

A NOTE ON AUTOMORPHIC FORMS OF WEIGHT ONE AND WEIGHT THREE

BY

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ABSTRACT. In this paper the author develops an interesting relationship between classical automorphic forms of weights one and three, and the solutions of certain second order differential equations related to elliptic (modular) surfaces. In particular for a cusp form of weight three, it is shown that the special values of the associated Dirichlet series can be determined from the periods of an inhomogeneous differential equation, or what is the same thing, the monodromy of an associated third order differential equation. Explicit examples are provided for principal congruence subgroups $\Gamma(N)$ with $N \equiv 0 \pmod{4}$.

1. Forms of weight one and differential equations. In this paper we wish to explain an interesting relationship between classical automorphic forms of weights one and three, and the solutions of certain second order differential equations related to elliptic surfaces.

Let $\mathfrak{h} = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$ denote the complex upper half-plane. We wish to consider automorphic forms of weight k with respect to a Fuchsian group of the first kind $\Gamma \subset \operatorname{SL}_2(\mathbb{R})$ so that $\mathfrak{h}^*/\Gamma = X$ is a compact Riemann surface, where \mathfrak{h}^* is \mathfrak{h} union the cusps of Γ (see Shimura [17]). Recall that an automorphic form of weight k with respect to Γ is a meromorphic function f on \mathfrak{h} such that

$$f(\gamma z) = (cz + d)^k f(z), \quad z \in \mathfrak{h},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \operatorname{SL}_2(\mathbb{R})$ acts by linear fractional transformation on \mathfrak{h} . Moreover, f must be meromorphic at the cusps (see Shimura [17]).

Suppose we are given such a meromorphic function f of weight $k = 1$. (We limit ourselves to this case, but clearly the construction below carries through to higher weight.) Consider the function $g(z) = zf(z)$. This function transforms as

$$g(\gamma z) = (az + b)f(z).$$

If we select a Zariski open subset $X_0 \subset X$ appropriately, we can regard $g(x), f(x)$ as a pair of multivalued holomorphic functions on X_0 . Moreover, if we fix a base point

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$x_0 \in X_0$, then for a path $\rho \in \pi_1(X_0, x_0)$ we have that g and f analytically continue around ρ to

$$ag(x) + bf(x), \quad cg(x) + df(x), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

respectively, i.e. $(\frac{g}{f}) \rightarrow (\frac{a}{c} \frac{b}{d})(\frac{g}{f})$. Denote by $K(X)$ the function field of X (which is isomorphic to the field of modular functions on \mathfrak{h} with respect to Γ) and select $x \in K(X)$ nonconstant. Then:

THEOREM 1. *The functions g and f form a basis for the space of solutions of a second order homogeneous linear differential equation*

$$\Lambda = d^2/dx^2 + P d/dx + Q, \quad P, Q \in K(X),$$

on X . This equation will also have regular singular points.

PROOF. Set

$$P = -\frac{g d^2 f / dx^2 - f d^2 g / dx^2}{g df / dx - f dg / dx}, \quad Q = -\frac{(df / dx) d^2 g / dx^2 - (dg / dx) d^2 f / dx^2}{g df / dx - f dg / dx}.$$

Clearly both P and Q are single-valued and, moreover, because of the conditions at the cusps they are meromorphic on X . The regularity is also clear from the nature of the growth of f and g at the cusps. \square

Analogous classical results for elliptic modular functions can be found in Fricke and Klein [5], Bateman [1] and Ford [4]. Modern discussions of related differential operators are found in Katz [10], Serre [15], Weil [23] and Lang [12].

We remark that the choice of the parameter $x \in K(X)$ used in the derivation d/dx is purely auxiliary. Moreover, by choosing X_0 small enough we can assume it is a good local parameter on all of X_0 . We will somewhat abuse notation in using x to denote both the parameter $x \in K(X)$ and a variable point $x \in X_0$.

The expression $W = g df / dx - f dg / dx$ is known as the *Wronskian* of the differential equation. It is easily seen that $W = c \exp(-\int P dx)$ for some $c \in \mathbf{C} - \{0\}$ and, in our case, because $\Gamma \subset \mathrm{SL}_2(\mathbf{R})$ and because of the nature of the growth of g and f at the cusps, we have $W \in K(X)$.

Lastly, the representation

$$\pi_1(X_0, x_0) \rightarrow \Gamma \subset \mathrm{SL}_2(\mathbf{R}),$$

given by the analytic continuation of branches of g and f at x_0 , is called the *monodromy representation* of Λ .

2. K -equations. Now let us assume $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ and attempt to isolate the properties of this differential equation. We do this in some generality by defining the notion of K -equation on an arbitrary curve X .

Let X be any complete smooth connected algebraic curve over \mathbf{C} with function field denoted by $K(X)$. After fixing a parameter $x \in K(X)$, consider an algebraic differential equation on X

$$\Lambda f = d^2 f / dx^2 + P d f / dx + Q f = 0$$

with P and Q in $K(X)$ and f an unknown function.

DEFINITION 2. $\Lambda f = 0$ is called a *K-equation* if it possesses two solutions ω_1 and ω_2 , which are holomorphic nonvanishing multivalued functions on some Zariski open subset X_0 of X , satisfying:

- (i) ω_1 and ω_2 form a basis of solutions,
- (ii) for every closed path $\gamma \in \pi_1(X_0)$ the analytic continuation of (ω_1, ω_2) around γ is $M_\gamma(\omega_1, \omega_2)$ with $M_\gamma \in \mathrm{SL}_2(\mathbf{Z})$ (the monodromy representation), and
- (iii) $\mathrm{Im}(\omega_1/\omega_2) > 0$ on X_0 (positivity).

Such a pair of solutions is called a *K-basis*. In addition, since the monodromy is in $\mathrm{SL}_2(\mathbf{Z})$, the Wronskian $W = \exp(-\int P dx)$ is single-valued. As part of our definition we assume:

- (iv) $W \in K(X)$.

This last condition is, in fact, equivalent to Λ having regular singular points. \square

For example, when Γ is contained in $\mathrm{SL}_2(\mathbf{Z})$ and f is a classical automorphic form of weight one with respect to Γ , the differential equation determined by f as in Theorem 1 is a *K-equation* with the pair g, f serving as a *K-basis*.

The set of all *K-equations* on an arbitrary Riemann surface X has been completely classified. We briefly recall that classification below.

Let $\Lambda f = 0$ be a *K-equation* with *K-basis* ω_1 and ω_2 . Consider the function $\mathcal{J} = J \circ \omega_1/\omega_2$,

$$X_0 \xrightarrow{\omega_1/\omega_2} \mathfrak{h} \xrightarrow{J} \mathbf{C},$$

where J is the elliptic modular function on the upper half-plane \mathfrak{h} . This \mathcal{J} is a single-valued holomorphic function on $X_0 \subset X$.

PROPOSITION 3. $\mathcal{J} \in K(X)$.

PROOF. This is an application of a result which appears in Kodaira [11] as Theorem 7.3. \square

We now determine all *K-equations*. Fix a *K-equation* $\Lambda f = 0$ on X with *K-basis* ω_1, ω_2 and let $\mathcal{J} = J(\omega_1/\omega_2)$. Say

$$\Lambda f = d^2f/dx^2 + P df/dx + Qf = 0.$$

THEOREM 4. *There exists an algebraic function λ on X with $\lambda^2 \in K(X)$ such that*

$$P = \frac{(d\mathcal{J}/dx)^2 - \mathcal{J} d^2\mathcal{J}/dx^2}{\mathcal{J} d\mathcal{J}/dx} - \frac{d}{dx} \log \lambda^2,$$

$$Q = \frac{(d\mathcal{J}/dx)^2(31\mathcal{J}/144 - 1/36)}{\mathcal{J}^2(\mathcal{J} - 1)^2} - \frac{(d\mathcal{J}/dx)^2 - \mathcal{J} d^2\mathcal{J}/dx^2}{\mathcal{J} d\mathcal{J}/dx} \frac{d}{dx} \log \lambda - \frac{d^2\lambda/dx^2}{\lambda} + 2 \left[\frac{d\lambda/dx}{\lambda} \right]^2.$$

This is known as the *K-equation* $\Lambda_{(\mathcal{J}, \lambda)}$.

PROOF. See Stiller [19]. \square

Note that in the case of the *K-equation* $\Lambda_{(\mathcal{J}, \lambda)}$ coming from an automorphic form f of weight one with respect to $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ the function \mathcal{J} is none other than the elliptic modular function J .

It is shown in Stiller [19] that K -equations are precisely those differential equations which arise as the Gauss-Manin connections associated to elliptic surfaces. To each elliptic surface over a curve X we can associate a K -equation which is unique up to a simple transformation; conversely given a K -equation on X we can associate to it an elliptic surface over X unique up to generic isogeny. If, in addition, we specify a K -basis, then the surface will be uniquely determined. In the case of the K -equation coming from an automorphic form f of weight one, we have a K -basis available, namely g, f . Thus there is a corresponding elliptic surface and we shall determine in the course of this paper explicitly which surface it is. In fact, the same surface arises for each choice of f and the family of K -equations is just the family associated to that one elliptic surface as mentioned above.

3. The associated elliptic surface. Let Γ denote a subgroup of finite index in $\mathrm{SL}_2(\mathbf{Z})$ acting in the usual manner via linear fractional transformation on the upper half-plane \mathfrak{h} . We shall assume for simplicity that Γ contains no torsion elements. This includes, for example, the case when $\Gamma = \Gamma(N)$ is the principal congruence subgroup of level N , $N \geq 3$, in $\mathrm{SL}_2(\mathbf{Z})$. The quotient $\mathfrak{h}/\Gamma = X_0$ is then a Zariski open subset of the compact Riemann surface X gotten by removing the cusps. Moreover, the natural projection $\mathfrak{h} \rightarrow \mathfrak{h}/\Gamma = X_0$ serves to uniformize X_0 , i.e. it is the universal covering and we can identify $\pi_1(X_0)$ with Γ .

Our purpose in this section is to construct an elliptic surface E over the compact curve X and to study the Gauss-Manin connection (or Picard-Fuchs equation) of this family of elliptic curves.

For every element $\gamma \in \Gamma$ let

$$f_\gamma(z) = (c_\gamma z + d_\gamma)^{-1} \quad \text{where } z \in \mathfrak{h}, \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix};$$

then we have $f_{\gamma\beta}(z) = f_\gamma(\beta z)f_\beta(z)$. Now, given a triple (γ, n_1, n_2) , with $\gamma \in \Gamma$ and $n_1, n_2 \in \mathbf{Z}$, define an automorphism $g(\gamma, n_1, n_2)$ of $\mathfrak{h} \times \mathbf{C}$ by

$$(z, t) \xrightarrow{g(\gamma, n_1, n_2)} (\gamma z, f_\gamma(z)(t + n_1 z + n_2)).$$

We have

$$\begin{aligned} g(\gamma, n_1, n_2)g(\beta, m_1, m_2)(z, t) \\ = g(\gamma\beta, a_\beta n_1 + c_\beta n_2 + m_1, b_\beta n_1 + d_\beta n_2 + m_2)(z, t), \end{aligned}$$

where

$$\beta = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}.$$

The energetic reader will verify that this set of automorphisms is actually a group \mathcal{G} with this composition law. Its action is proper discontinuous fixed point free, so that the quotient $(\mathfrak{h} \times \mathbf{C})/\mathcal{G}$ is a complex manifold which is clearly a family of elliptic curves over $\mathfrak{h}/\Gamma = X_0$ locally given by the lattice $z, 1$. This is essentially Kodaira's basic construction of an elliptic surface with a global section and prescribed functional and homological invariants (see Kodaira [11]).

We can apply Kodaira's basic results to compactify this surface to get a proper smooth elliptic surface E over the compact curve X unique up to a biregular fiber-preserving map over X . The surface will have degenerate fibers of type I_b (I_b^*), $b \geq 1$, over each regular (irregular) cusp where $(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix})$ $(\begin{smallmatrix} -1 & b \\ 0 & 1 \end{smallmatrix})$) generates the isotropy group of the cusp after conjugating the cusp to " ∞ " in $\mathrm{SL}_2(\mathbf{Z})$. This surface will possess a section s

$$E \quad \xrightarrow[s]{\pi} \quad X$$

(so that $\pi \circ s = 1_X$, where π is the projection of E onto X) which makes the generic fiber E^{gen} naturally an elliptic curve over the function field $K(X)$. Moreover, the functional invariant is just the elliptic modular function J thought of as an element $J \in K(X)$. The homological invariant can be viewed as the locally constant $\mathbf{Z} \oplus \mathbf{Z}$ sheaf on X_0 corresponding to the natural representation of $\pi_1(X_0, x_0)$ on $H^1(E_{x_0}, \mathbf{Z})$, the integral cohomology of the fiber at some base point $x_0 \in X_0$. The representation in this case is essentially the natural identification of $\pi_1(X_0, x_0)$ with $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$.

The surface above is called an *elliptic modular surface* and a full description can be found in Shioda [18]. We have repeated the construction so as to make clear some of the properties we shall be using later.

To the representation of $\pi_1(X_0, x_0) \rightarrow \Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ we can associate an algebraic vector bundle on X together with an integrable algebraic connection having regular singular points (Deligne [2], Griffiths [6]), which is known as the Gauss-Manin connection (Katz and Oda [9]). It can also be expressed as a second order algebraic differential equation

$$\Lambda = d^2/dx^2 + P d/dx + Q, \quad P, Q \in K(X),$$

on X having regular singular points. Using the results in Fricke and Klein [5] and Stiller [20], we have

THEOREM 5. *For each choice of a Weierstrass model*

$$Y^2 = 4X^3 - g_2 X - g_3, \quad g_2, g_3 \in K(X),$$

of the generic fiber of E over X , the differential equation, which annihilates the periods of dX/Y and expresses the Gauss-Manin connection, is the K -equation $\Lambda_{(J, \lambda)}$, where

$$\lambda^{-4} 27J/(J-1) = g_2 \quad \text{and} \quad \lambda^{-6} 27J/(J-1) = g_3.$$

Here J is the J -invariant of the elliptic curve and $\lambda^2 \in K(X)$. \square

From this we see that it is possible to select a K -basis ω_1, ω_2 so that under analytic continuation around $\rho \in \pi_1(X_0, x_0)$, corresponding to $\gamma \in \Gamma$, the monodromy matrix M_ρ in $\mathrm{SL}_2(\mathbf{Z})$ is just γ . One should be cautious here and point out that the equation $\Lambda_{(J, \lambda)}$ may have singularities on X_0 . However, the only true singularities (i.e. where the local monodromy is not $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$) will occur at the cusps. At singular points in X_0 the equation will possess local meromorphic solutions which will be single-valued. Such a point is called *cosingular*.

THEOREM 6. Consider the natural projection $\mathfrak{h} \rightarrow \mathfrak{h}/\Gamma = X_0$ and lift $\omega_2(x)$ to a single-valued meromorphic function $\omega_2(z)$ on \mathfrak{h} . The function $\omega_2(z)$ is an automorphic form of weight one with respect to Γ . (For a classical special case of this result see Jordan [8].)

PROOF. Let $\rho \in \pi_1(X_0, x_0)$ be a path corresponding to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then under analytic continuation around ρ we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \rightarrow M_\rho \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

where M_ρ is the monodromy matrix for the K -basis ω_1, ω_2 of $\Lambda_{(\mathcal{J}, \lambda)}$. Because we are working with the differential equation associated to an elliptic modular surface, we have

$$M_\rho = \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Moreover, $\omega_1(z)/\omega_2(z) = z$ because $\omega_1(z)$ and $\omega_2(z)$ are the periods for the lattice which gives the elliptic curve over each point in X_0 (minus, perhaps, a finite set where ω_1 and ω_2 are meromorphic) for the elliptic modular surface E over X .

We see that $\omega_2(x)$ continues to $c\omega_1(x) + d\omega_2(x)$ around ρ or

$$\omega_2(\gamma z) = c\omega_1(z) + d\omega_2(z) = (cz + d)\omega_2(z).$$

Thus ω_2 satisfies the correct transformation law and, finally, it will be meromorphic at the cusps because of the growth estimates on the solutions of equations with regular singularities. \square

So, given $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index and no torsion, we can construct a canonical elliptic surface E over $\mathfrak{h}^*/\Gamma = X$, called an elliptic modular surface, and by choosing a model for the generic fiber we can produce an automorphic form of weight one as a solution of the Picard-Fuchs equation for this surface.

Note that ω_2 is uniquely picked out as a solution of $\Lambda_{(\mathcal{J}, \lambda)}$ up to a constant multiple. Namely, at the point $x \in X$ corresponding to the cusp at ∞ , the K -equation $\Lambda_{(\mathcal{J}, \lambda)}$ will have exponents (n, n) (or $(n/2, n/2)$ if ∞ is irregular) for some $n \in \mathbb{Z}$ (see Stiller [21]). This tells us that the global monodromy is conjugate in $\mathrm{GL}_2(\mathbb{C})$ to the local monodromy which is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (or $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ in the irregular case), where locally a basis of solutions will be

$$\begin{aligned} u_2 &= t^n(1 + \dots) \quad (\text{or } t^{n/2}(1 + \dots)), \\ u_1 &= u_2((1/2\pi i) \log t + \text{holomorphic}), \end{aligned}$$

t being a suitable local parameter. It follows immediately that ω_2 is a multiple of u_2 since we wish to select ω_1, ω_2 so that the lifts to \mathfrak{h} transform according to the homological invariant of E over X . This fixes ω_2 as some constant multiple of the invariant solution at the point x corresponding to the cusp ∞ (and similarly for the irregular case).

Lastly, given one model

$$Y^2 = 4X^3 - g_2X - g_3, \quad g_2, g_3 \in K(X),$$

for the generic fiber of E over X , any other model is of the form

$$Y^2 = 4X^3 - g^{-4}g_2X - g^{-6}g_3, \quad g \in K(X), g \neq 0.$$

It follows that if $\Lambda_{(\mathcal{J}, \lambda)}$ is the equation for the first model, then $\Lambda_{(\mathcal{J}, g\lambda)}$ is the equation for the second, and the basis ω_1, ω_2 in the first corresponds to $g\omega_1, g\omega_2$ in the second. In other words, the periods multiply and pointwise the lattices are homothetic except where g has poles.

Of course when $g = c$ is constant, $\Lambda_{(\mathcal{J}, \lambda)}$ and $\Lambda_{(\mathcal{J}, c\lambda)}$ are equal. Thus, as we change the model we run through all nonzero multiples of ω_2 by $g \in K(X)$ which means we get all automorphic forms of weight one.

In the last section we will explain how to choose $\lambda^2 \in K(X)$ so that $\Lambda_{(\mathcal{J}, \lambda)}$ corresponds to the elliptic modular surface E over X for Γ . Remember that λ is not unique as we can multiply by any nonzero modular function $g \in K(X)$. We will also explain how to tell if the resulting automorphic form is holomorphic or a cusp form and discuss computing q -expansions.

4. Forms of weight three and parabolic cohomology. Let Γ be a subgroup of finite index in $SL_2(\mathbf{Z})$ with no torsion and let X be the compact Riemann surface \mathfrak{h}^*/Γ . We will denote by $K(X)$ the function field of X and by $E \xrightarrow{\pi} X$ the elliptic modular surface for Γ . We shall consider one of the second order linear homogeneous differential equations

$$(1) \quad \Lambda = d^2/dx^2 + P d/dx + Q, \quad P, Q \in K(X),$$

which represents the Gauss-Manin connection for E over X . As we have seen such an equation is a K -equation, i.e. $\Lambda = \Lambda_{(\mathcal{J}, \lambda)}$, where $\mathcal{J} \in K(X)$ corresponds to the elliptic modular function $J(z)$ on \mathfrak{h} and λ is an algebraic function whose square λ^2 is in $K(X)$. We denote by $X_0 = \mathfrak{h}/\Gamma$ the Zariski open subset of X obtained by deleting the cusps and select a base point $x_0 \in X_0$ together with branches of ω_1 and ω_2 at x_0 as before.

DEFINITION 7. A rational function $Z \in K(X)$ will be said to satisfy the *parabolic residue condition* for Λ if the differential forms

$$(-\omega_2 Z/W) dx \quad \text{and} \quad (\omega_1 Z/W) dx$$

have zero residue at every point in X_0 (note that this is completely independent of the choice of basis ω_1, ω_2 because any solution of Λ is a linear combination $a_1\omega_1 + a_2\omega_2$, $a_i \in \mathbf{C}$, of ω_1, ω_2 , and $((a_1\omega_1 + a_2\omega_2)Z/W) dx$ will also have zero residue at every point in X_0), and if at every regular cusp $s \in S = X - X_0$ (where the local monodromy is conjugate to $(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix})$, $n > 0$, $n \in \mathbf{Z}$) the differential form $(-\tilde{\omega}_2 Z/W) dx$ has zero residue, where $\tilde{\omega}_2$ is a local invariant solution at s . We denote by L_λ^{para} the subspace of $Z \in K(X)$ which satisfies the parabolic residue condition for Λ . \square

In what follows, we shall be interested in the inhomogeneous equations formed from the operator Λ as given in (1) above. Thus, for $Z \in L_\lambda^{\text{para}} \subset K(X)$ consider

$$(2) \quad \Lambda f = Z, \quad f \text{ unknown.}$$

THEOREM 8. *We have that*

$$f = \left(\int_{x_0}^x \frac{-\omega_2 Z}{W} dx + c_1 \right) \omega_1 + \left(\int_{x_0}^x \frac{\omega_1 Z}{W} dx + c_2 \right) \omega_2$$

as a multivalued meromorphic function on X_0 . Here $c_1, c_2 \in \mathbf{C}$ and $W = \omega_1 d\omega_2/dx - \omega_2 d\omega_1/dx$ is the Wronskian of Λ . Note that we have somewhat abused notation in using x to denote both the parameter $x \in K(X)$ and a variable point $x \in X_0$.

PROOF. Apply variation of parameters to (2). \square

Note that the parameter x need not be a good local parameter everywhere. However, if we replace the parameter $x \in K(X)$ by $t \in K(X)$ the new homogeneous equation is

$$\Lambda^{\text{new}} f = \frac{d^2 f}{dt^2} + \left(P \frac{dx}{dt} - \frac{d}{dt} \log \frac{dx}{dt} \right) \frac{df}{dt} + Q \left(\frac{dx}{dt} \right)^2 f = 0,$$

where P and Q are then original coefficients (see (1)). Moreover, we have for the inhomogeneous case

$$\Lambda^{\text{new}} f = Z(dx/dt)^2,$$

where f is the original inhomogeneous solution for (2) as in the theorem. Note that the Wronskian W , which clearly depends on a choice of derivation d/dx , becomes $W dx/dt$ when we change parameter. We can thus see directly from the formula in Theorem 8 that the expressions

$$(-\omega_2 Z/W) dx \quad \text{and} \quad (\omega_1 Z/W) dx$$

are naturally independent of parameter since

$$\frac{-\omega_2 Z(dx/dt)^2}{W(dx/dt)} dt = \frac{-\omega_2 Z}{W} dx \quad \text{and} \quad \frac{\omega_1 Z(dx/dt)^2}{W(dx/dt)} dt = \frac{\omega_1 Z}{W} dx.$$

We now return to consideration of the inhomogeneous equation.

Clearly we can continue the solution f of the inhomogeneous equation in an unrestricted way throughout X_0 . For $\rho \in \pi_1(X_0, x_0)$ corresponding to $\gamma \in \Gamma$ we have

$$f \rightarrow f + m_\gamma \omega_1 + n_\gamma \omega_2, \quad m_\gamma, n_\gamma \in \mathbf{C},$$

because any two solutions of the inhomogeneous equation differ by a solution of the homogeneous equation.

Lifting f to the upper half-plane viewed as the universal cover \mathbb{h} of $X_0 = \mathbb{h}/\Gamma$ we have a meromorphic single valued function (because of the residue condition) which we also denote by f and which transforms as

$$f(\gamma z) = f(z) + m_\gamma \omega_1(z) + n_\gamma \omega_2(z), \quad m_\gamma, n_\gamma \in \mathbf{C}, z \in \mathbb{h}, \gamma \in \Gamma \subset \text{SL}_2(\mathbf{Z}).$$

Note that f satisfies growth conditions near the cusps similar to those for ω_1, ω_2 which are solutions of Λ , an equation with regular singular points.

As before we consider the inhomogeneous equation

$$\Lambda f = Z, \quad Z \in L_\Lambda^{\text{para}} \subset K(X),$$

and the solution as in Theorem 8

$$f = \left(\int_{x_0}^x \frac{-\omega_2 Z}{W} dx + c_1 \right) \omega_1 + \left(\int_{x_0}^x \frac{\omega_1 Z}{W} dx + c_2 \right) \omega_2,$$

where $x_0 \in X_0$ is a base point and ω_1 and ω_2 are fixed branches at x_0 . Because of our assumptions, f is meromorphic multivalued on X_0 as is the function

$$F = \frac{f}{\omega_2} = \left(\int_{x_0}^x \frac{-\omega_2 Z}{W} dx + c_1 \right) \omega + \left(\int_{x_0}^x \frac{\omega_1 Z}{W} dx + c_2 \right), \quad \omega = \frac{\omega_1}{\omega_2}.$$

We define $F(z)$, a meromorphic single-valued function on the upper half-plane \mathfrak{h} , to be the lift of F via the universal cover $\mathfrak{h} \rightarrow X_0 = \mathfrak{h}/\Gamma$.

Note that for $\gamma \in \Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ corresponding to $\rho \in \pi_1(X_0, x_0)$ we have

$$(3) \quad F(\gamma z) = (c_\gamma z + d_\gamma)^{-1} (F(z) + m_\gamma z + n_\gamma), \quad z \in \mathfrak{h},$$

where $m_\gamma, n_\gamma \in \mathbf{C}$ are called the *periods* of F around γ or ρ , and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is the monodromy matrix for the continuation of ω_1, ω_2 around the corresponding path. Note that $\omega = \omega_1/\omega_2 = z$ in this case. Again we remark that regularity imposes growth conditions on F near the cusps.

As in Eichler [3], for each meromorphic F on the upper half-plane \mathfrak{h} satisfying (3), there is a meromorphic function g on \mathfrak{h} satisfying

$$g(\gamma z) = (c_\gamma z + d_\gamma)^3 g(z), \quad z \in \mathfrak{h}, \gamma \in \Gamma \subset \mathrm{SL}_2(\mathbf{Z}), \quad \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix},$$

which is the monodromy as before. As such, g is an automorphic form of weight three with respect to Γ . These functions are related by the formulas

$$(4) \quad F(z) = \int_{z_0}^z g(t)(z-t) dt + c_1 z + c_2,$$

$$(5) \quad g(z) = \frac{1}{\pi i} \oint_{\gamma} F(t)(t-z)^{-3} dt,$$

where the path is taken around the boundary of a small disc about $z \in \mathfrak{h}$, and $z_0 \in \mathfrak{h}$ lies over our base point $x_0 \in X_0$.

LEMMA 9. $g(z) = \omega_2(z)^3 Z(z)/W(z)^2$, all functions being understood as pull-backs via the canonical map $\mathfrak{h} \rightarrow X_0$ of the corresponding functions on X_0 .

PROOF. We analyze the situation locally near x_0 . Here

$$F(x) = \frac{f(x)}{\omega_2(x)} = \left(\int_{x_0}^x \frac{-\omega_2 Z}{W} dx + c_1 \right) \omega_1(x) + \left(\int_{x_0}^x \frac{\omega_1 Z}{W} dx + c_2 \right).$$

When we lift to the upper half-plane we get

$$F(z) = \left(\int_{z_0}^z \frac{-\omega_2(z) Z(z)}{W(z)} \left(\frac{dx}{dz} \right) dz + c_1 \right) z + \left(\int_{z_0}^z \frac{\omega_1(z) Z(z)}{W(z)} \left(\frac{dx}{dz} \right) dz + c_2 \right).$$

Formula (5) is equivalent to

$$g(z) = (d^2/dz^2) F(z),$$

so we compute

$$\begin{aligned} \frac{d}{dz} F(z) &= \left(\int_{z_0}^z \frac{-\omega_2(z)Z(z)}{W(z)} \left(\frac{dx}{dz} \right) dz \right) \\ &\quad + \frac{-\omega_2(z)Z(z)}{W(z)} \left(\frac{dz}{dx} \right) + \frac{\omega_1(z)Z(z)}{W(z)} \left(\frac{dz}{dx} \right). \end{aligned}$$

But these last two terms add to zero as $\omega_1/\omega_2 = z$. Thus

$$g(z) = \frac{-\omega_2(z)Z(z)}{W(z)} \left(\frac{dx}{dz} \right)$$

and, as $z = \omega(x)$, we have

$$\frac{dz}{dx} = \frac{d}{dx} \omega(x) = \frac{d}{dx} \left(\frac{\omega_1(x)}{\omega_2(x)} \right) = \frac{-W(x)}{\omega_2(x)^2}.$$

On \mathfrak{h} this gives $g(z) = \omega_2(z)^3 Z(z)/W(z)^2$. \square

Note that the automorphic form is completely independent of the choice of a particular solution to $\Lambda f = Z$, i.e. it is independent of c_1, c_2 as well as the base point $x_0 \in X_0$. Moreover, it is independent of the parameter chosen since, as remarked above, Z becomes $Z(dx/dt)^2$ and W^2 becomes $W^2(dx/dt)^2$.

Recall that on \mathfrak{h} the function f transforms as

$$f(\gamma z) = f(z) + m_\gamma \omega_1(z) + n_\gamma \omega_2(z), \quad \gamma \in \Gamma, m_\gamma, n_\gamma \in \mathbf{C}.$$

THEOREM 10.

$$\begin{aligned} [m_\gamma, n_\gamma] &= \left[\int_\rho \frac{-\omega_2 Z}{W} dx, \int_\rho \frac{\omega_1 Z}{W} dx \right] \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\quad + [c_1, c_2] \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \end{aligned}$$

where $\rho \in \pi_1(X_0, x_0)$ corresponds to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \text{SL}_2(\mathbf{Z})$, which is the monodromy matrix for the continuation of ω_1, ω_2 around ρ . On the upper half-plane this becomes

$$\begin{aligned} [m_\gamma, n_\gamma] &= \left[\int_{z_0}^{\gamma(z_0)} g(z) dz, \int_{z_0}^{\gamma(z_0)} -g(z)z dz \right] \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\quad + [c_1, c_2] \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \end{aligned}$$

and it follows that the assignment

$$\begin{aligned} \gamma &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [m_\gamma, n_\gamma]' \in \mathbf{C}^2 \\ &= \left[\begin{array}{c} \int_{z_0}^{\gamma(z_0)} g(z)z dz \\ \int_{z_0}^{\gamma(z_0)} g(z) dz \end{array} \right] + \left[I - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} \end{aligned}$$

is a group cocycle which is determined by f , or by g up to coboundary. Note that all the integrals are well defined because Z satisfies the residue conditions, i.e. $Z \in L^{\text{para}}$.

PROOF. Let f be a solution to $\Lambda f = Z$ which by Theorem 8 has the form

$$f = \left(\int_{x_0}^x \frac{-\omega_2 Z}{W} dx + c_1 \right) \omega_1 + \left(\int_{x_0}^x \frac{\omega_1 Z}{W} dx + c_2 \right) \omega_2.$$

Now under analytic (meromorphic) continuation around ρ we have that

$$\int_{x_0}^x \frac{-\omega_2 Z}{W} dx \rightarrow \int_{x_0}^x \frac{-(c\omega_1 + d\omega_2)Z}{W} dx + \int_{\rho} \frac{-\omega_2 Z}{W} dx$$

and likewise for $\int_{x_0}^x \omega_1 Z/W dx$. An easy computation now gives that f analytically continues to $f + m_{\gamma}\omega_1 + n_{\gamma}\omega_2$ with m_{γ} and n_{γ} given by the stated formula. Lifting to the upper half-plane and using Lemma 9, which says that $g = \omega_2^3 Z/W^2$, and the fact that $dz/dx = d\omega(x)/dx = -W/\omega_2^2$ gives the remainder of the result. \square

In fact, the periods m_{γ} and n_{γ} give a parabolic cocycle for Γ (see Shimura [16]). In order to see this, we must show that for any parabolic element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbf{C}^2$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [m_{\gamma}, n_{\gamma}]' = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Without loss of generality, we can assume that the parabolic element γ fixes the cusp ∞ and so takes the form

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbf{Z}.$$

(In the case $\text{trace } \gamma = -2$ there is nothing to prove.) The path $\rho \in \pi_1(X_0, x_0)$ corresponding to γ can be taken to go from x_0 to near the cusp, then to go around the cusp a number of times, and finally to return to x_0 along the original path. Calculating $[m_{\gamma}, n_{\gamma}]$ via Theorem 10, we find that $m_{\gamma} = 0$ because $(-\omega_2 Z/W) dx$ is single-valued at the cusp and has zero residue when $Z \in L_{\Lambda}^{\text{para}}$. A simple calculation then shows that a_1, a_2 can be chosen to satisfy the desired condition.

Thus $Z \in L_{\Lambda}^{\text{para}}$ and the corresponding automorphic form g determine a parabolic cohomology class, and this class arises naturally from the nature of the analytic continuation of the solutions to an inhomogeneous differential equation.

DEFINITION 11. Let g be any meromorphic automorphic form of weight three with respect to $\Gamma \subset \text{SL}_2(\mathbf{Z})$, Γ torsion-free of finite index. We say g is of the *second kind* if $g = \omega_2^3 Z/W^2$ for some $Z \in L_{\Lambda}^{\text{para}}$ and *exact* if $Z = \Lambda Z'$ for some $Z' \in K(X)$. \square

It is easy to see that any exact form is of the second kind and that Z (of f or g) gives the trivial parabolic cohomology class if and only if $Z = \Lambda Z'$ for some $Z' \in K(X)$ (i.e. g is exact).

In a subsequent paper it will be shown that this correspondence induces bijections

$$\frac{L_{\Lambda}^{\text{para}}}{\Lambda K(X)} \simeq \frac{\text{automorphic forms of second kind}}{\text{exact}} \simeq H_p^1(\Gamma, \mathbf{C}^2)$$

with the parabolic cohomology group as in Shimura [16] and Eichler [3]. These bijections are closely related to, but by no means easily derived from, Shimura's isomorphism

$$S_3(\Gamma) \simeq H_p^1(\Gamma, \mathbf{R}^2),$$

where $S_3(\Gamma)$ is the space of usual cusp forms of weight three for Γ .

Lastly, we remark that the periods, which are often of arithmetic significance (see Shimura [16]), can be computed by certain mixed abelian and hypergeometric integrals over a punctured sphere (see Stiller [22] for examples of such computations).

5. q -expansions and computations. In this final section we wish to show how one can use the differential equation to compute q -expansions, how to find the factor λ appearing in the formula for the differential equation, and how to tell when the corresponding automorphic form of weight one is holomorphic or a cusp form.

Let f be an automorphic form of weight one with respect to Γ as before. We have seen that f can be regarded as a multivalued solution of a K -equation $\Lambda_{(\mathcal{J}, \lambda)}$ on $X = \mathbb{H}^*/\Gamma$. In order to compute the q -expansion of f at a cusp, we first shift the cusp to ∞ by a linear fractional transformation of the upper half-plane \mathbb{H} . The effect of this is to conjugate Γ . Without any loss of generality, we thus assume that we are working with the cusp at ∞ . Here f corresponds to the invariant solution of $\Lambda_{(\mathcal{J}, \lambda)}$ up to a constant multiple (at least for a regular cusp). Since our equation has regular singular points we can apply well-known techniques to determine the solutions (see Ince [7], Griffiths [6], or Deligne [2]).

At this point, we need to calculate the local behavior of $\Lambda_{(\mathcal{J}, \lambda)}$ at a point in $X = X_0$. By Theorem 4

$$\Lambda_{(\mathcal{J}, \lambda)} f = d^2 f / dx^2 + P df / dx + Qf = 0,$$

where

$$\begin{aligned} P &= \frac{(d\mathcal{J}/dx)^2 - \mathcal{J} d^2\mathcal{J}/dx^2}{\mathcal{J} d\mathcal{J}/dx} - \frac{d}{dx} \log \lambda^2, \\ Q &= \frac{(d\mathcal{J}/dx)^2 (31\mathcal{J}/144 - 1/36)}{\mathcal{J}^2 (\mathcal{J} - 1)^2} - \frac{(d\mathcal{J}/dx)^2 - \mathcal{J} d^2\mathcal{J}/dx^2}{\mathcal{J} d\mathcal{J}/dx} \frac{d}{dx} \log \lambda \\ &\quad - \frac{d^2\lambda/dx^2}{\lambda} + 2 \left(\frac{d\lambda/dx}{\lambda} \right)^2. \end{aligned}$$

Now the “parameter” x does not provide a good local parameter everywhere and could therefore introduce singularities. This, however, is *not* the case; if t is a local parameter at any point in X , then when we express $\Lambda_{(\mathcal{J}, \lambda)}$ locally it retains the above form with t in place of x , i.e.

$$\begin{aligned} P &= \frac{(d\mathcal{J}/dt)^2 - \mathcal{J} d^2\mathcal{J}/dt^2}{\mathcal{J} d\mathcal{J}/dt} - \frac{d}{dt} \log \lambda^2, \\ Q &= \frac{(d\mathcal{J}/dt)^2 (31\mathcal{J}/144 - 1/36)}{\mathcal{J}^2 (\mathcal{J} - 1)^2} - \frac{(d\mathcal{J}/dt)^2 - \mathcal{J} d^2\mathcal{J}/dt^2}{\mathcal{J} d\mathcal{J}/dt} \frac{d}{dt} \log \lambda \\ &\quad - \frac{d^2\lambda/dt^2}{\lambda} + 2 \left(\frac{d\lambda/dt}{\lambda} \right)^2. \end{aligned}$$

We see that the global “parameter” x plays no role in determining the singularities. Thus the above form can be used in local calculations. In particular, we can take $t = q = e^{2\pi iz/n}$ at the point corresponding to the cusp at ∞ where $\pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma$ generates the isotropy group at ∞ .

We will begin by calculating the local behavior of the K -equation $\Lambda_{(\mathcal{J}, 1)}$ (i.e. assume $\lambda = 1$) when \mathcal{J} has a pole of order $n \geq 1$. In terms of local coordinate t , $\mathcal{J} = c_{-n}/t^n + \dots$. An easy calculation gives

$$P = 1/t + \text{holomorphic}, \quad Q = 0/t^2 + q_{-1}/t + \text{holomorphic}.$$

The indicial equation is

$$I(v) = v^2 + (p_{-1} - 1)v + q_{-2},$$

where $P = p_{-1}/t + \dots$ and $Q = q_{-2}/t^2 + \dots$. In this case $I(v) = v^2$ and the exponents are 0, 0. Thus the equation has a basis of solutions of the form

$$u_2 = 1 + \dots, \quad u_1 = u_2((1/2\pi i)\log t + \text{holomorphic}),$$

so that the local monodromy is $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. To reintroduce λ is simple. $\Lambda_{(\mathcal{J}, \lambda)}$ will have solutions λu_1 and λu_2 . Now $\lambda^2 \in K(X)$, so if $\lambda^2 = t^r \cdot (\text{holomorphic nonvanishing})$, then the new exponents are $r/2, r/2$ and the local monodromy is $\pm(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. Note that u_1, u_2 need not be a K -basis. Using Kodaira's classification for the monodromy at the singular fibers of an elliptic surface one can show that if ω_1, ω_2 is a K -basis, then after "shifting the cusp to ∞ " we have that the continuation of ω_1, ω_2 around the singularity gives $\pm(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix})$, where $n > 0$ is the order of the pole of \mathcal{J} —the negative case corresponding to an irregular cusp; the positive to a regular cusp.

Note that the elliptic modular surface defined in §3 for Γ has good fibers at every point in $X_0 = \mathbb{h}/\Gamma$; thus the only other possible singularities of Λ will be cosingular, i.e. the exponents must be distinct integers (see Stiller [21]) and the solutions will be meromorphic. Also because $\omega_1/\omega_2 = z$ the order of ω_2 (corresponding to a form f of weight one) will equal the lower exponent and the difference between the exponents will be 1 at a cosingular point. We see that in order to compute the q -expansion up to a constant multiple we need to know the q -expansion of J , the elliptic modular function which is \mathcal{J} in $\Lambda_{(\mathcal{J}, \lambda)}$, and the q -expansion for $\lambda^2 \in K(X)$.

It is clear from the above discussion that if all the lower exponents are ≥ 0 (including at the cosingular points), then the automorphic form f will be holomorphic and if, in addition, the lower exponent (in fact, both exponents) are $\geq \frac{1}{2}$ at the cusps, the resulting form will be a cusp form. We state this as

THEOREM 12. *If f is an automorphic form of weight one coming from the K -equation $\Lambda_{(\mathcal{J}, \lambda)}$, then f is holomorphic if and only if the lower exponent at each singularity of $\Lambda_{(\mathcal{J}, \lambda)}$ is ≥ 0 and, in addition, f will be a cusp form if the lower exponent at each cusp is $\geq \frac{1}{2}$. \square*

We can also say when the inhomogeneous equations

$$d^2f/dx^2 + P df/dx + Qf = \Lambda f = Z, \quad Z \in L_\Lambda^{\text{para}},$$

give rise to cusp forms g of weight three for Γ .

THEOREM 13. *Let $\Gamma \subset \text{SL}_2(\mathbb{Z})$ be a subgroup of finite index having no torsion and Λ one of the second order differential equations representing the Gauss-Manin connection of the canonical elliptic modular surface E over the modular curve $X = \mathbb{h}^*/\Gamma$. An inhomogeneous equation*

$$\Lambda f = Z, \quad Z \in L_\Lambda^{\text{para}} \subset K(X),$$

gives rise to a cusp of weight three for Γ if and only if $Z \in L(\mathfrak{A}_0) = \{Z \in K(X) \text{ s.t. } \text{div } Z + \mathfrak{A} > 0\}$, where \mathfrak{A}_0 is the divisor

$$\mathfrak{A}_0 = \text{div}(dx)^2 + \sum_{\substack{x \in X \\ s_x \in \mathbf{Z}}} (-s_x + 1)[x] + \sum_{\substack{x \in X \\ s_x \notin \mathbf{Z}}} \left(-s_x + \frac{3}{2}\right)[x],$$

and s_x is the largest root of the indicial equation for Λ at x .

PROOF. As we have seen above the two roots of the indicial equation for Λ at $x \in X$ are integers except at the irregular cusps where they are half-integers. Since for most points the roots are 0, 1, we have $s_x = 1$ for all but finitely many points and the divisor \mathfrak{A}_0 is well defined.

To see that the automorphic form associated to

$$\Lambda f = Z, \quad Z \in L(\mathfrak{A}_0),$$

is a cusp form is purely a local calculation on X . For example, if t is a local parameter at $x \in X$ and x is a regular cusp, then in terms of t we have $\Lambda f = Z(dx/dt)^2$, with

$$\omega_2 = t^{s_x} \cdot (\text{holomorphic nonvanishing}),$$

$$W = t^{2s_x - 1} \cdot (\text{holomorphic nonvanishing}).$$

As g is the pull-back of $\omega_2^3 Z(dx/dt)^2/W^2$ we see that it will vanish at the cusp if and only if $\text{ord}_x(Z(dx/dt)^2) \geq s_x - 1$. The same analysis at the irregular cusps and cosingular points gives the result. \square

We remark that it follows easily from Shimura [16] or Shioda [18] that no nonzero $Z \in L(\mathfrak{A}_0)$ corresponds to an exact form g , and hence the map $S_3(\Gamma) \rightarrow H_p^1(\Gamma, \mathbf{C}^2)$ is injective.

Lastly we wish to make some remarks on choosing $\lambda^2 \in K(X)$ when Γ is a principal congruence subgroup $\Gamma(N)$, $N \geq 3$. If one examines all the singularities of the equation $\Lambda_{(\mathcal{J}, 1)}$ with \mathcal{J} the elliptic modular function J , one sees that it has the following singularities:

- (1) At the cusps the exponents are $(0, 0)$.
- (2) At the $\mu/3$ points where $J = 0$ the exponents are $-\frac{1}{2}, \frac{1}{2}$. (Here μ is the index of $\Gamma \cdot \{\pm I\}$ in $\text{SL}_2(\mathbf{Z})$.)
- (3) At the $\mu/2$ points where $J = 1$ the exponents are $\frac{1}{2}, \frac{3}{2}$.
- (4) At all other points Λ is holomorphic and the exponents are 0, 1.

Thus we must choose λ^2 to have odd order at each of the $5\mu/6$ points in cases (2) and (3), and to have even order elsewhere. Once one λ is found all others are of the form $g\lambda$ with $g \in K(X)$.

Now when $\Gamma = \Gamma(N)$ with $N \equiv 0 \pmod{4}$ we can actually find λ . Note that when N is even, $\Gamma(N) \subset \Gamma(2)$ and the Legendre λ -function (which we denote by L to avoid confusion with λ above) is a modular function for $\Gamma(N)$. Recall (Robert [14]) that

$$(A) \quad J = \frac{4}{27} \frac{(1 - L + L^2)^3}{L^2(1 - L)^2}$$

and we have the maps

$$\mathfrak{h}^*/\Gamma(N) \xrightarrow{\alpha} \mathfrak{h}^*/\Gamma(2) \xrightarrow{\beta} \mathfrak{h}^*/\mathrm{SL}_2(\mathbf{Z}),$$

where α is unramified except over the cusps and β is ramified to order 2 at $0, 1, \infty$ over ∞ , to order 3 at $e^{2\pi i/6}, e^{-2\pi i/6}$ over 0, and to order 2 at $-1, 2, \frac{1}{2}$ over 1.

If we let

$$(B) \quad \begin{aligned} \lambda^2 &= \frac{(\alpha^*(L) - e^{2\pi i/6})(\alpha^*(L) - e^{-2\pi i/6})}{(\alpha^*(L) + 1)(\alpha^*(L) - 2)(\alpha^*(L) - 1/2)} \\ &= \alpha^*\left(\frac{L^2 - L + 1}{(L+1)(L+2)(L-1/2)}\right), \end{aligned}$$

where α^* means view L (or whatever) as a modular function for $\Gamma(N)$, then the equation $\Lambda_{(\mathcal{J}, \lambda)}$ will be holomorphic on $\mathfrak{h}/\Gamma(N) = X_0$, so the exponents will be 0, 1.

In order to determine the behavior at the cusps, observe that $\Gamma(N)$ has μ/N cusps and that α maps $\mu/3N$ of them to each of the points $0, 1, \infty \in \mathfrak{h}^*/\Gamma(2)$ with ramification $N/2$. At the $2\mu/3N$ cusps over 0, 1 the function λ^2 is holomorphic nonvanishing, so $\Lambda_{(\mathcal{J}, \lambda)}$ will have exponents $(0, 0)$, but at each of the $\mu/3N$ cusps over ∞ the function λ^2 has a zero of order $N/2$ so that $\Lambda_{(\mathcal{J}, \lambda)}$ will have exponents $(N/4, N/4)$.

Thus if $N \equiv 0 \pmod{4}$ the function λ^2 above will serve to produce one of the desired K -equations. This equation then determines (up to a constant multiple) an automorphic form f of weight one with respect to $\Gamma(N)$. Moreover, f will be holomorphic nonvanishing on \mathfrak{h} , will vanish to order $N/4$ at $\mu/3N$ cusps, and will be holomorphic nonvanishing at the $2\mu/3N$ remaining cusps. One can easily compute its q -expansion in terms of the q -expansions of the elliptic modular function J and the Legendre λ -function which we have denoted by L . In addition, it is easy to see that f can be normalized to have Fourier coefficients in \mathbf{Q} since both J and L have rational Fourier coefficients.

For example, for $\Gamma = \Gamma(4)$ the principal congruence subgroup of level four, the differential equation above is

$$\Lambda_{\Gamma(4)} = \frac{d^2}{dx^2} + \frac{x^4 - 4x^2 - 1}{x(x^2 - 1)(x^2 + 1)} \frac{d}{dx} + \frac{4}{(x^2 - 1)^2},$$

where $X = \mathfrak{h}^*/\Gamma(4)$ is identified with the Riemann x -sphere by choosing a modular function $x(z)$ which satisfies

$$L(z) = \left(\frac{x(z)^2 + 1}{x(z)^2 - 1} \right)^2,$$

suitably normalized (see Petersson [13] and Stiller [22]). The map

$$\frac{\mathfrak{h}^*}{\Gamma(4)} \rightarrow \frac{\mathfrak{h}^*}{\Gamma(2)}, \quad x \rightarrow \left(\frac{x^2 + 1}{x^2 - 1} \right)^2 = L$$

shows that $\pm i$ go to 0, ± 1 go to ∞ , and 0, ∞ go to 1, and as is easily checked the

exponents are $(0, 0)$ at $i, 0, \infty$ and $(1, 1)$ at ± 1 as observed above. The q -expansion of the associated normalized holomorphic form of weight one at $i\infty$ (which corresponds to i on the x -sphere) can then be calculated (see Stiller [22]).

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