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CONSTANT TERM IDENTITIES EXTENDING THE q-DYSON THEOREM

BY

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ABSTRACT. Andrews [1] has conjectured that the constant term in a certain product is equal to a q-multinomial coefficient. This conjecture is a q-analogue of Dyson's conjecture [5], and has been proved, combinatorically, by Zeilberger and Bressoud [15]. In this paper we give a combinatorial proof of a master theorem, that the constant term in a similar product, computed over the edges of a nontransitive tournament, is zero. Many constant terms are evaluated as consequences of this master theorem including Andrews' q-Dyson theorem in two ways, one of which is a q-analogue of Good's [6] recursive proof.

1. Introduction. The q-Dyson theorem states that the constant term (with respect to $x_1,\ldots,x_n$) in

$$\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\left(1-q\frac{x_i}{x_j}\right)\cdots\left(1-q^{a_i-1}\frac{x_i}{x_j}\right)\left(1-q\frac{x_j}{x_i}\right)\cdots\left(1-q^{a_j}\frac{x_j}{x_i}\right)$$

is equal to

$$\frac{(1-q)(1-q^2)\cdots(1-q^{a_1+\cdots+a_n})}{(1-q)(1-q^2)\cdots(1-q^{a_1})\cdots(1-q)(1-q^2)\cdots(1-q^{a_n})}.$$

This was conjectured and verified for $n=1,2,3$ by Andrews [1] in 1975. It was proved for $n=4$ by Kadell [8] in 1983, and for all n by Zeilberger and Bressoud [15], later that year.

This theorem has its origins in work in statistical mechanics by Dyson [5] in 1962. Specifically, in describing the statistical properties of a finite Coulomb gas of N particles, he found it necessary to evaluate the integral

$$\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{j<k}|e^{i\theta_j}-e^{i\theta_k}|^{2z}\,d\theta_1\cdots d\theta_N,$$

which he conjectured to equal $(2\pi)^N\Gamma(Nz+1)/\Gamma(z+1)^N$. His approach to this integral was to recognize that its evaluation is equivalent to showing that the constant term in $\prod_{i<j}(1-x_i/x_j)^z(1-x_j/x_i)^z$ is $\Gamma(Nz+1)/\Gamma(z+1)^N$. This constant term is easily verified for $N=2$ by the binomial theorem and for $N=3$ by a classical result of Dixon. These results led Dyson to conjecture the following generalization, in which the operator [1] indicates the constant term (coefficient of 1) in the polynomial in $x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}$ to which it is applied.

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THEOREM 1.1. *For nonnegative integers* $a_1, \ldots, a_n$,

$$[1] \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} \left(1 - \frac{x_j}{x_i}\right)^{a_j} = \frac{(a_1 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \quad \square$$

Theorem 1.1 was proved independently by Gunson [7] and Wilson [**13**] later in 1962. In 1970, Good [**6**] gave a very elementary proof using recursions. This proof is discussed in §5. Zeilberger [**14**] gave a radically different combinatorial proof in 1982.

If we introduce the notation

$$(x)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x), \qquad (x)_0 = 1, \qquad (x)_{-1} = (1 - xq^{-1})^{-1}$$

for positive integers n, then the q-Dyson theorem can be more succinctly stated, as follows.

THEOREM 1.2. *For nonnegative integers* $a_1, \ldots, a_n$,

$$[1] \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j} = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}}. \quad \square$$

It is readily verified that Theorem 1.2 becomes Theorem 1.1 in the limit as q approaches 1. For $n = 1, 2, 3$, Theorem 1.2 is either trivial or classical. Kadell's proof for $n = 4$ is modelled on Good's proof of Theorem 1.1, and the Zeilberger-Bressoud proof for general n is an extension of Zeilberger's proof of Theorem 1.1.

In this paper we use the Zeilberger-Bressoud approach to prove a master theorem, which is stated in terms of tournaments. A *tournament* T on n vertices is a set of ordered pairs (i, j) such that $1 \le i \ne j \le n$ and $(i, j) \in T$ if and only if $(j, i) \notin T$. Equivalently, T can be thought of as a directed graph with vertices $1, \ldots, n$ and edges directed from i to j for all $(i, j) \in T$. Thus we write $i \to j$ if $(i, j) \in T$. The tournament T is *transitive* if the relation $\to$ is transitive. Equivalently, T is transitive if it contains no cycles $(i \to j \to k \to i)$. Otherwise, T is *nontransitive*.

THEOREM 1.3 (MASTER THEOREM). *If T is a nontransitive tournament on n vertices and $a_1, \ldots, a_n$ are nonnegative integers, then*

$$[1] \prod_{(i,j) \in T} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j - 1} = 0. \quad \square$$

§2 is devoted to consequences of our master theorem, which include deducing the q-Dyson theorem directly from it. Before summarizing them, we explain our notation.

The set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{N}_n$. If T is a tournament on n vertices, then the *in-degree* of vertex j, $1 \le j \le n$, is the cardinality of $\{i \in \mathcal{N}_n | (i, j) \in T\}$, and the *out-degree* of vertex j is the cardinality of $\{i \in \mathcal{N}_n | (j, i) \in T\}$. Let S_n denote the set of permutations on $\mathcal{N}_n$. If T is transitive, then T defines a permutation $\sigma \in S_n$, called the *winner permutation*, by $\sigma(i) = \sigma_i$ and equals the vertex with in-degree $i - 1$. Thus if we say that $i \to j$ means that i beats j, then σ_1 beats everyone, σ_2 beats everyone but σ_1, and σ_n is beaten by everyone. The identity permutation is denoted by e_n (or e

if there is no ambiguity about the value of n). The transitive tournament corresponding to e_n (e) is denoted by E_n (E). The set of *inversions* of σ is denoted by $I(\sigma) = \{(\sigma_i, \sigma_j) | \sigma_i > \sigma_j \text{ and } i < j\}$. If $S \subseteq T$ for any tournament T, then $T\bar{S}$ is the tournament obtained from T by reversing the edges of S.

The vector $(a_1, \ldots, a_n)$ is denoted by $\mathbf{a}$, while $(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$ is denoted by $(a_1, \ldots, \hat{a}_k, \ldots, a_n)$ for $k = 1, \ldots, n$. The unit vector in the ith direction is denoted by $\boldsymbol{\delta}_i$. For $\sigma \in S_n$, $\sigma(\mathbf{a})$ denotes $(a_{\sigma_1}, \ldots, a_{\sigma_n})$. For any statement A, $\chi(A)$ is defined to be 1 if A is true and 0 otherwise. For any set X of integers, we denote $\{(i, j) | i < j, i, j \in X\}$ by $\binom{X}{2}$.

Finally, for compactness, let T be a tournament and define

$$P(T; \mathbf{a}) = \prod_{(i,j) \in T} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j - 1}, \qquad C(T; \mathbf{a}) = [1] P(T; \mathbf{a}),$$

$$r(\mathbf{a}) = \prod_{i=1}^{n} (1 - q^{a_i})(1 - q^{a_1 + \cdots + a_i})^{-1}.$$

The most general results of §2 are: If $\mathscr{A} \subseteq \binom{\mathscr{N}_n}{2}$ and $\mathscr{R}(A) = \{\sigma \in S_n | \text{ if } (j, i) \in I(\sigma), \text{ then } (i, j) \notin \mathscr{A}\}$, then

$$\text{(1.4)} \quad \begin{aligned} &[1] \prod_{1 \le i < j \le n} \left\{ \left(\frac{x_j}{x_i}\right)^{\chi((i,j) \notin \mathscr{A})} - \frac{x_i}{x_j} \right\} \left(q\frac{x_i}{x_j}\right)_{a_i - 1} \left(q\frac{x_j}{x_i}\right)_{a_j - 1} \\ &= \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}} \sum_{\sigma \in \mathscr{R}(\mathscr{A})} (-1)^{|I(\sigma)|} r(\sigma(\mathbf{a})). \end{aligned}$$

Furthermore, if $\mathscr{B} \subseteq \binom{\mathscr{N}_n}{2}$ with $\mathscr{A} \cap \mathscr{B} = \varnothing$, then

$$\text{(1.5)} \quad \begin{aligned} &[1] \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i} q^{\chi((i,j) \notin \mathscr{B})}\right)_{a_j - \chi((i,j) \in \mathscr{A})} \\ &= \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}} \sum_{\sigma \in \mathscr{R}(\mathscr{A})} q^{\Sigma^* a_i} r(\sigma(\mathbf{a})), \end{aligned}$$

where Σ^* is the sum over pairs $(i, j) \in I(\sigma)$ such that $(j, i) \notin \mathscr{B}$. □

The main results of §2 are the corollaries of these general results that involve choices of $\mathscr{A}$ and $\mathscr{B}$ for which the summations on the RHS of (1.4) and (1.5) can be expressed explicitly as a product. Many of these corollaries have been conjectured by Kadell **[8]**.

§3 introduces the notion of a *tournamented statistic* on a word. This is a generalization of the z-statistic used in the Zeilberger-Bressoud proof of the q-Dyson theorem, and is central to the proof of the master theorem, which is given in §4.

In §5 we demonstrate how the master theorem can be used to prove the q-Dyson theorem by a method analogous to Good's proof of Theorem 1.1. This section concludes with a summary of the conjectures and open problems suggested by our work.

It should be mentioned that Theorems 1.1 and 1.2 are only the simplest cases of several families of integral evaluations and constant term identities indexed by root systems of Lie algebras, this larger context being first observed by Macdonald [**10**] (see also Morris [**12**], Andrews [**3**] and Askey [**4**]). This is a field that is characterized by having far more conjectures than theorems.

2. Consequences of the master theorem. In this section we deduce various constant-term identities from the master theorem. Frequent use is made of the following result.

PROPOSITION 2.1. *Let T be a tournament with $S \subseteq T$. Then*

$$P(T;\mathbf{a}) \prod_{(i,j)\in S}\left(-\frac{x_j}{x_i}\right) = P(T\bar{S};\mathbf{a}).$$

PROOF. First note that

$$\left(-\frac{x_j}{x_i}\right)\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1} = \left(-\frac{x_j}{x_i}\right)\left(1-\frac{x_i}{x_j}\right)\left(q\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-1}$$
$$= \left(\frac{x_j}{x_i}\right)_{a_j}\left(q\frac{x_i}{x_j}\right)_{a_i-1}.$$

Applying this for all $(i, j) \in S$ gives

$$P(T;\mathbf{a})\prod_{(i,j)\in S}\left(-\frac{x_j}{x_i}\right) = \prod_{(i,j)\in T-S}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1}\prod_{(i,j)\in S}\left(\frac{x_j}{x_i}\right)_{a_j}\left(q\frac{x_i}{x_j}\right)_{a_i-1}$$
$$= P(T\bar{S};\mathbf{a}), \quad \text{as required.} \quad \square$$

Now we are ready to give the transitive analogue of the master theorem.

THEOREM 2.2. *If T is a transitive tournament with winner permutation σ, then for nonnegative integers $a_1,\ldots,a_n$,*

$$[1]\prod_{(i,j)\in T}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{i=1}^{n}\frac{(1-q^{a_i})}{(1-q^{a_{\sigma(1)}+\cdots+a_{\sigma(i)}})}.$$

PROOF. We first prove this result for $\sigma = e$. Now

$$\left(q\frac{x_n}{x_i}\right)_{a_n-1} = \left(q\frac{x_n}{x_i}\right)_{a_n-2}\left(1-q^{a_n-1}\frac{x_n}{x_i}\right),$$

and applying this for $i = 1,\ldots,n-1$ gives, with $a_n \geq 1$,

$$P(e;\mathbf{a}) = P(e;\mathbf{a}-\boldsymbol{\delta}_n)\prod_{i=1}^{n-1}\left(1-q^{a_n-1}\frac{x_n}{x_i}\right)$$
$$= \sum_{S\subseteq\mathcal{N}_{n-1}\times\{n\}} q^{|S|\cdot(a_n-1)}P\left(E\bar{S};\mathbf{a}-\boldsymbol{\delta}_n\right).$$

The only nonzero contributions to the constant terms in the above expression will arise from S such that $E\bar{S}$ is a transitive tournament, from the master theorem. The only such S are of the form $\{(k, n), (k+1, n), \ldots, (n-1, n)\}$ for $k = 1, \ldots, n$, for which $|S| = n - k$ and $E\bar{S}$ has winner permutation $1 \cdots k-1\, n\, k \cdots n-1$. Thus, equating the constant term on both sides of the above expression gives

$$C(e; \mathbf{a}) = \sum_{k=1}^{n} q^{(n-k)(a_n - 1)} C(1 \cdots k-1\, n\, k \cdots n-1; \mathbf{a} - \boldsymbol{\delta}_n).$$

But $C(\sigma; \mathbf{a}) = C(e; \sigma(\mathbf{a}))$, so for $a_n \geq 1$,

$$C(e; \mathbf{a}) = \sum_{k=1}^{n} q^{(n-k)(a_n - 1)} C(e; (a_1, \ldots, a_{k-1}, a_n - 1, a_k, \ldots, a_{n-1})),$$

giving a recurrence equation for $C(e; \mathbf{a})$, the LHS of the result to be proved. The RHS satisfies this recurrence equation by Proposition A1 of the Appendix, dividing both sides by $(q)_{a_1 + \cdots + a_n}/(q)_{a_1} \cdots (q)_{a_{n-1}}(q)_{a_n - 1}$ and setting $y_i = q^{a_i - \chi(i=n)}$.

To obtain initial conditions for $C(e; \mathbf{a})$, note that

$$\left(q\frac{x_k}{x_i}\right)_{-1} = -\frac{x_i}{x_k}\left(1 - \frac{x_i}{x_k}\right)^{-1},$$

so for $a_k = 0$,

$$P(e_n; \mathbf{a}) = \left\{\prod_{i<k}\left(-\frac{x_i}{x_k}\right)\left(1 - \frac{x_i}{x_k}\right)^{-1}\left(\frac{x_i}{x_k}\right)_{a_i} \prod_{j>k}\left(q\frac{x_j}{x_k}\right)_{a_j - 1}\right\}$$
$$\times P(e_{n-1}; (a_1, \ldots, \hat{a}_k, \ldots, a_n)).$$

Every term in this expression has negative exponents for x_k if $k \geq 2$, and the only terms which are constant in x_k are in $1 \cdot P(e_{n-1}; a_2, \ldots, a_n)$ for $k = 1$. Thus initial conditions are

$$C(e_n; \mathbf{a}) = \begin{cases} 0, & \text{if } a_k = 0 \text{ for } k = 2, \ldots, n, \\ C(e_{n-1}; (a_2, \ldots, a_n)), & \text{if } a_1 = 0. \end{cases}$$

But these initial conditions are clearly satisfied by the RHS of the result to be proved, and the result follows for $\sigma = e$. In general, the result is obtained by applying $C(\sigma; \mathbf{a}) = C(e; \sigma(\mathbf{a}))$. □

Theorem 2.2 has been conjectured by Kadell [**8**]. The expansion that we have used in its proof can also be used to deduce that the master theorem for a nontransitive tournament with some vertex with out-degree = 0 follows from the master theorem for nontransitive tournaments in which every vertex has out-degree ≥ 1.

From Theorem 2.2 and the master theorem we now deduce the q-Dyson theorem, conjectured by Andrews [**1**] and proved by Zeilberger and Bressoud [**15**].

THEOREM 2.3. *For nonnegative integers* $a_1, \ldots, a_n$,

$$[1] \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j} = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}}.$$

PROOF. Expanding $(qx_j/x_i)_{a_j}$ for all $i < j$, we have

$$\prod_{i<j} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j} = P(e; \mathbf{a}) \prod_{i<j} \left(1 - q^{a_j}\frac{x_j}{x_i}\right)$$

$$= \sum_T q^{\sum_{i<j,(j,i)\in T} a_j} P(T; \mathbf{a}),$$

where the summation is over all tournaments T, from Proposition 2.1. Now denote the LHS of the required result by $F(\mathbf{a})$. Equating constant terms in the above expression yields, from the master theorem and Theorem 2.2,

$$F(\mathbf{a}) = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}} \sum_{\sigma \in S_n} q^{\sum_{(i,j)\in I(\sigma)} a_i} r(\sigma(\mathbf{a})).$$

Now let $G(\mathbf{a})$ denote the inner summation in the above expression for $F(\mathbf{a})$. The q-Dyson theorem is equivalent to showing that $G(\mathbf{a}) = 1$. We prove this by induction on n. Clearly $G(a_1) = 1$ and the result is true for $n = 1$. For $n > 1$, we partition the sum over σ into those σ with $\sigma_n = k$, $k = 1, \ldots, n$. This yields immediately

$$G(a_1, \ldots, a_n) = \sum_{k=1}^{n} \frac{(1 - q^{a_k}) q^{\sum_{i=k+1}^{n} a_i}}{1 - q^{a_1 + \cdots + a_n}} G(a_1, \ldots, \hat{a}_k, \ldots, a_n)$$

$$= (1 - q^{a_1 + \cdots + a_n})^{-1} \sum_{k=1}^{n} (1 - q^{a_k}) q^{\sum_{i=k+1}^{n} a_i}, \quad \text{by the induction hypothesis}$$

$$= (1 - q^{a_1 + \cdots + a_n})^{-1} \left\{ \sum_{k=1}^{n} q^{\sum_{i=k+1}^{n} a_i} - \sum_{k=1}^{n} q^{\sum_{i=k}^{n} a_i} \right\}$$

$$= (1 - q^{a_1 + \cdots + a_n})^{-1} \{1 - q^{a_1 + \cdots + a_n}\} = 1,$$

so $G(\mathbf{a}) = 1$ for all $\mathbf{a}$, and the result follows. □

The constant terms considered in Theorems 2.2 and 2.3 are the extreme cases $\mathscr{A} = \binom{\mathscr{N}_n}{2}$ and $\mathscr{A} = \varnothing$ of the following result.

PROPOSITION 2.4. *If* $A \subseteq \binom{\mathscr{N}_n}{2}$, *then*

$$[1] \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j - \chi((i,j)\in\mathscr{A})}$$

$$= \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}} \sum_{\sigma \in \mathscr{R}(\mathscr{S})} q^{\sum_{(i,j)\in I(\sigma)} a_i} r(\sigma(\mathbf{a})),$$

where $\mathscr{R}(\mathscr{S})$ *consists of those* $\sigma \in S_n$ *such that if* $(j, i) \in I(\sigma)$, *then* $(i, j) \notin \mathscr{A}$.

PROOF. We have

$$\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-\chi((i,j)\in\mathscr{A})} = P(e;\mathbf{a})\prod_{\substack{(i,j)\notin\mathscr{A}\\ i<j}}\left(1-q^{a_j}\frac{x_j}{x_i}\right)$$

and the result follows from Proposition 2.1 and Theorem 2.2. □

Proposition 2.4 expresses the constant term in a large class of polynomials as a single sum of at most $n!$ (independent of the size of the a_i's) terms, and does therefore give the constant term in a reasonable form. However, we are most interested in finding $\mathscr{A}$ for which the summation can be determined explicitly as a single term. A general class of $\mathscr{A}$ for which this can be carried out is considered in the following result.

THEOREM 2.5. *Let* $0=\alpha_0<\beta_1\le\alpha_1<\beta_2\le\cdots\le\alpha_{k-1}<\beta_k\le\alpha_k=n$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$, $\boldsymbol{\beta}=(\beta_1,\dots,\beta_k)$, $k\ge1$, *and*

$$\mathsf{U}(\boldsymbol{\alpha},\boldsymbol{\beta})=\left\{\dot{\bigcup_{i=1}^{k}}\binom{\mathscr{N}_{\alpha_i}-\mathscr{N}_{\beta_i}}{2}\right\}\cup\left\{\dot{\bigcup_{i=1}^{k}}(\mathscr{N}_{\beta_i}-\mathscr{N}_{\alpha_{i-1}})\times(\mathscr{N}_{\alpha_i}-\mathscr{N}_{\beta_i})\right\}.$$

Then

$$[1]\quad \prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-\chi((i,j)\in\mathsf{U}(\boldsymbol{\alpha},\boldsymbol{\beta}))}$$
$$=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{i=1}^{k}\prod_{j=\beta_i+1}^{\alpha_i}\frac{1-q^{a_j}}{1-q^{a_{\alpha_{i-1}+1}+\cdots+a_j}}.$$

PROOF. Let $\rho\in S_m$ and $u,v\ge0$. We first consider

$$f(m,u,v,\rho;\mathbf{a})=\sum_{\alpha\in\mathscr{T}(m,u,v,\rho)}q^{\sum_{(i,j)\in I(\sigma)}a_i}r(\sigma(\mathbf{a})),$$

where $\mathscr{T}(m,u,v,\rho)$ consists of all permutations σ in S_{m+u+v} such that elements $1,\dots,m$ appear internally in the fixed order $\rho(1)\cdots\rho(m)$, elements $m+1,\dots,m+u$ appear in any order, but all appear to the left of elements $m+u+1,\dots,m+u+v$, which themselves appear in increasing order. For example, 5346127 $\in\mathscr{T}(3,2,2,312)$. Now define $g(m,u,v,\rho;\mathbf{a})$ by

$$f(m,u,v,\rho;\mathbf{a})=g(m,u,v,\rho;\mathbf{a})q^{\sum_{(i,j)\in I(\rho)}a_i}\prod_{j=1}^{m}\frac{(1-q^{a_j})}{(1-q^{a_{\rho_1}+\cdots+a_{\rho_j}})}.$$

We prove, by induction on $m+u+v$, that

$$g(m,u,v,\rho;\mathbf{a})=\prod_{i=1}^{v}\frac{(1-q^{a_{m+u+i}})}{(1-q^{a_{m+1}+\cdots+a_{m+u+i}})}.$$

First, the result is clearly true for $u=v=0$, since the only summand corresponds to $\sigma=\rho$, so $g(m,0,0,\rho;\mathbf{a})=1$ by definition.

For $u \geq 1$, by considering whether the final element of σ is $\rho(m)$ or $m + i$ for $i = 1, \ldots, u$, we obtain

$$g(m, u, 0, \rho; \mathbf{a}) = \frac{1}{1 - q^{a_1 + \cdots + a_{m+u}}}$$

$$\times \Big\{ g(m-1, u, 0, \rho(1) \cdots \rho(m-1); a_1 \cdots \hat{a}_{\rho(m)} \cdots a_{m+u})$$

$$\times q^{a_{m+1} + \cdots + a_{m+u}} (1 - q^{a_1 + \cdots + a_m})$$

$$+ \sum_{i=1}^{u} g(m, u-1, 0, \rho; a_1 \cdots \hat{a}_i \cdots a_{m+u}) q^{a_{m+i+1} + \cdots + a_{m+u}} (1 - q^{a_{m+i}}) \Big\}$$

$$= \frac{1}{1 - q^{a_1 + \cdots + a_{m+u}}} \Big\{ q^{a_{m+1} + \cdots + a_{m+u}} (1 - q^{a_1 + \cdots + a_m})$$

$$+ \sum_{i=1}^{u} q^{a_{m+i+1} + \cdots + a_{m+u}} (1 - q^{a_{m+i}}) \Big\},$$

by induction hypothesis

$$= \frac{1}{1 - q^{a_1 + \cdots + a_{m+u}}} \Big\{ q^{a_{m+1} + \cdots + a_{m+u}} - q^{a_1 + \cdots + a_{m+u}}$$

$$+ \sum_{i=2}^{u+1} q^{a_{m+i} + \cdots + a_{m+u}} - \sum_{i=1}^{u} q^{a_{m+i} + \cdots + a_{m+u}} \Big\}$$

$$= \frac{1}{1 - q^{a_1 + \cdots + a_{m+u}}} \{ q^{a_{m+1} + \cdots + a_{m+u}} - q^{a_1 + \cdots + a_{m+u}} + 1 - q^{a_{m+1} + \cdots + a_{m+u}} \}$$

$$= 1$$

and the result is true for $v = 0$.

For $v \geq 1$, by considering whether the final element of σ is $\rho(m)$ or $m + u + v$, we obtain

$$g(m, u, v, \rho; \mathbf{a}) = \frac{1}{1 - q^{a_1 + \cdots + a_{m+u+v}}}$$

$$\times \{ g(m-1, u, v, \rho(1) \cdots \rho(m-1); a_1 \cdots \hat{a}_{\rho(m)} \cdots a_{m+u+v})$$

$$\times q^{a_{m+1} + \cdots + a_{m+u+v}} (1 - q^{a_1 + \cdots + a_m})$$

$$+ g(m, u, v-1, \rho; a_1 \cdots a_{m+u+v-1})(1 - q^{a_{m+u+v}}) \}$$

$$= \frac{1}{1 - q^{a_1 + \cdots + a_{m+u+v}}} \prod_{i=1}^{v} \frac{(1 - q^{a_{m+u+i}})}{(1 - q^{a_{m+1} + \cdots + a_{m+u+i}})}$$

$$\times \{ q^{a_{m+1} + \cdots + a_{m+u+v}} (1 - q^{a_1 + \cdots + a_m} + (1 - q^{a_{m+1} + \cdots + a_{m+u+v}})) \},$$

by induction hypothesis

$$= \prod_{i=1}^{v} \frac{(1 - q^{a_{m+u+i}})}{(1 - q^{a_{m+1} + \cdots + a_{m+u+i}})},$$

and the result is true for all $v \geq 0$.

Now the required result, from Proposition 2.4, is

$$\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \sum_{\rho_1\in\mathscr{T}(0,\beta_1,\alpha_1-\beta_1,\varnothing)} \sum_{\rho_2\in\mathscr{T}(\alpha_1,\beta_2-\alpha_1,\alpha_2-\beta_2,\rho_1)} \cdots \sum_{\rho_k\in\mathscr{T}(\alpha_{k-1},\beta_k-\sigma_{k-1},\alpha_k-\beta_k,\rho_{k-1})} q^{\sum_{(i,j)\in I(\rho_k)} a_i}\, r(\rho_k(\mathbf{a}))$$
$$= \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \prod_{j=1}^{k} g\left(\alpha_{j-1}, \beta_j-\alpha_{j-1}, \alpha_j-\beta_j, e_{\alpha_{j-1}}; a_1\cdots a_{\alpha_{j-1}}\right)$$

and the result follows immediately. □

Theorem 2.5 specializes to Theorem 2.2 by choosing $k=1$, $\beta_1=1$, and to the q-Dyson theorem (Theorem 2.3) by choosing $k=1$, $\beta_1=n$. The complete case $k=1$ has been conjectured by Kadell **[8]** and is given by the following result.

Corollary 2.6. *For* $2\le m\le n+1$, $0\le t\le n-1$, *and nonnegative integers* $a_1,\ldots,a_n$,

(1)
$$[1]\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-\chi(j\ge m)} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{i=m}^{n}\frac{(1-q^{a_i})}{(1-q^{a_1+\cdots+a_i})},$$

(2)
$$[1]\prod_{1\le i<j\le n}\left(q\frac{x_i}{x_j}\right)_{a_i-\chi(i\le t)}\left(\frac{x_j}{x_i}\right)_{a_j} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{i=1}^{t}\frac{(1-q^{a_i})}{(1-q^{a_i+\cdots+a_n})}.$$

Proof. (1) In Theorem 2.5, set $k=1$, $\beta_1=m-1$, so $(i,j)\in \mathcal{U}(\boldsymbol{\alpha},\beta)$ if and only if $j\ge m$.

(2) In (1), replace x_i and a_i by x_{n+1-i} and a_{n+1-i}, respectively, for $i=1,\ldots,n$, and set $t=n+1-m$. □

Another constant term that can be determined in a compact form is given by the following result, which has been conjectured by Kadell (private communication).

Theorem 2.7. *For nonnegative integers* $a_1,\ldots,a_n$,

$$[1]\prod_{1\le i<j\le n}\left(\frac{x_j}{x_i}-\frac{x_i}{x_j}\right)\left(q\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-1} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{1\le i<j\le n}\frac{(q^{a_i}-q^{a_j})}{(1-q^{a_i+a_j})}.$$

PROOF. The LHS of this result can be expressed as

$$[1]\prod_{1\le i<j\le n}\left\{\left(1-\frac{x_i}{x_j}\right)-\left(1-\frac{x_j}{x_i}\right)\right\}\left(q\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-1}$$

$$=[1]\prod_{1\le i<j\le n}\left\{\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1}-\left(\frac{x_j}{x_i}\right)_{a_j}\left(q\frac{x_i}{x_j}\right)_{a_i-1}\right\}$$

$$=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}h_n(\mathbf{a}),$$

where $h_n(\mathbf{a})=\sum_{\sigma\in\mathscr{S}_n}(-1)^{|I(\sigma)|}r(\sigma(\mathbf{a}))$, from Theorems 1.3, 2.2 and 2.3. Thus the required result is equivalent to proving that

$$h_n(\mathbf{a})=\prod_{1\le i<j\le n}\frac{(q^{a_i}-q^{a_j})}{(1-q^{a_i+a_j})}.$$

We prove this by induction on n.

First, it clearly is true for $n=1$. For $n>1$, by considering that the final element is equal to k for some $k=1,\ldots,n$, we obtain

$$h_n(\mathbf{a})=\frac{1}{1-q^{a_1+\cdots+a_n}}\sum_{k=1}^{n}(-1)^{n-k}(1-q^{a_k})h_{n-1}(a_1,\ldots,\hat{a}_k,\ldots,a_n)$$

$$=\frac{1}{1-q^{a_1+\cdots+a_n}}\left\{\prod_{1\le i<j\le n}\frac{(q^{a_i}-q^{a_j})}{(1-q^{a_i+a_j})}\right\}$$

$$\times\sum_{k=1}^{n}(1-q^{a_k})\prod_{\substack{i=1\\ i\ne k}}^{n}\frac{(1-q^{a_i+a_k})}{(q^{a_i}-q^{a_k})},\quad\text{by induction hypothesis}$$

$$=\frac{1}{1-q^{a_1+\cdots+a_n}}\left\{\prod_{1\le i<j\le n}\frac{(q^{a_i}-q^{a_j})}{(1-q^{a_i+a_j})}\right\}(1-q^{a_1+\cdots+a_n}),$$

from Proposition A2, with $y_i=q^{a_i}$, $i=1,\ldots,n$, and the result is true for all $n\ge 1$. □

The constant term identities given in this section up to this point have been ones in which we have used Proposition 2.1 together with Theorems 1.3 and 2.2 to express the constant term as a q-multinomial coefficient multiplied by a weighted sum of $r(\sigma(\mathbf{a}))$'s. Furthermore, we have been able to simplify these summations to single terms. There are a number of constant terms that we can treat in this way, but in which we cannot simplify the summation. Examples of these are given in the following result.

THEOREM 2.8. *Let* $r(\mathbf{a})=\prod_{i=1}^{n}(1-q^{a_i})(1-q^{a_1+\cdots+a_i})^{-1}$. *Then*

(1) $$[1]\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(\frac{x_j}{x_i}\right)_{a_j}=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\sum_{\sigma\in\mathscr{S}_n}r(\sigma(\mathbf{a})).$$

If $\mathscr{A}, \mathscr{B} \subseteq \binom{\mathscr{N}_n}{2}$ *with* $\mathscr{A} \cap \mathscr{B} = \varnothing$, *then*

$$(2)\quad [1] \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q^{\chi((i,j)\notin\mathscr{B})}\frac{x_j}{x_i}\right)_{a_j-\chi((i,j)\in\mathscr{A})} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \sum_{\sigma\in\mathscr{R}(\mathscr{A})} q^{\Sigma^* a_i} r(\sigma(\mathbf{a})),$$

where Σ^* *is the sum over pairs* $(i, j) \in I(\sigma)$ *such that* $(j, i) \notin \mathscr{B}$.

$$(3)\quad [1] \prod_{1\le i<j\le n} \left\{\left(\frac{x_j}{x_i}\right)^{\chi((i,j)\notin\mathscr{A})} - \frac{x_i}{x_j}\right\} \left(q\frac{x_i}{x_j}\right)_{a_i-1} \left(q\frac{x_j}{x_i}\right)_{a_j-1} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \sum_{\sigma\in\mathscr{R}(\mathscr{A})} (-1)^{|I(\sigma)|} r(\sigma(\mathbf{a})).$$

PROOF. (1) The product on the LHS can be written as

$$P(e;\mathbf{a}) \prod_{1\le i<j\le n} \left(1 - \frac{x_j}{x_i}\right) = \sum_T P(T;\mathbf{a})$$

from Proposition 2.1, where the summation is over all tournaments T. The result follows immediately from Theorems 1.3 and 2.2 by equating the constant terms.

(2) The product on the LHS can be written as

$$P(e;\mathbf{a}) \prod_{(i,j)\notin\mathscr{S}} \left(1 - \frac{x_j}{x_i} q^{a_j\chi((i,j)\notin\mathscr{B})}\right)$$

and the result follows from Proposition 2.1, Theorems 1.3 and 2.2.

(3) The product on the LHS can be written as

$$\prod_{1\le i<j\le n} \left\{\left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j-1} - \chi((i,j)\notin\mathscr{A}) \left(\frac{x_j}{x_i}\right)_{a_j} \left(q\frac{x_i}{x_j}\right)_{a_i-1}\right\}$$

and the result follows from Proposition 2.1, Theorems 1.3 and 2.2. □

Theorem 2.8(1), whose LHS at first glance seems to be the most natural q-analogue of Dyson's conjecture, has been conjectured by Kadell (private communication).

It must be pointed out that, though large classes of constant terms can be deduced from the master theorem, there are many similar constant terms on which our method sheds no light. For example,

$$\prod_{1\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i+1} \left(q\frac{x_j}{x_i}\right)_{a_j-1} = P(e;\mathbf{a}) \prod_{1\le i<j\le n} \left(1 - q^{a_i}\frac{x_i}{x_j}\right)$$

but now Proposition 2.1 does not apply to any of the $2^{\binom{n}{2}}$ terms in the expansion of $\prod_{1\le i<j\le n}(1 - q^{a_i}x_i/x_j)$ except for the leading term 1. Thus the methods of this section do not yield any expression for

$$[1] \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i+1} \left(q\frac{x_j}{x_i}\right)_{a_j-1}.$$

Similarly, consider the problem of deducing Theorem 2.2 from Theorem 2.3. We can write

$$\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1}$$
$$=\left\{\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-1}\right\}\prod_{1\le i<j\le n}\left(1-q^{a_i-1}\frac{x_i}{x_j}\right)$$

and expand the second product on the RHS as

$$\prod_{1\le i<j\le n}\left(1-q^{a_i-1}\frac{x_i}{x_j}\right)=\sum_{\alpha\subseteq\binom{\mathcal{N}_n}{2}}\prod_{(i,j)\in\alpha}\left(-q^{a_i-1}\frac{x_i}{x_j}\right).$$

But this does us no good since we have no knowledge of

$$[1]\left\{\prod_{(i,j)\in\alpha}\left(-q^{a_i-1}\frac{x_i}{x_j}\right)\right\}\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-1}$$

unless $\alpha=\varnothing$, in which case we know the constant term by the q-Dyson theorem.

Note also that in results 2.3 to 2.8, we have considered constant terms in products over $1\le i<j\le n$ which is equivalent to $(i,j)\in T$ where $T=E$ is the transitive tournament with winner permutation e. Clearly, to obtain analogous constant terms when T is the transitive tournament with winner permutation ρ, we need only to replace $\mathbf{a}$ by $\rho(\mathbf{a})$ on the RHS. Of course, in the case of the q-Dyson theorem (Theorem 2.3) this results in no change, since the RHS is symmetric in $\mathbf{a}$.

However, if T is a nontransitive tournament, then we have analogous weighted sums of $r(\sigma(\mathbf{a}))$'s to replace results 2.4 and 2.8, but, in general, we know of no closed form RHS's analogous to results 2.3, 2.5, 2.6, 2.7.

Finally, we note that our results for constant terms are equivalent in some cases to results for nonconstant terms in related products. For example, we have the following result.

THEOREM 2.9. *Let $Q\subseteq T$, where T is a tournament. Then*

$$\left[\prod_{(l,k)\in Q}\frac{x_l}{x_k}\right]\prod_{(i,j)\in T}\left(\frac{x_i}{x_j}\right)_{a_i}\left(q\frac{x_j}{x_i}\right)_{a_j-1}$$
$$=\begin{cases}0, & \text{if } T\overline{Q} \text{ is nontransitive},\\ (-1)^{|Q|}\dfrac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\displaystyle\prod_{i=1}^{n}\frac{(1-q^{a_{\sigma_i}})}{(1-q^{a_{\sigma_1}+\cdots+a_{\sigma_i}})}, & \\ \qquad \text{if } T\overline{Q} \text{ is transitive, with winner permutation } \sigma. & \end{cases}$$

PROOF. The LHS of this result is equal to

$$[1]\left\{\prod_{(l,k)\in Q}\frac{x_k}{x_l}\right\}P(T;\mathbf{a})=[1](-1)^{|Q|}P(T\overline{Q};\mathbf{a})$$

from Proposition 2.1, and the result follows from Theorems 1.3 and 2.2. □

3. Tournamented statistics. Let $W = W_1W_2 \cdots W_r \in M(\mathbf{a})$, where $M(\mathbf{a})$ is the set of words with a_1 1's,...,a_n n's, and $r = a_1 + \cdots + a_n$. Two of the most useful statistics for studying such words were introduced by MacMahon [**11**], the *major index*

$$\mathrm{MAJ}(W) = \sum_{k=1}^{r-1} k\chi(W_k > W_{k+1})$$

and the *inversion number*

$$\mathrm{INV}(W) = \sum_{1 \le i < j \le r} \chi(W_i > W_j).$$

For the proof of the master theorem, given in §4, we shall need generalizations of these statistics indexed by a tournament T on n vertices.

The first of these statistics is the *tournamented major index* $\mathrm{MAJ}_T(W)$ defined by

$$\mathrm{MAJ}_T(W) = \sum_{1 \le i < j \le n} \mathrm{MAJ}_T(W_{ij}),$$

where W_{ij} is the subword of W consisting of all i's and j's in W, and $\mathrm{MAJ}_T(W_{ij})$ is given by

$$\mathrm{MAJ}_T(W_{ij}) = \sum_{k=1}^{a_i + a_j - 1} k\chi((V_{k+1}, V_k) \in T),$$

where $W_{ij} = V_1V_2 \cdots V_{a_i+a_j}$. Thus, if $W = 1322132 \in M(2,3,2)$, then $W_{12} = 12212$, $W_{13} = 1313$, $W_{23} = 32232$. If T is the transitive tournament $1 \to 3 \to 2$, $1 \to 2$, then $\mathrm{MAJ}_T(1322132) = 3 + 2 + 3 = 8$. If T' is the nontransitive tournament $1 \to 2 \to 3 \to 1$, then $\mathrm{MAJ}_{T'}(1322132) = 3 + 4 + 5 = 12$. Note that MAJ_E is precisely the z-statistic used by Zeilberger and Bressoud in their proof of the q-Dyson theorem [**15**].

The second new statistic is the *tournamented inversion number* $\mathrm{INV}_T(W)$, defined by

$$\mathrm{INV}_T(W) = \sum_{1 \le i < j \le n} \mathrm{INV}_T(W_{ij}),$$

where

$$\mathrm{INV}_T(W_{ij}) = \sum_{1 \le k < l \le a_i + a_j} \chi((V_l, V_k) \in T),$$

and $W_{ij} = V_1V_2 \cdots V_{a_i+a_j}$. Thus for T given by $1 \to 3 \to 2$, $1 \to 2$ and T' given by $1 \to 2 \to 3 \to 1$, we have $\mathrm{INV}_T(1322132) = 2 + 1 + 2 = 5$ and $\mathrm{INV}_{T'}(1322132) = 2 + 3 + 4 = 9$. Note that INV_E is simply MacMahon's inversion number.

The generating functions for MAJ_T and INV_T are trivially deducible from the classical case $T = E$ when T is transitive and are given in the following result.

PROPOSITION 3.1. *If T is a transitive tournament, then*

$$\sum_{W \in M(\mathbf{a})} q^{\mathrm{INV}_T(W)} = \sum_{W \in M(\mathbf{a})} q^{\mathrm{MAJ}_T(W)} = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}}.$$

PROOF. Let the required generating functions be denoted by $\Phi_T(\mathbf{a})$ (for INV_T) and $\Psi_T(\mathbf{a})$ (for MAJ_T). If T has winner permutation σ, then clearly $\Phi_T(\mathbf{a}) = \Phi_E(\sigma(\mathbf{a}))$ and $\Psi_T(\mathbf{a}) = \Psi_E(\sigma(\mathbf{a}))$. But from MacMahon [**11**] and Zeilberger and Bressoud [**15**], respectively, we know

$$\Phi_E(\mathbf{a}) = \Psi_E(\mathbf{a}) = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}},$$

and the result follows since the RHS of this result is symmetric in the elements of $\mathbf{a}$. □

However, for nontransitive tournaments, the generating functions for INV_T and MAJ_T are not identical. For example, if T is given by $1 \to 2 \to 3 \to 1$, then

$$\sum_{W\in M(2,1,1)} q^{\mathrm{INV}_T(W)} = q + 5q^2 + 5q^3 + q^4,$$

$$\sum_{W\in M(2,1,1)} q^{\mathrm{MAJ}_T(W)} = 2q + 4q^2 + 4q^3 + 2q^4.$$

In general, these generating functions are not symmetric in the elements of $\mathbf{a}$. It is easy to see (by considering the words $W = W_1 \cdots W_r$ read in reverse order $W_r \cdots W_1$) that they are both *reciprocal* (F is reciprocal if $[q^i]F = [q^{m+1-i}]F$ for $i = 1,\ldots,[m/2]$, where $m \geq$ degree (F): in this case $m = \binom{n+1}{2}$). Moreover, we conjecture that, for any tournament, the generating functions are unimodal.

Following the notation of [**15**], associate with each word $W \in M(\mathbf{a})$ a permutation $\pi \in S_n$ defined by: $\pi(1)$ is the last letter of W. For $i = 2,\ldots,n$, $\pi(i)$ is the last letter of W not equal to $\pi(1),\ldots,\pi(i-1)$. Thus if $W = 12214334414$, then $\pi(1) = 4$, $\pi(2) = 1$, $\pi(3) = 3$, $\pi(4) = 2$, so $\pi = 4132$. For $\pi \in \mathscr{S}_n$, we denote by $M_\pi(\mathbf{a})$ the subset of $M(\mathbf{a})$ consisting of words associated with fixed π.

We cannot find a representation as a rational product for the generating functions of $M(\mathbf{a})$ with respect to INV_T and MAJ_T for nontransitive T. However, we can find such a representation for the generating function of $M_\pi(\mathbf{a})$ with respect to MAJ_T, which is the result about tournamented statistics needed in §4.

THEOREM 3.2. *For any tournament T, and nonnegative $a_1,\ldots,a_n$,*

$$\sum_{W\in M_\pi(\mathbf{a})} q^{\mathrm{MAJ}_T(W)} = q^{C_T(\pi,\mathbf{a})}\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}\prod_{i=1}^{n}\frac{(1-q^{a_i})}{(1-q^{a_{\pi(i)}+\cdots+a_{\pi(n)}})},$$

where

$$C_T(\pi,\mathbf{a}) = \sum_{(i,j)\in T} a_j\chi(\pi^{-1}(i) < \pi^{-1}(j)) = \sum_{(i,j)\in T} a_j\chi(\pi_{ij} = i),$$

and $\pi_{ij} = i$ if $\pi^{-1}(i) < \pi^{-1}(j)$, and $= j$ otherwise.

PROOF. When $T = E$, this result is Sublemma 4.1.1 in [**15**], and we precisely parallel that proof here.

The theorem is trivially true for $n = 1$, and if it is true for $n = 1$, then it is true for n when at least one of the a_i's is zero. Thus we must show only that each side of the result satisfies the same recursion.

Let $\pi^{(i)} = \pi(2)\cdots\pi(i)\pi(1)\pi(i+1)\cdots\pi(n)$ for $i = 1,\ldots,n$ (so $\pi^{(1)} = \pi$). Given a word $W \in M_\pi(\mathbf{a} + \boldsymbol{\delta}_{\pi(1)})$, consider removing the last letter (which must be $\pi(1)$). Then we are left with a word in $M_{\pi^{(i)}}(\mathbf{a})$ for some $i = 1,\ldots,n$, and the contribution to $\mathrm{MAJ}_T(W)$ which has been made by this last letter is

$$\sum_{r=2}^{i} (a_{\pi(1)} + a_{\pi(r)})\chi((\pi(1),\pi(r)) \in T).$$

Thus, if $F_{\pi,T}(\mathbf{a})$ is the generating function on the LHS of the theorem, we obtain the recurrence equation

$$F_{\pi,T}(\mathbf{a} + \boldsymbol{\delta}_{\pi(1)}) = \sum_{i=1}^{n} q^{\sum_{r=2}^{i}(a_{\pi(1)}+a_{\pi(r)})\chi((\pi(1),\pi(r))\in T)} F_{\pi^{(i)},T}(\mathbf{a}).$$

To show that the RHS satisfies the same recurrence equation, we must show that

$$q^{c_T(\pi,\mathbf{a}+\boldsymbol{\delta}_{\pi(1)})} \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1-1}\cdots(q)_{a_{\pi(1)}}\cdots(q)_{a_n-1}} G(\pi)$$
$$= \sum_{i=1}^{n} q^{H_i} G(\pi^{(i)}) \frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1-1}\cdots(q)_{a_n-1}},$$

where

$$G(\pi) = \prod_{j=2}^{n} (1 - q^{a_{\pi(j)}+\cdots+a_{\pi(n)}})^{-1}$$

and

$$H_i = c_T(\pi^{(i)},\mathbf{a}) + \sum_{r=2}^{i} (a_{\pi(1)} + a_{\pi(r)})\chi((\pi(1),\pi(r)) \in T).$$

Multiplying by $(q)_{a_1-1}\cdots(q)_{a_n-1}/(q)_{a_1+\cdots+a_n-1}$ makes this equivalent to

$$(*)\qquad q^{c_T(\pi,\mathbf{a}+\boldsymbol{\delta}_{\pi(1)})} \frac{(1 - q^{a_1+\cdots+a_n})}{(1 - q^{a_{\pi(1)}})} G(\pi) = \sum_{i=1}^{n} q^{H_i} G(\pi^{(i)}).$$

But we have

$$c_T(\pi,\mathbf{a}) = \sum_{(i,j)\in T} a_j \chi(\pi^{-1}(i) < \pi^{-1}(j))$$
$$= \sum_{i\neq j} a_j \chi(\pi^{-1}(i) < \pi^{-1}(j)) \chi((i,j) \in T)$$
$$= \sum_{i\neq j} a_{\pi(j)} \chi(i<j) \chi((\pi(i),\pi(j)) \in T)$$
$$= \sum_{i<j} a_{\pi(j)} \chi((\pi(i),\pi(j)) \in T),$$

so

$$c_T(\pi^{(i)},\mathbf{a}) = c_T(\pi,\mathbf{a}) - \sum_{r=2}^{i} a_{\pi(r)}\chi((\pi(1),\pi(r)) \in T)$$
$$+ \sum_{r=2}^{i} a_{\pi(1)}\chi((\pi(r),\pi(1)) \in T),$$

and thus $H_i = c_T(\pi, \mathbf{a}) + (i-1)a_{\pi(1)}$. Moreover, $c_T(\pi, \mathbf{a} + \boldsymbol{\delta}_{\pi(1)}) = c_T(\pi, \mathbf{a})$, so $(*)$ is equivalent to

$$\frac{(1 - q^{a_1 + \cdots + a_n})}{(1 - q^{a_{\pi(1)}})} G(\pi) = \sum_{i=1}^{n} q^{(i-1)a_{\pi(1)}} G(\pi^{(i)}),$$

which is proved in [15] at the end of the proof of Sublemma 4.1.1. Thus the RHS satisfies the same recursion as the LHS, and the result follows. □

4. Proof of the master theorem. We shall prove Theorem 1.3 by exactly following the proof of the q-Dyson theorem given by Zeilberger and Bressoud [15] with appropriate modifications for the fact that instead of taking the product over pairs (i, j), $1 \le i < j \le n$, the product is taken over pairs $(i, j) \in T$, where T will be an arbitrary tournament. Perhaps surprisingly, the arguments are no more difficult when T is an arbitrary tournament. As we shall see at the end of this section, Theorem 2.2, which takes its product over transitive tournaments, is a corollary of the argument we shall use to prove Theorem 1.3. It is assumed that the reader has access to the Zeilberger-Bressoud proof [15], hereafter referred to as Z.-B., to which we shall frequently refer.

Let us initially assume that T is an arbitrary tournament on n labelled vertices and begin as in Z.-B. by expanding the product under consideration by use of the q-binomial theorem [2, p. 36], yielding

$$(y)_a(qy^{-1})_b = \sum \frac{(-1)^k q^{\binom{k+1}{2}} (q)_{a+b} y^{-k}}{(q)_{a+k}(q)_{b-k}}, \qquad -a \le k \le b.$$

This gives us

$$\prod_{(i,j)\in T} \left(\frac{x_i}{x_j}\right)_{a_i} \left(q\frac{x_j}{x_i}\right)_{a_j - 1}$$

$$(4.1) \qquad = \prod_{(i,j)\in T} \sum_{k_{ij}} \frac{(-1)^{k_{ij}} q^{\binom{k_{ij}+1}{2}} (q)_{a_i + a_j - 1} x_i^{-k_{ij}} x_j^{-k_{ji}}}{(q)_{a_i + k_{ij}} (q)_{a_j + k_{ji} - 1}}$$

$$= \sum_{K \in \mathscr{K}'} \frac{(-1)^{\Sigma^* k_{ij}} q^{\binom{k_{ij}+1}{2}}}{\prod_{i \ne j} (q)_{a_i + k_{ij} - \chi((j,i)\in T)}} \prod_{i=1}^{n} x_j^{-\Sigma_{j\ne i} k_{ij}} \prod_{i<j} (q)_{a_i + a_j - 1},$$

where $k_{ji} = -k_{ij}$, $\mathscr{K}'$ is the set of integer matrices $K = (k_{ij})$ satisfying $k_{ij} = -k_{ji}$, and Σ^* is the sum over all pairs $(i, j) \in T$. We get the constant term if we restrict our matrices to the subset $K \subseteq \mathscr{K}'$, those whose row sums are zero.

As in Z.-B., the expression

$$\sum_{K \in \mathscr{K}} \frac{(-1)^{\Sigma^* k_{ij}} q^{\binom{k_{ij}+1}{2}}}{\prod_{i \ne j} (q)_{a_i + k_{ij} - \chi((j,i)\in T)}}$$

is the generating function for matrices of partitions $P = (P_{ij})$, where P_{ii} is empty. If $\#(P_{ij})$ denotes the number of parts in partition P_{ij} then

$$\sum_{\substack{j=1\\ j\ne i}}^{n} \#(P_{ij}) = (n-1)a_i - \sum_{\substack{j=1\\ j\ne i}}^{n} \chi((j,i) \in T),$$

and

$$\#(P_{ij}) + \#(P_{ji}) = a_i + a_j - 1.$$

Moreover, if $|P_{ij}|$ denotes the sum of the parts in P_{ij} then the weight assigned to P is

$$(-1)^{\Sigma^* k_{ij}} q^{\Sigma^* \binom{k_{ij}+1}{2} + \Sigma_{i \neq j} |P_{ij}|},$$

where $k_{ij} = \#(P_{ij}) - a_i + \chi((j, i) \in T)$.

We apply Algorithm 3.2 of Z.-B. to our matrix P with the following modifications: In step 2, order i, j so that $(i, j) \in T$ and then as before define

$$s_{ij} = \begin{cases} i & \text{if } B_{ij}^{(1)} + k_{ij} \geq B_{ji}^{(1)}, \\ j & \text{if } B_{ij}^{(1)} + k_{ij} < B_{ji}^{(1)}. \end{cases}$$

In steps 5 and 6, replace the pairs (i, j) such that $i < j$ with pairs $(i, j) \in T$. The implication in step 6 becomes: If $(W_{ij}^{(l+1)}, W_{ij}^{(l)}) \in T$, then $\overline{Q}_{ij}^{(l+1)}$ is strictly less than $\overline{Q}_{ij}^{(l)}$. The same Bijection M, with the winner between i and j in T considered to be the smaller number, yields a new partition Q_{ij} satisfying $|\overline{Q}_{ij}| = |Q_{ij}| + \mathrm{MAJ}_T(W_{ij} t_{ij})$, where MAJ_T is the tournamented major index defined in §3.

In step 3, case 3a holds if $(l, i) \in T$ while 3b applies if $(i, l) \in T$.

In step 4, the implication again should read that if $(W_{ij}^{(l+1)}, W_{ij}^{(l)}) \in T$, then $\overline{Q}_{ij}^{(l+1)} < \overline{Q}_{ij}^{(l)}$ and Bijection M yields Q_{ij} satisfying $|\overline{Q}_{ij}| = |Q_{ij}| + \mathrm{MAJ}_T(W_{ij})$.

The key observation about this modification of Algorithm 3.2 is the following lemma.

Lemma 4.1. *If $\#(P_{ij}) = a_i + k_{ij} - \chi((j, i) \in T)$ where $k_{ij} = -k_{ji}$, and $\Sigma k_{ij} = 0$, $1 \leq j \leq n$, $j \neq i$, and if T is nontransitive, then step 2 of the modified Algorithm 3.2 eventually produces a nontransitive tournament S.*

Proof. We first observe that until step 5, the number of parts in B_{ij} is given by

$$\#(B_{ij}) = a_i - c_i + k_{ij} - \chi((j, i) \in T).$$

For fixed i, this quantity is nonnegative for all $j \neq i$ and thus c_i is less than or equal to a_i and equality is possible only if $k_{ij} = 0$ and $(i, j) \in T$ for all $j \neq i$.

If vertex i has out-degree in T which is neither 0 nor $n - 1$, then the sequence $(\#(P_{i1}), \#(P_{i2}), \ldots, \#(P_{in}))$ cannot be constant because for any choice of $\{k_{ij}\}_{j=1, ji}^{n}$ satisfying $\Sigma k_{ij} = 0$, $1 \leq j \leq n$, $j \neq i$, at least one P_{ij} has at least a_i parts and at least one has at most $a_i - 1$ parts. Let us fix a j such that $\#(P_{ij})$ is the minimum of the set of values in the sequence. From the definition of s_{ij} in step 2 of Algorithm 3.2, once B_{ij} is empty the ith row cannot be further emptied unless

(1) $(i, j) \in T$ and B_{ji} is empty, or

(2) all partitions in the jth row are empty.

Since B_{ij} and B_{ji} are empty:

$$0 = a_i - c_i + k_{ij}, \qquad 0 = a_j - c_j + k_{ji} - 1.$$

Adding these equations yields $0 = a_i + a_j - (c_i + c_j + 1)$ and thus $c_i = a_i$ or $c_j = a_j$. If $(i, j) \in T$, then c_j cannot equal a_j and so $c_i = a_i$ which implies that $k_{im} = 0$ for all m and $(i, m) \in T$ for all $m \neq i$. Since we assumed that i does not have out-degree

$n-1$, this cannot happen. Thus $(j, i) \in T$ and we must be in the second case: all partitions in the jth row are empty.

Thus no row can be completely vacated unless there is a row with out-degree $n-1$ in T and that row has been completely vacated first. Once a row has been completely vacated we eliminate the corresponding node from all subsequent tournaments S. Since T is nontransitive we must eventually get down to a subtournament in which no vertex beats every other vertex and thus no more rows can be vacated. Since we can never completely empty all partitions in P, eventually the tournament S must be nontransitive. □

Let us now look at what comes out of Algorithm 3.2 modified. It will always be an element of $\mathscr{B}$ as defined in §3 of Z.-B. with the following modifications:

(ii) We shall call our nontransitive tournament defined by the algorithm S to avoid confusion with the initial specified tournament T.

(v) $k_{ij} = r_{ij} + \chi(s_{ij} = i) - \chi((j, i) \in T) + c_i - a_i$.

As in Z.-B. this implies that

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \chi(s_{ij} = i) = (n-1)(a_i - c_i) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(r_{ij} - \chi((j, i) \in T)\right)$$

and thus the score vector for S is completely determined by B, T, $(a_1, \ldots, a_n)$ and $(c_1, \ldots, c_n)$.

(vii) Replace the pair (i, j), $1 \leq i < j \leq n$, by $(i, j) \in T$.

The weight of an element $(W, S; Q, B) \in \mathscr{B}$ is now given by

$$\text{(4.3)} \quad \text{Weight}(W, S; Q, B) = (-1)^{\Sigma^* k_{ij}} q^{**\left[\sum_{i \neq j} |B_{ij}| + \sum^* \left(|Q_{ij}| + \binom{k_{ij}+1}{2} - k_{ij}\left(c_i + \chi(s_{ij} = i)\right) + \mathrm{MAJ}_T(W_{ij} s_{ij})\right)\right]},$$

where $k_{ij} = r_{ij} + \chi(s_{ij} = i) - \chi((j, i) \in T) + c_i - a_i$ and Σ^* is the sum over all pairs $(i, j) \in T$.

We shall fix Q and B and sum this weight over pairs (W, S) for which $(W, S; Q, B) \in \mathscr{B}$. Theorem 1.3 will be proved if we can show this sum to be zero.

As in Z.-B., let GAR denote anything which is constant with respect to the pairs (W, S) in the summation. Let us write $k_{ij} = b_{ij} + \chi(s_{ij} = i)$ and therefore

$$\text{(4.4)} \quad \begin{aligned} &\tfrac{1}{2}k_{ij}(k_{ij}+1) - k_{ij}\left(c_i + \chi(s_{ij} = i)\right) \\ &= \tfrac{1}{2}b_{ij}^2 + b_{ij}\chi(s_{ij} = i) + \tfrac{1}{2}\chi(s_{ij} = i) + \tfrac{1}{2}b_{ij} + \tfrac{1}{2}\chi(s_{ij} = i) \\ &\quad - \left(b_{ij}c_i + \chi(s_{ij} = i)c_i + b_{ij}\chi(s_{ij} = i) + \chi(s_{ij} = i)\right) \\ &= \mathrm{GAR} - c_i\chi(s_{ij} = i) = \mathrm{GAR} + c_i\chi(s_{ij} = j). \end{aligned}$$

Thus we have that

(4.5)

$$\text{Weight}(W, S; Q, B) = \text{GAR}(-1)^{\Sigma^* \chi(s_{ij}=j)} q^{**}\left[\textstyle\sum^*\left(c_i \chi(s_{ij}=j) + \text{MAJ}_T(W_{ij})\right)\right].$$

It is thus sufficient to prove that

$$\sum (-1)^{\Sigma^* \chi(s_{ij}=j)} q^{**}\left[\textstyle\sum^*\left(c_i \chi(s_{ij}=j) + \text{MAJ}_T(W_{ij}s_{ij})\right)\right] = 0, \tag{4.6}$$

where the outer sum is over all words $W \in M(c_1, \ldots, c_n)$ and S is over all nontransitive tournaments which have a fixed score vector and for which the last letter of W is a "spoiler" (see §1 of Z.-B. for definition of "spoiler"). Note that if the word W is empty, then the sum is over all nontransitive tournaments with fixed score vector and equation (4.6) is trivial.

We now specify the permutation associated to a given word and rewrite equation (4.6) as

$$\begin{aligned}\sum_{\pi \in S_n} \sum_{S \in \text{NonTran}(n;\, w;\, \pi(1))} & (-1)^{\Sigma^* \chi(s_{ij}=j)} q^{\Sigma^* c_i \chi(s_{ij}=j)} \\ \times \sum_{w \in M_\pi(c_1, \ldots, c_n)} & q^{\Sigma^* \text{MAJ}_T(W_{ij}s_{ij})} = 0.\end{aligned} \tag{4.7}$$

For $(i, j) \in T$, we have that

$$\text{MAJ}_T(W_{ij}s_{ij}) = \text{MAJ}_T(W_{ij}) + (c_i + c_j)\chi((s_{ij}, \pi_{ij}) \in T). \tag{4.8}$$

By Theorem 3.2, we know that

$$\begin{aligned}\sum_{w \in M_\pi(c_1, \ldots, c_n)} q^{\Sigma^* \text{MAJ}_T(W_{ij})} &= q^{\Sigma^* c_j \chi(\pi_{ij}=i)} \frac{(q)_{c_1 + \cdots + c_n - 1}}{(q)_{c_1 - 1} \cdots (q)_{c_n - 1}} \\ &\quad \times \prod_{i=2}^{n} \left(1 - q^{c_{\pi(i)} + \cdots + c_{\pi(n)}}\right)^{-1},\end{aligned}$$

and therefore it is sufficient to prove that

$$\begin{aligned}0 = \sum_{\pi \in S_n} \sum_{S \in \text{NonTrans}(n;\, \bar{w};\, \pi(1))} & (-1)^{\Sigma^* \chi(s_{ij}=j)} \\ & \times q^{**}\left[c_i \chi(s_{ij}=j) + (c_i + c_j)\chi((s_{ij}, \pi_{ij}) \in T) + c_j \chi(\pi_{ij} = 1)\right] \\ & \times \prod_{i=2}^{n} \left(1 - q^{c_{\pi(i)} + \cdots + c_{\pi(n)}}\right)^{-1}.\end{aligned} \tag{4.9}$$

Equivalently, replacing q^{c_i} by Y_i, we must prove that

$$\begin{aligned}0 = \sum_{\pi \in S_n} \sum_{S \in \text{NonTrans}(n;\, \bar{w};\, \pi(1))} & (-1)^{\Sigma^* \chi(s_{ij}=j)} \\ & \times \frac{\prod_{(i,j) \in T} Y_i^{\chi(s_{ij}=j)} (Y_i Y_j)^{\chi((s_{ij}, \pi_{ij}) \in T)} Y_j^{\chi(\pi_{ij}=i)}}{\prod_{i=2}^{n} \left(1 - Y_{\pi(i)} \cdots Y_{\pi(n)}\right)}.\end{aligned} \tag{4.10}$$

If we can prove Observation 5.1.1 in Z.-B. for the modified weight function

$$\text{weight}(\pi, s) = (-1)^{\Sigma^* \chi(s_{ij}=j)} \frac{\prod_{(i,j)\in T} Y_i^{\chi(s_{ij}=j)} (Y_i Y_j)^{\chi((s_{ij}, \pi_{ij})\in T)} Y_j^{\chi(\pi_{ij}=i)}}{\prod_{i=2}^{n} (1 - Y_{\pi(i)} \cdots Y_{\pi(n)})},$$

then the remainder of the proof in Z.-B. goes through without any modifications required. We are thus reduced to proving the following lemma.

LEMMA 4.1.1. *Let $r = \pi(1)$, let π' satisfy $\pi = r\pi'$, and let σ be the transitive tournament obtained when the spoiler $r = \pi(1)$ is removed from S. We then have that*

$$|\text{weight}(\pi, s)| = \frac{|\text{weight}(\pi', \sigma)| (Y_1 Y_2 \cdots \hat{Y}_r \cdots Y_n) Y_r^{L_r S}}{1 - Y_{\pi(2)} \cdots Y_{\pi(n)}}.$$

As before, $L_r S$ is the in-degree of vertex r in tournament S.

PROOF.

$$|\text{weight}(\pi, S)| = \frac{|\text{weight}(\pi', \sigma)|}{(1 - Y_{\pi(2)} \cdots Y_{\pi(n)})} \prod_{(i,r)\in T} Y_i^{\chi(s_{ir}=r)} (Y_i Y_r)^{\chi((s_{ir}, \pi_{ir})\in T)}$$

$$\times Y_r^{\chi(\pi_{ir}=i)} \prod_{(r,j)\in T} Y_r^{\chi(s_{rj}=j)} (Y_r Y_j)^{\chi((s_{rj}, \pi_{rj})\in T)} Y_j^{\chi(\pi_{rj}=r)}$$

$$= \frac{|\text{weight}(\pi'\sigma)|}{(1 - Y_{\pi(2)} \cdots Y_{\pi(n)})} \prod_{(i,r)\in T} Y_i^{\chi(s_{ir}=r)} (Y_i Y_r)^{\chi(s_{ir}=i)} \prod_{(r,j)\in T} Y_r^{\chi(s_{rj}=j)} Y_j^1$$

$$= \frac{|\text{weight}(\pi', \sigma)|}{(1 - Y_{\pi(2)} \cdots Y_{\pi(n)})} (Y_1 \cdots \hat{Y}_r \cdots Y_n) Y_r^{L_r S}.$$

This concludes the proof of Theorem 1.3. It should be pointed out that we have effectively also proved Theorem 2.2, for if T is transitive the only piece of the proof which is affected is Lemma 4.1. From the proof of Lemma 4.1 we see that we can completely vacate our original matrix of partitions only if T is transitive and K is the zero matrix. If T corresponds to the permutation σ, then for all i, $1 < i < n$, row $\sigma(i-1)$ must be completely vacated before row $\sigma(i)$ can be vacated. The word that is created will have $a_{\sigma(1)}$ $\sigma(1)$'s, $a_{\sigma(2)}$ $\sigma(2)$'s, $\ldots, a_{\sigma(n-1)}$ $\sigma(n-1)$'s but only $a_{\sigma(n)} - 1$ $\sigma(n)$'s. If we append $\sigma(n)$ to the end of this word we necessarily get a word in $M_\pi(\mathbf{a})$ where $\pi(i) = \sigma(n+1-i)$, $1 \le i \le n$. Within those constraints, the word is arbitrary.

Thus, if T is transitive, then the partition matrices P which are not taken to an element of $\mathscr{B}$ by Algorithm 3.2 modified correspond to a Cartesian product of an arbitrary word $\in M_\pi(\mathbf{a})$ weighted by the tournamented major index and an upper triangular matrix of partitions with $a_i + a_j - 1$ parts in position (i, j), $i < j$. By Theorem 3.2, the generating function for words in $M_\pi(\mathbf{a})$ weighted by $q^{\mathrm{MAJ}_T(W)}$ with $\pi(i) = \sigma(n+1-i)$ is

$$\frac{(q)_{a_1 + \cdots + a_n - 1}}{(q)_{a_1 - 1} \cdots (q)_{a_n - 1}} \prod_{j=2}^{n} (1 - q^{\sigma(n+1-j) + \cdots + \sigma(1)})^{-1},$$

concluding the proof of Theorem 2.2. □

5. Alternative methods of proof. In this section we examine other methods of proof for constant term identities. First we give Good's [6] proof of the Dyson conjecture. Then a q-analogue of this proof will be obtained to give an alternative proof of the q-Dyson theorem.

Good's proof of Dyson's conjecture. Let $f(\mathbf{a};\mathbf{x}) = \prod_{i,j=1,i\neq j}^{n}(1 - x_i/x_j)^{a_i}$ and $F(\mathbf{a}) = [1]f(\mathbf{a};\mathbf{x})$. Now note that

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\frac{x_i}{x_j}\right) = \sum_{k=1}^{n}\left\{\prod_{\substack{j=1\\ j\neq k}}^{n}\left(1-\frac{x_k}{x_j}\right)\right\}\prod_{\substack{i,j=1\\ i\neq k\\ i\neq j}}^{n}\left(1-\frac{x_i}{x_j}\right).$$

Since both sides are polynomials of degree $\leq n-1$ in x, and have the same value for the n distinct choices $x = x_1,\dots,x_n$. Now let $x = 0$, so

$$(*)\qquad \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\frac{x_i}{x_j}\right) = \sum_{k=1}^{n}\prod_{\substack{i,j=1\\ i\neq k\\ i\neq j}}^{n}\left(1-\frac{x_i}{x_j}\right).$$

Multiplying $(*)$ by $f(\mathbf{a}-\mathbf{1};\mathbf{x})$, we obtain $f(\mathbf{a};\mathbf{x}) = \sum_{k=1}^{n} f(\mathbf{a}-\boldsymbol{\delta}_k;\mathbf{x})$ for $\mathbf{a}\geq\mathbf{1}$, and equating constant terms gives

$$F(\mathbf{a}) = \sum_{k=1}^{n} F(\mathbf{a}-\boldsymbol{\delta}_k), \qquad \mathbf{a}\geq\mathbf{1}.$$

Also $F(\mathbf{a}) = F(a_1,\dots,\hat{a}_k,\dots,a_n)$ if $a_k = 0$, and $F(0) = 1$. These initial conditions and the above recurrence uniquely identify $F(\mathbf{a})$ as the multinomial coefficient

$$\begin{bmatrix} a_1+\cdots+a_n \\ a_1,\dots,a_n\end{bmatrix} = (a_1+\cdots+a_n)!/a_1!\cdots a_n!. \quad \square$$

We now give a q-analogue of this proof to obtain the q-Dyson theorem. Like the proof of Theorem 2.3, this proof relies on the master theorem. However, it does not require Theorem 2.2.

Alternate proof of Theorem 2.3. Let $F(\mathbf{a}) = [1]\prod_{1\leq i<j\leq n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}$. We prove that $F(\mathbf{a}) = (q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}$ by induction on $a_1+\cdots+a_n$. Clearly

$$F(\mathbf{a}) = F(a_1,\dots,\hat{a}_k,\dots,a_n) \quad \text{if } a_k = 0,\ k = 1,\dots,n,$$

and $F(0) = 1$, so the result is true for $a_1+\cdots+a_n = 0$. Now for $a_1,\dots,a_n\geq 1$ we have $F(\mathbf{a}) = [1]P(e;\mathbf{a})\prod_{1\leq i<j\leq n}(1 - q^{a_j}x_j/x_i)$. But

$$(**)\qquad \prod_{1\leq i<j\leq n}\left(1-q^{a_j}\frac{x_j}{x_i}\right) = \sum_{k=1}^{n}\left\{\prod_{j=k+1}^{n} -q^{a_j}\frac{x_j}{x_k}\right\}\prod_{\substack{1\leq i<j\leq n\\ i,j\neq k}}\left(1-q^{a_j}\frac{x_j}{x_i}\right)$$
$$+\sum_{T\in\mathscr{P}_n}\prod_{\substack{(j,i)\in T\\ j>i}}\left(-q^{a_j}\frac{x_j}{x_i}\right),$$

where $\mathscr{P}_n$ consists of all tournaments on n vertices such that no vertex has in-degree equal to $n-1$. This follows by identifying each of the $2^{\binom{n}{2}}$ terms in the expansion of the LHS with a tournament, in which choosing the "1" from $1 - q^{a_i}x_j/x_i$ corresponds to the edge (i, j), and choosing "$-q^{a_i}x_j/x_i$" corresponds to the edge (j, i). The kth term in the RHS of $(**)$ corresponds to the tournaments in which vertex k has in-degree equal to $n-1$. These tournaments are disjoint for $k = 1,\dots,n$ and $\mathscr{P}_n$ contains all other tournaments on n vertices.

Thus, multiplying both sides of $(**)$ by $P(e;\mathbf{a})$ and equating constant terms, we have

$$F(\mathbf{a}) = \sum_{k=1}^{n} q^{\sum_{j=k+1}^{n} a_j} F(a_1,\dots,a_{k-1}, a_{k+1},\dots,a_n, a_k - 1) + \sum_{T\in\mathscr{P}_n} \left\{ \prod_{\substack{(j,i)\in T \\ j>i}} q^{a_i} \right\} [1] P(T;\mathbf{a}).$$

But all tournaments T in $\mathscr{P}_n$ are nontransitive, so $[1]\ P(T;\mathbf{a}) = 0$ from the master theorem, and

$$\begin{aligned} F(\mathbf{a}) &= \sum_{k=1}^{n} q^{\sum_{j=k+1}^{n} a_j} F(a_1,\dots,a_{k-1}, a_{k+1},\dots,a_n, a_k - 1) \\ &= \frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1}\cdots(q)_{a_n}} \sum_{k=1}^{n} q^{\sum_{j=k+1}^{n} a_j}(1 - q^{a_k}), \quad \text{by induction hypothesis} \\ &= \frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1}\cdots(q)_{a_n}} \left\{ \sum_{k=1}^{n+1} q^{\sum_{j=k}^{n} a_j} - \sum_{k=1}^{n} q^{\sum_{j=k}^{n} a_j} \right\} \\ &= \frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1}\cdots(q)_{a_n}} \{1 - q^{a_1+\cdots+a_n}\} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}; \end{aligned}$$

the result is true for all $\mathbf{a}$. □

Kadell [**8**] has obtained the case $n = 4$ of the q-Dyson theorem by a q-analogue of Good's method, without using the master theorem. He has also shown [**9**] by a q-analogue of Good's method that the q-Dyson theorem is true for all n if the required constant term can be assumed to be a symmetric function of the a_i's. However, this symmetry has never been established independently of actually proving the q-Dyson theorem, as Z.-B. and we have done. It is one of the most tantalizing features of the q-Dyson theorem that a constant term which seems most asymmetric is in fact symmetric. A proof of this symmetry which is not equivalent to actually proving the q-Dyson theorem would be very desirable.

Now we apply Good's method to prove the case $q = 1$ of Theorems 1.3 and 2.2. This will allow us to consider the possibility of finding a q-analogue of these proofs, which would lead to an algebraic proof of the master theorem, and hence all of our results.

Alternative proof of Theorems 1.3 *and* 2.2 *when* $q = 1$. Multiplying $P(T, \mathbf{a} - \mathbf{1})|_{q=1}$ by $(*)$ yields

$$P(T;\mathbf{a})|_{q=1} = \sum_{k=1}^{n} P(T;\mathbf{a} - \boldsymbol{\delta}_k)|_{q=1}.$$

Equating the constant terms in this equation gives the recurrence equation, where $\mathbf{a} \geq \mathbf{1}$,

$$C(T;\mathbf{a})|_{q=1} = \sum_{k=1}^{n} C(T;\mathbf{a}-\delta_k)|_{q=1}.$$

The argument given in the proof of Theorem 2.2 enables us to obtain initial conditions. If $a_k = 0$, then

$$C(T;\mathbf{a})|_{q=1} = \begin{cases} C(T-k;a_1,\ldots,\hat{a}_k,\ldots,a_n)|_{q=1}, \\ \qquad\qquad \text{if vertex } k \text{ has in-degree} = 0 \text{ in } T, \\ 0, \quad \text{otherwise,} \end{cases}$$

where $T-k$ is the tournament on vertices $\{1,\ldots,\hat{k},\ldots,n\}$ obtained by deleting vertex k and all incident edges. Also $C(T;0)|_{q=1} = 1$, where T is the (empty) tournament on 1 vertex.

It is easy to verify that the unique solution to this recurrence is (we have already done this for transitive T in Theorem 2.2)

$$[1] \prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}$$

$$= \begin{cases} 0 \\ \begin{bmatrix} a_1+\cdots+a_n \\ a_1,\ldots,a_n \end{bmatrix} \dfrac{a_1a_2\cdots a_n}{a_{\sigma_1}(a_{\sigma_1}+a_{\sigma_2})\cdots(a_{\sigma_1}+\cdots+a_{\sigma_n})} \\ \qquad\qquad \text{if } T \text{ is transitive with winner permutation } \sigma. \end{cases} \quad \square$$

The transitive case of the above result has been given by Kadell [**8**]. Now we consider the possibility of finding a q-analogue of the above proof to obtain a proof of Theorems 1.3 and 2.2. The q-analogue of $(*)$ which allowed us to prove the q-Dyson theorem is obtained by multiplying $(**)$ by $\prod_{1\leq i<j\leq n}(1-x_i/x_j)$, and is

$$(***) \qquad \prod_{1\leq i<j\leq n}\left(1-\frac{x_i}{x_j}\right)\left(1-q^{a_j}\frac{x_j}{x_i}\right)$$

$$= \sum_{k=1}^{n} q^{a_{k+1}+\cdots+a_n}\left\{\prod_{\substack{i=1\\ i\neq k}}^{n}\left(1-\frac{x_i}{x_j}\right)\right\}\prod_{\substack{1\leq i<j\leq n\\ i,j\neq k}}\left(1-\frac{x_i}{x_j}\right)\left(1-q^{a_j}\frac{x_j}{x_i}\right)$$

$$+ \sum_{T\in\mathscr{P}_n} q^{\sum_{(j,i)\in T,\, j>i}a_j}\prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right).$$

Suppose we replace a_j by a_j-1, $j=1,\ldots,n$, multiplying both sides of $(***)$ by

$$\prod_{1\leq i<j\leq n}\left(q\frac{x_i}{x_j}\right)_{a_i-1}\left(q\frac{x_j}{x_i}\right)_{a_j-2},$$

and then equate constant terms. Then on the LHS we have $C(e;\mathbf{a})$, which is considered in Theorem 2.2. However, none of the summands on the RHS are recognizable as constant terms that we have dealt with.

Similarly, we might try proving the master theorem by constructing an analogue of (***) in which the product on the LHS is taken over $(i, j) \in T$ for a nontransitive tournament T. However, we have been unable to find such an expansion in which the terms on the RHS yield recognizable constant terms. It certainly would be useful to find an analogue of (*) which allows us to deduce the master theorem, though it appears to be a difficult task.

A final approach that we mention is to try to prove that the constant term in the master theorem is equal to the negative of itself, and hence must be zero. This "asymmetry" approach can be carried out for special values of $\mathbf{a}$, as given in the following result.

THEOREM 5.1. *The master theorem is true for* $\mathbf{a}$ *such that* $a_k = a_l$ *for some pair of vertices* k *and* l *with equal out-degrees in* T.

PROOF. Suppose, w.l.o.g. that $k < l$ and $(k, l) \in T$. There are unique vertices $v_1, \dots, v_{m-1}$ and $u_1, \dots, u_m$ for some $m = 1, \dots, [(n-1)/2]$, such that $(k, v_j) \in T$ and $(l, v_j) \notin T, j = 1, \dots, m-1$, and $(k, u_j) \notin T$ and $(l, u_j) \in T, j = 1, \dots, m$. Now let $S = \{(k, l), (k, v_1), \dots, (k, v_{m-1}), (v_1, l), \dots, (v_{m-1}, l), (u_1, k), \dots, (u_m, k), (l, u_1), \dots, (l, u_m)\}$. Clearly $\prod_{(i, j) \in S}(-x_j/x_i) = (-1)^{|S|} = -1$, so

$$-P(T; \mathbf{a}) = P(T; \mathbf{a}) \prod_{(i, j) \in S} \left(-\frac{x_j}{x_i} \right) = P(T\bar{S}; \mathbf{a})$$

from Proposition 2.1, since $S \subseteq T$. Thus equating constant terms gives $-C(T; \mathbf{a}) = C(T\bar{S}; \mathbf{a})$.

But the tournament TS is isomorphic to T, under interchanging vertices k and l. Thus

$$-C(T; a_1, \dots, a_k, \dots, a_l, \dots, a_n) = C(T; a_1, \dots, a_l, \dots, a_k, \dots, a_n)$$

and the result follows immediately. □

The choice of S in the above proof was found by N. Alon (private communication). One corollary of this result is worth mentioning.

COROLLARY 5.2. *The master theorem is true when* $a_1 = a_2 = \cdots = a_n$.

PROOF. The result follows from Theorem 5.1 since a nontransitive tournament must have at least one pair of vertices with equal out-degrees. The a_i's corresponding to such a pair of vertices must be equal since all a_i's are equal. □

Note that Corollary 5.2 does *not* imply the truth of the q-Dyson theorem for $a_1 = \cdots = a_n$, at least not by the techniques of this paper.

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Appendix.

PROPOSITION A1. *For* $n \geq 1$,

$$(1 - y_n) \sum_{k=1}^{n} y_n^{n-k} \prod_{i=1}^{k-1} (1 - y_1 \cdots y_i)^{-1} \prod_{i=k}^{n} (1 - y_1 \cdots y_{i-1} y_n)^{-1} = \prod_{i=1}^{n-1} (1 - y_1 \cdots y_i)^{-1}.$$

PROOF. Let the expression on the LHS of the above result be denoted by $L(y_1, \ldots, y_n)$. We prove the identity by induction on n. Clearly

$$L(y_1) = (1 - y_1)(1 - y_1)^{-1} = 1$$

and the result is true for $n = 1$. Now

$$L(y_1, \ldots, y_n) = \frac{(1 - y_n) y_n}{(1 - y_1 \cdots y_n)} \sum_{k=1}^{n-1} y_n^{n-1-k} \prod_{i=1}^{k-1} (1 - y_1 \cdots y_i)^{-1}$$
$$\times \prod_{i=k}^{n-1} (1 - y_1 \cdots y_{i-1} y_n)^{-1} + (1 - y_n) \prod_{i=1}^{n} (1 - y_1 \cdots y_i)^{-1}$$
$$= y_n (1 - y_1 \cdots y_n)^{-1} L(y_1, \ldots, y_{n-2}, y_n) + (1 - y_n) \prod_{i=1}^{n} (1 - y_1 \cdots y_i)^{-1}$$
$$= \prod_{i=1}^{n} (1 - y_1 \cdots y_i)^{-1} \{ y_n (1 - y_1 \cdots y_{n-1}) + (1 - y_n) \},$$

by the induction hypothesis

$$= \prod_{i=1}^{n-1} (1 - y_1 \cdots y_i)^{-1},$$

and the result is true for $n \geq 1$. □

PROPOSITION A2. *For* $n \geq 1$,

$$\sum_{k=1}^{n} (1 - y_k) \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{1 - y_i y_k}{y_i - y_k} = 1 - y_1 \cdots y_n.$$

PROOF. Denote the LHS of the required result by $L(y_1, \ldots, y_n)$. Then

$$L(y_1, \ldots, y_n) = \sum_{k=1}^{n} \frac{1}{1 + y_k} \prod_{i=1}^{n} (1 - y_i y_k) \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{y_i - y_k}$$
$$= \sum_{k=1}^{n} \left(\frac{1}{1 + y_k} \left\{ \prod_{i=1}^{n} (1 - y_i y_k) - (-1)^n y_1 \cdots y_n y_k^{n-1} (1 + y_k) \right\} \right.$$
$$\left. + (-1)^n y_1 \cdots y_n y_k^{n-1} \right) \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{y_i - y_k}$$
$$= \sum_{k=1}^{n} f(y_k) \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{y_i - z}{y_i - y_k} - y_1 \cdots y_n \sum_{k=1}^{n} g(y_k^{-1}) \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{y_k (y_i w - 1)}{y_i - y_k},$$

where

$$f(x)=\prod_{i=1}^{n}(1+y_i)^{-1}\left\{\prod_{i=1}^{n}(1-y_ix)-(-1)^n y_1\cdots y_n(x^n+x^{n-1})\right\},$$

$z=-1$, $g(x)=1$, $w=0$. But $f(x)$ and $g(x)$ are polynomials in x of degree at most $n-1$. Thus

$$f(x)=\sum_{k=1}^{n} f(y_k)\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{y_i-x}{y_i-y_k}$$

since both sides are equal for the n choices $x=y_1,\ldots,y_n$. Similarly

$$g(x)=\sum_{k=1}^{n} g(y_k^{-1})\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{y_k(y_ix-1)}{y_i-y_k}$$

since both sides are equal for the n choices $x=y_1^{-1},\ldots,y_n^{-1}$. Combining these results, we have $L(y_1,\ldots,y_n)=f(-1)-x_1\cdots x_n g(0)=1-x_1\cdots x_n$, as required. □

References

1. G. E. Andrews, *Problems and prospects for basic hypergeometric functions*, Theory and Application of Special Functions (R. A. Askey, ed.), Academic Press, New York, 1975, pp. 191–224.

2. ______, *The theory of partitions*, vol. 2 of Encyclopedia of Mathematics and Its Applications (G.-C. Rota, ed.), Addison-Wesley, Reading, Mass., 1976.

3. ______, *Notes on the Dyson conjecture*, SIAM J. Math. Anal. **11** (1980), 787–792.

4. R. A. Askey, *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J. Math. Anal. **11** (1980), 938–951.

5. F. J. Dyson, *Statistical theory of the energy levels of complex systems*. I, J. Math. Physics **3** (1962), 140–156.

6. I. J. Good, *Short proof of a conjecture of Dyson*, J. Math. Phys. **11** (1970), 1884.

7. J. Gunson, *Proof of a conjecture by Dyson in the statistical theory of energy levels*, J. Math. Phys. **3** (1962), 752–753.

8. K. W. Kadell, *Andrews' q-Dyson conjecture*: $n=4$, Trans. Amer. Math. Soc. (to appear).

9. ______, *Andrews' q-Dyson conjecture*. II: *Symmetry* (preprint).

10. I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.

11. P. A. MacMahon, *Combinatory analysis*, vol. 1, Cambridge Univ. Press, Cambridge, 1915; reprinted by Chelsea, New York, 1960.

12. W. G. Morris II, *Constant term identities for finite and affine root systems: conjectures and theorems*, Ph. D. Dissertation, University of Wisconsin-Madison, January, 1982.

13. K. Wilson, *Proof of a conjecture by Dyson*, J. Math. Phys. **3** (1962), 1040–1043.

14. D. Zeilberger, *A combinatorial proof of Dyson's conjecture*, Discrete Math. **41** (1982), 317–322.

15. D. Zeilberger and D. M. Bressoud, *A proof of Andrews' q-Dyson conjecture*, Discrete Math. (to appear).

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