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ON THE SPECTRA OF COMPACT NILMANIFOLDS

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ABSTRACT. We show the equivalence of the Howe-Richardson multiplicity formula for compact nilmanifolds and the formula obtained by Corwin and Greenleaf using the Selberg trace formula.

Introduction. Let G be a connected simply connected nilpotent Lie group and suppose G contains a discrete cocompact subgroup Γ. Let $\rho = \mathrm{ind}_\Gamma^G(1)$. Then ρ is a direct sum of irreducible representations each occurring with finite multiplicity; we will write $\rho = \bigoplus m(\pi)\pi$. A basic problem in representation theory is to determine $m(\pi)$ and give a criterion for $m(\pi)$ not to be zero. Moore first studied this problem in **[M]** and later Howe **[H]** and Richardson **[R]** independently gave a closed formula for $m(\pi)$ that generalized the classical Frobenius reciprocity formula for finite groups. Using the Poisson summation and Selberg trace formulas, Corwin and Greenleaf **[C-G]** gave a formula for $m(\pi)$ that depended only on the coadjoint orbit in $\mathfrak{g}^*$ corresponding to π via Kirillov theory and the structure of Γ, but the connection between the two formulas was not clear. In §1 we consider the case when Γ is a lattice subgroup of G, i.e. $\log(\Gamma)$ is an additive subgroup of the Lie algebra of G, and show there is a simple relationship between the two. In §2 we show how Frobenius reciprocity can be used to reduce the general case to the lattice subgroup case.

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1. Let G be a connected simply connected nilpotent Lie group. We denote the Lie algebra of G by $\mathfrak{g}$ and the dual of g by g^*. Let $\exp\colon \mathfrak{g} \to G$ be the exponential map and $\log\colon G \to \mathfrak{g}$ its inverse. We let Ad be the adjoint action of G on $\mathfrak{g}$ and Ad^* the coadjoint on $\mathfrak{g}^*$. If π is an irreducible unitary representation we write $\mathcal{O}(\pi) \subseteq \mathfrak{g}^*$ for the coadjoint orbit associated to π via Kirillov theory. Let $\Gamma \subseteq G$ be a discrete cocompact subgroup of G. If Q denotes the rational numbers, then Γ determines a Q structure on $\mathfrak{g}$ by $\mathfrak{g}_Q = \mathrm{span}_Q\{\log(\Gamma)\}$. We say $g \in G$ is rational iff $g = \exp(X)$ with $X \in \mathfrak{g}_Q$ and let G_Q denote the set of rational points. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we say $\mathfrak{h}$ is rational if $\mathfrak{h} \cap \mathfrak{g}_Q$ contains a basis of $\mathfrak{h}$ over $\mathbf{R}$. This is equivalent to $\exp(\mathfrak{h}) = H$ having $H \cap \Gamma$ for a discrete cocompact subgroup. If $f \in \mathfrak{g}^*$, say, f is

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rational if $f(\mathfrak{g}_Q) \subseteq Q$. For a very complete and detailed discussion of this see [C-G]. One should also note that if one views G as an affine algebraic group, then the existence of a cocompact Γ is equivalent to G being defined over Q and the notion of a rational point in the sense of algebraic groups is equivalent to the definition given above.

For the rest of this section we suppose that $\Lambda = \log(\Gamma)$ is an additive subgroup of $\mathfrak{g}$. Let $\Lambda^{\perp} = \{f \in \mathfrak{g}^* | f(\Lambda) \subseteq \mathbf{Z}\}$. Let π be an irreducible representation of G and $\mathcal{O} \subseteq \mathfrak{g}^*$ the coadjoint orbit corresponding to π. According to [M], we have $m(\pi) > 0$ if and only if $\mathcal{O} \cap \Lambda^{\perp} \neq \varnothing$, so we will suppose this intersection is nonempty. Since $\mathcal{O} \cap \Lambda^{\perp}$ is $\mathrm{Ad}^*(\Gamma)$ invariant we can write it as a union of $\mathrm{Ad}(\Gamma)$ orbits. To each such orbit $\Omega \subseteq \mathcal{O} \cap \Lambda^{\perp}$ one can associate a number $C(\Omega)$ as follows:

Let $f \in \Omega$ and $\mathfrak{g}(f) = \{X \in \mathfrak{g} | \mathrm{ad}^*(X)f = 0\}$. Then $\mathfrak{g}(f)$ is a rational subalgebra, so the $\mathbf{Z}$-rank of $\mathfrak{g}(f) \cap \Lambda$ is equal to $\dim(\mathfrak{g}(f))$. Choose a $\mathbf{Z}$-basis $X_1, \dots, X_n$ of Λ (consequently it is an $\mathbf{R}$ basis for $\mathfrak{g}$) such that $X_1, \dots, X_s$ span $\mathfrak{g}(f)$ over $\mathbf{R}$. Let A_f be the matrix with entries $f([X_i, X_j])$, $s < i, j \leqslant n$. Then $|\det(A_f)|$ is independent of the basis satisfying the above conditions and depends only on the Γ orbit of f in $\mathcal{O} \cap \Lambda^{\perp}$. Set $C(\Omega) = |\det(A_f)|^{-1/2}$. Then $C(\Omega)$ is a positive rational number. The multiplicity formula of Corwin and Greenleaf can be written as

$$m(\pi) = \sum_{\Omega \in \mathcal{O} \cap \Lambda^{\perp}/\mathrm{Ad}^*(\Gamma)} C(\Omega).$$

For details see [C-G].

We now describe the Howe-Richardson formula for $m(\pi)$. If $m(\pi) > 0$ there exists a rational element $f \in \mathcal{O}(\pi)$, fix such an element f. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a rational polarization for f and $H = \exp(\mathfrak{h})$ the connected subgroup of G with Lie algebra $\mathfrak{h}$. For $\mathfrak{h} \in H$ let $\chi_f(\mathfrak{h}) = \exp(2\pi i f(\log(\mathfrak{h})))$. Then χ_f is a character of H and π is equivalent to $\mathrm{ind}_H^G(\chi_f)$. Let $X = \{(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) | g \in G\}$ and define an equivalence relation on X by $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) \sim (\mathrm{Ad}(g_1)\mathfrak{h}, \mathrm{Ad}^*(g_1)f)$ iff $\mathrm{Ad}(g)\mathfrak{h} = \mathrm{Ad}(g_1)\mathfrak{h}$ and $\mathrm{Ad}^*(g)f|_{\mathrm{Ad}(g)\mathfrak{h}} = \mathrm{Ad}^*(g_1)f|_{\mathrm{Ad}(g)\mathfrak{h}}$. Let $C(\mathfrak{h}, f)$ denote the set of equivalence classes. Define $L(\mathfrak{h}, f) \subseteq C(\mathfrak{h}, f)$ by $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) \in L(\mathfrak{h}, f)$ iff $\mathrm{Ad}(g)\mathfrak{h}$ is a rational subalgebra of $\mathfrak{g}$ and $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1} \cap \Gamma} \equiv 1$. There is a natural action of Γ on $L(\mathfrak{h}, f)$ by

$$\gamma \cdot \big(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f\big) = \big(\mathrm{Ad}(\gamma g)\mathfrak{h}, \mathrm{Ad}^*(\gamma g)f\big)$$

and the number of Γ orbits in $L(\mathfrak{h}, f)$ is the multiplicity $m(\pi)$. In what follows we will assume that $\mathfrak{h}$ and f have been chosen such that $\mathfrak{h}$ is rational and $\chi_f|_{H \cap \Gamma} \equiv 1$.

Suppose $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1} \cap \Gamma} \equiv 1$. Then $\mathrm{Ad}^*(g)f(\mathrm{Ad}(g)\mathfrak{h} \cap \Lambda) \subseteq \mathbf{Z}$: thus we can find $\phi \in (\mathrm{Ad}(g)\mathfrak{h})^{\perp}$ such that $\mathrm{Ad}^*(g)f + \phi \in \Lambda^{\perp}$. From $\mathrm{Ad}^*(gHg^{-1})(\mathrm{Ad}^*(g)f) = \mathrm{Ad}^*(g)f + (\mathrm{Ad}(g)\mathfrak{h})^{\perp}$ we have $\mathrm{Ad}^*(g)f + \phi = \mathrm{Ad}^*(y)(\mathrm{Ad}^*(g)f) = \mathrm{Ad}^*(yg)f$ for some $y \in gHg^{-1}$. Since $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ is equivalent to $(\mathrm{Ad}(yg)\mathfrak{h}, \mathrm{Ad}^*(yg)f)$ we have that every equivalence class in $L(\mathfrak{h}, f)$ has a representative $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ such that $\mathrm{Ad}^*(g)f \in \Lambda^{\perp}$. We can now define a surjective map α: $\mathcal{O} \cap \Lambda^{\perp} \to L(\mathfrak{h}, f)$. If $\phi \in \Lambda^{\perp}$, then both ϕ and f are rational points in $\mathcal{O}$; it follows [**B** or **M**] that there exists $g \in G_Q$, the rational points of G, such that $\mathrm{Ad}^*(g)f = \phi$. Define $\alpha(\phi) = \alpha(\mathrm{Ad}^*(g)f) = (\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$. Since g is rational, $\mathrm{Ad}(g)\mathfrak{h}$ is a rational subalgebra of $\mathfrak{g}$ and $\phi \in \Lambda^{\perp}$ automatically says $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1} \cap \Gamma} \equiv 1$. Let $G(f) = \{g \in G | \mathrm{Ad}^*(g)f = f\}$; since $G(f) \subseteq H$ for any polarization H we see that α is well defined. Since α is a Γ equivariant map we can write the Corwin-Greenleaf formula as follows:

$$m(\pi) = \sum_{\omega \in L(\mathfrak{h}, f)/\mathrm{Ad}^*(\Gamma)} \sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega).$$

The equivalence of the Howe-Richardson formula and the Corwin-Greenleaf formula for Γ a lattice subgroup follows from above once we show

THEOREM 1. *With notation as above* $\sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega) = 1$.

Before we begin with the proof of Theorem 1 we need the following lemmas from [**H** and **C-G**].

LEMMA 1. *Let* $\Gamma \subseteq G$ *be a discrete cocompact subgroup of* G, *suppose* $Z =$ *center of* $\mathfrak{g}$ *is one-dimensional and let* $z \in \log(\Gamma) \cap Z$ *be a generator. Then there exists* $y \in \mathfrak{g}$ *such that if* $W =$ *span of* y *and* z *over* $\mathbf{R}$, *then* y *and* z *generate* $\log(\Gamma) \cap W$. *Let* $\mathfrak{g}_1 =$ *the centralizer of* y *in* $\mathfrak{g}$. *Then* $\mathfrak{g}_1$ *is rational and of codimension* 1 *in* $\mathfrak{g}$. $\mathfrak{g}_1$ *is the Kirillov codimension* 1 *subalgebra* [**K**]. *There is* $x \in \log(\Gamma)$ *such that if* $\Gamma_1 = \Gamma \cap \exp(\mathfrak{g}_1)$, *then* Γ_1 *and* $\exp(x)$ *generate* Γ. *If* $L =$ *span over* $\mathbf{R}$ *of* x, y, z, *then* L *is a three-dimensional Heisenberg algebra and* $\exp(x)$, $\exp(y)$, $\exp(z)$ *generate* $\exp(L) \cap \Gamma$. *Finally, there exists* $a \in \mathbf{Z}$, $a \neq 0$, *so that* $[x, y] = az$ *and* a *is independent of the choice of* x *satisfying the above conditions.*

DEFINITION 1. Let Γ be a discrete torsion-free nilpotent group. A weak Malcev basis for Γ is a set $\{d_1, \dots, d_p\} \subseteq \Gamma$ such that:

(i) For any $d \in \Gamma$ there is a decomposition $d \equiv d_1^{n_1} \cdots d_p^{n_p}$, where $n_i \in \mathbf{Z}$.

(ii) The set $\Gamma_i = d_1^{\mathbf{Z}} \cdots d_i^{\mathbf{Z}}$ is a subgroup with Γ_{i-1} normal in Γ_i for $i = 1, 2, \dots, p$.

(iii) $\Gamma_i/\Gamma_{i-1} \approx \mathbf{Z}$ for $i = 2, \dots, p$.

DEFINITION 2. Let G be a simply connected nilpotent Lie group. A weak Malcev basis for G is a set $\{X_1, \dots, X_p\} \subseteq \mathfrak{g}$ such that:

(i) For $X \in G$ $\exists t_i \in \mathbf{R}$, $i - 1, \dots, p$, such that $X = \gamma_1(t_1) \cdots \gamma_p(t_p)$, where $\gamma_i(t) = \exp(tX_i)$.

(ii) The set $G_i = \gamma_1(\mathbf{R}) \cdots \gamma_i(\mathbf{R})$ is a closed subgroup of G with G_{i-1} normal in G_i for each i.

(iii) $G_i/G_{i-1} \approx \mathbf{R}$.

Weak Malcev basis is an adaptation of Malcev's coordinates of the 2nd kind as necessitated by inducing from nonnormal subgroups in the Kirillov model [**Ma**].

If $\Gamma \subseteq G$ is a discrete cocompact subgroup of G we say a weak Malcev basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ is subordinate to Γ if $\{\exp(X_1), \dots, \exp(X_n)\}$ is a weak Malcev basis for Γ.

LEMMA 2. *Let Γ be a discrete cocompact subgroup of G; if $\{d_1,\ldots,d_n\}$ is a weak Malcev basis for Γ, then $\{X_i = \log(d_i) | i = 1,\ldots,n\}$ is a weak Malcev basis of G subordinate to Γ.*

LEMMA 3. *Let G and Γ be as above, $M \subseteq G$ a closed connected subgroup of G such that $M/M \cap \Gamma$ is compact. If $\{X_1,\ldots,X_s\}$ is a weak Malcev basis of M subordinate to $M \cap \Gamma$, then it can be extended to a weak Malcev basis of G subordinate to Γ_0.*

LEMMA 4. *If Γ is a lattice subgroup of G and $\{X_1,\ldots,X_n\}$ is a weak Malcev basis of G subordinate to Γ, then $\{X_1,\ldots,X_n\}$ forms a $\mathbf{Z}$ basis of $A = \log(\Gamma)$.*

PROOF OF THEOREM 1. We proceed by induction on $\dim(G)$. If $\dim(G) = 1$ the statement is trivial and therefore suppose $\dim(G) > 1$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$; if $\dim(\ker(f) \cap \mathfrak{z}) > 1$ we can divide out the corresponding central subgroup and proceed to a lower dimensional case. Consequently we can assume $\dim(\mathfrak{z}) = 1$, and if x, y, z and $\mathfrak{g}_1$ are as in Lemma 1, then $f(z) = \lambda \neq 0$. Note that for $f_1, f_2 \in \mathcal{O}(\pi)$, $f_1(z) = \mathrm{Ad}^*(g)f_2(z) = f_2(z) = \lambda$.

We can assume $\omega = \Gamma \cdot (\mathfrak{h}, f)$ simply by relabeling. To find Γ orbits in $\mathcal{O} \cap \Lambda^\perp$ such that $\alpha(\Omega) = \omega$ we proceed as follows: The point $(H, f) \in \omega$ has $\Gamma \cap H$ for its stability group; consequently $\Gamma \cap H$ preserves the fiber $\alpha^{-1}((H, f)) = (f + \mathfrak{h}^\perp) \cap \Lambda^\perp$. A set of representatives for Γ orbits in $\mathcal{O} \cap \Lambda^\perp$ that map into ω via α is given by a set of representatives of $\Gamma \cap H$ orbits in $(f + \mathfrak{h}^\perp) \cap \Lambda^\perp$.

As usual, there are two cases we must consider.

Case I. Suppose $\mathfrak{h} \subseteq \mathfrak{g}_1$. Let $\bar{f}$ be the restriction of f to $\mathfrak{g}_1$. Then $\mathfrak{h}$ is a rational polarization for $\bar{f}$ and $\bar{\pi} = \mathrm{ind}_H^{G_1}(\chi_{\bar{f}})$ occurs in $L^2(G_1/\Gamma_1)$ by the criterion in [**C-G** or **M**], i.e., $\bar{f} \in \Lambda_1^\perp \subseteq \mathfrak{g}_1^*$. Let $\mathfrak{h}^{\perp 1} = \{\phi \in \mathfrak{g}_1^* | \phi(x) = 0\ \forall X \in \mathfrak{h}\}$ and let $r\colon \mathfrak{g}^* \to \mathfrak{g}_1^*$ be the restriction map—so r is Ad^* equivariant. If $L = (f + \mathfrak{h}^\perp) \cap \Lambda^\perp$ and $L_1 = (\bar{f} + h^{\perp 1}) \cap \Lambda_1^\perp$, then $r|_L\colon L \to L_1$ is a $\Gamma \cap H = \Gamma_1 \cap H$ equivariant surjective map. If $S_1 \subseteq L_1$ is a set of $\Gamma \cap H$ orbit representatives, then $(r|_L)^{-1}(S_1)$ contains a set of $\Gamma \cap H$ orbit representatives in L, say S. Let $\bar{\phi} \in S_1, \phi \in (r|_L)^{-1}(\bar{\phi})$, $G_1(\underline{\phi})$ = stability group of $\bar{\phi}$ in G, and $G(\phi)$ be the stability group of ϕ in G. Then $G_1(\bar{\phi}) = G(\phi) \cdot \{\exp(tY) | t \in \mathbf{R}\}$. Consequently a $\Gamma \cap G_1(\bar{\phi})$ orbit in $(r|_L)^{-1}(\bar{\phi})$ is the same as a $\Gamma \cap \{\exp(tY) | t \in \mathbf{R}\} = \{\exp(nY) | n \in \mathbf{Z}\}$ orbit in $(r|_L)^{-1}(\bar{\phi})$. An element $\phi \in (r|_L)^{-1}(\bar{\phi})$ is determined by its value on $x \in \mathfrak{g}$. Suppose $\phi(x) = b$. Then $\mathrm{Ad}^*(\exp(nY))(\phi)(x) = b + na\lambda$ and we see there are $|a\lambda| \{\exp(nY) | n \in \mathbf{Z}\}$ orbits in $(r|_L)^{-1}(\bar{\phi})$.

Let $\phi \in (r|_L)^{-1}(\bar{\phi})$ for some $\bar{\phi} \in S_1$. We want to compute $C(\Gamma \cdot \phi)$. To do this we need a basis $X_1,\ldots,X_n$ of $\mathfrak{g}$ such that the $\mathbf{Z}$ span is Λ and $X_1,\ldots,X_s$ span $\mathfrak{g}(\phi)$ over $\mathbf{R}$. Using Lemmas 3 and 4 we can find a basis $X_1 = z$, $X_2 = y,\ldots,X_{n-1}$ of $\mathfrak{g}_1$ such that $X_1,\ldots,X_{n-1}$ span Λ_1 over $\mathbf{Z}$, $X_1, X_2,\ldots,X_s$ span $\mathfrak{g}_1(\bar{\phi})$, and $X_1, X_3, X_4,\ldots,X_s$ span $\mathfrak{g}(\phi) \subseteq \mathfrak{g}_1(\bar{\phi})$. We have $X_1, X_3,\ldots,X_{n-1}\, y, x$ span Λ over $\mathbf{Z}$ and $X_1, X_3,\ldots,X_s$ span $\mathfrak{g}(\phi)$. Using this basis we can compute $C(\Gamma \circ \phi) = C(\phi)$. Recall that $C(\phi) = |\det(A(\phi))|^{1/2}$, where $A(\phi) = (\phi([x_i, x_j]))$, $s \leqslant i, j \leqslant n$. Since $[Y, \mathfrak{g}_1] = 0$, if we expand $A(\phi)$ on that row and column, using $\phi([x, y]) = \lambda a$, we get $\det(A(\phi)) = |\lambda a|^2 \det(A(\bar{\phi}))$. Consequently, we have $C(\phi) = |\lambda a|^{-1} \cdot C(\bar{\phi})$ for all $\bar{\phi} \in S_1$.

By our induction hypothesis $\Sigma_{\bar{\phi}\in S_1} C(\bar{\phi}) = 1$, so we get

$$\sum_{\phi\in S} C(\phi) = \sum_{\bar{\phi}\in S_1}\left(\sum_{\phi\in(r|_L)^{-1}(\bar{\phi})\cap S} C(\phi)\right)$$

$$= \sum_{\bar{\phi}\in S_1} C(\bar{\phi})\left(\sum_{\phi\in(r|_L)^{-1}(\bar{\phi})\cap S} |\lambda a|^{-1}\right) = \sum_{\bar{\phi}\in S_1} C(\bar{\phi}) = 1.$$

Thus Case I is verified.

Case II. Now suppose $\mathfrak{h} \not\subseteq \mathfrak{g}_1$ and set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_1$, $\bar{\mathfrak{h}} = \operatorname{span}(\mathfrak{h}_0, Y)$. Then $\bar{\mathfrak{h}} \subseteq \mathfrak{g}_1$ and is a polarization for f. As before, let $r\colon g^* \to \mathfrak{g}_1^{*\prime}$ be the restriction map. Then $r|_{f+\mathfrak{h}^\perp}\colon f + \mathfrak{h}^\perp \to \mathfrak{g}_1^*$ is injective and $r(f + \mathfrak{h}^\perp) = \bigcup_{s\in\mathbf{R}}(f + sy^* + \bar{\mathfrak{h}}^{\perp 1})$. Consequently we have $r((f + \mathfrak{h}^\perp) \cap \Lambda^\perp) = \bigcup_{n\in\mathbf{Z}}(f + ny^* + \mathfrak{h}^{\perp 1}) \cap \Lambda_1^\perp$. A set S of $\Gamma \cap H$ orbit representatives in $(f + \mathfrak{h}^\perp) \cap \Lambda^\perp$ can be written as $\bigcup_{0\leqslant b<|\lambda a|} S_b$, where S_b is a set of $\Gamma \cap \bar{H}$ orbit representatives in $(f + by^* + \bar{\mathfrak{h}}^{\perp 1}) \cap \Lambda_1^\perp$ and $b \in \mathbf{Z}$. As before, if $\phi \in r^{-1}(S_b) \cap \Lambda^\perp$, then $C(\phi) = |\lambda a|^{-1} C(\bar{\phi})$, where $\bar{\phi} = r(\phi)$. Thus we have

$$\sum_{\phi\in S} C(\phi) = \sum_{0\leqslant b<|\lambda a|}\left(\sum_{\phi\in r^{-1}(S_b)\cap S} C(\phi)\right)$$

$$= \sum_{0\leqslant b<|\lambda a|} |\lambda a|^{-1}\left(\sum_{\bar{\phi}\in S_b} C(\bar{\phi})\right) = \sum_{0\leqslant b<|\lambda c|} |\lambda a|^{-1} = 1. \qquad \text{Q.E.D.}$$

COROLLARY 1 (MOORE [M]). $m(\pi) \leqslant \#\{\mathcal{O} \cap \Lambda^\perp/\mathrm{Ad}^*(\Gamma)\}$.

PROOF. By the above proof, $C(\phi)^{-1} = |\lambda a| C(\bar{\phi})$; thus we can reason by induction to conclude that $C(\phi)^{-1}$ is an integer. Q.E.D.

In [**R**] Richardson constructed a polarization for f such that in the inductive reasoning one never has Case II occurring. We note that if $\mathfrak{h}$ is a Richardson polarization, then the above proof shows that the $C(\Omega)$s are the same for every Ω such that $\alpha(\Omega) = \omega$. Since $\Sigma_{\alpha(\Omega)=\omega} C(\Omega) = 1$, $C(\Omega)$ equals the number of $\Gamma \cap H$ orbits in $f + \mathfrak{h}^\perp \cap \Lambda^\perp$. This observation was pointed out to me by Larry Corwin.

2. If π is a unitary representation of G on the Hilbert space $H(\pi)$, we let $H^\infty(\pi) = \{u \in H(\pi) | q \to \pi(q)u \text{ is a } C^\infty\text{-mapping}\}$. There is a representation of $\mathfrak{g}$, the Lie algebra of G, on $H^\infty(\pi)$. If $X \in \mathfrak{g}$, $u \in H^\infty(\pi)$ define

$$\pi(X)u = \frac{d}{dt}\,(\pi(\exp(tX)u))|_{t=0}.$$

Then $X \to \pi(X)$ is a Lie algebra representation of $\mathfrak{g}$, so it extends to a representation of $\mathfrak{U}(\mathfrak{g})$. Given $a \in \mathfrak{U}(\mathfrak{g})$ define a seminorm on $H^\infty(\pi)$ by $\rho_a(u) = \|\pi(a)u\|$. Then $H^\infty(\pi)$ has the structure of a Fréchet space with respect to these seminorms and we let $H^{-\infty}(\pi)$ be the topological dual of $H^\infty(\pi)$. For details see [**P-1**]. We write π^∞ for the restriction of π to $H^\infty(\pi)$ and $\pi^{-\infty}$ for the dual representation of G on $H^{-\infty}(\pi)$. The following is in [**P-1**].

THEOREM 2.1 (PENNEY). *Let G be a Lie group, Γ a discrete cocompact subgroup and π an irreducible representation of G. Then*

$$\operatorname{Hom}_G\big(\pi, \operatorname{ind}_\Gamma^G(1)\big) \simeq \operatorname{Hom}_\Gamma(1, \pi^{-\infty}).$$

If we set $\Gamma(\pi) = \{D \in H^{-\infty}(\pi) | \pi^{-\infty}(\lambda)D = D\ \forall \lambda \in \Gamma\}$, then the above theorem says $\dim(\operatorname{Hom}_G(\pi, \operatorname{ind}_\Gamma^G(1))) = \dim(\Gamma(\pi))$.

When G is a connected simply connected nilpotent Lie group, the lift maps of Richardson can be viewed as providing a basis of $\Gamma(\pi)$ **[R]**. If $\pi = \operatorname{ind}_H^G(\chi_f)$, then $H^\infty(\pi)$ corresponds to all Schwartz functions on G/H **[K]** and for each Γ orbit in $L(\mathfrak{h}, f)$ one can construct an element of $\Gamma(\pi)$. If $(\operatorname{Ad}(g)\mathfrak{h}, \operatorname{Ad}^*(g)f)$ is a point of $L(\mathfrak{h}, f)$, then we can construct $D_g \in H^{-\infty}(\pi)$ as follows: For $\phi \in H^\infty(\pi)$ let $D_g(\phi) = \sum_{\gamma \in \Gamma/\Gamma \cap gHg^{-1}} \phi(\gamma g)$. The D_g's are linearly independent for g's in different $\Gamma : H$ double cosets and they span $\Gamma(\pi)$ **[P-2, F]**.

Now suppose $\Gamma_0 \subseteq \Gamma$ is a normal subgroup of finite index and we know the truth of the Howe-Richardson formula for Γ_0.

Let $L_0(\mathfrak{h}, f)$ be defined using Γ_0 and $L(\mathfrak{h}, f)$ be defined using Γ. Of course $L(\mathfrak{h}, f) \subseteq L_0(\mathfrak{h}, f)$, so we will suppose $L_0(\mathfrak{h}, f)$ is not empty. Let $D_g \in \Gamma_0(\pi) \subseteq H^{-\infty}(\pi)$. For $\gamma \in \Gamma$, $\phi \in H^\infty(\pi)$ we have

$$\big(\pi^{-\infty}(\gamma) D_g\big)(\phi) = \sum_{\delta \in \Gamma_0/\Gamma_0 \cap gHg^{-1}} \phi(\gamma\delta g) = \sum_{\delta \in \Gamma_0/\Gamma_0 \cap \gamma gHg^{-1}\gamma^{-1}} \phi(\delta\gamma g).$$

Thus $\pi^{-\infty}(\gamma)$ stabilizes $\mathbf{C} \cdot D_g$ iff $\gamma \in \Gamma_0 \cdot (\Gamma \cap gHg^{-1})$, then

$$\pi^{-\infty}(\gamma) D_g = \chi_f\big(g^{-1}\gamma^{-1}g\big) D_g = \overline{\chi}_{g \cdot f}(\gamma) \cdot D_g$$

(where $\overline{\chi}_{g \cdot f}$ extends to a character of $\Gamma_0 \cdot (\Gamma \cap gHg^{-1})$ by being trivial on Γ_0). If we set $W_g = \operatorname{span}\{\pi^{-\infty}(\gamma)D_g | \gamma \in \Gamma\}$, then we see that $\pi^{-\infty}|_\Gamma$ acting on W_g is exactly $\operatorname{ind}^\Gamma_{\Gamma_0 \cdot (\Gamma \cap gHg^{-1})}(\overline{\chi}_{q \cdot f})$. Let S be a set of representatives for Γ orbits in $L_0(\mathfrak{h}, f)$, so given $g \in S$ we get $(\operatorname{Ad}(g)\mathfrak{h}, \operatorname{Ad}^*(g)f)$ or equivalently a $D_g \in H^{-\infty}(\pi)$. From above we see that the representation $\pi^{-\infty}|_\Gamma$ acting on $\Gamma_0(\pi)$ is

$$\bigoplus_{g \in S} \operatorname{ind}^\Gamma_{\Gamma_0 \cdot (\Gamma \cap gHg^{-1})}\big(\overline{\chi}_{g \cdot f}\big).$$

Since $\Gamma(\pi) \subseteq \Gamma_0(\pi)$, we have that

$$nn\operatorname{Hom}_\Gamma(1, \pi^{-\infty}) \approx \bigoplus_{g \in S} \operatorname{Hom}_\Gamma\Big(1, \operatorname{ind}^\Gamma_{\Gamma_0 \cdot (\Gamma \cap gHg^{-1})}\big(\overline{\chi}_{q \cdot f}\big)\Big)$$
$$\approx \bigoplus_{g \in S} \operatorname{Hom}_{\Gamma_0 \cdot (\Gamma \cap gHg^{-1})}\big(1, \overline{\chi}_{g \cdot f}\big).$$

Thus

$$\dim\Big(\operatorname{Hom}_\Gamma\Big(1, \operatorname{ind}^\Gamma_{\Gamma_0 \cdot (\Gamma \cap gHg^{-1})}\big(\overline{\chi}_{g \cdot f}\big)\Big)\Big) = \begin{cases} 1 & \text{if } \chi_{g \cdot f}|_{gHg^{-1} \cap \Gamma} \equiv 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally those $g \in S$ such that $\chi_{g \cdot f}|_{gHg^{-1} \cap \Gamma} \equiv 1$ are parametrized by $L(\mathfrak{h}, f)$. Thus we have

PROPOSITION. *If $\Gamma_0 \subseteq \Gamma$ is a normal subgroup and the Howe-Richardson formula for Γ_0 is known, then the Howe-Richardson formula for Γ is true.*

REMARK. It is shown in [**M** or **C-G**] that for a given Γ it is always possible to find a normal Γ_0 such that Γ_0 is a lattice subgroup.

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