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UNITARY STRUCTURES ON COHOMOLOGY

BY

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ABSTRACT. Let $\mathbf{C}^{p+q}$ be endowed with a hermitian form H of signature (p, q). Let M_r be the manifold of r-dimensional subspaces of $\mathbf{C}^{p+q}$ on which H is positive-definite and let E be the determinant bundle of the tautological bundle on M_r. We show (starting from the oscillator representation of $\mathrm{SU}(p, q)$) that there is an invariant subspace of $H^{r(p-r)}(M_r, \mathcal{O}(E(p+k)))$ which defines a unitary representation of $\mathrm{SU}(p, q)$. For $W \in M_p$, $\mathrm{Gr}(r, W)$ is the subvariety of r-dimensional subspaces of W. Integration over $\mathrm{Gr}(r, W)$ associates to an $r(p-r)$-cohomology class α, a function $P(\alpha)$ on M_p. We show that this map is injective and provides an intertwining operator with representations of $\mathrm{SU}(p, q)$ on spaces of holomorphic functions on Siegel space

0. Introduction. Let $\mathbf{C}^{p+q}$ be endowed with a hermitian form H of signature $(p, q) \cdot U(p, q)$ is the invariance group of this form and $\mathrm{SU}(p, q)$ the subgroup of linear transformations of determinant one. In this article we are concerned with a commutative diagram:

$$\begin{tikzcd}
 & \text{oscillator representation} \arrow[dl, "\alpha"'] \arrow[dr, "\Phi"] & \\
\text{cohomology of line bundles on } M_r \arrow[rr, "P"] & & \text{sections of holomorphic bundles on Siegel space}
\end{tikzcd}$$

Here, $r < p$ and M_r is the manifold of r-dimensional subspaces of $\mathbf{C}^{p+q}$ on which H is positive-definite. Siegel space is the same as M_p. The map Φ is the intertwining operator first studied by Gross and Kunze [**10**], elaborated on by Howe [**11**], and completely explained by Kashiwara and Vergne [**13**]. The question of the existence of nontrivial Hilbert spaces in the lower left corner was studied in detail by many authors in the late 70's, culminating in the paper of Rawnsley, Schmid and Wolf [**21**]. We refer to Appendix A of that paper for a detailed account of this history. Their paper studies such cohomology spaces in great generality, showing when they are nontrivial and how to put an invariant Hilbert space structure on them. In §13 of that paper their general techniques are applied to $\mathrm{SU}(p, q)$. Thus our construction is elementary and independent, but it is not new. It is clearly stated in their work that

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the representations arising in this way are highest weight modules, and therefore intertwine somehow with representations in spaces in the lower right corner. Furthermore, the results of Jakobsen and Vergne [**15**] indicate that the oscillator representation is lying somewhere in the background. In the present work (originally due to Patton and summarized in [**18**]) the oscillator representation is brought into the foreground, the operator α is introduced, and the existence of nontrivial Hilbert spaces on the lower left becomes elementary. Furthermore, the action of $\mathrm{SU}(p, q)$ is just that introduced by Bargmann [**2, 3**], viewed in a new light. That new light is provided by a realization of the oscillator representation in cohomology on $\mathbf{C}^{p+q}$: this cohomology was first constructed by Carmona [**7**] and Satake [**26**]. In [**5**] Blattner and Rawnsley give a reconstruction of this cohomology, and in [**15**] Mantini uses the Blattner-Rawnsley construction to define and study the map Φ much as we do here; in fact, she does more: she calculates the differential operators which define the image of Φ.

The diagram finally is filled in by the map P which we call the Penrose transform: it is a generalization of the transform developed by Penrose [**20**] for the group $\mathrm{SU}(2,2)$. We prove an injectivity theorem which follows the lines of Eastwood, Penrose and Wells [**8**] and is an application, via a calculation, of the Borel-Weil-Bott theorem. In [**9**], Eastwood also generalizes [**8**] and obtains these results for the case $r = 1$. The commutativity of the diagram (the fact that α, Φ and P give intertwining operators) now follows explicitly from the formulae of Bargmann, or from the naturality of the geometric actions. The case $r > 1$ (discussed in §6) follows more or less directly from the case $r = 1$.

We now describe the oscillator representation so that we may expose the results of this article in detail. Write $R^{2n} = R^n \oplus R^n$ as the space of pairs of n-vectors, and let ω be the usual symplectic form

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{0.1}$$
$$\omega(x, y) = {}^t y J_0 x = {}^t y_2 x_1 - {}^t y_1 x_2.$$

Let J be a complex structure on R^{2n} compatible with ω:

$$J^2 = -I, \qquad \omega(Jx, Jy) = \omega(x, y).$$

Then (R^{2n}, J) is a complex vector space with hermitian form $h_J(x, y) = \omega(Jx, y) + i\omega(x, y)$. We shall let $\mathcal{O}(R^{2n}, J)$ denote the space of functions holomorphic on (R^{2n}, J), and

$$\mathcal{F}^J = \left\{ f \in \mathcal{O}(R^n, J);\ \int |f|^2 \exp(-h_J(x, x))\, dx < \infty \right\}, \tag{0.2}$$

where by dx we mean normalized Lebesgue measure:

$$\int \exp(-h_J(x, x))\, dx < \infty. \tag{0.3}$$

Of course, (0.2) and (0.3) make sense only if h_J is positive-definite. Let

$$B_q = \{ J;\ h_J \text{ has signature } (n - q, q) \}.$$

Then $\mathcal{F} = \bigcup_J \mathcal{F}^J$ is a Hilbert space bundle over B_0. The invariance group of ω, $\mathrm{Sp}(n, R)$, acts transitively on B_0 by $g \cdot J = gJg^{-1}$, and the geometric action on

functions ($f \to f \circ g^{-1}$) defines an isometry of $\mathscr{F}^J$ with $\mathscr{F}^{g \circ J}$. Thus $\mathscr{F} \to B_0$ is a homogeneous $\mathrm{Sp}(n, R)$-bundle.

Now let $p + q = n$, and

$$I_{p,q} = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q\times q} \end{pmatrix}, \qquad J_q = \begin{pmatrix} 0 & -I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}. \tag{0.4}$$

Then $J_q \in B_q$ and $U(p, q)$ is the isotropy subgroup of J_q (for details see [**24**]). Since $U(n)$ is the isotropy group of J_0, the above-described action gives a unitary representation of $U(n)$ on $\mathscr{F}^{J_0}$ (we discuss this further in §2). The analog for general $U(p, q)$ is harder to describe. In fact, there are two ways to so define a unitary representation of $U(p, q)$, and it is the interrelation of these two ways which underlies §§3 and 4.

The "classical way" is to realize $\mathscr{F}^J$ as the representation space for the Heisenberg group H at the positive polarization. If J_1 is another such polarization, by the Stone-von Neumann theorem $\mathscr{F}^{J_1}$ and $\mathscr{F}^J$ are equivalent as H-modules, so there is an intertwining operator

$$B^{\theta}_{J_1,J}: \mathscr{F}^{J_1} \cong \mathscr{F}^J, \tag{0.5}$$

uniquely determined up to a scalar θ. (These are the BKS-operators of [**4**], but were first explicitly given by Bargmann [**2**].) For $g \in \mathrm{Sp}(n, R)$, the maps

$$U_g: \mathscr{F}^J \to \mathscr{F}^{g\cdot J} \to \mathscr{F}^J,$$

$$f \to f \circ g^{-1} \to B^{\theta}_{gJ,J}$$

define a projective unitary representation of $\mathrm{Sp}(n, R)$, whose restriction to $\mathrm{SU}(p, q)$ we call the *oscillator representation*.

On the other hand, we can work directly on B_q. For $J \in B_q$, Carmona [**7**] has described a representation of H on a space $\mathscr{H}^J$ of square-integrable $(0, q)$-forms on (R^{2n}, J). Given Carmona's explicit description, we easily observe (§3; see also [**5**]) that the $\mathscr{H}^J$ carry a Hilbert space structure which is invariant under the geometric action of $\mathrm{Sp}(n, R)$. Thus $\mathscr{H} = \bigcup_J \mathscr{H}^J$ is an $\mathrm{Sp}(n, R)$-bundle over B_q and the oscillator representation is this action restriction to $U(p, q)$, as the isotropy group of J_q. (Blattner and Rawnsley further show that $\mathscr{H}^J$ represents $\bar{\partial}$-cohomology on (R^{2n}, J).) This approach shows us that the oscillator representation is a genuine (not just projective) unitary representation.

Now, fix $C^{p+q} = (\mathbf{R}^{2n}, J_q)$ and write the hermitian form h_{J_q} as h and H^{J_q} as H^q. A splitting of $\mathbf{C}^{p+q}$ is a pair of complementary subspaces $S = (V, W)$ such that $h|V \gg 0$ and $h|W \ll 0$. For S a splitting, let $\tilde{J}_S = J_q|V \oplus -J_q|W$. Then $\tilde{J}_S \in B_0$. Let $\mathscr{F}^S = \mathscr{F}^{\tilde{J}_S}$. Carmona's theorem is that $H^q \cong \mathscr{F}^s$ as H_n-modules. Now, for $g \in \mathrm{SU}(p, q)$, $g \circ S = (g \circ V, g \circ W)$ is another splitting, and by the irreducibility of this H_n-module we must necessarily have that the composition $\mathscr{F}^{g\cdot S} \to \mathscr{H}^q \to \mathscr{F}^S$ is Bargmann's intertwining operator $B^{\theta}_{g\cdot S,S}$ for some θ. This observation is the basis for the main result of §4 which intertwines the oscillator representation with the geometric action on vector bundles on Siegel space. In [**15**] Mantini obtains the same results and more: she gives explicit descriptions of the parameters of the irreducible representations which result. Our result goes as follows.

Let $M_p = \{V \subset \mathbf{C}^{p+q};\ \dim V = p,\ h|V \gg 0\}$. For $V \in M_p$, let $\mathscr{F}^V$ be the Hilbert space of holomorphic functions, square-integrable in the measure $\exp(-h(x, x))\,dx$, and π_V the orthogonal projection onto $\mathscr{F}^V$. $\mathscr{E} = \bigcup \mathscr{F}^V$ is a Hilbert space bundle over M_p. For any splitting S, $\mathscr{F}^S$ is contained in the domain of π_V, and thus there is a map Φ^S from $\bar{\mathscr{F}}^S$ to sections of the dual of $\mathscr{E}$ defined by $\Phi^S(h)(V) = \pi_V(h|V)$. In §4 we show that Φ^S defines an intertwining operator from $\bar{\mathscr{H}}^q$ to sections of $\mathscr{E}'$.

For $k \geqslant 0$, let $\mathscr{F}_k^S$ be the space of functions h in $\mathscr{F}^S$ such that

$$f(e^{i\theta}x) = e^{ik\theta}f(x).$$

These are the irreducible components of $\mathscr{F}^S$. Let

$$M_1 = \{L \in \mathbf{P}(\mathbf{C}^{p+1});\ h|L \gg 0\}.$$

In §5 we describe a natural map of $\bar{\mathscr{F}}_k^S$ into $H^{p-1}(M_1, \mathcal{O}(E(-k-p)))$, where E is the hyperplane section bundle. We show in §5 that the geometric action of $\mathrm{SU}(p, q)$ on M_1 leaves the image invariant and intertwines with the oscillator representation. In §6 we generalize this to

$$M_r = \{V \in \mathrm{Gr}(r, \mathbf{C}^{p+1});\ h|V \gg 0\}$$

by examining the oscillator representation of $\mathrm{SU}(\mathbf{C}^{(p+q)\times r})$ restricted to the dual pair $\mathrm{SU}(p+q, \mathbf{C}) \times \mathrm{SU}(r)$. The representation occurs in $H^{r(p-r)}(M_r, \mathcal{O}(E(-k-p)))$, where E is the dual of the determinant bundle of the tautological bundle.

Now, let $Y_r = \{(V, W);\ \dim V = r,\ W \in M_p,\ V \subset W\}$ and let $\mu\colon Y_r \to M_r$ and $\nu\colon Y_r \to M_p$ be the projections onto the first and second factors, respectively. Let

$$\theta(k) = R^{r(p-r)}\nu_*\mu^*\mathcal{O}(E(-k-p)),$$

the $r(p-r)$th direct image under ν of the lift of $E(-k-p)$ by μ. There is an induced map

$$P\colon H^{r(p-r)}(M_r, \mathcal{O}(E(-k-p))) \to H^0(M_p, \theta(k))$$

which generalizes the Penrose correspondence (and is given, along the fibers, by Serre duality). In §7 we verify that P is injective and commutes with the $\mathrm{SU}(p, q)$-action.

The central line of reasoning and conception of the results is due to the first author and were reported in his article [**18**]. The second author became involved at the time of revision and contributed the proof of §7. We are indebted to Nolan Wallach for help with the criteria of the Borel-Weil-Bott theorem. Both authors benefited from discussions with others, notably V. Bargmann, R. Howe and W. Schmid.

I. Notational conventions and other preliminaries. $\mathbf{C}^p$ is the space of (column) vectors of p-tuples of complex numbers, $\mathbf{C}^{p+q}$ is the direct sum of $\mathbf{C}^p$ and $\mathbf{C}^q$, and $\mathbf{C}^{m\times n}$ is the space of m by n complex matrices. We shall use the usual multi-index notation

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}, \quad m = \begin{pmatrix} m_1 \\ \vdots \\ m_p \end{pmatrix}, \quad z^{(m)} = z_1^{m_1} \cdots z_p^{m_p},$$

$$m! = m_1! \cdots m_p!, \qquad |m| = m_1 + \cdots + m_p.$$

If X is a complex manifold, $\mathcal{O}(X)$ or $\mathcal{O}$ is its sheaf of holomorphic functions. If $V \to X$ is a holomorphic vector bundle, $\mathcal{O}(V)$ is its sheaf of holomorphic sections and $\mathscr{E}(V)$ its sheaf of smooth sections. T_X is the holomorphic tangent bundle of X and Λ_X^p is the bundle of holomorphic p-forms on X, with sheaves of sections θ_X and Ω_X^p, respectively. $\Omega^p(V)(\mathscr{E}^{p,q}(V))$ is the sheaf of holomorphic p-forms (smooth (p, q)-(forms)) with values in the bundle V. We shall write $\mathscr{E}^q$ for $\mathscr{E}^{0,q}$.

$U(p, q)$ is the group of complex linear transformations of $\mathbf{C}^{p+q}$ which leave invariant the form

$$v, w \to {}^t\overline{w} I_{p,q} v, \tag{1.1}$$

where $I_{p,q}$ is given by (0.4). $\mathrm{SU}(p, q)$ is the group of those transformations in $U(p, q)$ of determinant one. We shall denote $\mathrm{SU}(p, 0)$ by $\mathrm{SU}(p)$, although in this paper it is often instructive to think of this group as $\mathrm{SU}(p, q)$ with $q = 0$. For W a topological vector space, we let W' represent its dual. A hermitian product on W describes a complex isomorphism of $\overline{W}$ with W'.

$\mathbf{P}^n$ is the projective space of complex one-dimensional subspaces of $\mathbf{C}^{n+1}$. Let $\mathbf{C}_*^{n+1} = \mathbf{C}^{n+1} - \{0\}$ and let π: $\mathbf{C}_*^{n+1} \to \mathbf{P}^n$ be the natural projection. $E(-1)$, the tautological bundle of $\mathbf{P}^n$, is the bundle associated to this principal bundle. It is tautological in the sense that it associates to each $p \in \mathbf{P}^n$ itself as a subspace of $\mathbf{C}^{n+1}$. We define the bundle Q by the exact bundle sequence on $\mathbf{P}^n$

$$0 \to E(-1) \to \mathbf{C}^{n+1} \to Q \to 0. \tag{1.2}$$

The *hyperplane section bundle* is the dual E of $E(-1)$. For k a positive integer, $E(k)$ $(= \odot^k E)$ is the bundle of homogeneous polynomials of degree k on the tautological bundle. Its space of sections is the space $\mathscr{F}_k$ of homogeneous polynomials of degree k in $n + 1$ variables. If $V \to \mathbf{P}^n$ is a holomorphic vector bundle we shall denote $V \otimes E(k)$ by $V(k)$.

If $z_0, \ldots, z_n$ are coordinates for $\mathbf{C}^{n+1}$, the ratios $z_0 : \cdots : z_n$ are "homogeneous" coordinates for $\mathbf{P}^n$. The vector field

$$\Gamma = \sum z_i \partial/\partial z_i$$

is tangent to the fibers of π: $\mathbf{C}_*^{n+1} \to \mathbf{P}^n$, so $\pi^{-1}(T_{\mathbf{P}^n})$ is the quotient of $\mathbf{C}^{n+1}$ by the span of Γ (as bundles on $\mathbf{C}_*^{n+1}$). Since the coefficients of Γ are homogeneous of degree 1, this descends to the exact bundle sequence on $\mathbf{P}^n$:

$$0 \to \mathbf{C} \to E(1)^{n+1} \to T_{\mathbf{P}^n} \to 0. \tag{1.3}$$

Comparing this with (1.2), we have that $Q \cong T_{\mathbf{P}^n}(-1)$. Taking determinants in (1.3) and dualizing, we obtain the canonical bundle

$$\Lambda_{\mathbf{P}^n}^n \cong E(-(n+1)). \tag{1.4}$$

A function f on an open set in $\mathbf{C}^{n+1}$ is homogeneous of degree k if and only if $\Gamma(f) = kf$. This condition thus describes the lifted sheaf $\pi^{-1}(\mathcal{O}(E(k)))$. This leads to the following description on $\mathbf{C}_*^{n+1}$ of the cohomology of $E(k)$ on $\mathbf{P}^n$. Let $\Lambda_k^{p,q}$ be the space of smooth (p, q)-(forms) α defined on $\mathbf{C}_*^{n+1}$ such that:

$$\begin{aligned} &\text{(i)} && \Gamma \lrcorner \alpha = \overline{\Gamma} \lrcorner \alpha = 0, \\ &\text{(ii)} && \Gamma \lrcorner d\alpha = 0, \text{ and} \\ &\text{(iii)} && \Gamma \lrcorner d\alpha = k\alpha. \end{aligned} \tag{1.5}$$

Then $\bar{\partial}\colon \Lambda_k^{p,q} \to \Lambda_k^{p,q+1}$ and

$$H^{p,q}(\mathbf{P}^n, \mathcal{O}(E(k))) \cong \frac{\ker \bar{\partial}\colon \Lambda_k^{p,q} \to \Lambda_k^{p,q+1}}{\bar{\partial}\Lambda_k^{p,q-1}}. \tag{1.6}$$

Let

$$\eta = \Gamma \lrcorner\, dz_0 \wedge \cdots \wedge dz_n = \sum (-1)^i z_i\, dz_0 \cdots \widehat{dz_j} \cdots dz_n. \tag{1.7}$$

η defines a nowhere vanishing n-form on $\mathbf{C}^{n+1}$ and, since η is homogeneous of degree $n+1$, it descends to a holomorphic section of $\Lambda^n_{\mathbf{P}^n}(n+1)$ which is, by (1.4), the trivial sheaf. Thus, in fact, η generates $H^{n,0}(\mathbf{P}^n, E(n+1))$. Finally,

$$dV = \|z\|^{-2}\eta \wedge \bar{\eta} \tag{1.8}$$

is an invariant volume form on $\mathbf{P}^n$.

$G(r, n)$ is the Grassmanian manifold of r-planes in $\mathbf{C}^n$. Let V' denote the tautological bundle: V' assigns to each point itself as a subspace of $\mathbf{C}^n$. We define the bundle Q on $G(r, n)$ by the exact sequence

$$0 \to V' \to \mathbf{C}^n \to Q \to 0. \tag{1.9}$$

The dual V is the bundle of linear functions on V' and $\odot^k V$ is the bundle of homogeneous polynomials on the fibers. Once again

$$H^0(G(r, n), \odot^k V) \cong p_k, \tag{1.10}$$

the space of homogeneous polynomials of degree k on $\mathbf{C}^n$. Let $E = \Lambda^r V$, the determinant bundle of E. We shall write $E(k)$ for E^k. Let $\mathbf{C}_*^{n\times r}$ be the space of $n \times r$ matrices with independent columns:

$$\mathbf{C}_*^{n\times r}\colon \{(z_1, \ldots, z_r);\, z_1 \wedge \cdots \wedge z_r \neq 0\}. \tag{1.11}$$

We have the natural projections

$$\pi_0\colon \mathbf{C}_*^{n\times r} \to G(r, n), \qquad \pi\colon \mathbf{C}_*^{n\times r} \to E(-1), \tag{1.12}$$

$$\pi_0(z_1, \ldots, z_r) = \operatorname{span}\{z_1, \ldots, z_r\}, \quad \pi(z_1, \ldots, z_r) = z_1 \wedge \cdots \wedge z_r.$$

$\mathrm{GL}(r, \mathbf{C})$ acts on $\mathbf{C}_*^{n\times r}$ on the right; the fibers of π_0 are the orbits and the fibers of π are the $\mathrm{SL}(r, \mathbf{C})$-orbits. We now describe the analog of (1.6).

Let Γ be the Euler field on $E(-1)$ (the generator of the $\mathbf{C}^*$-action). For $z = (z_i^j) \in \mathbf{C}^{n\times r}$, let

$$\Gamma_0 = \sum z_i^j \partial/\partial z_i^j.$$

Note that Γ_0 is $\mathrm{SL}(r, \mathbf{C})$-invariant and $\pi_* \Gamma_0 = \Gamma$. Since $\mathrm{SL}(r, \mathbf{C})$ acts freely on $\mathbf{C}_*{}^{n\times r}$, its Lie algebra can be realized by an $\mathrm{SL}(r, \mathbf{C})$-invariant space of holomorphic vector fields on $\mathbf{C}_*^{n\times r}$, tangent to the orbits. Let $\Gamma_1, \ldots, \Gamma_{r^2-1}$ be a basis for this space. We take $\Lambda_k^{p,q}$ to be the space of smooth (p, q)-forms α on $\mathbf{C}_*^{n\times r}$ which vanish on tangents to the fibers to π, and are $\mathrm{SL}(r, \mathbf{C})$-invariant and homogeneous of degree kr along the orbits of Γ_0. What is the same,

$$\begin{aligned}
&\text{(i)} && \Gamma_i \lrcorner\, \alpha = \bar{\Gamma}_i \lrcorner\, \alpha = 0,\ 0 \leqslant i \leqslant r^2 - 1,\\
&\text{(ii)} && \Gamma_i \lrcorner\, d\alpha = \bar{\Gamma}_i \lrcorner\, d\alpha = 0,\ 1 \leqslant i \leqslant r^2 - 1,\\
&\text{(iii)} && \bar{\Gamma}_0 \lrcorner\, d\alpha = 0, \text{ and}\\
&\text{(iv)} && \Gamma_0 \lrcorner\, d\alpha = kr\alpha.
\end{aligned} \tag{1.13}$$

Then $\bar{\partial}\colon \Lambda_k^{p,q} \to \Lambda_k^{p,q+1}$ and

$$H^{p,q}(G(r,n), \mathcal{O}(E(k))) \cong \frac{\ker \bar{\partial}\colon \Lambda_k^{p,q} \to \Lambda_k^{p,q+1}}{\bar{\partial}\Lambda_k^{p,q-1}}. \tag{1.14}$$

Let

$$\eta = (\Gamma_0 \wedge \cdots \wedge \Gamma_{r^2-1}) \lrcorner (dz_1^1 \wedge \cdots \wedge dz_n^r). \tag{1.15}$$

η is a nowhere vanishing holomorphic $nr - r^2 = r(n-r)$-form which is homogeneous of degree nr and so defines a generator of $H^{r(n-r),0}(G(r,n), \mathcal{O}(E(n)))$. Finally, the invariant volume form on $G(r,n)$ is

$$dV = \det({}^t\bar{z}z)^{-1}\eta \wedge \bar{\eta}. \tag{1.16}$$

II. The compact case SU(p). The results of this section are well known and are reviewed in a form useful to later sections. It is illustrative of our constructions to think of SU(p) as SU(p, 0), the "dengenerate case".

We endow $\mathbf{C}^p$ with the usual Euclidean inner product

$$\langle z, w\rangle = {}^t\bar{w}z = \sum z_i\bar{w}_i, \qquad \|z\|^2 = \langle z, z\rangle.$$

Let $d\mu$ represent Lebesgue measure, normalized so that

$$\int \exp(-\langle z, z\rangle)\, d\mu(z) = 1.$$

Let

$$\mathscr{F} = \left\{ f \in \mathcal{O}(\mathbf{C}^p);\ \|f\|^2 = \int |f(z)|^2 \exp(-\langle z, z\rangle)\, d\mu(z) < \infty \right\}. \tag{2.1}$$

$\mathscr{F}$ is a Hilbert space in which the polynomials are dense. For $f \in \mathscr{F}$, we have the estimate

$$|f(z)| \leqslant \exp(\|z\|^2/2)\|f\|^2. \tag{2.2}$$

SU(p) acts on $\mathscr{F}$ by the geometric action

$$(g \cdot f)(z) = f(g^{-1}z), \tag{2.3}$$

and clearly the spaces $\mathscr{F}_k$ (of homogeneous polynomials of degree k) are invariant and irreducible SU(p)-modules.

Let $M_1 = \mathbf{P}^{p-1}$. Then $E(k) \to M_1$ is an SU(p)-homogeneous bundle, and the restriction of functions on $\mathbf{C}^p$ to lines induces isomorphisms

$$\alpha_0\colon \mathscr{F}_k \cong H^0(M_1, \mathcal{O}(E(k))) \tag{2.4}$$

which intertwine the SU(p)-action.

Now, SU(p) acts unitarily on $\bar{\mathscr{F}}_k$, the space of conjugate holomorphic polynomials of degree k. Let $\bar{\eta}$ be the $(0, p-1)$-form on M_1 defined by (1.7). To $\bar{f} \in \bar{\mathscr{F}}_k$ we make correspond the form

$$\alpha(\bar{f}) = ((k+p-1)!/(2i)^p)\bar{f}(z)(\|z\|^{-2(k+p)})\bar{\eta}. \tag{2.5}$$

2.6. PROPOSITION. *α induces an isomorphism*

$$\bar{\mathscr{F}}_k \to H^{p-1}(M_1, \mathscr{O}(E(-k-p))) \tag{2.7}$$

which intertwines the SU(p)*-action.*

PROOF. This follows from (1.13) and (1.14), but we can also argue directly. Since f is homogeneous of degree k, $\bar{f}\eta$ is homogeneous of degree $k+p$, so $f\bar{\eta}$ defines a global $(0, p-1)$-form on M_1 with values in $E(k+p)$. Considering $\|z\|^2$ as a section of $E(1) \otimes \bar{E}(1)$, we view $\alpha(\bar{f})$ as a global $(0, p-1)$-form with values in $E(-k-p)$, and thus as a class on the right side of (2.7). Now, the Hilbert space pairing of $\mathscr{F}$ defines an isomorphism of $\bar{\mathscr{F}}_k$ with $\mathscr{F}_k'$ and, by Serre duality, integration on M_1 defines a pairing of right sides of (2.4) and (2.7). These pairings are obviously SU(p)-invariant, so it remains only to show that they are preserved by α_0, α. Equivalently, we need to verify that α_0 is an isometry. But this is a straightforward calculation in polar coordinates: for f and g homogeneous polynomials of degree k,

$$\begin{aligned}\langle f, g\rangle &= \int_{\mathbf{C}^p} f\bar{g}\exp(-\|z\|^2)\, d\mu = \frac{(k+p-1)!}{(2i)^p}\int_{M_1}\frac{f\bar{g}}{\|z\|^{2(k+p)}}\eta\wedge\bar{\eta} \\ &= \langle \alpha_0(f), \alpha(\bar{g})\rangle,\end{aligned} \tag{2.8}$$

where the last line refers to the pairing of Serre duality.

Now let $M_r = G(r, p)$; $\odot^k V \to M_r$ (see (1.9)) is an SU(p)-homogeneous bundle, and the restriction of functions on $\mathbf{C}^p$ to r-planes induces an SU(p)-invariant isomorphism α_r of $\mathscr{F}_k$ with sections of $\odot^k V$ (see (1.10)). If we let

$$M_{1,r} = \{(L, \Pi);\ L \subset \Pi, \dim L = 1, \dim \Pi = r\}$$

and let μ: $M_{1,r} \to M_1$ and ν: $M_{1,\nu} \to M_\nu$ be the projections, we obtain an intertwining operator

$$\nu_*\mu^*: H^0(M_1, \mathscr{O}(E(k))) \to H^0(M_r, \mathscr{O}(\odot^k V)).$$

But this is precisely $\alpha_r \circ \alpha_0^{-1}$, so it gives us nothing new. Nevertheless, it is useful to review this situation, for these ideas underlie our discussion in §5.

To obtain new representations, we replace $\mathbf{C}^p$ by $\mathbf{C}^{p\times r}$, endowed with the tensor inner product

$$\langle z, w\rangle = tr\,{}^t\bar{w}z = \sum_{i,j} z_i^j\bar{w}_i^j. \tag{2.9}$$

Let $\mathscr{F}$ and $\mathscr{F}_k$ be defined as in (2.1), (2.2), but on $\mathbf{C}^{p\times r}$. SU(p) $\times$ SU(r) acts unitarily on $\mathbf{C}^{p\times r}$ by multiplication on the left by the first factor and on the right by the second. The geometric action

$$[(g\times h)\cdot f](z) = f(g^{-1}zh)$$

gives a unitary representation on $\mathscr{F}$ with the $\mathscr{F}_k$ as invariant subspaces. Let $\mathscr{F}^0$ and $\mathscr{F}_k^0$ be the SU(r)-invariants of $\mathscr{F}$ and $\mathscr{F}_k$, respectively.

For $f \in \mathscr{F}^0$, f is (by Weyl's unitary trick) SL(r, C)-invariant, so $f|\mathbf{C}_*^{p\times r}$ descends (as in §1) to a function $\tilde{f}$ holomorphic on $E(-1) - Z$ (Z is the zero section and

$E = \Lambda^r V$, the determinant bundle of E, as before). If $f \in \mathscr{F}_k^0$, f is also a homogeneous polynomial, so $\tilde{f}$ extends to all of $E(-1)$, and we must necessarily have that k is a multiple of r (recall (1.13), (1.14)).

2.10. PROPOSITION. *$f \to \tilde{f}$ induces an isomorphism*

$$\mathscr{F}_{kr}^0 \to H^0(M_r, \mathcal{O}(E(k)))$$

which intertwines the SU(p)*-action.*

PROOF. The SU(p)-covariance is obvious since the map π: $\mathbf{C}_*^{p\times r} \to E(-1)$ is SU(p)-covariant. The injectivity is obvious. Finally, if $\hat{f} \in H^0(M_r, \mathcal{O}(E(k)))$, $\hat{f} \circ \pi$ is a holomorphic function on $\mathbf{C}_*^{p\times r}$, homogeneous of degree kr and SU(r)-invariant. Since $\hat{f} \circ \pi$ is locally bounded in $\mathbf{C}^{p\times r}$, it extends to an element $f \in \mathscr{F}_{kr}^0$, and clearly $\hat{f} = \tilde{f}$.

Now, just as for projective space, we can dualize to top cohomology. For $f \in \mathscr{F}_{kr}^0$, define

$$\alpha(\bar{f}) = c_k \bar{f}(\|z\|^{-2(k+p)})\bar{\eta}, \tag{2.11}$$

where η is given by (1.15). Since $f\eta$ is homogeneous of degree $r(p+k)$, $\alpha(\bar{f})$ is a $(0, r(p-r))$-form with values in $E(-k-p)$.

2.12. PROPOSITION. *α defines an* SU(p)*-covariant isometry*

$$\bar{\mathscr{F}}_{kr}^0 \to H^{r(p-r)}(M_r, \mathcal{O}(E(-k-p))). \tag{2.13}$$

PROOF. The proof is, as in the case of projective space, a consequence of Serre duality and a calculation that the map of (2.11) is an isometry.

III. The oscillator representation of SU(p, q). We consider $\mathbf{C}^{p+q}$ as endowed with the hermitian form $\langle \cdot, \cdot \rangle$ given by (1.1). We begin by reviewing the construction by Carmona [**7**] (see also [**23, 25**]) of certain L^2-cohomology spaces. We show that there is a natural unitary action of SU(p, q) defined on these spaces and via the formulae of Bargmann [**2, 3**], that this is the oscillator representation. This picture appears at the same time in Blattner and Rawnsley [**5**], and Mantini [**15**]. For the general setting, see [**19, 21, 22**].

Let $\mathscr{E}^r$ be the space of smooth $(0, r)$-forms on $\mathbf{C}^{p+q}$, and $\mathscr{E}_0^r$ those of compact support. Let L^r be the Banach space of $(0, r)$-forms whose coefficients are square-integrable in the measure $\exp(-\langle z, z\rangle)\, d\mu(z)$. As a topological vector space, L^r is invariant under any coordinate change leaving the form $\langle \cdot, \cdot \rangle$ invariant, i.e., L^r is SU(p, q)-invariant.

3.1. DEFINITION. A *splitting* S of $\mathbf{C}^{p+q}$ is a pair of orthogonal complementary subspaces W_+, W_- with W_+ positive-definite and W_- negative-definite.

This means that:

(i) $\mathbf{C}^{p+q} = W_+ \oplus W_-$, $\dim W_+ = p$, $\dim W_- = q$,
(ii) $\langle v, v\rangle \geqslant 0$ for $v \in W_+$,
(iii) $\langle v, v\rangle \leqslant 0$ for $v \in W_-$, and
(iv) $\langle v, w\rangle = 0$ for $v \in W_+$, $w \in W_-$.

$S_0 = (\mathbf{C}^p, \mathbf{C}^q)$ is a splitting. If $S = (W_+, W_-)$ is a splitting and $g \in \mathrm{SU}(p, q)$, then $g \cdot S = (g(W_+), g(W_-))$ is another splitting. $\mathrm{SU}(p, q)$ acts transitively on the space of splittings.

Let $S = (W_+, W_-)$ be a splitting. To S we associate a new complex structure J_S by

$$J_S(v) = iv, \quad v \in W_+; \qquad J_S(v) = -iv, \quad v \in W_-, \tag{3.2}$$

and a hermitian (with respect to both J and J_S) inner product $\langle\ ,\ \rangle_S$ defined by

$$\begin{aligned} \langle v, w\rangle_S &= \langle v, w\rangle, \quad v, w \in W_+, \\ \langle v, w\rangle_S &= -\langle v, w\rangle, \quad v, w \in W_-, \end{aligned} \qquad W_+ \perp W_-.$$

We can extend this inner product to the tangent bundle of $\mathbf{C}^{p+q}$, making $\mathbf{C}^{p+q}$ a hermitian manifold with metric

$$H_S = \exp(-\langle z, z\rangle)\langle dz, dz\rangle_S. \tag{3.3}$$

Then the hermitian form on L^r

$$(\omega, \theta)_S = \int_{\mathbf{C}^{p+q}} H_S(\omega, \theta)\, d\mu \tag{3.4}$$

makes L^r into a Hilbert space. This Hilbert space structure is definitely dependent on the splitting and thus is *not* $\mathrm{SU}(p, q)$-invariant.

3.5. Definition. Let S be a splitting for $\mathbf{C}^{p+q}$. The *formal adjoint* to $\bar\partial\colon \mathscr{E}^r \to \mathscr{E}^{r+1}$ *relative to* S is the differential operator $\vartheta_S\colon \mathscr{E}^r \to \mathscr{E}^{r-1}$ which is adjoint to $\bar\partial$ in the metric (3.4):

$$(\bar\partial\theta, \phi)_S = (\theta, \vartheta_S\phi)_S, \qquad \theta \in \mathscr{E}_0^{r-1}, \phi \in \mathscr{E}^r.$$

The *W-norm* on $\mathscr{E}_0^r$ is defined by

$$\|\phi\|_w^2 = \|\phi\|_S^2 + \|\bar\partial\phi\|_S^2 + \|\vartheta_S\phi\|_S^2, \tag{3.6}$$

and we take W_S^r to be the completion of $\mathscr{E}_0^r$ in this norm. Note that the operators 'inclusion' $\bar\partial$, ϑ_S, as defined on $\mathscr{E}_0^r$, are bounded in the W-norm, so extend continuously to W^r to i, $\bar\partial$, ϑ_S. It is easily verified that i is still injective.

3.7. Definition. $H_S^r = \{\omega \in W_S^r;\ \bar\partial\omega = \vartheta_S\omega = 0\}$.

Note that $i\colon H_S^r \to L^r$ is an isometry, so we may also consider H_S^r as a subspace of L^r. We now give Carmona's description of H_S^r.

3.8. Definition. Real linear $\mathbf{C}$-valued functions $z_1, \ldots, z_p$, $\xi_1, \ldots, \xi_q$ are *adapted* to a given splitting S if:

(i) z_i, ξ_j are J_S-complex linear,

(ii) $W_+ = \{\xi_1 = 0, \ldots, \xi_q = 0\}$, and

(iii) $\langle v, v\rangle_S = \Sigma|z_i(v)|^2 + \Sigma|\xi_i(v)|^2$.

Note that z_i and $\bar\xi_j$ are complex linear on $\mathbf{C}^{p+q}$ and

$$\langle v, v\rangle = \sum|z_i(v)|^2 - \sum|\xi_i(v)|^2.$$

Thus the transformation from one set of adapted coordinates to another is in $\mathrm{SU}(p, q)$.

3.9. Definition. $\mathscr{F}^S$ is the space of J_S-holomorphic functions f such that

$$\|f\|^2 = \int_{\mathbf{C}^{p+q}} |f|^2 \exp(-\langle v, v\rangle_S)\, d\mu(z) < \infty.$$

3.10. THEOREM [7, 21]. (a) $H_S^r = \{0\}$ *for* $r \neq q$. (b) $H_S^q \neq \{0\}$. *In fact,* $\mathscr{F}^S$ *and* H_S^q *are isomorphic via the map*

$$\alpha_S : \mathscr{F}^S \to L^q, \qquad \alpha_S(f) = f(z, \xi)\exp\left(-\Sigma|\xi_i|^2\right) d\xi_1 \wedge \cdots \wedge d\xi_q, \tag{3.11}$$

in any coordinates adapted to S.

Thus, $\bigcup_S H_S^q$ forms a Hilbert space bundle over the space of splittings and there is a natural action of $\mathrm{SU}(p, q)$ on this bundle. To make a unitary representation of this we need an $\mathrm{SU}(p, q)$-invariant realization of the H_S^q.

Now, the $\mathrm{SU}(p, q)$-invariant form $\langle \cdot , \cdot \rangle$ defines a hermitian form $(\, , \,)$ on L^q as in (3.3), (3.4): first we define

$$H = \exp(-\langle z, z\rangle)\langle dz, dz\rangle$$

on the tangent bundle and extend it to forms, defining

$$(\omega, \theta) = \int_{\mathbf{C}^{p+1}} H(\omega, \theta)\, d\mu.$$

This form is $\mathrm{SU}(p, q)$-invariant, but is *not* a metric.

3.12. DEFINITION. (a) K^q is the kernel of the closure of $\bar\partial$:

$$K^q = \left\{\phi \in L^q;\ \phi = \lim \phi_\nu,\ \phi_\nu \in \mathscr{E}_0^q,\ \bar\partial\phi_\nu \to 0\right\}.$$

(b) $H^q = \{\omega \in K^q;\ (\omega, \bar\partial\theta) = 0 \text{ for all } \theta \in \mathscr{E}_0^{q-1}\}$.

H^q is an $\mathrm{SU}(p, q)$-module which has the $\mathrm{SU}(p, q)$-invariant form $(\cdot, \cdot)$. We now verify that the form *is* definite on H^q by defining an isometry of H_S^q with H^q.

Let S be a splitting with adapted coordinates z_i and ξ_j.

3.13. LEMMA. *There is a constant* $C > 0$ *such that for* $\phi \in \mathscr{E}_0^p$

$$\|\vartheta_S\phi\|_S^2 \leqslant C\left(\|\bar\partial\phi\|_S^2 + \|\phi\|_S^2\right). \tag{3.14}$$

PROOF. This is because $[\vartheta_S, \bar\partial]$ is a 0th order operator with constant coefficients or, equivalently, apply (3.11) of [25, p. 201], and add.

By this lemma, if $\phi \in K^q$ and ϕ_ν are as in 3.12(a), then $\{\phi_\nu\}$ is Cauchy in W_S^q and so $\phi \in W_S^q$ as well.

3.15. LEMMA. $H_S^q = K^q \cap (\bar\partial\mathscr{E}_0^{q-1})^{\perp_S}$.

PROOF. If $\omega \in H_S^q$, then $\omega \in K^q$. Let $\phi \in \mathscr{E}_0^{q-1}$. Then

$$(\omega, \bar\partial\phi)_S = (\vartheta_S\omega, \phi)_S = 0,$$

so ω is on the right-hand side. Conversely, suppose $\omega \in K^q$ and ω is orthogonal to the image of $\bar\partial$. Then, as observed above, $\omega \in W_S^q$ and

$$(\vartheta_S\omega, \phi)_S = (\omega, \bar\partial\phi)_S = 0 \quad \text{for all } \phi \in \mathscr{E}_0^{q-1},$$

so $\vartheta_S\omega = 0$ also.

Finally, we note that in the adapted coordinates if

$$\omega = \sum a_{IJ} d\bar z_I \wedge d\xi_J, \qquad \phi = \sum b_{IJ} d\bar z_I \wedge d\xi_J,$$

then

$$(\omega, \phi)_S = \sum \int a_{IJ}\bar b_{IJ} \exp(-\langle z, z\rangle)\, d\mu$$

and

$$(\omega, \phi) = \sum (-1)^{|J|} \int a_{IJ} \bar{b}_{IJ} \exp(-\langle z, z \rangle)\, d\mu.$$

Thus, for $\phi \in H_S^q$, by Theorem 3.10, if $\phi \in L_S^q$,

$$(\omega, \phi) = (-1)^q (\omega, \phi)_S. \tag{3.16}$$

3.17. THEOREM. *The S-orthogonal projection π_S: $K^q \to H_S^q$ defines an isomorphism of H^q with H_S^q such that*

$$(-1)^q (\omega, \phi) = (\pi_S \omega, \pi_S \phi)_S.$$

Thus SU(p, q) *acts unitarily on H^q in this inner product.*

PROOF. By Lemma 3.15, $\phi - \pi_S \phi$ is in the closure of the image of $\bar{\partial}$ for $\phi \in K^q$. Thus, by (3.16) if $\omega \in H^q$, $(\omega, \phi - \pi_S \phi) = 0$. Thus

$$(\omega, \phi) = (\omega, \pi_S \phi) = (-1)^q (\omega, \pi_S \phi)_S = (-1)^q (\pi_S \omega, \pi_S \phi)_S.$$

REMARK. In [5] it is further proven that the closure of $\bar{\partial}$ has closed range, that

$$K^q = H^q \oplus \left(K^q \cap (\bar{\partial} L^{q-1}) \right),$$

and H^q represents cohomology in the sense that each L^2 $\bar{\partial}$-closed form is cohomologous to precisely one form in H^q.

This representation is explicitly related to Bargmann's realization of the oscillator representation as follows. Let $g \in$ SU(p, q), and for S a splitting let $S' = g \cdot S$. Let $B_{g^{-1}}^{\theta}$ be the operator (0.5) with $J_1 = J_{S'}$ and $J = J_S$. We have the diagram:

$$\begin{tikzcd}
\mathscr{F}^S \arrow[r, "g\cdot"] \arrow[d, "\alpha_S"] & \mathscr{F}^{S'} \arrow[r, "B_{g^{-1}}^{\theta}"] \arrow[d, "\alpha_{S'}"] & \mathscr{F}^S \arrow[d, "\alpha_S"] \arrow[dd, bend left, "\beta_S"] \\
H_S^q \arrow[r, "g\cdot"] & H_{S'}^q \arrow[r, phantom, "\beta_{S'}"] & H_S^q \\
H^q \arrow[r, "g\cdot"] \arrow[u, "\pi_S"] & H^q \arrow[u, "\pi_{S'}"] \arrow[ur, "\pi_S"] & H^q \arrow[l]
\end{tikzcd} \qquad (\beta_S = \pi_S^{-1} \circ \alpha_S). \tag{3.18}$$

The top row describes the oscillator representation and the bottom row the representation on H^q. To relate these representations we have to verify the commutativity of this diagram. Since the action on the left squares is always the geometric action, we easily have commutativity there. In the next section we shall explicitly verify the commutativity on the right. But this also follows from consideration of the Heisenberg group H_n: β_S and β_S' intertwine representations of H_n by Carmona's theorem, and so does B_g^{θ}; so by the irreducibility there is a choice of θ making the right side commutative.

Finally, we give the formula for B_g. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let z and ξ be coordinates adapted to S. Then z' and ξ' are adapted to S', where

$$z' = az + b\bar{\xi}, \qquad \xi' = \bar{c}\bar{z} + \bar{d}\xi,$$
$$z = a^* z' - c^* \xi', \qquad \xi = -{}^t b \bar{z} + \bar{d} \xi'.$$

Let θ be such that

$$|\theta|^2|\det_{\mathbf{C}}(a^*)\det_{\mathbf{C}}(d^*)|. \tag{3.19}$$

Then, for $f \in \mathscr{F}^{S'}$

$$B_g^\theta f(z', \xi') = \theta^{-1}\int \exp[\cdots] f(z, \xi)\, d\mu(z, \xi),$$

where

$$\begin{aligned}[\cdots] = -\tfrac{1}{2}(\|z'\|^2 + \|\xi'\|^2) + z'(a^{*-1}c^*)\bar{\xi}' + \bar{z}'a^{*-1}z \\ + \bar{\xi}'^t d^{-1}\xi - z({}^tc^t d^{-1})\xi - \tfrac{1}{2}(\|z\|^2 + \|\xi\|^2).\end{aligned}$$

IV. Realization in the holomorphic series. Let M_p be the space of p-dimensional subspaces of $\mathbf{C}^{p+q}$ on which the hermitian form $\langle\ ,\ \rangle$ is positive-definite. M_p is an open subset of $\mathrm{Gr}(p, \mathbf{C}^{p+q})$, which is referred to as *Siegel space*. As in §1, $V' \to M_p$ is the tautological bundle. For $V' \in M_p$, let $\mathscr{F}_V$ be the space (2.1) of square-integrable holomorphic functions on V'. For our purposes, we want to think of elements of $\mathscr{F}_V$ as p-forms; f is replaced by $f dz_1 \wedge \cdots \wedge dz_p$.

Let $\mathscr{B}\mathscr{F} \to M_p$ be the bundle whose fiber at V' is the Hilbert space $\mathscr{F}_V$. $\mathrm{SU}(p, q)$ acts naturally on this bundle: we shall refer to the induced action on the space of sections as the *geometric action*. We identify (recall (1.10)) $\odot^k V$ as the space $\mathscr{F}_{k,V}$ of homogeneous polynomials of degree k on V and obtain the $\mathrm{SU}(p, q)$-invariant direct sum decomposition

$$\mathscr{B}\mathscr{F} = \bigoplus_{k \geqslant 0} \odot^k V. \tag{4.1}$$

We now relate the oscillator representation as defined in §3 with geometric action on $\mathscr{B}\mathscr{F}$.

Now $dZ_1 = dZ \wedge \cdots \wedge dZ_{p+q}$ is an $\mathrm{SL}(\mathbf{C}^{p+q})$-invariant holomorphic form of top degree, and thus allows us to define an $\mathrm{SU}(p, q)$-covariant conjugate linear bundle map δ: $\Lambda^{q,0} \to \Lambda^{p,0}$ by

$$\delta(\omega) \wedge \theta = \langle \theta, \omega \rangle dZ \quad \text{for all } \theta, \omega \in \Lambda_x^{q,0}. \tag{4.2}$$

4.3. Definition. Let L^q be the Hilbert space of $(0, q)$-forms defined in §3. We define a complex linear transformation Φ from L^{-q} to (locally square-integrable) sections of $\mathscr{B}\mathscr{F}'$ (the space of functionals on $\mathscr{B}\mathscr{F}$) by

$$\Phi(\omega)|_V(T) = \langle T, i_v^*\delta_v(\omega)\rangle_v, \qquad T \in \mathscr{F}_v, \tag{4.4}$$

where i_v^* is the pull-back induced by the inclusion $V' \to \mathbf{C}^{p+q}$.

In other words, if H_v is the Hilbert space projection to $\mathscr{F}_v$, $\Phi(\omega)|_v$ is the linear functional defined by $H_v i_v^* \delta(\omega)$.

Strictly speaking, the identification of $\mathscr{F}_V$ as holomorphic p-forms on V depends on the choice of (adapted) coordinates. But a linear change of coordinates replaces the Hilbert space structure by a scalar multiple:

$$dz_1'' \wedge \cdots \wedge dz_p'' = \big(\partial(z_1'', \ldots, z_p'')/\partial(z_1', \ldots, z_p')\big)\, dz_1' \wedge \cdots \wedge dz_p',$$

so the holomorphic projection H_V and the map Φ it defines, are well defined. $\Phi|H^q$ is the intertwining operator we seek.

4.5. LEMMA. *Φ intertwines the geometric actions of* $\mathrm{SU}(p, q)$.

PROOF. We have not verified that Φ takes values in locally square-integrable sections; that will be apparent in the next lemma. Here we make a formal calculation. Let $g \in \mathrm{SU}(p, q)$. Since g acts holomorphically and isometrically, it commutes with restriction and holomorphic projection: for ω a $(p, 0)$-form

$$H_V i_V^*(g \cdot \omega) = g \cdot H_{g^{-1}V} i_{g^{-1}V}^*(\omega).$$

But now, for $\omega \in L^q$ and $\theta \in \Lambda_x^{q,0}$, $x \in V'$,

$$\begin{aligned}\delta(g \circ \omega) \wedge \theta|_x &= \langle \theta, g \circ \omega \rangle dZ = \langle g^{-1}\theta, \omega \rangle dZ \\ &= \delta(\omega) \wedge g^{-1}\theta|_{g^{-1}x} = g \circ \delta(\omega) \wedge \theta|_x.\end{aligned}$$

Now, let $S = (V_+, V_-)$ be a splitting with adapted coordinates z and ξ. Let $\Phi^S = \Phi \circ \beta_S$ and $\Phi_k^S = \Phi^S | \bar{\mathscr{F}}_k^S$. ($\beta_s$ is as defined in (3.18)). We shall make (4.4) explicit in coordinates.

For $A \in \mathbf{C}^{q \times p}$, let $V_A = \{\bar{\xi} = Az\}$. V_A is in M_p if and only if $I - A^*A$ is positive-definite, and this open subset of $\mathbf{C}^{p \times q}$ coordinates M_p. If we consider z as a map from $\mathbf{C}^{p+p}$ to V_+, then $\pi_A = z|V_A$ is an isomorphism of V_A with V_+, trivializing all bundles over M_p. For $h \in \bar{\mathscr{F}}_k^S$, by (3.11), (3.18) and (4.2)

$$\delta(\beta_S(h)) = \bar{h}(z, \xi) \exp\left(-\|\xi\|^2\, dz_1 \wedge \cdots \wedge dz_p\right),$$

$$i_A^* \delta(\beta_S(h)) = \bar{h}(z, \bar{A}z|) \exp\left(-\|Az\|^2\, dz_1 \wedge \cdots \wedge dz_p\right).$$

Thus, if T is a homogeneous polynomial in $z_1, \ldots, z_p$ of degree k, we have

$$\Phi_k^S(h)(A)(T) = \int_{V^+} T(z) h(z, Az) \exp(-\|z\|^2)\, d\mu(z), \tag{4.6}$$

since $\langle\ ,\ \rangle_{V_A}$ is, in these coordinates, given by $\|z\|^2 - \|Az\|^2$.

4.7. LEMMA. *Φ_k^S is well defined, takes values in $H^0(M_p, \odot^k V)$, and is injective.*

PROOF. We have to show that the integral on the right of (4.6) converges to a function holomorphic in A. For $h \in \bar{\mathscr{F}}_k^S$, by (2.2),

$$|h(z, \xi)| \leqslant \exp \tfrac{1}{2}(\|z\|^2 + \|\xi\|^2) \cdot \|h\|_S.$$

Thus the integral in (4.6) is dominated by

$$\|h\|_S \int |T(z)| \exp(-\|z\|^2/2 + \|Az\|^2/2)\, d\mu(z). \tag{4.8}$$

But $\|z\|^2 - \|Az\|^2 = {}^t\bar{z}(I - A^*A)z$ and, since T is homogeneous of degree k, we evaluate (4.8) by polar coordinates to obtain

$$\left|\Phi_k^S(h)(A)(T)\right| \leqslant C_{kp} \|T\|_{V_+} \|h\|_S \left|\det(I - A^*A)\right|^{-k/2-1}. \tag{4.9}$$

Since h is a limit in $\|\cdot\|_S$ of polynomials, the integral in (4.6) is locally, uniformly in A, the limit of $\langle T, \Phi_k^S(p) \rangle$ for polynomials P. But for polynomials the integral is obviously holomorphic in A, so it is for any $h \in \bar{\mathscr{F}}_k^S$.

To show the injectivity, we first must verify that the Taylor coefficients of $\Phi_k^S(h)$ at $A = 0$ are given by continuous linear functionals on $\bar{\mathscr{F}}_k^S$.

Let $A = (\mu_{rs})$ be the coordinates in $\mathbf{C}^{q\times p}$. For $\mu = (\mu_{rs})$ a matrix of nonnegative integers, let

$$D^{(\mu)} = \prod_{r,s}\left(\frac{\partial}{\partial a_{rs}}\right)^{\mu_{rs}}, \qquad \hat{D}^{(\mu)} = \prod_{r,s}(z_r\xi_s)^{\mu_{rs}}.$$

4.10. LEMMA. *For* $h \in \bar{\mathscr{F}}_k^S$, $T \in F_{V_+}$, k,

$$D^{(\mu)}\Phi_k^S(h)(A)(T)|_{A=0} = \langle T\hat{D}^{(\mu)}, \bar{h}\rangle.$$

PROOF. It suffices to prove the lemma for $T = z^{(d)}$, $h = \bar{z}^{(e)}\bar{\xi}^{(f)}$, $|e| - |f| = k$:

$$\langle \Phi_k^S(h)(A)(T)\rangle = \int z^{(d)}\bar{z}^{(e)}(Az)^{(f)}e^{-\|z\|^2}\,d\mu(z).$$

The coefficient, in the Taylor expansion about $A = 0$, of $a^{(\mu)}$ is 0 unless $\sum_r \mu_{rs} = f_s$, in which case it is

$$I_e = \int z^{(d)}\bar{z}^{(e)}z^{(\mu)}e^{-\|z\|^2}\,d\mu(z).$$

On the other hand,

$$\begin{aligned}\langle z^{(d)}\hat{D}^{(\mu)}, \bar{h}\rangle &= \int_{\mathbf{C}^{p+q}} z^{(d)}\left(\prod_{r,s} z_r\xi_s\right)^{\mu_{rs}}\bar{z}^{(e)}\bar{\xi}^{(f)}\exp(-\|z\|^2 - \|\xi\|^2)\,d\mu(z,\xi)\\ &= I_e\int_{\mathbf{C}^q}\left(\prod_{r,s}\xi_s\right)^{\mu_{rs}}\bar{\xi}^{(f)}e^{-\|\xi\|^2}\,d\mu(\xi),\end{aligned}$$

which is 0 unless $f_s = \sum_r \mu_{rs}$, in which case it is $(\mu!)^{-1}$. Thus

$$\Phi^{(\mu)}\Phi_k^S(h)(A)(T)|_{A=0} = I_e/\mu! = \langle z^{(d)}\hat{D}^{(\mu)}, \bar{h}\rangle.$$

Now, if $\Phi_k^S(h) = 0$, then $\bar{h}$ is orthogonal to all monomials of the form $z^{(d)}\hat{D}^{(\mu)}$, obviously a basis for $\mathscr{F}_k^S$. Thus $h = 0$.

4.11. THEOREM. Φ_k: $\bar{H}_k^q \to H^0(M_p, \mathcal{O}(\odot^k V)')$ *intertwines the oscillator representation of* SU(p, q) *with the geometric action.*

PROOF. This follows directly from the preceding lemmas. We also give a proof based on the computations of Bargmann which avoids Carmona's realization of the Schrödinger representation of the Heisenberg group.

Let S_1 be a splitting with adapted coordinates z_1 and ξ_1 and for $g \in$ SU(p, q) let $S_2 = g \circ S_1$ with adapted coordinates z_2 and ξ_2. Fix a $V \in M_p$ and let z_3 and ξ_3 be coordinates adapted to a splitting (V, W). That Φ_k is an intertwining operator amounts to showing that there exists a choice of θ such that

$$\Phi_k^{S_1} = \Phi_k^{S_2} \circ B_{G^{-1}}^{\theta}.$$

(Recall diagram (3.18).)

Let $\pi_i(z_i, \bar{\xi}_i) = (z_i, 0)$, $i = 1, 2$. We have to show, for $T = z_3^{(d)}$, $|d| = k$ and $h \in \bar{\mathscr{F}}^{S_1}$, that

$$\int (Th)\circ\pi_1^{-1}\exp(-\|z_1\|^2)\,d\mu(z_1) = \int\left(TB_{g^{-1}}^{\theta}h\right)\pi_2^{-1}\exp(-\|z_2\|^2)\,d\mu(z_2). \tag{4.12}$$

Let

$$z_2 = az_1 + b\bar{\xi}_1, \qquad z_3 = a_1 z_2 + b_1\bar{\xi}_2 = a_2 z_1 + b_2\bar{\xi}_1,$$
$$\xi_2 = c\bar{z}_1 + d\xi_1, \qquad \xi_3 = c_1\bar{z}_2 + d_1\xi_2 = c_2\bar{z}_1 + d_2\xi_1.$$

The right-hand side then becomes

$$\det\left(a_2^* a_1^{*^{-1}}\right)\theta^{-1}\int\left(a_1^{*^{-1}}z_2\right)^{(d)}\exp\left(\bar{z}_2 a^{*^{-1}}{}^t c b_1^* a_1^{*^{-1}} z_2\right)[\cdots]\exp\left(-\|z_2\|^2\right)d\mu(z_2),$$

where $[\cdots] =$

$$\int\exp\left[\bar{z}_2 a^{*^{-1}}z_1 + \bar{\xi}_1 d^{-1}b_1^* a_1^{*^{-1}}z_2 - \bar{\xi}_1 d^{-1}c\xi_1 - \|z_1\|^2 - \|\xi_1\|^2\right]h(z_1,\xi_1)\,d\mu(\bar{z}_1,\xi_1).$$

Now, we perform the ξ_1-integration and rearrange. Let $L = (I - a^{*^{-1}}c\bar{b}_1 a_1^{*^{-1}})$ The right-hand side now becomes

$$\det\left(a_2^* a_1^{*^{-1}}\right)\theta^{-1}\int\left(a_1^{*^{-1}}z_2\right)^{(d)}\exp(-\bar{z}_2 L z_2)[\cdots]\,d\mu(z_2),$$

where now

$$[\cdots] = \int\exp\left(\bar{z}_2 a^{*^{-1}}z_1 - \|z_1\|^2\right)h\left(\bar{z}_1, d^{-1}\left(b_1^* a_1^{*^{-1}}z_2 - \bar{c}z_1\right)\right)d\mu(z_1).$$

By analytic continuation from the case where L is hermitian (Bargmann [3]), we obtain

$$\det\left(a_2^* a_1^{*^{-1}}\right)\theta^{-1}\det L^{-1}\int\left(a_1^{*^{-1}}L^{-1}a^{*^{-1}}z_1\right)^d e^{-\|z_1\|^2}\,d\mu(z_1)$$
$$= h\left(\bar{z}_1, d^{-1}\left(b_1^* a_1^{*^{-1}}L^{-1}a^{*^{-1}}z_1 - \bar{c}z_1\right)\right).$$

Note that $a^* L a_1^* = a_2^*$, $b_2^* a_2^{*^{-1}} = d^{-1}[b_1^* a_1^{*^{-1}} L a^{*^{-1}} - c]$ and that, in particular, L is invertible. Thus the right-hand side may now be written

$$(\det a^*)\theta^{-1}\int\left(a_2^{*^{-1}}z_1\right)^{(d)}h\left(\bar{z}_1, b_2^* a_2^{*^{-1}}z_1\right)e^{-\|z_1\|^2}\,d\mu(z_1).$$

Thus, if we choose $\theta = \det a^{*-1}$, this is the left-hand side of (4.12). The condition (3.19) on θ is guaranteed since $|\det(a^*)^{-1}\det(d^*)| = 1$.

REMARK. Since we already know the oscillator representation is unitary, Φ_k serves to unitarize the geometric action on sections of $\odot^k V$. This is Manitini's point of view [**15**]. She explicitly identifies these bundles on Siegel space in terms of the irreducible representations they realize, and gives the differential equations describing the image of Φ_k. We shall give an abstract prescription for these in §7.

V. Representation in cohomology: projective space. Let M_1 be the collection of positive-definite lines in $\mathbf{C}^{p+q}$: M_1 is the image of

$$\mathbf{C}_+^{p+q} = \{v \in \mathbf{C}^{p+q};\ \langle v, v\rangle > 0\}$$

in $\mathbf{P}^{p+q-1}\cdot\mathbf{C}_+^{p+q} \to M_1$ is a principal $\mathbf{C}^*$-bundle whose associated line bundle is $E(-1)$ (recall (1.2)). To each $V' \in M_p$ corresponds a compact submanifold $P(V')$ of M_1 (the collection of lines of V) of dimension $p-1$. This is the maximum

dimension of complex subvarieties of M_1, and we thus expect to find cohomology in degree $p-1$.

We represent cohomology on M_1 by homogeneous data in $\mathbf{C}_+^{p+q}$, as in (1.5), (1.6). Let Λ_k^r be the space of smooth $(0, r)$-forms α defined on $\mathbf{C}_+^{p+q}$ satisfying (1.5), where Γ is the Euler field of $\mathbf{C}^{p+q}$. Then $\bar\partial\colon \Lambda_k^{r-1} \to \Lambda_k^r$ and

$$H^r(M_1, \mathcal{O}(E(k))) \cong \frac{\ker \bar\partial\colon \Lambda_k^r \to \Lambda_k^{r+1}}{\bar\partial \Lambda^{r-1}}. \tag{5.1}$$

For $V' \in M_p$ we have the conjugate linear (Serre duality) isomorphism (2.7):

$$\mathcal{S}\colon H^{p-1}(\mathbf{P}(V'), \mathcal{O}(E(-k-p))) \cong \bar{\mathcal{F}}_{V,k} \cong \theta^k V', \tag{5.2}$$

and thus a transformation

$$P\colon H^{p-1}(M_1, \mathcal{O}(E(-p-k))) \to \mathcal{E}^0({}^kV'), \tag{5.3}$$

given by

$$P(\alpha)(V') = S([\alpha|V']).$$

Now, let S be a splitting with adapted coordinates z and ξ. Let

$$\eta = -\sum_{i=1}^{p} (-1)^i z_i\, dz_1 \wedge \cdots \wedge \hat{dz}_i \wedge \cdots \wedge dz_p.$$

5.4. Definition. For $f \in \bar{\mathcal{F}}_k^S$, let $\alpha_S(f) \in \Lambda_{-(k+p)}^{p-1}$:

$$\alpha_S(f) = \big((k+p-1)!/(2i)^p\big) f(z, \xi)\big(\|z\|^{-2(k+p)}\big)\bar\eta. \tag{5.5}$$

5.6. Proposition. $\bar\partial \alpha_S(f) = 0$.

Proof. Since f is conjugate holomorphic in z and holomorphic in $\bar\xi$,

$$\bar\partial f = \Sigma\big(\partial f/\partial \bar z_j\big)\, d\bar z_j.$$

Thus, by the homogeneity of f

$$\bar\partial(f\bar\eta) = \Gamma(f)\, d\bar z_1 \wedge \cdots \wedge d\bar z_p = kf d\bar z_1 \wedge \cdots \wedge d\bar z_p.$$

Similarly,

$$\partial\big(\|z\|^{-2(k+p)}\bar\eta\big) = -(k+p)\|z\|^{-2(k+p)}\, d\bar z_1 \wedge \cdots \wedge d\bar z_p$$

and $\overline{\partial\eta} = p d\bar z_1 \wedge \cdots \wedge d\bar z_p$. Substituting this into the formula for $\bar\partial\alpha_S(f)$ obtained from (5.5) we obtain $\bar\partial\alpha_S(f) = 0$.

Let $[\alpha_S]\colon \bar{\mathcal{F}}_k^S \to H^{p-1}(M_1, \mathcal{O}(E-k-p))$ be the cohomology class of α_S. Consider the following diagram which is easily seen to be commutative:

$$\begin{tikzcd}
 & \mathcal{F}_k^S \arrow[dl, "\Phi_k^S"'] \arrow[dr, "{[\alpha_S]}"] & \\
E^0(\odot^k V') & & H^{p-1}(M_1, \mathcal{O}(E(-k-p))) \arrow[ll, "P"']
\end{tikzcd}$$

5.7 Lemma. $P \circ [\alpha_S] = \Phi_k^S$.

Proof. Let T be a homogeneous polynomial of degree k in $z_1, \ldots, z_p$, $h \in \bar{\mathcal{F}}_k^S$ and $\alpha = \alpha_S(h)$. Then for $V \in M_p$, we can evaluate the Serre duality on V as:

$$\langle T, P(\alpha)(V)\rangle = \int_{P(V)} P(\alpha)(V) \wedge T\eta = \int_{P(V)} i_V^*\alpha \wedge T\eta$$

which, by putting (5.5) into (2.8), is

$$\int_{\mathbf{C}^p} Th \exp(-\|z\|^2)\, d\mu(z) = \Phi_k^S(h)(V)(T) \tag{5.8}$$

by (4.6). This is true for all T, so the lemma is proved.

5.9. THEOREM. *There is a subspace of* $H^{p-1}(M_1, \mathcal{O}(E(-k-p)))$ *which is a unitary representation space for* $\mathrm{SU}(p,q)$ *and an operator intertwining* $\overline{H}^q$ *with this subspace.*

PROOF. We have to show that $[\alpha_S] \circ \pi_S^{-1}\colon \overline{H}^q \to H^{p-1}(M_1, \mathcal{O}(E(-k-p)))$ is independent of S, and is the desired intertwining operator. For this we need the injectivity of P, which we shall prove in §7.

Let $g \in \mathrm{SU}(p,q)$ and $f \in \overline{\mathscr{F}}_k^S$. Since g acts linearly, $g \cdot \alpha_S(f) = \alpha_{g\cdot S}(g \cdot f)$. Thus

$$\begin{aligned} P(g[\alpha_S](f)) &= P\big([\alpha_{g\cdot S}](g\cdot f)\big) = \phi_k^{g\cdot S}(g\cdot f) \\ &= \phi_k^S\big(B_{g^{-1}}^\theta(g\cdot f)\big) = P\Big([\alpha_S]\big(B_{g^{-1}}^\theta(g\cdot f)\big)\Big). \end{aligned}$$

Since P is injective, $g[\alpha_S](f) = [\alpha_S](B_{g^{-1}}^\theta(g\cdot f))$, so $[\alpha_S](\overline{\mathscr{F}}_k^S)$ is $\mathrm{SU}(p,q)$-invariant, and the theorem follows from (3.18).

VI. Representations in cohomology, Grassmanians. Let M_r be the collection of r-dimensional subspaces of $\mathbf{C}^n$ $(n = p+q)$ on which $\langle\ ,\ \rangle$ is positive-definite. M_r is an open subset of $\mathrm{Gr}(r,n)$. Let

$$\mathbf{C}_+^{n\times r} = \big\{(z^1,\dots,z^r) \in \mathbf{C}^{n\times r};\ \mathrm{span}(z^1,\dots,z^r) \in M_r\big\}$$

and $\pi_0\colon \mathbf{C}_+^{n\times r} \to M_r$, $\pi\colon \mathbf{C}_+^{n\times r} \to E(-1)$ as introduced by (1.12).

Extend $\langle\ ,\ \rangle$ to a hermitian form on $\mathbf{C}^{n\times r}$ by

$$\langle z, w\rangle = \sum_{j=1}^r \langle z^j, w^j\rangle = \sum_{j=1}^r {}^t\overline{w}^j I_{pq} z^j = tr\, {}^t\overline{w} I_{pq} z. \tag{6.1}$$

This hermitian form has signature (pr, qr).

Let $S = \{V_+, V_-\}$ be a splitting of $\mathbf{C}^n$ with adapted coordinates z_i and ξ_i. Associated to S is the product splitting $\hat{S} = \{V_+^r, V_-^r\}$ with adapted coordinates

$$z_i^j = z_i \quad (j\text{th column}), \qquad \xi_i^j = \xi_i \quad (j\text{th column}).$$

Let $\mathscr{F}_k^{\hat{S}}$ be as defined in §4 for the oscillator representation on $\mathbf{C}^{n\times r}$ and let M_{pr} be the space of positive-definite pr-dimensional subspaces of $\mathbf{C}^{n\times r}$. We then have the intertwining operator of Theorem 4.11:

$$\Phi_k^S\colon \overline{\mathscr{F}}_k^{\hat{S}} \to H^0\big(M_{pr}, \odot^k V'\big), \tag{6.2}$$

where $\odot^k V$ is the space of homogeneous polynomials of degree k on the tautological bundle of $\mathrm{Gr}(pr, nr)$.

Now, we want to factor out the $\mathrm{SL}(r,\mathbf{C})$-action on the right. Given a splitting $\hat{S}$ arising us above from a splitting on $\mathbf{C}^n$, $\hat{S}$ is $\mathrm{SL}(r,\mathbf{C})$-invariant, and thus $\mathrm{SL}(r,\mathbf{C})$ acts complex linearly in the J_S structure, and $\mathrm{SU}(r)$ preserves $\langle\ ,\ \rangle_S$.

6.3. DEFINITION. $\mathscr{F}_{0,k}^{\hat{S}}$ is the space of $\mathrm{SL}(r,\mathbf{C})$-invariant functions in $\mathscr{F}_k^{\hat{S}}$.

$\mathscr{F}_{0,0}^{\hat{S}}$ is spanned by all polynomials in the entries of $z^t\xi$, and $\mathscr{F}_{0,kr}^{\hat{S}}$ is the module over $\mathscr{F}_{0,0}^{\hat{S}}$ generated by the homogeneous polynomials of degree k in the functions

$$d_{i_1\cdots i_r}(z) = \det\begin{pmatrix} z_{i_1} \\ \vdots \\ z_{i_r}\end{pmatrix} \tag{6.4}$$

(here z_j is the jth row of z).

Now (6.2) takes $\bar{\mathscr{F}}_{0,kr}^{\hat{S}}$ to the $\mathrm{SL}(r,\mathbf{C})$-invariants of the right-hand side. Since

$$\mathrm{Gr}(pr, nr)/\mathrm{SL}(r,\mathbf{C}) \cong \mathrm{Gr}(p,n),$$

we have $M_{pr}/\mathrm{SL}(r,\mathbf{C}) \cong M_p$, and the $\mathrm{SL}(r,\mathbf{C})$-invariants of $H^0(M_{pr}, \odot^{kr} V)$ is the space of sections on M_p of a bundle $\theta(k)$. $\theta(k)$ is trivialized by the global sections (6.4). (6.2) descends to the intertwining operator

$$\psi_S\colon \bar{\mathscr{F}}_{0,kr}^{S} \to H^0(M_p, \theta(k)). \tag{6.5}$$

Now, we return to $\pi\colon \mathbf{C}_+^{n\times r} \to E(-1) \to M_r$. As in (1.13), (1.14), if Λ_k^s represents the space of smooth 0, s-forms on $\mathbf{C}_+^{n\times r}$ satisfying (1.13), then $\bar\partial\colon \Lambda_k^s \to \Lambda_k^{s+1}$ and

$$H^s(M_r, \mathcal{O}(E(k))) \cong \frac{\ker \bar\partial\colon \Lambda_k^s \to \Lambda_k^{s+1}}{\bar\partial\Lambda_k^{s-1}}. \tag{6.6}$$

Let η be as in (1.15) so that η is a holomorphic $r(p-r)$-form, and $\Gamma_0 \lrcorner d\eta = rp\eta$. Let

$$G(z,\xi) = \int_{\mathrm{GL}(k,\mathbf{C})} \det({}^t g\bar g)^{p+k} \exp(-tr\, {}^t g\, {}^t z \bar z g)\, dg,$$

so that for $g \in \mathrm{GL}(k,\mathbf{C})$

$$G((z,\xi)\cdot g) = (\det {}^t g\bar g)^{-(p+k)} G(z,\xi). \tag{6.7}$$

6.8. DEFINITION. For $f \in \bar{\mathscr{F}}_{0,kr}^{\hat{S}}$, let $\alpha_S(f) = fG\bar\eta$.

6.9. PROPOSITION. $\alpha_S(f) \in \Lambda_{-(k+p)}^{r(p-r)}$ *and* $\bar\partial\alpha_S(f) = 0$.

PROOF. By hypotheses, $\Gamma_0(f) = 0$ and $\bar\Gamma_0(f) = krf$. By (6.7), $\Gamma_0(G) = \bar\Gamma_0(G) = -(p+k)rG$. Thus, for $\alpha = \alpha_S(f)$

$$\Gamma_C \lrcorner d\alpha = \Gamma_0 \lrcorner d(fG\bar\eta) = -(p+k)fG\bar\eta + fG(\Gamma_0 \lrcorner d\bar\eta) = -(p+k)\alpha,$$

since $\bar\eta$ is conjugate holomorphic. Now, if we let $\nu = (rp)^{-1} d\bar\eta$, we find (as in 5.6) that

$$\bar\partial\alpha = \bar\Gamma_0(fG)\nu + fGd\bar\eta = krfG\nu - (p+k)rfG\nu + rpfG\nu = 0.$$

This leads us to the diagram

$$\begin{tikzcd} & \bar{\mathscr{F}}_{0,kr}^{S} \arrow[dl, "\psi_S"'] \arrow[dr, "{[\alpha_S]}"] & \\ H^0(M_p, \theta(k)) & & H^{r(p-r)}(M_r, \mathcal{O}(E(-k-p))) \arrow[ll, "P"] \end{tikzcd} \tag{6.10}$$

where P is defined as follows.

For $V \in M_p$, let $\mathrm{Gr}(r, V)$ be the set of r-dimensional subspaces of V. $\mathrm{Gr}(r, V)$ is a submanifold of M_r of dimension $r(p-r)$, and by (2.12) we have the conjugate linear isomorphism (Serre duality)

$$\mathscr{S}: H^{r(p-r)}(\mathrm{Gr}(r, V), \mathcal{O}(E(-k-p))) \cong \theta(k)_V,$$

the fiber of $\theta(k)$ at V. P is defined by

$$P(\alpha)(V) = \mathscr{S} \circ [\alpha]|_{\mathrm{Gr}(r,V)}.$$

Now, the argument in §5 generalizes without any modification, save to verify the analogue of (5.8) which is

$$\int_{\mathrm{Gr}(r, V_A)} P(\alpha_S(f)) \wedge \eta = K(p, k) \int_{V_+} (Pf) \circ \pi_A^{-1} \exp\left[-tr({}^t z\bar{z})\right] d\mu(z) \tag{6.11}$$

for some constant $K(p, k)$. We obtain, using again the injectivity of §7,

THEOREM 6.12. *The diagram* (6.10) *is commutative. There is a subspace of* $H^{p-1}(M_1, \mathcal{O}(E(-k-p)))$ *which is a unitary representation space for* $\mathrm{SU}(p, q)$, *and an operator intertwining this with with a subrepresentation of the oscillator representation on* $\mathbf{C}^{(p+1)\times r}$.

VII. Injectivity of the Penrose correspondence. In this section we shall generalize Theorem 5.1 of [**8**], concluding that the maps P of §§5 and 6 are injective. First, we need to describe the general set-up of the Penrose correspondence.

Let μ: $Y \to X$ be a holomorphic fibration of complex manifolds ($d\mu$ is everywhere surjective). We consider $d\mu$: $T_Y \to \mu^*(T_X)$ as a map of holomorphic bundles on Y. Let $K = \ker d\mu$; K is the bundle of vertical tangent vectors and has rank $d = \dim Y - \dim X$. Let Λ^p be the bundle of p-forms on K and let Ω^p be the sheaf of holomorphic sections of Λ^p.

For $S \to X$ a sheaf we let $\mu^{-1}(S)$ be the topological lift of S of Y, and when S is an $\mathcal{O}_X$-sheaf

$$\mu^*(S) = \mu^{-1}(S) \otimes_{\mu^{-1}(\mathcal{O}_X)} \mathcal{O}_Y.$$

Note that $\mu^*(\mathcal{O}_X) = \mathcal{O}_Y$.

7.1. LEMMA. *Exterior differentiation induces an exact sequence of sheaves*

$$0 \to \mu^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \xrightarrow{d} \Omega^1 \to \cdots \to \Omega^d \to 0. \tag{7.2}$$

PROOF. This is a standard fact, easily verified by using local coordinates. One checks that Ω^p is defined by the exact sequence

$$0 \to \mu^*(\Omega_X^{p-1}) \wedge \Omega_Y^1 \to \Omega_Y^p \to \Omega^p \to 0$$

and obviously $d^2 = 0$. Exactness is proven by the usual Poincaré lemma, treating the horizontal coordinates as parameters.

7.3. LEMMA. *Let*

$$0 \to S \to S_0 \to S_1 \to \cdots \to S_N \to 0$$

be an exact sheaf sequence on Y and suppose

$$H^s(Y, S_r) = 0, \qquad s < s_0, 1 \leqslant r \leqslant N.$$

Then

$$H^{s_0}(Y, S) \to H^{s_0}(Y, S_0).$$

is injective.

PROOF. This is the standard Leray theorem.

If ν: $Y \to M$ is a proper holomorphic map and $S \to Y$ is a coherent analytic sheaf, then the direct image sheaves $R^s\nu_*(S)$, defined by the presheaves

$$R^s\nu_*(S)(U) = H^s(\nu^{-1}(U), S),$$

are coherent analytic sheaves, and if M is a Stein manifold,

$$H^s(Y, S) \cong H^0(M, R^s\nu_*(S)) \tag{7.4}$$

under the natural map. These facts comprise the direct image theorem of Grauert; however, in our case S will be locally free with $H^p(\nu^{-1}(m), S|\nu^{-1}(m))$ of constant dimension, so we need only appeal to the theorem as proven by Kodaira and Spencer.

Now, suppose we are given a pair of fibrations

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ X & & M \end{tikzcd}$$

with ν proper and μ having connected fibers. Let $E \to X$ be a holomorphic vector bundle and tensor (7.2) with $\mu^{-1}(\mathcal{O}(E))$, obtaining an exact sequence of sheaves on Y:

$$0 \to \mu^{-1}(\mathcal{O}(E)) \to \mu^*(\mathcal{O}(E)) \xrightarrow{d} \Omega^1(E) \to \cdots \to \Omega^d(E) \to 0. \tag{7.5}$$

Associated to this, we have the maps

$$H^s(Y, \mu^{-1}(\mathcal{O}(E))) \to H^s(Y, \mu^*(\mathcal{O}(E))) \tag{7.6}$$

and

$$H^s(Y, \mu^*(\mathcal{O}(E))) \xrightarrow{d_s} H^s(Y, \Omega^1(E)) \tag{7.7}$$

with (7.6) maping onto the kernel of d_s. Now, it is easily verified (as in [**8**, §3]) that the natural map

$$H^0(X, \mathcal{O}(E)) \to H^0(Y, \mu^{-1}(\mathcal{O}(E))) \tag{7.8}$$

is an isomorphism.

7.9. DEFINITION. The *Penrose correspondence* is the map

$$P = (7.4) \circ (7.6) \circ (7.8), \qquad P\colon H^s(X, \mathcal{O}(E)) \to H^0(M, R^s\nu_*(\mathcal{O}(\mu^{-1}(E)))).$$

P maps into the kernel of d_s. Since (7.8) and (7.4) are isomorphisms, in view of Lemma 7.3 applied to (7.5), we have

7.10. PROPOSITION. *In the given situation, P is injective at $s = s_0$ if*

$$H^s(\nu^{-1}(m), \Omega^r(E)|\nu^{-1}(m)) = 0 \tag{7.11}$$

for all $m \in M$, $s < s_0$ and $1 \leqslant r \leqslant d$.

PROOF. The hypotheses imply that $R^s\nu_*\Omega^r(E) = 0$ in the given ranges, so that by (7.4)

$$H^s(Y, \Omega^r(E)) = 0, \qquad s \leqslant s_0, 1 \leqslant r \leqslant d.$$

But then, by Lemma 7.3, P is injective.

In our case, we consider the pair of fibrations

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M_r & & M_p \end{tikzcd}$$

with $1 \leqslant r \leqslant p$, where

$$Y = \{(W, V);\ \dim W = r,\ V \in M_p,\ W \subset V\},$$

and μ and ν are the corresponding projections. For $V \in M_p$, $\mu(\nu^{-1}(M_p)) = \mathrm{Gr}(r, V)$, so for E the hyperplane section bundle on $\mathrm{Gr}(r, p+q)$, the P defined in §§5 and 6 is just the Penrose correspondence defined above applied to $E(-k-p)$ with $s_0 = r(p-r)$. Because of the homogeneity we need only verify (7.11) for one $m \in M_p$ to apply (7.10). We take $m = \mathbf{C}^p$. We shall calculate the sheaves $\Omega^s(E(-\nu))$ over $\mathrm{Gr}(r, \mathbf{C}^p)$ and show that the Borel-Weil-Bott theorem applies to conclude (7.11) for $\nu \geqslant p$.

Let z and ξ be coordinates adapted to the splitting $(\mathbf{C}^p, \mathbf{C}^q)$ so that $\mathbf{C}^{q\times p}$ parametrizes M_p with $A = 0$ corresponding to $\mathbf{C}^p$. Let $W \in \mathrm{Gr}(r, \mathbf{C}^p)$. Then $\xi = 0$ on W, so $W \subset V_A$ if and only if $A|W = 0$. That is, $\mu^{-1}(W)$ is parametrized by

$$\{A \in \mathbf{C}^{p\times p};\ A|W = 0\} \tag{7.12}$$

with $A = 0$ corresponding to $(W, \mathbf{C}^p) \in Y$. Since (7.12) is a linear space, it is the space of coordinates on the tangent K space to $\mu^{-1}(W)$ at $(W, \mathbf{C}^p)$ as well; that is, (7.12) is $\Omega^1_{(W,\mathbf{C}^p)}$. We thus have the exact bundle sequence on $\mathrm{Gr}(r, \mathbf{C}^p)$

$$0 \to \Omega^1 \to (\mathbf{C}^p)^q \to V^q \to 0, \tag{7.13}$$

where V is the dual of the tautological bundle of $\mathrm{Gr}(r, \mathbf{C}^p)$. Comparing (7.13) with (1.10) we have $\Omega^1 \cong (Q')^q$, and thus

$$\Omega^r = \bigoplus_{i_1 + \cdots + i_1 = r} \Lambda^{i_1}(Q') \otimes \cdots \otimes \Lambda^{i_q}(Q').$$

7.14. THEOREM. *Let Q on $\mathrm{Gr}(r, p)$ be defined as in* (1.10). *Then, for $\nu \geqslant p$, $s < r(p-r)$ and $i_1 \geqslant 0, \ldots, i_q \geqslant 0$,*

$$H^s(\mathrm{Gr}(r, p), \Lambda^{i_1}(Q') \otimes \cdots \otimes \Lambda^{i_q}(Q')) \otimes E(-\nu) = 0.$$

PROOF. Write $\mathrm{Gr}(r, p) = \mathrm{SL}(p, \mathbf{C})/0$, where

$$P = M \cdot N, \quad M = \begin{pmatrix} A_{r\times r} & 0 \\ 0 & B_{p-r,p-r} \end{pmatrix}, \quad N = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}.$$

Let

$$\mathfrak{h} = \left\{ \left(H = \begin{matrix} h_1 & & \\ & \ddots & \\ & & h_p \end{matrix} \right) \right\}$$

be a Cartan subalgebra and let ε_i be the roots $\varepsilon_i(H) = h_i$. We calculate, from (1.9), the following weights for $\mathfrak{h}$ on the relevant modules:

(7.15)

$$\begin{aligned} \text{tautological bundle; } V'\text{:} \quad & \varepsilon_1, \ldots, \varepsilon_r, \\ E(-1) = \Lambda^r V'\text{:} \quad & \varepsilon_1^+ \cdots + \varepsilon_r, \\ Q\text{:} \quad & \varepsilon_{r+1}, \ldots, \varepsilon_p, \\ \Lambda^i Q'\text{:} \quad & \text{of the form } -\Sigma n_j \varepsilon_{r+j}, \\ \Lambda^{i_1}(Q') \otimes \cdots \otimes \Lambda^{i_q}(Q') \otimes E(-\nu)\text{:} \quad & \text{of the form } -\Sigma n_j \varepsilon_{r+j} + \nu(\varepsilon_1 + \cdots + \varepsilon_k). \end{aligned}$$

Let us refer to the module in question (7.15) as B. Let

$$\rho = \tfrac{1}{2}\Sigma(p + 1 - 2i)\varepsilon_i.$$

Now, the Borel-Weil-Bott theorem asserts the following. Let L^Λ be a finite-dimensional irreducible representation of P with highest weight Λ and let B^Λ be the corresponding holomorphic vector bundle on $G(r, p)$. Then

$$H^s(G(r, p), B^\Lambda) = 0$$

if there is no w in the Weyl group such that:

(i) $l(w) = s$,

(ii) $w\{\varepsilon_i - \varepsilon_j; i < j\} \supset \{\varepsilon_i - \varepsilon_j; i < j \leqslant r\} \cup \{\varepsilon_i - \varepsilon_j; r < i < j\}$, and

(iii) $\langle w(\Lambda - \rho), \varepsilon_i - \varepsilon_j \rangle < 0$ for $i \leqslant r, j > r$.

Now the components of B are of the form L^Λ with

$$\Lambda = -\Sigma n_j \varepsilon_{r+j} + \nu(\varepsilon_1 + \cdots + \varepsilon_k).$$

Thus, for $i \leqslant r, j > r$,

$$\begin{aligned} \langle (\Lambda - \rho), \varepsilon_i - \varepsilon_j \rangle &= n_j + \nu - \tfrac{1}{2}[(p + 1 - 2i) - (p + 1 - 2j)] \\ &= n_j + \nu + i - j \geqslant n_j + \nu + 1 - p > 0 \end{aligned}$$

if $\nu \geqslant p$.

Now suppose a w exists satisfying (i)–(iii). w is a permutation of p letters, and $l(w)$ is the number of pairs whose order is reversed. Thus (in view of (ii)), if $l(w) < r(p - r)$, there are an $i \leqslant r$ and a $j > r$ such that $w(i) \leqslant r$ and $w(j) > r$, and then

$$\langle w(\Lambda - \rho), \varepsilon_i - \varepsilon_j \rangle = \langle \Lambda - \rho, \varepsilon_{w(i)} - \varepsilon_{w(j)} \rangle > 0.$$

Thus we can have only $l(w) = r(p - r)$.

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