. The present paper is almost completely independent of [**1**].

1. Background. Let $\mathbf{x}$ be a subset of an l-group F. F *is free on* $\mathbf{x}$ if every function from $\mathbf{x}$ into an arbitrary l-group H can be extended uniquely to an l-homomorphism from F into H. Any two l-groups free on sets of the same cardinality η are l-isomorphic.

F is *free* if it is free on some subset $\mathbf{x}$. The *rank* of F means the cardinality of $\mathbf{x}$, which is well defined by [**1**, Proposition 1].

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[1] This is an expanded version of one aspect of results presented in 1982 at the Special Session on Ordered Groups in Cincinnati, Ohio [**13**].

[2] The answers are "trivial", "yes", "no", "no".

Thus for each cardinality η, there is a unique free l-group F_η of rank η. As is well known, F_1 is the direct sum $\mathbf{Z} \boxplus \mathbf{Z}$ of two copies of the integers $\mathbf{Z}$. Most of our results (including the answers to all four introductory questions) fail for $\eta = 1$, and we shall usually assume that $\eta > 1$.

When dealing with F_η, we shall always envision a fixed $\mathbf{x}$ on which F_η is free. The subgroup generated by $\mathbf{x}$ is a free group on $\mathbf{x}$, the free group G_η of rank η. If $\mathbf{x}' \subseteq \mathbf{x}$, then the l-subgroup $l(\mathbf{x}')$ generated by $\mathbf{x}'$ is the free l-group on $\mathbf{x}'$. Thus when $\eta' < \eta$, $F_{\eta'}$ is an l-subgroup of F_η.

A substitution for F_η in $A(\Omega)$, Ω a chain and $A(\Omega)$ the l-group of all automorphisms of Ω, is simply a function $\mathbf{x} \to A(\Omega)$, an assignment to each free generator x of an o-automorphism $\hat{x}$ of Ω; or equivalently, an l-homomorphism $w \to \hat{w}$ from F_η into $A(\Omega)$, known also as an *action* of F_η on Ω. When the action is a representation (i.e., faithful) and when there is no danger of confusion, we shall sometimes denote $\hat{x}$ by x, i.e., we shall speak of x as actually being an o-automorphism of Ω.

Each $w \in F$ can be put in a standard form $w = \bigvee_i \bigwedge_j w_{ij}$, a finite sup of finite infs of elements of G, i.e, of reduced group words in the elements of $\mathbf{x}$. Of course, this standard form is far from unique.

Our fundamental tool is the notion of a *diagram for* w [**9**], which we now define. Each group word w_{ij} has the form $x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1}$ ($n \geqslant 0$). The *points* of the diagram are the initial segments $x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1}$ ($k \geqslant 0$) of the w_{ij}'s. (Of course, a given point may arise from several w_{ij}'s). For each ordered pair (α, β) of points such that $\alpha x_i^{\pm 1} = \beta$, the diagram includes an x_i-*arrow* from α to β if the exponent on x_i is $+1$, otherwise from β to α. The remaining aspect of the diagram is a total order on the set D of points which is consistent with the arrows in that if there are x_i-arrows from α_1 to β_1 and from α_2 to β_2 (same i for both), then $\alpha_1 \leqslant \alpha_2$ iff $\beta_1 \leqslant \beta_2$.

An x_i-arrow from α to β may alternately be described as an x_i^{-1}-arrow from β to α. We emphasize that a diagram is necessarily *connected*, meaning that for all points α, β, there must be at least one sequence of arrows leading from α to β; and *loop-free*, meaning that there cannot be more than one such sequence. (The latter property is essential for some of the later results, though not for Theorem 1.)

Given any substitution $x \to \hat{x}$ for F_η in $A(\Omega)$, and any $\alpha \in \Omega$, there arises a diagram for w (with $x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1} \leftrightarrow \alpha\hat{x}_{i_1}^{\pm 1} \cdots \hat{x}_{i_k}^{\pm 1}$, and with an $x_{i_k}^{\pm 1}$-arrow from $\alpha\hat{x}_{i_1}^{\pm 1} \cdots \hat{x}_{i_{k-1}}^{\pm 1}$ to $\alpha\hat{x}_{i_1}^{\pm 1} \cdots \hat{x}_{i_k}^{\pm 1}$) provided the points $\alpha\hat{x}_{i_1}^{\pm 1} \cdots \hat{x}_{i_k}^{\pm 1}$ are distinct. For any $e \neq w \in F_\eta$ and any $\alpha \in \mathbf{Q}$ ($\mathbf{Q}$ the rational numbers), there is a substitution for F_η in $A(\mathbf{Q})$ from which there arises a diagram for w which shows $e \neq w$ by having $\alpha \neq \max_i \min_j \alpha\hat{w}_{ij} = \alpha\hat{w}$ [**9**].

2. The pathologically o-2-transitive representation of F_η. A lattice-ordered permutation group (l-permutation group) (F, Ω) is o-2-*transitive* if for all $\alpha < \beta$ and $\gamma < \delta$ in Ω, there exists $f \in F$ such that $\alpha f = \gamma$ and $\beta f = \delta$. If in addition, no $e \neq f \in F$ has bounded support, (F, Ω) is called *pathological*.

MAIN THEOREM 1 (GLASS - MCCLEARY). *The free l-group F_η $(\eta > 1)$ has a (faithful) pathologically o-2-transitive representation.*

REMARKS ABOUT INFINITE η. This case is basically due to Glass [6, Corollary 2 or 7, p. 137], who established the result for nonregular η with the aid of the Generalized Continuum Hypothesis. Glass' proof was slightly modified by the present author to get rid of these restrictions. Since Glass' proof serves as a starting point for the present proof, and since the modification just referred to appears only in the Errata for [7], we give the modified proof here:

PROOF FOR INFINITE η. F_η can be (faithfully) represented on $\Omega = \Omega_\eta$ for some ordered field Ω_η of cardinality η [7, p. 332], and hence also on any open interval Δ of Ω (since Δ is o-isomorphic to Ω). We select a set $\mathscr{D}$ of pairwise disjoint open intervals of Ω such that there is no greatest one and their union is cofinal in Ω, and select a representation of F_η on each $\Delta \in \mathscr{D}$. Since η is infinite, we can establish a one-to-one correspondence between the free generating set $\mathbf{x}$ and the set of ordered pairs $((\alpha, \beta), (\gamma, \delta))$ of Ω with $\alpha < \beta$ and $\gamma < \delta$. The desired representation of F_η is obtained by using the multiple transitivity of Ω to arrange two things about the image $\hat{x}$ of an arbitrary free generator x (corresponding to $((\alpha, \beta), (\gamma, \delta))$):

(a) $\alpha\hat{x} = \gamma$ and $\beta\hat{x} = \delta$; and

(b) for each $\Delta \in \mathscr{D}$ such that α, β, γ, $\delta < \inf \Delta \in \overline{\Omega}$, $\hat{x}$ acts according to the selected representation on Δ.

Clearly the image $(\hat{F}_\eta, \Omega)$ is o-2-transitive. For each $e \neq w \in F_\eta$, w involves only finitely many free generators, so $e \neq \hat{w}$ on a cofinal set of Δ's. Thus the representation is faithful and pathological.

PROOF FOR FINITE η. We shall represent F_η on $\mathbf{Q}$. For each $x \in \mathbf{x}$, the action of its image $\hat{x}$ on $\mathbf{Q}$ will be specified at enough points to guarantee the desired results.

F_η is countable, and we enumerate its nonidentity elements: $w_0, w_1, \ldots$. In the interval $[0, 1]$, we lay out a copy of a diagram for $w_0 = \bigvee_i \bigwedge_j \prod_k x_{ijk}^{\pm 1}$ showing $e \neq w_0$, with the smallest point r_0 of the diagram taken to be 0 and the largest point t_0 taken to be 1. We specify about the $\hat{x}$'s that the point (corresponding to) $x_{ij1}^{\pm 1} \cdots x_{i,j,k-1}^{\pm 1}$ be sent by $\hat{x}_{ijk}^{\pm 1}$ to the point (corresponding to) $x_{ij1}^{\pm 1} \cdots x_{i,j,k-1}^{\pm 1} x_{ijk}^{\pm 1}$. Similarly, in each interval $[2n, 2n + 1]$, we lay out such a diagram for w_n and make such specifications.

This is enough to give a faithful action of F_η on $\mathbf{Q}$, but we need more. We want all the points in the various diagrams to lie in the same orbit of $\hat{F}_\eta$. For this it suffices to arrange that for each $n = 0, 1, \ldots,$ the points $2n + 1$ and $2(n + 1)$ lie in the same orbit, and now we construct the appropriate "bridges".

We begin with the interval $[1, 2]$. In the original diagram for w_0, t_0 ($\leftrightarrow$ 1) must have been moved by at least one free generator, say x_{t_0}; and in the diagram for w_1, r_1 ($\leftrightarrow$ 2) must have been moved by some free generator x_{r_1}. We decree that

(a_1) $1\hat{x}_{t_0} = \frac{4}{3}$ if t_0 was moved up by x_{t_0},

(a_2) $1\hat{x}_{t_0}^{-1} = \frac{4}{3}$ if t_0 was moved down by x_{t_0},

(b_1) $\frac{5}{3}\hat{x}_{r_1} = 2$ if r_1 was moved up by x_{r_1},

(b_2) $\frac{5}{3}\hat{x}_{r_1}^{-1} = 2$ if r_1 was moved down by x_{r_1}.

To connect $\frac{4}{3}$ and $\frac{5}{3}$, we further decree that $\frac{4}{3}\hat{x}_{t_0} = \frac{5}{3}$ if (a_1) obtains (or that $\frac{4}{3}\hat{x}_{t_0}^{-1} = \frac{5}{3}$ if (a_2) obtains); except that this may conflict with (b_2) or (b_1) if $x_{t_0} = x_{r_1}$, so in that case we pick any other $x \in \mathbf{x}$ ($\eta > 1$) and decree that $\frac{4}{3}\hat{x} = \frac{5}{3}$.

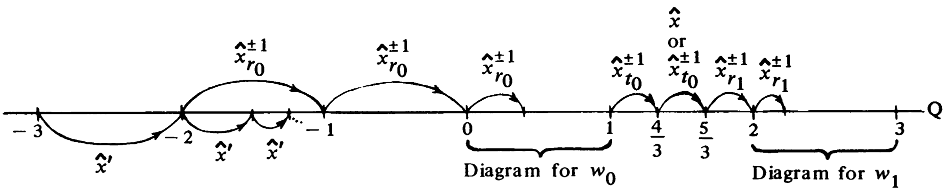

FIGURE 1

We build similar bridges in the other intervals $[2n + 1, 2n + 2]$, $n = 1, 2, \ldots$.

We decree also that $(-2)\hat{x}_{r_0}^{\pm 1} = -1$ and $(-1)\hat{x}_{r_0}^{\pm 1} = 0$; and, picking any $x' \neq x_{r_0}$, that $-(n+1)\hat{x}' = -n$, $n = 2, 3, \ldots$, and that $(-1 - 1/n)\hat{x}' = -1 - 1/(n+1)$, $n = 1, 2, \ldots$. Now all diagram points, and all negative (as well as positive) integers, and all points of the form $-1 - 1/n$ do indeed lie in some one orbit Ω' of $\hat{F}_\eta$, and the action of $\hat{F}_\eta$ on Ω' is faithful. Because of $\hat{x}'$, the stabilizer subgroup of -1 has a (convex) orbit which includes all points in Ω' that are less than -1, forcing the action of $\hat{F}_\eta$ on Ω' to be o-2-transitive.

Finally, we modify the above procedure so that each $e \neq w \in F_\eta$ is used for infinitely many of the intervals $[2n, 2n + 1]$, and then the action of $\hat{F}_\eta$ on Ω' becomes pathological as well. Since Ω' is countable, dense in itself, and lacks end points, Ω' is o-isomorphic to **Q**.

Now we use Theorem 1 to answer the four questions raised in the abstract.

COROLLARY 2 (MEDVEDEV [**14**], and [**1**]). F_η $(\eta > 1)$ *has trivial center.*

PROOF. Every o-2-transitive l-permutation group (F, Ω) has trivial center. For let $e \neq f \in F$. Then $\alpha \neq \alpha f$ for some $\alpha \in \Omega$, and we pick $g \in F$ such that $\alpha g = \alpha$ but $(\alpha f)g \neq \alpha f$. Then $\alpha(gf) = \alpha f \neq \alpha(fg)$.

The next corollary gives a negative answer to Question 2.

COROLLARY 3 (ARORA AND McCLEARY [**1**]). F_η $(\eta > 1)$ *is finitely subdirectly irreducible.*

PROOF. No transitive l-permutation group can have two nontrivial l-ideals whose intersection is trivial: If H and K are l-ideals, and if $e < h \in H$ and $e < k \in K$, then some conjugate k^f of k fails to be disjoint from h, and $e < h \wedge k^f \in H \cap K$.

COROLLARY 4. F_η $(\eta > 1)$ *has no basic elements.*

PROOF. No o-2-transitive l-permutation group (F, Ω) has basic elements. Let $e < f \in F$, and pick $\alpha \in \Omega$ such that $\alpha < \alpha f$. Since Ω is dense in itself, we may pick $\beta_1, \beta_2 \in \Omega$ such that $\alpha < \beta_1 < \beta_2 < \alpha f$. Now pick $g_1, g_2 \in G$ such that $\beta_1 g_1 = \beta_1$ but $\beta_2 g_1 > \beta_2$, and $\beta_2 g_2 = \beta_2$ but $\beta_1 g_2 > \beta_1$. Let $k_i = (g_i \wedge f) \vee e$, so that $e < k_i < f$. Then k_1 and k_2 are incomparable, so f is not basic.

Now we can answer a question raised in [**1**].

COROLLARY 5. *The free group* G_η $(\eta > 1)$ *has no finite subset S for which there is a unique right ordering of* G_η *making all elements of S positive.*

PROOF. The existence of such an S is equivalent to the existence of a basic element in F_η [**1**, Proposition 20].

COROLLARY 6. *F_η ($\eta > 1$) has the following properties*:

(a) *F_η is not completely distributive.*

(b) *The distributive radical $D(F_\eta)$ is F_η itself.*

(c) *F_η has no proper closed prime subgroups.*

(d) *F_η has no proper closed l-ideals.*

Moreover, every nontrivial l-ideal of F_η enjoys these same four properties.

PROOF. These properties are enjoyed by all pathologically o-2-transitive l-permutation groups. The first three are part of [**12**, Theorem 1] and the fourth is part of [**11**, Theorem 9]. Furthermore, [**12**, Theorem 6] says that every nontrivial l-ideal of a pathologically o-2-transitive l-permutation group is itself pathologically o-2-transitive.

An element f of an l-group F is *singular* if $e < f$ and if $e \leqslant g \leqslant f$ implies that $fg^{-1} \wedge g = e$ ($g \in G$). It has been thrice proved that the free abelian l-group A_η ($\eta > 1$) has no singular elements [**5**, **2**, **4**].

COROLLARY 7. *F_η ($\eta > 1$) has no singular elements.*

PROOF. No o-2-transitive l-permutation group (F, Ω) can have singular elements, for Ω is necessarily dense in itself. Thus if $e < f$, we have $\alpha < \alpha f$ for some $\alpha \in \Omega$; and picking β such that $\alpha < \beta < \alpha f$, and $h \in F$ such that $\alpha h = \beta$, $g = (h \wedge f) \vee e$ violates the singularity condition.

COROLLARY 8. *Every nonidentiy element of F_η ($\eta > 1$) has at least $2^{\aleph_0}$ values.*

PROOF. Let $e \neq w \in F_\eta$. Pick an o-2-transitive representation (F_η, Ω). For each $\bar{\alpha} \in \bar{\Omega}$ moved by w, the stabilizer $(F_\eta)_{\bar{\alpha}}$ omits w and is a maximal proper convex l-subgroup of F_η since (F_η, Ω) is o-2-transitive [**8**, Theorem 4.1.1]. Hence $(F_\eta)_{\bar{\alpha}}$ is a value of w. Since every Dedekind complete chain which is dense in itself has cardinality at least $2^{\aleph_0}$, and since these stabilizers are distinct [**12**, Lemma 2], we have exhibited $2^{\aleph_0}$ values for w.

COROLLARY 9. *Let $e < w \in F_\eta$ ($\eta > 1$). Then the infimum of all conjugates of w in F_η is e.*

PROOF. Pick an o-2-transitive representation (F_η, Ω). For any $\alpha < \beta \in \Omega$, some conjugate of w must move α to β.

In contrast to the present paper, [**1**] was based on the other important technique for studying free l-groups F_η, the Conrad representation using the various right orderings of the free group G_η [**3**]. For any one right ordering, G_η acts on itself via the right regular representation, and this action is extended to F_η in the unique possible way. Kopytov [**10**, Corollary 1] has shown that if F_η has any transitive representation whatsoever, then in the Conrad representation of F_η, some one right ordering of G_η will suffice! He applied this to obtain the following result for countable η, and of course he could have stated the result for other infinite η. Now we complete the picture:

COROLLARY 10. *In the Conrad representation of* F_η $(\eta > 1)$, *there exists some one right ordering of the free group* G_η *on which the (transitive) representation is faithful.*

Corollaries 3 and 10 hint at the possibility that F_η might be (infinitely) subdirectly irreducible. That would mean that F_η would have a smallest nontrivial l-ideal L. Pick $e \neq w \in L$. Then for *any* substitution in $A(\mathbf{Q})$ making *any* $\hat{u} = e$ $(e \neq u \in F_\eta)$, $\hat{w}$ would be e (since otherwise the kernel of the substitution would be a nontrivial l-ideal not containing L). Such a w "should not" exist, and we conclude with an argument due to Charles Holland showing that this intuition is correct.

THEOREM 11 (HOLLAND). F_η *is not subdirectly irreducible.*

PROOF. Suppose $e \neq w \in F_\eta$. We shall produce a substitution in $A(\mathbf{R})$, $\mathbf{R}$ the real numbers, for which $\hat{w} \neq e$, but $\hat{u} = e$ for some $e \neq u \in F_\eta$.

We lay out on $\mathbf{R}$ a copy of the diagram for w in which $0 \neq 0\hat{w}$. Since the diagram is loop-free, no x fixes any point in the diagram. What matters about an $\hat{x}$ is only what it does to points in the diagram; *it can be changed at other points* without interfering with the desired "$0 \neq 0\hat{w}$". Hence it can be further arranged that between any two consecutive points of the diagram (and above the largest point, and below the smallest), $\hat{x}$ have at most one fixed point.

We do this for two free generators x and y (if $\eta = 1$, the theorem is obvious), and arrange further that $\hat{y}$ move each fixed point of $\hat{x}$ to a point not fixed by $\hat{x}$. Then $|\hat{x}|$ and $\hat{y}^{-1}|\hat{x}|\hat{y}$ have no common fixed points, and thus $|\hat{x}| \vee \hat{y}^{-1}|\hat{x}|\hat{y}$ has no fixed points at all. Moreover, since the points of the diagram are all contained in some bounded interval, we can arrange that some $t_{1/m} \leqslant |\hat{x}| \vee \hat{y}^{-1}|\hat{x}|\hat{y}$, where $m \in \mathbf{Z}^+$ and $t_{1/m}$ denotes the translation $r \to r + 1/m$. Finally, we can arrange that $\hat{y} \leqslant t_n$ for some $n \in \mathbf{Z}^+$. Then $\hat{y} \leqslant (|\hat{x}| \vee \hat{y}^{-1}|\hat{x}|\hat{y})^{mn}$. Letting $u = (|x| \vee y^{-1}|x|y)^{mn}y^{-1} \wedge e$, we have a substitution making $\hat{w} \neq e$ but $\hat{u} = e$. Finally, $u \neq e$ as an element of F_η since $y \leqslant (|x| \vee y^{-1}|x|y)^{mn}$ is not an l-group identity (it fails in $\mathbf{Z}$).

Since F_η is finitely subdirectly irreducible but not subdirectly irreducible, we have

COROLLARY 12. F_η $(\eta > 1)$ *has no minimal proper l-ideal.*

3. Other transitive representations of F_η. A transitive l-permutation group (F, Ω) is *o-primitive* if it has no proper convex congruences. If in addition, the centralizer of F_η in $A(\overline{\Omega})$ is a cyclic group $\langle z \rangle$ with $e < z$ and with the orbits of $\langle z \rangle$ coterminal in $\overline{\Omega}$, (F, Ω) is called *periodically o-primitive*. Every o-primitive (F, Ω) is o-2-transitive, the regular representation of a subgroup of $\mathbf{R}$, or periodically o-primitive [**8**, Theorem 4.3.1]. The results of this section and the next apply even when $\eta = 1$ because F_1, being abelian but not totally ordered, has no transitive representation.

PROPOSITION 13. *Every o-primitive representation of* F_η *is pathologically o-2-transitive.*

PROOF. By Corollary 6, F_η is not completely distributive, and among o-primitive l-permutation groups, it is precisely those which are pathologically o-2-transitive which fail to be completely distributive [**12**, Theorem 1].

LEMMA 14. (a) *For finite* η, $|x_1| \vee \cdots \vee |x_\eta|$ *is a strong order unit of* F_η, *and thus in any transitive action* (F_η, Ω), *must move every* $\bar{\omega} \in \bar{\Omega}$.

(b) *For infinite* η, F_η *has no strong order units.*

PROOF. Let $w = |x_1| \vee \cdots \vee |x_\eta|$. Any group word g in $x_1, \ldots, x_\eta$ is exceeded by w^p, where p is the number of occurrences of letters in g, because each $x_i^{\pm 1} \leqslant w$. But for infinite η, no power of any w could exceed any letter x not involved in the spelling of w—just choose a substitution making the involved letters be e but $x > e$.

For any convex congruence $\mathscr{C}$ of a transitive l-permutation group (F, Ω), F acts (not necessarily faithfully) on the chain $\Omega/\mathscr{C}$ of $\mathscr{C}$-classes (also known as o-blocks), via $\Delta \to \Delta f$ (the image of the set Δ in (F, Ω)). The next theorem tells us that every transitive representation of F_η (η finite) is "almost" pathologically o-2-transitive, being obtained from a pathologically o-2-transitive representation by blowing the points up to o-blocks. In particular, this holds for the representations of Corollary 10.

THEOREM 15. *Let* η *be finite, and let* (F_η, Ω) *be any transitive representation of* F_η. *Then* (F_η, Ω) *has a largest convex congruence* $\mathscr{C} \neq \{\Omega\}$, *the chain* $\Omega/\mathscr{C}$ *is o-isomorphic to* $\mathbf{Q}$, *and the action of* F_η *on* $\Omega/\mathscr{C}$ *is faithful and pathologically o-2-transitive.*

PROOF. There must be a largest $\mathscr{C} \neq \{\Omega\}$. Otherwise, picking any $\alpha \in \Omega$, there would be an o-block $\Delta \neq \Omega$ containing α and all of the finitely many points αx ($x \in \mathbf{x}$), forcing $\Delta F_\eta = \Delta$ and contradicting the transitivity of the representation.

Suppose w lies in the kernel of the action of F_η on $\Omega/\mathscr{C}$. Since $|x_1| \vee \cdots \vee |x_\eta|$ moves all $\bar{\beta} \in \bar{\Omega}$ (Lemma 14), and in particular moves up all o-blocks in $\mathscr{C}$, all of its conjugates exceed $|w|$, forcing $w = e$ by Corollary 9. Therefore, F_η acts faithfully on $\Omega/\mathscr{C}$, and also o-primitively since any proper convex congruence would yield a proper convex congruence of (F_η, Ω) exceeding $\mathscr{C}$. By Proposition 13, this representation of F_η on $\Omega/\mathscr{C}$ must be pathologically o-2-transitive. Hence $\Omega/\mathscr{C}$ is dense in itself, and being countable because F_η acts transitively on it, must be o-isomorphic to $\mathbf{Q}$.

A prime subgroup P of an l-group F is called a *representing* subgroup of F if the action $(Ph)f = P(hf)$ of F on the chain F/P of right cosets Ph is a (faithful) representation, i.e., if the intersection of all conjugates of P in F is $\{e\}$. In the root system $\mathscr{P}$ of prime subgroups of F, the set of representing subgroups is downward closed. For free l-groups F_η, much more is true of $\mathscr{P}_\eta$.

COROLLARY 16. *For finite* η, *every branch of* $\mathscr{P}_\eta$ *has a largest element, and any one branch consists either entirely of representing subgroups or entirely of nonrepresenting subgroups.*

PROOF. Let $P \neq F_\eta$ be a prime subgroup of F_η. Whether or not the action of F_η on F_η/P is faithful, the proof of the theorem guarantees the existence of a largest $\mathscr{C} \neq \{\Omega\}$. For the $\mathscr{C}$-class Δ containing the point P, the stabilizer $(F_\eta)_\Delta$ is a largest proper prime subgroup containing P (since the o-blocks containing a point α are in one-to-one order-preserving correspondence with the prime subgroups containing $(F_\eta)_\alpha$ via $\Delta \leftrightarrow (F_\eta)_\Delta$, by [**8**, Theorem 1.6.2]). Moreover, if $P \subset Q \neq F_\eta$, with P

representing and Q a prime subgroup, then since the action of F_η on $\Omega/\mathscr{C}$ is faithful and since $Q \subseteq (F_\eta)_\Delta$, $(F_\eta)_\Delta$ and thus also Q are representing.

THEOREM 17. *Let η be infinite. Then*

(1) *For every chain Γ of cardinality at most η, F_η has a transitive representation in which the tower of covering pairs of convex congruences is o-isomorphic to Γ.*

(2) *If a transitive representation (F_η, Ω) has a largest convex congruence $\mathscr{C} \neq \{\Omega\}$, then the action of F_η on $\Omega/\mathscr{C}$ must be o-2-transitive (but need not be pathological).*

PROOF. For (1), let Λ be a totally ordered field of cardinality η, as in the proof of Theorem 1. Form the generalized wreath product with index chain Γ and all factors equal $(A(\Lambda), \Lambda)$, with base point $0 \in \Lambda$ (see [**8**]). Let Ω be the subchain of the wreath chain consisting of all points ρ such that the γth coordinate $\rho(\gamma) = 0$ for all but finitely many $\gamma \in \Gamma$, so that card$(\Omega) = \eta$. We shall represent F_η on Ω. For each free generator x_i of F_η, we shall specify an element of the wreath product such that for each point, only finitely many of its coordinates are changed by the relevant o-primitive components of x_i. Thus the restriction of x_i to Ω will give an o-automorphism of Ω.

First suppose Γ has no largest element. Pick $0 < \sigma < \tau \in \Lambda$, pick a representation φ of F_η on the interval (σ, τ), and for $w \in F_\eta$ let $w\psi \in A(\Lambda)$ act like $w\varphi$ on (σ, τ) and be the identity elsewhere. For each $\gamma \in \Gamma$, let $\Delta_\gamma \subseteq \Omega$ be the interval $(\sigma_\gamma, \tau_\gamma)$, where σ_γ is the element of Ω having $\sigma_\gamma(\gamma) = \sigma$ and all other components zero, and similarly for τ_γ. The set $\mathscr{D}$ of these Δ_γ's is pairwise disjoint and its union is cofinal in Ω. Partition $\mathbf{x}$ into subsets $\mathbf{x}_1$ and $\mathbf{x}_2$ whose cardinalities are also η. Establish one-to-one correspondences between $\mathbf{x}_1$ and Ω, and between $\mathbf{x}_2$ and $\{(\gamma, (\lambda, \mu)) \mid \gamma \in \Gamma,\ 0 > \lambda,\ \mu \in \Lambda\}$. The desired representation is obtained by arranging about the o-primitive components $x_{\gamma,\omega}$ (at the level $\gamma \in \Gamma$ for the point $\omega \in \Omega$) of each free generator x that:

(a_1) If $x \in \mathbf{x}_1$ and $x \leftrightarrow \rho$, then $0x_{\gamma,0} = \rho(\gamma)$ for all γ's at which $\rho(\gamma) \neq 0$. (The subscript 0 refers to the point $0 \in \Omega$ such that $0(\gamma) = 0 \in \Lambda$ for all γ.)

(a_2) If $x \in \mathbf{x}_2$ and $x \leftrightarrow (\gamma, (\lambda, \mu))$, then $0x_{\gamma,0} = 0$ and $\lambda x_{\gamma,0} = \mu$.

(b) When γ' exceeds the largest γ in (a_1), or the γ in (a_2), then $x_{\gamma',0} = x\psi$.

(c) All other o-primitive components of x be the identity.

Then (a_1) assures that the action of F_η is transitive on Ω; (a_2) assures that between any covering pair $(\mathscr{C}_\gamma, \mathscr{C}^\gamma)$ of convex congruences of the wreath product (and thus also of (F_η, Ω)), there are no other convex congruences of (F_η, Ω), so that the chain of covering pairs of convex congruences of (F_η, Ω) is just Γ; and (b) and (c) assure that the action of F_η on Ω is faithful.

Now suppose that Γ has a largest element $\bar{\gamma}$. Choose a pairwise disjoint set $\mathscr{D}$ of intervals of Λ, all to the right of 0, whose union is cofinal in Λ. For each $\Delta \in \mathscr{D}$ pick a representation φ_Δ of F_η on Δ. Arrange as before the correspondences involving $\mathbf{x}_1$ and $\mathbf{x}_2$ and the conditions (a_1), (a_2), and (c). Also arrange

(b) For all Δ (in case (a_1), all Δ for which $\rho < \inf \Delta$), the one $\bar{\gamma}$-component $x_{\bar{\gamma}}$ act on Δ in accordance with φ_Δ, and act like the identity elsewhere.

For (2), if the o-primitive action $(\hat{F}_\eta, \Omega/\mathscr{C})$ were regular or periodically o-primitive, there would exist $u \in F_\eta$ such that for any $\hat{w} \in \hat{F}_\eta$ there would exist $p \in \mathbf{Z}^+$

such that $\Delta\hat{u}^p > \Delta\hat{w}$ for every $\Delta \in \Omega/\mathscr{C}$. (In the periodically o-primitive case, any $\hat{u}$ exceeding the period z would do.) Then u would be a strong order unit of F_η, violating Lemma 14.

To see that the o-2-transitive action of F_η on $\Omega/\mathscr{C}$ need not be pathological, apply the proof of (1) to a 2-element chain Γ, but choose each Δ within a natural o-block $\{\omega \in \Omega \mid \omega(\bar{\gamma}) = \lambda\}$, $\lambda \in \Lambda$, permitting the action on $\Omega/\mathscr{C}$ of all x_i's to have bounded support.

4. Extensions of pathologically o-2-transitive representations.

THEOREM 18. *Every pathologically o-2-transitive representation of F_η (η finite) on a chain Ω can be extended to a pathologically o-2-transitive representation of F_{ω_0} on Ω. (Here F_η is to be taken to be $l(x_1,\dots,x_\eta)$, where $\mathbf{x} = \{x_1, x_2,\dots\}$ is a free generating set for F_{ω_0}.)*

To prove the theorem, we need a technical lemma. By a *diagram* $\mathscr{D}$ in $x_1,\dots,x_\eta$ (without reference to any $w \in F_\eta$), we mean a finite totally ordered set D, together with a (finite) set of "x_i-arrows" (ordered pairs of elements of D, each labelled with some x_i, $1 \leqslant i \leqslant \eta$) which is consistent with the order on D in the sense of §1, with $\mathscr{D}$ connected and loop-free. Deleting the connectedness requirement gives us a *multidiagram*.

LEMMA 19. *Let (F_η, Ω) be a pathologically o-2-transitive representation of F_η, and let $[\beta, \gamma]$ be an interval of Ω. Then*

(1) *Every diagram $\mathscr{D}$ in $x_1,\dots,x_\eta$ arises in (F_η, Ω) as a diagram for some $w \in F_\eta$; and if $\mathscr{D}$ is a diagram for a particular $w \in F_\eta$, $\mathscr{D}$ arises for that same w.*

(2) *Every multidiagram in $x_1,\dots,x_\eta$ arises in (F_η, Ω).*

Moreover, it can be arranged that either all the points of the (multi)diagram lie below β or all lie above γ.

PROOF. Let α be any point of $\mathscr{D}$. We shall make $\alpha \leftrightarrow \square$ (the empty word). For each way of beginning at α and following a sequence of arrows (perhaps stopping before running out of arrows), we get a group word g_j. Since $\mathscr{D}$ is loop-free, we obtain only finitely many g_j's in this way. For these g_j's the points αg_j are distinct since $\mathscr{D}$ is loop-free, and they include all points of the diagram since $\mathscr{D}$ is connected. Let $w = \bigwedge\{g_k g_j^{-1} \mid \alpha g_j < \alpha g_k\} \vee e$. Laying out a copy of this diagram in $\mathbf{Q}$, we see that $\alpha < \alpha\hat{w}$ for some substitution in $A(\mathbf{Q})$, so that $e < w$ in F_η. Pick $\alpha' \in \Omega$ so that in the given representation (F_η, Ω), $\alpha' < \alpha' w$. Then $\alpha' g_j < \alpha' g_k$ iff $\alpha g_j < \alpha g_k$, so that $\mathscr{D}$ arises in (F_η, Ω). This proves the first part of (1). For the second part, simply choose $\square$ as α. Even though $\bigwedge\{g_k g_j^{-1} \mid \alpha g_j < \alpha g_k\} \neq w$, a diagram for it will also be a diagram for w.

Now let $\mathscr{M}$ be a multidiagram. We claim that $\mathscr{M}$ can be augmented to create a (connected) diagram $\mathscr{D}$ by adding some extra points and extra x_1- and x_2-arrows. (Since F_η has a transitive representation, $\eta > 1$.) To establish the claim, we give a procedure for reducing the number of connected components.

Pick a component. Since $\mathscr{M}$ is loop-free, we may pick a point μ_1 in that component which is an end point of at most one arrow, and thus with no loss of generality not of an x_1-arrow. While maintaining the consistency of the multidiagram, add an $x_2^{\pm 1}$-arrow whose tail is at μ_1 and whose head μ_2 is a new point exceeding at least one more point of the multidiagram than does μ_1. Proceed in this fashion, alternating between $x_1^{\pm 1}$- and $x_2^{\pm 1}$-arrows, until a point μ_1 is obtained which exceeds all points of the current multidiagram. Repeat this for another component, obtaining ν_j. This makes $\mu_i < \nu_j$ the *largest two points* in the current multidiagram. Let the last arrow used to reach μ_i be, say, an $x_1^{\pm 1}$-arrow. Finally, add an $x_1^{\pm 1}$-arrow from μ_i to ν_j; except that this will be impossible if the last arrow used to reach ν_j was also an $x_1^{\pm 1}$-arrow, in which case we connect μ_i to ν_j with an x_2-arrow. This proves the claim.

Now apply (1) to $\mathscr{D}$, and from a copy of $\mathscr{D}$ arising in (F_η, Ω), delete all points and arrows involved in augmenting $\mathscr{M}$ to make $\mathscr{D}$.

Now for the last sentence of the lemma: Since (F_η, Ω) has no nonidentity element of bounded support, either all nonidentity elements have support unbounded above, or dually; suppose the former. To make all points in the copy just produced of $\mathscr{D}$ (or $\mathscr{M}$) lie above γ, we choose α' so that $\alpha' > \gamma g_j^{-1}$ (and thus $\alpha' g_j > \gamma$) for each j.

Proof of Theorem 18. Our task is to specify $x_{\eta+1}, x_{\eta+2}, \ldots$ so that the resulting action of F_{ω_0} on Ω is faithful—and thus automatically o-2-transitive since F_η already acts o-2-transitively on Ω, and pathological by Proposition 13.

Let $e \neq w \in F_{\omega_0}$. Pick a diagram showing $w \neq e$. Delete from this diagram all arrows for x's other than $x_1, \ldots x_\eta$ (but without deleting any points), obtaining a multidiagram involving some subset of $x_1, \ldots, x_\eta$. By Lemma 19, this multidiagram arises in the given representation of F_η. We then specify the remaining letters $x_{\eta+1}$, $x_{\eta+2}, \ldots$ occurring in the spelling of w at the appropriate points of this multidiagram so as to restore it to the original diagram showing $w \neq e$. We have shown that for any one $e \neq w \in F_\eta$, the given representation of F_η can be extended to an action of F_{ω_0} on Ω in which $w \neq e$.

Now we enumerate the nonidentity elements of F_{ω_0}: $w_0, w_1, \ldots$. As above, we specify finitely many of $x_{\eta+1}, x_{\eta+2}, \ldots$ at finitely many points so as to make $w_0 \neq e$. Inductively, we do the same for w_i, using Lemma 19 to choose the multidiagram for w_i so that all its points lie above (or all lie below) all points involved in the previous multidiagrams. This makes it possible to meet all the specifications simultaneously, so that every $w_i \neq e$.

Corollary 20. *There exists a pathologically o-2-transitive representation of F_{ω_0} on $\mathbf{Q}$ in which some element moves every $r \in \mathbf{R} = \overline{\mathbf{Q}}$.*

Proof. By Theorem 1, there exists a pathologically o-2-transitive representation of F_2 on $\mathbf{Q}$. By Lemma 14, the strong order unit $|x_1| \vee |x_2|$ moves every element of $\overline{\mathbf{Q}}$. Now apply Theorem 18.

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