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CONJUGACY PROBLEM IN $\mathrm{GL}_2(\mathbf{Z}[\sqrt{-1}])$ AND UNITS OF QUADRATIC EXTENSIONS OF $\mathbf{Q}(\sqrt{-1})$

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ABSTRACT. A highly efficient procedure for deciding if two given elements of $\mathrm{GL}_2(\mathbf{Z}[\sqrt{-1}])$ are conjugate or not will be presented. It makes use of a continued fraction algorithm in $\mathbf{Z}[\sqrt{-1}]$ and gives a fundamental unit of any given quadratic extension of $\mathbf{Q}(\sqrt{-1})$.

(1) *Introduction.* A solution to the conjugacy problem in the group $G = \mathrm{GL}_2(\mathbf{Z}[\sqrt{-1}])$ is included in the result of Grunewald [**3**]. But for a nice group like this there ought to be a simpler solution which makes use of the special nature of G. On the other hand, since G is not an amalgam of simpler groups, we should not expect too easy a solution. In this paper we present a straightforward procedure for deciding if two given elements of G are conjugate or not. It is based on a continued fraction algorithm in the ring $\mathbf{Z}[\sqrt{-1}]$ and a module theoretic consideration. It combines the ideas used in [**1** and **2**]. As the examples show it is highly efficient. A similar solution can be given for the group $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of any imaginary quadratic field, but in order to fix our attention we shall deal with the case when $\mathcal{O} = \mathbf{Z}[i]$, $i = \sqrt{-1}$.

(2) Actually what we solve is the similarity problem for the 2×2 matrices over $\mathcal{O} = \mathbf{Z}[i]$; given two such matrices A and B the problem is to decide if there is an $R \in \mathrm{GL}_2(\mathcal{O})$ such that $RAR^{-1} = B$. Our solution gives an explicit R if there is one. It also gives an effective characterization of the centralizer

$$Z(A) = \{R \in \mathrm{GL}_2(\mathcal{O}) | RA = AR\}$$

for a given A, so that we can find all $R \in \mathrm{GL}_2(\mathcal{O})$ such that $RAR^{-1} = B$. The characterization of $Z(A)$ is obtained by finding a fundamental unit of an order in a quadratic extension of $F = \mathbf{Q}(i)$; our method generates a fundamental unit.

(3) Given 2×2 matrices A and B over $\mathcal{O}$, call $A \sim B$ *similar* if $RAR^{-1} = B$ for some $R \in \mathrm{GL}_2(\mathcal{O})$. If $A \sim B$, then A and B have the same characteristic polynomial f over $\mathcal{O}$. Given a monic quadratic polynomial f over $\mathcal{O}$, let $M(f)$ denote the set of 2×2 matrices over $\mathcal{O}$ whose characteristic polynomials are equal to f. In deciding if $A \sim B$, we may assume that A and $B \in M(f)$ for some f. When f is reducible over F, deciding if $A \sim B$ is easy and we discuss it in the Appendix.

(4) Assume that f is irreducible over F. Put

$$f(t) = t^2 - qt + r, \qquad \Delta = q^2 - 4r.$$

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Given

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M(f),$$

put $\lambda = (q+\sqrt{\Delta})/2$, where $\mathrm{Im}(\sqrt{\Delta}) > 0$ or $\sqrt{\Delta} > 0$. The number λ is an eigenvalue of A. Put

$$\phi(A) = \alpha = (\lambda - d)/b = (a - d + \sqrt{\Delta})/2b.$$

Since f is irreducible, $bc \neq 0$. The column vector $(\alpha, 1)^T$ is an eigenvector of A belonging to λ and $A\alpha = \alpha$ (under the projective action of A on $\mathbf{C}$).

(5) Put $K = F(\sqrt{\Delta})$. Given $\xi \in K$, let ξ' denote its conjugate over F. Given $A \in M(f)$, if $\alpha = \phi(A)$ then

$$A = \begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix}^{-1}.$$

Thus the map $\phi\colon M(f) \to K$ is injective (for a given f). For any $R \in \mathrm{GL}_2(\mathcal{O})$,

$$RAR^{-1} = \left(R\begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix}\right) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \left(R\begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix}\right)^{-1}.$$

Thus by injectivity of ϕ on $M(f)$, we have $\phi(RAR^{-1}) = R\phi(A)$. Given α and $\beta \in K - F$, call $\alpha \sim \beta$ if $R\alpha = \beta$ for some $R \in \mathbf{GL}_2(\mathcal{O})$. From the discussion above, we see that, given A and $B \in M(f)$,

$$A \sim B \quad \text{iff} \quad \phi(A) \sim \phi(B).$$

Thus the problem is transformed to this: Given α and $\beta \in K - F$, decide if $\alpha \sim \beta$.

(6) Given α and $\beta \in K$, let $\langle \alpha, \beta \rangle$ denote the module over $\mathcal{O}$ generated by α and β. In this paper, by a *module* we shall mean a finitely generated full module over $\mathcal{O}$ contained in K. Every module is of the form $\langle \alpha, \beta \rangle$ and (α, β) is a basis of this module over $\mathcal{O}$. For example, if $\alpha \in K - F$, then $\langle \alpha, 1 \rangle$ is a module. Given modules U and V, call $U \sim V$ *similar* if $U = \lambda V$ for some $\lambda \in K^\times$.

(7) Given α and $\beta \in K - F$, put $U = \langle \alpha, 1 \rangle$ and $V = \langle \beta, 1 \rangle$. Then

$$\alpha \sim \beta \quad \text{iff} \quad U \sim V.$$

In fact, if $\alpha \sim \beta$, say $R\alpha = \beta$, $R \in \mathrm{GL}_2(\mathcal{O})$, then $R\binom{\alpha}{1} = \lambda\binom{\beta}{1}$ for some $\lambda \in K^\times$ and hence $U = \lambda V$, i.e., $U \sim V$. Going backward we get the converse. Thus the problem is now transformed to the following: Given modules U and V, decide if $U \sim V$.

(8) Let $U = \langle \alpha, \beta \rangle$ be a module. An element $\xi = x\alpha + y\beta$ of U, where it is understood that x and $y \in \mathcal{O}$, is called *primitive* if $(x, y) = 1$, i.e., x and y are coprime. The primitiveness of an element of U does not depend on the choice of a basis (α, β) of U. A member of a basis is primitive. It is easy to see that if ρ is a primitive element of U, then $U = \langle \sigma, \rho \rangle$ for some $\sigma \in U$. A module U is called *normalized* if 1 is a primitive element of U so that $U = \langle \alpha, 1 \rangle$ for some $\alpha \in K - F$. Given modules U and V, call $U \equiv V$ if $U = cV$ for some $c \in F^\times$.

(9) For any module U, there is a unique normalized module V such that $U \equiv V$.

PROOF. $U \cap \mathcal{O}$ is a nonzero fractional ideal of $\mathcal{O}$ and hence $U \cap \mathcal{O} = (b)$ for some $b \in F^\times$ and b has to be a primitive element of U. Thus $U = \langle \alpha, b \rangle$ for

some α. $V = b^{-1}U = \langle \alpha b^{-1}, 1\rangle$ is normalized and $U \equiv V$. To see the uniqueness, suppose that U and V are normalized modules such that $U \equiv V$, say $U = cV$, $c \in F^\times$, $U = \langle \alpha, 1\rangle$, and $V = \langle \beta, 1\rangle$. Then $\langle \alpha, 1\rangle = \langle c\alpha, c\rangle$ and hence there is an $R \in \mathrm{GL}_2(\mathcal{O})$ such that $R\binom{\alpha}{1} = c\binom{\beta}{1}$. Since $c \in F$ and $\alpha \notin F$, R has to be of the form

$$R = \begin{pmatrix} x & y \\ 0 & c \end{pmatrix}$$

with $xc = \det R \in \mathcal{O}^\times = \{\pm 1, \pm i\}$. Thus $c \in \mathcal{O}^\times$ and $U = V$.

(10) Given α and $\beta \in K - F$, call $\alpha \equiv \beta$ if

$$\begin{pmatrix} \varepsilon & c \\ 0 & 1 \end{pmatrix} \alpha = \varepsilon\alpha + c = \beta$$

for some $\varepsilon \in \mathcal{O}^\times$ and $c \in \mathcal{O}$. From the proof of (9), it is clear that, given normalized modules $U = \langle \alpha, 1\rangle$ and $V = \langle \beta, 1\rangle$,

$$U = V \quad \text{iff} \quad \alpha \equiv \beta.$$

We assume that, given α and $\beta \in K - F$, recognizing if $\alpha \equiv \beta$ is instantaneous. For example, if

$$\alpha = (e_1 + \sqrt{\Lambda})/2b_1 \quad \text{and} \quad \beta = (e_2 + \sqrt{\Delta})/2b_2,$$

where $e_1, b_1, e_2, b_2 \in \mathcal{O}$, then $\alpha \equiv \beta$ iff $\varepsilon b_1 = b_2$ for some $\varepsilon \in \mathcal{O}^\times$ and $e_1 \equiv e_2 \pmod{2b_1}$.

(11) Let U be a module. A nonzero element ρ of U is called a *convergent* of U if 0 is the only element ξ of U such that

$$|\xi| < |\rho| \quad \text{and} \quad |\xi'| < |\rho'|.$$

Note that given α and $\beta \in K$, $|\alpha| = |\beta|$ iff $|\alpha'| = |\beta'|$. (This can be easily proved by looking at $\gamma = \alpha/\beta$ and its complex conjugate $\overline{\gamma}$ and their norms.) For any $\lambda \in K^\times$, as ρ ranges over the convergents of U, $\lambda\rho$ ranges over the convergents of λU.

(12) Let $U = \langle \alpha, \beta\rangle$. If $\xi = x\alpha + y\beta \in U$, then $\xi' = x\alpha' + y\beta'$ and

$$x = (\xi\beta' - \xi'\beta)/(\alpha\beta' - \alpha'\beta) \quad \text{and} \quad y = (\alpha\xi' - \alpha'\xi)/(\alpha\beta' - \alpha'\beta).$$

Thus if $|\xi|$ and $|\xi'|$ are bounded, then $|x|$ and $|y|$ are bounded. Thus for any c_1 and $c_2 > 0$, U contains only a finite number of element ξ such that $|\xi| < c_1$ and $|\xi'| < c_2$. This shows that there are convergents of U.

(13) Let ρ be a convergent of U. Then ρ is a primitive element of U and $U = \langle \sigma, \rho\rangle$ for some σ and $\rho^{-1}U = \langle \sigma\rho^{-1}, 1\rangle$ is normalized. The normalized module $\rho^{-1}U$ is called a *derived module* of U. Let $\mathcal{D}(U)$ denote the set of all derived modules of U, i.e., $\mathcal{D}(U) = \{\rho^{-1}U | \rho \text{ is a convergent of } U\}$.

(14) If $V \in \mathcal{D}(U)$, then $U \sim V$. Thus if $\mathcal{D}(U) \cap \mathcal{D}(V) \neq \varnothing$, then $U \sim V$. Conversely, suppose $U \sim V$, say $\lambda U = V$, $\lambda \in K^\times$. The relation $\lambda\rho = \sigma$ establishes a one-to-one correspondence between the convergents ρ of U and the convergents σ of V and $\rho^{-1}U = \sigma^{-1}V$. Thus $\mathcal{D}(U) = \mathcal{D}(V)$. In particular, given modules U and V, either $\mathcal{D}(U) = \mathcal{D}(V)$ or $\mathcal{D}(U) \cap \mathcal{D}(V) = \varnothing$ according as $U \sim V$ or not.

(15) By an argument similar to the one given in [**2**], we can show that $\mathcal{D}(U)$ is a finite set for any module U and such an argument indicates how to find all members

of $\mathcal{D}(U)$. In this paper we shall accomplish this by means of a continued fraction algorithm, which is more efficient.

(16) Given a module U, let $\mathcal{O}_U$ denote its coefficient ring; $\mathcal{O}_U$ consists of $\omega \in K$ such that $\omega\xi \in U$ for all $\xi \in U$. $\mathcal{O}_U$ is a module and $\mathcal{O} \subset \mathcal{O}_U \subset \mathcal{O}_K$, where$\mathcal{O}_K$ is the ring of integers of K. If $U \sim V$, then $\mathcal{O}_U = \mathcal{O}_V$. Given $\lambda \in K^\times$, $\lambda U = U$ iff $\lambda \in \mathcal{O}_U^\times$. If $\lambda \in \mathcal{O}_U^\times$, then as ρ ranges over the convergents of U, so does $\lambda\rho$.

(17) Given convergents ρ and σ of U, call $\rho \simeq \sigma$ if $\lambda\rho = \sigma$ for some $\lambda \in \mathcal{O}_U^\times$. We have $\rho \simeq \sigma$ iff $\rho^{-1}U = \sigma^{-1}U$. Thus the set $\mathcal{D}(U)$ of derived modules of U is equivalent to the set $\mathcal{C}(U)$ of equivalence classes (under $\simeq$) of the convergents of U. Given convergents ρ and σ of U, call $\rho \cong \sigma$ if $\varsigma\rho = \sigma$ for some root of unity ς in $\mathcal{O}_U^\times$. A root of unity in $\mathcal{O}_U^\times$ is usually a 4th root of unity, i.e., in $\mathcal{O}^\times$, but it could be an 8th root or a 12th root of unity. We assume that given convergents ρ and σ of U, recognizing if $\rho \cong \sigma$ is instantaneous. If $\rho \cong \sigma$, then $|\rho| = |\sigma|$ (and hence $|\rho'| = |\sigma'|$). But since ρ and σ are not necessarily integers, it is possible that $\rho \ncong \sigma$ and $|\rho| = |\sigma|$ (cf. Example 3 and (35)).

(18) Our main objective in the rest of the paper is to show that $\mathcal{C}(U)$ is finite and to see how we can systematically obtain a complete set of representatives of the equivalence classes in $\mathcal{C}(U)$. We are going to develop a continued fraction algorithm for these purposes. We start with a simplest version. Such an algorithm has an independent interest of its own (cf. [**4**, pp. 181–188]).

(19) Given $\alpha \in \mathbf{C}$, let $[\alpha]$ denote the element $p \in \mathcal{O}$ such that $\alpha - p$ is in the square

$$-\tfrac{1}{2} < x \leq \tfrac{1}{2} \quad \text{and} \quad -\tfrac{1}{2} < y \leq \tfrac{1}{2}$$

of the complex plane. Given $\alpha \in \mathbf{C}$, put $\alpha_0 = \alpha$ and having defined α_n for some $n \geq 0$, put $p_n = [\alpha_n]$ and $\alpha_{n+1} = 1/(\alpha_n - p_n)$ provided $\alpha_n \neq p_n$, i.e., $\alpha_n \notin \mathcal{O}$. Note that $|\alpha_n| \geq \sqrt{2}$ for $n > 0$. It is easily verified that $\alpha_n \in \mathcal{O}$ for some $n \geq 0$ iff $\alpha \in F$.

(20) Given $\alpha \in \mathbf{C}$, let p_n be as in (19) and put

$$P_n = \begin{pmatrix} p_n & 1 \\ 1 & 0 \end{pmatrix}, \quad A_0 = I \quad \text{and} \quad A_n = P_0 P_1 \cdots P_{n-1}.$$

Then we verify that

$$A_n = \begin{pmatrix} a_n & a_{n-1} \\ b_n & b_{n-1} \end{pmatrix},$$

where a_n and b_n are given by the recursions $a_0 = 1$, $a_1 = p_0$, $a_{n+1} = a_n p_n + a_{n-1}$, $b_0 = 0$, $b_1 = 1$, $b_{n+1} = b_n p_n + b_{n-1}$. Since $\det P_n = -1$, $\det A_n = (-1)^n$. In particular, $(a_n, b_n) = 1$.

(21) From the definition of α_n and p_n in (19), we have

$$P_n^{-1}\alpha_n = \alpha_{n+1} \quad \text{and} \quad P_n^{-1}\begin{pmatrix} \alpha_n \\ 1 \end{pmatrix} = \alpha_{n+1}^{-1}\begin{pmatrix} \alpha_{n+1} \\ 1 \end{pmatrix}.$$

Thus

$$A_n^{-1}\alpha = \alpha_n \quad \text{and} \quad A_n^{-1}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = (\alpha_1 \cdots \alpha_n)^{-1}\begin{pmatrix} \alpha_n \\ 1 \end{pmatrix}.$$

By looking at the second component of the second equality above, we get that $a_n - b_n\alpha = (-1)^n/(\alpha_1 \cdots \alpha_n)$. Since $|\alpha_n| \geq \sqrt{2}$ for $n > 0$, it follows that $|a_n - b_n\alpha| \leq$

$1/\sqrt{2^n}$. In particular, for $\alpha \notin F$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \alpha \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty.$$

(22) LEMMA. *With α_n and b_n as above for $\alpha \notin F$, $|b_n/(\alpha_1 \cdots \alpha_n)| < \sqrt{2}+1$ for all $n > 0$.*

PROOF. By looking at the second component of the equality

$$(\alpha_1 \cdots \alpha_n)\begin{pmatrix}\alpha\\1\end{pmatrix} = A_n\begin{pmatrix}\alpha_n\\1\end{pmatrix},$$

we get that $\alpha_1 \cdots \alpha_n = b_n\alpha_n + b_{n-1}$, and hence

$$|b_n/(\alpha_1 \cdots \alpha_n)| = |\alpha_n + b_{n-1}/b_n|^{-1}.$$

Call $n > 0$ good if $|b_n| > |b_{n-1}|$. Since $b_0 = 0$ and $b_1 = 1$, $n = 1$ is good. (It can be shown that all $n > 0$ are good but we do not need this. In any case since $b_n \to \infty$, there are infinitely many good n's.) If n is good, then

$$|\alpha_n + b_{n-1}/b_n| \geq |\alpha_n| - |b_{n-1}/b_n| > \sqrt{2}-1,$$

and hence

$$|b_n/(\alpha_1 \cdots \alpha_n)| < 1/(\sqrt{2}-1) = \sqrt{2}+1.$$

Suppose n is bad. Take the largest good $k < n$. Then since $n, n-1, \ldots, k+1$ are bad, $|b_n| \leq |b_{n-1}| \leq \cdots \leq |b_k|$, and hence

$$\left|\frac{b_n}{\alpha_1 \cdots \alpha_n}\right| = \left|\frac{b_k}{\alpha_1 \cdots \alpha_k}\right| \left|\frac{b_n}{b_k\alpha_{k+1} \cdots \alpha_n}\right| < \left(\sqrt{2}+1\right)\left|\frac{b_n}{b_k}\right| \leq \sqrt{2}+1.$$

(23) THEOREM (PERIODICITY). *Given $\alpha \in \mathbf{C} - F$, $\alpha_{k+l} = \alpha_k$ for some k and l with $l > 0$ iff α is quadratic over F.*

PROOF. Suppose $\alpha_{k+l} = \alpha_k$, $l > 0$. Then with $A = P_k \cdots P_{k+l-1} = A_k^{-1}A_{k+l}$, $\alpha_k = A\alpha_{k+l} = A\alpha_k$. Thus α_k is quadratic over F. Since $\alpha = A_k\alpha_k$, α is quadratic over F also.

Conversely, suppose α is quadratic over F, say $d\alpha^2 - e\alpha + c = 0$, where $d, e, c \in \mathcal{O}$ and $dc \neq 0$. Put $C = \left(\begin{smallmatrix} e & -2c \\ 2d & e\end{smallmatrix}\right)$. Then $C\alpha = \alpha$. Since $\alpha = A_n\alpha_n$, $A_n^{-1}CA_n\alpha_n = \alpha_n$. Computing

$$C_n = A_n^{-1}CA_n = \begin{pmatrix} e_n & -2c_n \\ 2d_n & -e_n \end{pmatrix}$$

modulo $\pm I$, we get that

$$d_n = da_n^2 - ea_nb_n + cb_n^2 \quad \text{and} \quad c_n = -d_{n-1}.$$

Put $a_n = b_n\alpha + \delta_n$ and substitute this into the expression for d_n above. We get that $d_n = (2d\alpha - e)b_n\delta_n + d\delta_n^2$. By (21) and (22), $|b_n\delta_n| < \sqrt{2}+1$ and $\delta_n \to 0$ as $n \to \infty$. Thus d_n are bounded by a constant (depending only on α). Then so are c_n. Since $e_n^2 - 4d_nc_n = e^2 - 4dc$, e_n are bounded also. Since $d_n\alpha_n^2 - e_n\alpha_n + c_n = 0$ and d_n, e_n, c_n are bounded, we conclude that there are only a finite number of distinct α_n. Thus $\alpha_{k+l} = \alpha_k$ for some k and l with $l > 0$ (cf. [**4**, p. 185] for another proof).

(24) Let α be quadratic over F and suppose $\alpha_{k+l} = \alpha_k$, $l > 0$. Then $|\alpha_n - \alpha'_n| \geq \sqrt{2} - 1$ for all $n \geq k$.

PROOF. Let the notations be as in (23) and put $\Delta = e^2 - 4dc$. Then $\alpha_n = (e_n + \sqrt{\Delta})/2d_n$ for all $n \geq 0$ (with $d_0 = d$, $e_0 = e$ and $c_0 = c$) and hence $\alpha_n - \alpha'_n = \sqrt{\Delta}/d_n$. Since $2d\alpha - e = \sqrt{\Delta}$, $d_n = \sqrt{\Delta} b_n \delta_n + d\delta_n^2$. Since $|b_n\delta_n| < \sqrt{2} + 1$ and $\alpha_n = \alpha_{n+ml}$ for all $n \geq k$ and $m \geq 0$ and $\delta_n \to 0$ as $n \to \infty$, we get that $|d_n| \leq (\sqrt{2}+1)|\sqrt{\Delta}|$ for all $n \geq k$ and hence

$$|\alpha_n - \alpha'_n| = |\sqrt{\Delta}/d_n| \geq 1/(\sqrt{2}+1) = \sqrt{2} - 1.$$

(25) Given $\alpha \in K - F$, put $U = \langle \alpha, 1 \rangle$. By means of the simple continued fraction algorithm developed above, we can find a unit $\lambda \in \mathcal{O}_U^\times$ such that $|\lambda| > 1$. In fact, compute α_n until we get $\alpha_{k+l} \equiv \alpha_k$, $l > 0$, and consider $U_n = \langle \alpha_n, 1 \rangle$. Since $A_n \in \mathrm{GL}_2(\mathcal{O})$ and $A_n \binom{\alpha_n}{1} = (\alpha_1 \cdots \alpha_n)\binom{\alpha}{1}$, $U_n = (\alpha_1 \cdots \alpha_n)U$. In particular, $U_n \sim U$ and $\mathcal{O}_{U_n} = \mathcal{O}_U$. Put $\lambda = \alpha_{k+1} \cdots \alpha_{k+l}$. Then $|\lambda| > 1$ and $\lambda U_k = U_{k+l} = U_k$ and hence $\lambda \in \mathcal{O}_U^\times$. (But λ may not be a fundamental unit of $\mathcal{O}_U$.)

(26) Given a module U, the norms (over F) of the convergents of U are bounded.

PROOF. We may assume that $U = \langle \alpha, 1 \rangle$. Let ρ be a convergent of U. If $|\rho| > 1$, then take $\lambda \in \mathcal{O}_U^\times$ such that $|\lambda\rho| \leq 1$ (cf. (25)). Then $\sigma = \lambda\rho$ is a convergent of U such that $|\sigma| \leq 1$. Since $|N\lambda| = 1$, $|N\sigma| = |N\rho|$. Thus we may assume that $|\rho| \leq 1$. Let a_n and b_n be as in (20) for α and consider the elements $\xi_n = a_n - b_n\alpha$ of U. Since $\xi_n = (-1)^n(\alpha_1 \cdots \alpha_n)^{-1}$ (cf. (21)) and $|\alpha_n| \geq \sqrt{2}$, ξ_n decreases to 0. Take $n > 0$ such that

$$|\xi_n| < |\rho| \leq |\xi_{n-1}| = |\xi_n|\,|\alpha_n|.$$

Since $\xi_n \in U$ and ρ is a convergent of U, $|\rho'| < |\xi'_n|$. Thus $|N\rho| < |N\xi_n|\,|\alpha_n|$. Since only a finite number of α_n are distinct, $|\alpha_n|$ are bounded. On the other hand, $N\xi_n = (N\alpha_1 \cdots N\alpha_n)^{-1}$. With the notations as in (23), $d_n\alpha_n^2 - e_n\alpha_n + c_n = 0$, and hence $N\alpha_n = c_n/d_n$. Since $c_n = -d_{n-1}$, $N\alpha_1 \cdots N\alpha_n = (-1)^n\, dd_n^{-1}$. Thus $N\xi_n = (-1)^n\, d_n d^{-1}$ and these are bounded. (A modification of the argument used in (11) of [**2**] gives another proof of this result via Minkowski Theorem.)

(27) Given $c_2 > c_1 > 0$, the number of convergents of U such that $c_2 > |\rho| > c_1$ is finite.

PROOF. Choose $c_0 > 0$ such that $|N\rho| < c_0$ for all convergents ρ of U (cf. (26)). Let ρ be a convergent of U such that $c_2 > |\rho| > c_1$. Since $|\rho|\,|\rho'| < c_0$, $|\rho'| < c_0|\rho|^{-1} < c_0c_1^{-1}$. Since there are only a finite number of elements ξ of U such that $|\xi| < c_2$ and $|\xi'| < c_0c_1^{-1}$ (cf. (12)), we get the result.

(28) THEOREM. *For every module U, the set $\mathcal{C}(U)$ is finite (cf. (17)).*

PROOF. Take $\lambda \in \mathcal{O}_U^\times$ such that $|\lambda| > 1$. Given a convergent ρ of U, take $n \in Z$ such that $|\lambda|^{n-1} < |\rho| \leq |\lambda|^n$. Then $|\lambda|^{-1} < |\rho\lambda^{-n}| \leq 1$. Since $\rho\lambda^{-n} \simeq \rho$, we get the result by (27).

(29) We now turn to the problem of finding a complete set of representatives of the equivalence classes in $\mathcal{C}(U)$, where $U = \langle \alpha, 1 \rangle$. First of all, we have to find a convergent of U to get started. Let $\alpha = (e + \sqrt{\Delta})/2b \in K$, where $b, e \in \mathcal{O}$. If $\xi = y\alpha - x \in U$, then $y = (b/\sqrt{\Delta})(\xi - \xi')$. Thus if $|\xi| < 1$ and $|\xi'| < 1$, then $|y| < 2|b|/|\sqrt{\Delta}|$. In particular, if $2|b| \leq |\sqrt{\Delta}|$, then $y = 0$ and 1 must be a convergent of U.

(30) Suppose $2|b|/|\sqrt{\Delta}| > 1$ but not too large (cf. (32)). Let y range, in some convenient order, over the nonzero elements of $\mathcal{O}$ in the first quadrant (including the real axis but not the imaginary axis) such that $|y| \leq 2|b|/|\sqrt{\Delta}|$. For each y, choose $x \in \mathcal{O}$ such that $|y\alpha - x| \leq 1$ and compute $|y\alpha' - x|$. If $|y\alpha' - x| \geq 1$ for all y and x, then 1 is a convergent of U.

(31) Assume that for some y as in (30), there is an $x \in \mathcal{O}$ such that

$$|y\alpha - x| < 1 \quad \text{and} \quad |y\alpha' - x| < 1.$$

For each y for which such an x exists, choose x such that $|y\alpha' - x|$ is least. There are at most two choices for such x, and if so, they differ by 1 or i, and choose the one of smaller modulus for the sake of definiteness. List the elements $\xi = y\alpha - x$ thus chosen as $\xi_1, \xi_2, \ldots, \xi_r$ so that $|\xi_1| \geq |\xi_2| \geq \cdots \geq |\xi_r|$. If there is an ambiguity in this order, i.e., if two or more ξ_i have the same modulus, then order them according to the given ordering of y's. If $|\xi_1'| \leq |\xi_j'|$ for all $j > 1$, then ξ_1 is a convergent of U. Suppose $|\xi_1'| > |\xi_j'|$ for some $j > 1$. Then take the first such j and consider the list $\xi_j, \ldots, \xi_r$ and repeat, i.e., if $|\xi_j'| \leq |\xi_k'|$ for all $k > j$, then ξ_j is a convergent of U, etc.

(32) In case $2|b|/|\sqrt{\Delta}|$ is large, the method of finding a convergent of U described in (30) and (31) is tedious and unsatisfactory. This is where the result of (24) comes to the rescue. Compute α_n as in (19) until we get $\alpha_{k+l} = \alpha_k$, $l > 0$, and consider $\beta = \alpha_k$ and $V = \langle \beta, 1 \rangle$. With the notations as in (23) and (24),

$$2|d_k|/|\sqrt{\Delta}| = 2/|\beta - \beta'| \leq 2\left(\sqrt{2} + 1\right).$$

Thus we can find a convergent σ of V as in (29), (30) and (31) without much trouble. (It is likely that 1 is a convergent of V.) Let

$$A_k = \begin{pmatrix} a_k & a_{k-1} \\ b_k & b_{k-1} \end{pmatrix}$$

be as in (20) for α and put $\gamma = a_k - b_k\alpha$. Then $U = \langle \alpha, 1 \rangle = \langle b_{k-1}\alpha - a_{k-1}, \gamma \rangle = \gamma\langle \beta, 1 \rangle = \gamma V$. Thus $\gamma\sigma$ is a convergent of U.

(33) We now have a way to find a convergent $\rho_1 = p - q\alpha$ of $U = \langle \alpha, 1 \rangle$. If $\rho_1 = 1$, then put $Q_1 = I$. In any case, find r and $s \in \mathcal{O}$ such that $ps - qr = 1$ and put $Q_1 = \left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$. Although it does not matter how we find such r and s, one definite way to find them is to apply the simple continued fraction algorithm to the "rational" element $\beta = p/q \in F^\times$. Compute a_n and b_n as in (20) for β. Then we arrive at $k \geq 0$ such that $qa_k - pb_k = \varepsilon \in \mathcal{O}^\times$. Put $r = \varepsilon^{-1}a_k$ and $s = \varepsilon^{-1}b_k$. Put $\alpha_1 = Q_1^{-1}\alpha$ and $U_1 = \langle \alpha_1, 1 \rangle$. Since $Q_1^{-1}\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) = \rho_1\left(\begin{smallmatrix} \alpha_1 \\ 1 \end{smallmatrix}\right)$,

$$Q_1\begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix} = \rho_1^{-1}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \quad \text{and} \quad U_1 = \rho_1^{-1}U.$$

Since ρ_1 is a convergent of U, 1 is a convergent of U_1.

(34) LEMMA. *If 1 is a convergent of $U = \langle \alpha, 1 \rangle$ and $\xi = y\alpha - x$ is a primitive element of U such that $|\xi| \leq 1$ and $|y| \geq 2$, then U contains a nonzero element β such that $|\beta| < 1$ and $|\beta'| < |\xi'|$.*

PROOF. Let $\xi = y\alpha - x$ be a primitive element of U such that $|\xi| \leq 1$ and $|y| \geq 2$. Choose $p \in \mathcal{O}$ such that $|\alpha - p| < 1$ and $|\alpha' - p|$ is least. Since 1 is a

convergent of U, $|\alpha' - p| > 1$. Choose $\varepsilon \in \mathcal{O}^\times$ such that $\beta = \varepsilon(\alpha - p)$ is in the first quadrant. Consider the half-planes

$$H_1\colon \mathrm{Re}(z) \le \tfrac{1}{2}, \quad H_2\colon \mathrm{Im}(z) \le \tfrac{1}{2}, \quad H_3\colon \mathrm{Re}(z) + \mathrm{Im}(z) \le 1.$$

By the choice of p we have the following implications:

If $|\beta - 1| < 1$, then $\beta' \in H_1$.

If $|\beta - i| < 1$, then $\beta' \in H_2$.

If $|\beta - 1 - i| < 1$, then $\beta' \in H_3$.

Let $A\colon |z| \ge 1$. Since β is in the first quadrant and $|\beta| < 1$, there are five cases:

(i) $|\beta - 1| < 1$ and $|\beta - i| < 1$: put $B = A \cap H_1 \cap H_2$.

(ii) $|\beta - 1| \ge 1$ and $|\beta - 1 - i| < 1$: put $B = A \cap H_2 \cap H_3$.

(iii) $|\beta - 1| \ge 1$ and $|\beta - 1 - i| \ge 1$: put $B = A \cap H_2$.

(iv) $|\beta - i| \ge 1$ and $|\beta - 1 - i| < 1$: put $B = A \cap H_1 \cap H_3$.

(v) $|\beta - i| \ge 1$ and $|\beta - 1 - i| \ge 1$: put $B = A \cap H_1$.

Since ξ is primitive, $(x, y) = 1$. Since $|\xi| \le 1$, $|\alpha - xy^{-1}| \le |y|^{-1}$. Put $r = |p - xy^{-1}|$. The inequality

$$|z - \varepsilon p| \ge |y|\,|z - \varepsilon xy^{-1}| = |yz - \varepsilon x|$$

on z defines a disk D of radius $r\sqrt{|y|}/(|y| - 1)$ with the center c on the line through εp and εxy^{-1} so that εxy^{-1} is between εp and c and $|\varepsilon xy^{-1} - c| = r/(|y| - 1)$. Since $|y| \ge 2$, in any of the five cases above, if B is defined as indicated, then we see that $(B + \varepsilon p) \cap D = \varnothing$. Since $\varepsilon\alpha' \in B + \varepsilon p$,

$$|\beta'| = |\varepsilon\alpha' - \varepsilon p| < |y\alpha' - x| = |\xi'|.$$

(35) Although it is possible for U to have two convergents ρ and σ such that $|\rho| = |\sigma|$ and $\rho \not\cong \sigma$, if ρ, σ and τ are convergents of U such that $|\rho| = |\sigma| = |\tau|$, $\rho \not\cong \sigma$ and $\rho \not\cong \tau$, then $\sigma \cong \tau$. In fact, by considering $\rho^{-1}U$, we may assume that $\rho = 1$. Let σ and τ be convergents of U such that $|\sigma| = |\tau| = 1$, $\sigma \not\cong 1$ and $\tau \not\cong 1$. Put $\sigma = y\alpha - x$. If $|y| \ge 2$, then there is $\beta \in U$ such that $\beta \ne 0$, $|\beta| < 1$ and $|\beta'| < |\sigma'| = 1$ by (34), which contradicts that 1 is a convergent of U. Thus we may assume that $\sigma = \alpha - x$ or $\sigma = (1 + i)\alpha - x$. Similarly, we may assume that $\tau = \alpha - y$ or $\tau = (1 + i)\alpha - y$ for some $y \in \mathcal{O}$. Since $|\sigma| = |\tau| = 1$, if $\sigma \ne \tau$, then we get that $\alpha \in \mathcal{O}$ or $\alpha \in \mathcal{O} + \varsigma$ for some 12th root of unity ς. Since $\alpha \notin \mathcal{O}$, we get that σ and τ are 12th roots of unity, and hence $\sigma \cong \tau$.

(36) Having chosen a convergent ρ_1 of $U = \langle \alpha, 1 \rangle$, we are going to choose convergents $\rho_2, \rho_3, \ldots$ of U so that $|\rho_1| \ge |\rho_2| \ge |\rho_3| \ge \ldots$, $\rho_n \not\cong \rho_{n+1}$ for any $n > 0$, at most two ρ_n's have the same modulus, and if ρ is a convergent of U such that $|\rho_i| \ge |\rho| > |\rho_j|$ for some $j > i > 0$, then $\rho \cong \rho_n$ for some n, $\max\{1, i - 1\} \le n < j$. (The possibility that $\rho \cong \rho_{i-1}$ occurs only if $|\rho_{i-1}| = |\rho_i| = |\rho|$ and $i > 1$.)

(37) Suppose we have found convergents $\rho_1, \ldots, \rho_n$ of $U = \langle \alpha, 1 \rangle$ satisfying the conditions stated in (36). We have done so for $n = 1$ (in which case the various conditions are vacuous). Moreover, assume that we have matrices $A_1, \ldots, A_n \in \mathrm{GL}_2(\mathcal{O})$ of determinant 1 such that with $A_n = \left(\begin{smallmatrix} a_n & c_n \\ b_n & d_n \end{smallmatrix}\right)$, $\rho_n = a_n - b_n\alpha$. For $n = 1$, $A_1 = Q_1$. (The meanings of a_n, b_n and A_n are now different from those in (20).) Put $\alpha_n = A_n^{-1}\alpha$ and $U_n = \langle \alpha_n, 1 \rangle$. Since $A_n^{-1}\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) = \rho_n\left(\begin{smallmatrix} \alpha_n \\ 1 \end{smallmatrix}\right)$,

$$A_n\begin{pmatrix} \alpha_n \\ 1 \end{pmatrix} = \rho_n^{-1}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \quad \text{and} \quad U_n = \rho_n^{-1}U.$$

Since ρ_n is a convergent of U, 1 is a convergent of U_n.

(38) To find ρ_{n+1}, choose $p \in \mathcal{O}$ such that $|\alpha_n - p| \le 1$ and $|\alpha'_n - p|$ is least. There are at most two choices for such p, and if so, choose the one of smaller modulus. If $n > 1$ and $|\rho_n^{-1}\rho_{n-1}| = 1$, then make sure that $|\alpha_n - p| < 1$. Choose $\varepsilon \in \mathcal{O}^\times$ such that

$$\left|\alpha_n - p - \frac{\varepsilon}{1+i}\right| \le \frac{1}{\sqrt{2}} \quad \text{and} \quad c = \left|\alpha'_n - p - \frac{\varepsilon}{1+i}\right| \text{ is least}$$

and put

$$\sigma_{n+1} = \begin{cases} p - \alpha_n & \text{if } \sqrt{2}c \ge |\alpha'_n - p|, \\ (1+i)p + \varepsilon - (1+i)\alpha_n & \text{if } \sqrt{2}c < |\alpha'_n - p|, \end{cases}$$

and then put $\rho_{n+1} = \rho_n \sigma_{n+1}$.

(39) To see that ρ_{n+1} produced in (38) is a next desired convergent of U, we claim that, if ξ is a primitive element of U_n such that $|\xi| \le 1$ and $|\xi'| < |\sigma'_{n+1}|$, then $\xi \cong 1$ or $\xi \cong \rho_n^{-1}\rho_{n-1}$ (only if $n > 1$ and $|\rho_{n-1}| = |\rho_n|$). In fact, let ξ be such an element, say $\xi = y\alpha_n - x$. If $|y| \ge 2$, then there is $\beta = \alpha_n - q \in U_n$ such that $|\beta| < 1$ and $|\beta'| < |\xi'|$ by (34). Since $|\xi'| < |\sigma'_{n+1}| \le |\alpha'_n - p|$, this contradicts the choice of p in (38). Thus $|y| < 2$, and we may assume that $\xi = \alpha_n - x$ or $\xi = (1+i)\alpha_n - x$. First suppose that $\xi = \alpha_n - x$. If $|\xi| < 1$, then $|\xi'| \ge |\alpha'_n - p|$ by the choice of p. But since $|\alpha'_n - p| \ge |\sigma'_{n+1}|$, this is impossible. Thus $|\xi| = 1$. Suppose $\xi \not\cong 1$. If $n = 1$ or $n > 1$ and $|\rho_{n-1}| > |\rho_n|$, then since $|\xi'| = 1$, $\xi \cong \alpha_n - p$ by the choice of p and $\sigma_{n+1} = \alpha_n - p$. But since $|\xi'| < |\sigma'_{n+1}|$, this is impossible. Thus $n > 1$ and $|\rho_{n-1}| = |\rho_n|$. Then since $|\rho_n^{-1}\rho_{n-1}| = |\xi| = 1$ and $\rho_n^{-1}\rho_{n-1} \not\cong 1$, $\xi \cong \rho_n^{-1}\rho_{n-1}$ by (35). On the other hand, if $\xi = (1+i)\alpha_n - x$, then it contradicts the choice of ε in (38). This proves the claim.

(40) Let ξ be any element of U_n such that $|\xi| < |\sigma_{n+1}|$ and $|\xi'| < |\sigma'_{n+1}|$. If $\xi \ne 0$, then we may assume that ξ is primitive. Since $|\sigma_{n+1}| \le 1$, $|\xi| = 1$ by (39), which is absurd. Thus $\xi = 0$ and σ_{n+1} is a convergent of U_n and hence ρ_{n+1} is a convergent of U. Clearly, $|\rho_n| \ge |\rho_{n+1}|$. Since $1 \not\cong \sigma_{n+1}$, $\rho_n \not\cong \rho_{n+1}$. If $n > 1$ and $|\rho_{n-1}| = |\rho_n|$, then $1 > |\sigma_{n+1}|$ and $|\rho_n| > |\rho_{n+1}|$. Let ρ be a convergent of U such that $|\rho_i| \ge |\rho| > |\rho_j|$, $0 < i < j \le n+1$. To see $\rho \cong \rho_k$ for some k, $\max\{1, i-1\} \le k < j$, we may assume that $|\rho_n| \ge |\rho| > |\rho_{n+1}|$. Then $\xi = \rho_n^{-1}\rho$ is a convergent of U_n such that $1 \ge |\xi| > |\sigma_{n+1}|$. Since ξ is a convergent, $|\xi'| < |\sigma'_{n+1}|$. Thus by (39), $\xi \cong 1$ or $\xi \cong \rho_n^{-1}\rho_{n-1}$, and hence $\rho \cong \rho_n$ or $\rho \cong \rho_{n-1}$. This completes the proof that ρ_{n+1} is a next desired convergent of U.

(41) Put

$$Q_{n+1} = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p & r \\ 1+i & 1 \end{pmatrix}$$

according as $\sigma_{n+1} = p - \alpha_n$ or $p - (1+i)\alpha_n$, where $r = (p-1)/(1+i)$. Note that in the second case, since $(p, 1+i) = 1$, $1+i$ divides $p-1$ and $r \in \mathcal{O}$. In either case, $\det Q_{n+1} = 1$. (Q_{n+1} is rarely of the second type.) Put

$$A_{n+1} = A_n Q_{n+1} \quad \text{and} \quad \alpha_{n+1} = Q_{n+1}^{-1}\alpha_n = A_{n+1}^{-1}\alpha.$$

Since $Q_{n+1}^{-1}\binom{\alpha_n}{1} = \sigma_{n+1}\binom{\alpha_{n+1}}{1}$,

$$A_{n+1}^{-1}\begin{pmatrix}\alpha \\ 1\end{pmatrix} = Q_{n+1}^{-1}A_n^{-1}\begin{pmatrix}\alpha \\ 1\end{pmatrix} = \rho_n Q_{n+1}^{-1}\begin{pmatrix}\alpha_n \\ 1\end{pmatrix} = \rho_{n+1}\begin{pmatrix}\alpha_{n+1} \\ 1\end{pmatrix},$$

and hence $\rho_{n+1} = a_{n+1} - b_{n+1}\alpha$.

(42) We now have a way to generate a sequence of convergents $\rho_1, \rho_2, \rho_3, \ldots$ of $U = \langle \alpha, 1 \rangle$ satisfying the conditions stated in (36). Moreover, corresponding to these convergents, we have matrices $Q_1, Q_2, Q_3, \ldots \in \mathrm{GL}_2(\mathcal{O})$ of determinant 1 such that, for each $n > 0$, if

$$A_n = \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix} = Q_1 \cdots Q_n,$$

then $\rho_n = a_n - b_n\alpha$ so that, with $\alpha_n = A_n^{-1}\alpha$, $A_n\binom{\alpha_n}{1} = \rho_n^{-1}\binom{\alpha}{1}$ and $U_n = \langle \alpha_n, 1 \rangle = \rho_n^{-1}U \in \mathcal{D}(U)$.

(43) Since $\mathcal{D}(U)$ is finite by (28), $U_{l+1} = U_k$ for some $l \geq k > 0$. Let l be the least such integer. Then we claim that $k = 1$. In fact, put $\lambda = \rho_k\rho_{l+1}^{-1}$. Since $\rho_k^{-1}U = U_k = U_{l+1} = \rho_{l+1}^{-1}U$, $U = \lambda U$ and $\lambda \in \mathcal{O}_U^\times$. Since $|\rho_k| \geq |\rho_{l+1}|$, $|\lambda| \geq 1$. Since $\lambda \in \mathcal{O}_K$, if $|\lambda| = 1$, then λ is a root of unity and $\rho_k \cong \rho_{l+1}$. Thus $|\lambda| > 1$. Given a convergent ρ of U, choose $n \in \mathbf{Z}$ such that $|\rho_k|\,|\lambda|^n \geq |\rho| > |\rho_k|\,|\lambda|^{n-1}$. Then

$$|\rho_k| \geq |\rho\lambda^{-n}| > |\rho_k\lambda^{-1}| = |\rho_{l+1}|.$$

Thus $\sigma = \rho\lambda^{-n}$ is a convergent of U such that $|\rho_k| \geq |\sigma| > |\rho_{l+1}|$, and hence $\rho = \sigma \cong \rho_n$ for some n, $\max\{1, k-1\} \leq n \leq l$. In particular, taking $\rho = \rho_1$, we get that $\rho_1 = \rho_n$, and hence $U_1 = U_n$ for some $n \leq l$. Thus $n = k = 1$ by the choice of l.

(44) Let l be the least integer > 0 such that $U_{l+1} = U_1$, or equivalently $\rho_{l+1} \simeq \rho_1$ or $\alpha_{l+1} \equiv \alpha_1$. In choosing a convergent σ_{l+2} of U_{l+1} to get ρ_{l+2}, since $U_{l+1} = U_1$, it would be nice if we can choose $\sigma_{l+2} = \sigma_2$ so that the sequence $\rho_1, \rho_2, \rho_3, \ldots$ looks like

$$\rho_1, \ldots, \rho_l, \lambda\rho_1, \ldots, \lambda\rho_l, \lambda^2\rho_1, \ldots,$$

where $\lambda = \rho_{l+1}\rho_1^{-1} \in \mathcal{O}_U^\times$. If $|\rho_l| > |p_{l+1}|$ or $1 > |\sigma_2|$, then we can choose $\sigma_{l+2} = \sigma_2$. But suppose $|\rho_l| = |\rho_{l+1}|$ and $1 = |\sigma_2|$. Then $\sigma_2 \cong \rho_{l+1}^{-1}\rho_l$ by (35) and we cannot choose σ_2 as σ_{l+2}. This anomaly can be easily remedied by skipping ρ_l. This amounts to taking $\sigma_l\sigma_{l+1}$ as σ_l, Q_lQ_{l+1} as Q_l and $l-1$ as l. If this is done, then we can take σ_2 as σ_{l+2}. This situation is well illustrated by Example 3 in the Appendix. The least integer $l > 0$ such that $U_{l+1} = U_1$ (after the adjustment above if applicable) is called the *period* of U or of α. From the discussion in (43), we get the following theorem.

(45) THEOREM. *If l is the period of U, then $\{\rho_1, \ldots, \rho_l\}$ is a complete set of representatives of the equivalence classes in $\mathcal{C}(U)$ or equivalently*

$$\mathcal{D}(U) = \{U_1, \ldots, U_l\}, \qquad U_n = \langle \alpha_n, 1 \rangle.$$

(46) THEOREM. *If l is the period of U, then $\lambda_0 = \rho_{l+1}\rho_1^{-1}$ is a fundamental unit of $\mathcal{O}_U^\times$, i.e., every $\lambda \in \mathcal{O}_U^\times$ is uniquely of the form $\lambda = \lambda_0^n\varsigma$, where $n \in \mathbf{Z}$ and ς is a root of unity.*

PROOF. Given $\lambda \in \mathcal{O}_U^\times$, choose $n \in \mathbf{Z}$ such that

$$|\rho_1| \geq |\rho_1\lambda\lambda_0^{-n}| > |\rho_1|\,|\lambda|^{-1} = |\rho_{l+1}|.$$

Since $\rho_1\lambda\lambda_0^{-1} \simeq \rho_1$, $\rho_1\lambda\lambda_0^{-n} \cong \rho_1$ and $\lambda\lambda_0^{-n} = \varsigma$ is a root of unity. The uniqueness is clear.

(47) Here is a summary of the procedure for deciding if $A \sim B$ for given A and $B \in M(f)$. First compute $\alpha = \phi(A)$ and α_n as in (33) and (42) until we get $\alpha_{l+1} \equiv \alpha_1$ for the first time. Next compute $\beta = \phi(B)$ and β_1 for β as in (33). Then $A \sim B$ iff $\beta_1 \equiv \alpha_n$ for some n, $1 \le n \le l$.

(48) Given A and $B \in M(f)$, suppose that $A \sim B$ so that $\beta_1 \equiv \alpha_n$, $1 \le n \le l$, as in (47), say $\beta_1 = \varepsilon\alpha_n + c$, $\varepsilon \in \mathcal{O}^\times$, $c \in \mathcal{O}$. Compute A_n for $\alpha = \phi(A)$ and B_1 $(= Q_1)$ for $\beta = \phi(B)$ and put

$$R_1 = B_1 \begin{pmatrix} \varepsilon & c \\ 0 & 1 \end{pmatrix} A_n^{-1}.$$

Then $R_1 \in \mathrm{GL}_2(\mathcal{O})$ and $R_1\alpha = \beta$, and hence $R_1 A R_1^{-1} = B$.

(49) Given A, put $Z(A) = \{R \in \mathrm{GL}_2(\mathcal{O}) | RA = AR\}$, the centralizer of A in $\mathrm{GL}_2(\mathcal{O})$. $Z(A)$ is a subgroup of $\mathrm{GL}_2(\mathcal{O})$. If $R_1 A R_1^{-1} = B$, then the coset $R_1 Z(A)$ consists of those $R \in \mathrm{GL}_2(\mathcal{O})$ such that $RAR^{-1} = B$.

(50) If $\alpha = \phi(A)$ and $U = \langle \alpha, 1\rangle$, then $Z(A)$ is canonically isomorphic to $\mathcal{O}_U^\times$.

PROOF. Let $R \in Z(A)$. Since $R\alpha = \alpha$, $R\binom{\alpha}{1} = \lambda\binom{\alpha}{1}$ for some $\lambda \in K^\times$. Then $U = \langle \alpha, 1\rangle = \langle \lambda\alpha, \lambda\rangle = \lambda U$ and hence $\lambda \in \mathcal{O}_U^\times$. This defines a map $R \mapsto \lambda\colon Z(A) \to \mathcal{O}_U^\times$, and it is clear that it is a homomorphism. Suppose $\lambda = 1$ for the image λ of $R \in Z(A)$. Then $R\left(\begin{smallmatrix} \alpha & \alpha' \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \alpha' \\ 1 & 1 \end{smallmatrix}\right)$, and hence $R = I$. Thus the map is injective. Let $\lambda \in \mathcal{O}_U^\times$. Then $U = \lambda U = \langle \lambda\alpha, \lambda\rangle$, and hence there is an $R \in \mathrm{GL}_2(\mathcal{O})$ such that $R\binom{\alpha}{1} = \lambda\binom{\alpha}{1}$. Then $R\alpha = \alpha$ and $R \in Z(A)$. Thus the map $R \mapsto \lambda$ is onto $\mathcal{O}_U^\times$.

(51) Let l be the period of $\alpha = \phi(A)$, say $\alpha_{l+1} = \varepsilon\alpha_1 + c$, $\varepsilon \in \mathcal{O}^\times$, $c \in \mathcal{O}$. Put $R_0 = A_{l+1}\left(\begin{smallmatrix} \varepsilon & c \\ 0 & 1 \end{smallmatrix}\right)A_1^{-1}$. Then $R_0 \in \mathrm{GL}_2(\mathcal{O})$ and

$$R_0\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \rho_1^{-1} A_{l+1} \begin{pmatrix} \varepsilon & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix} = \rho_1^{-1} A_{l+1} \begin{pmatrix} \alpha_{l+1} \\ 1 \end{pmatrix} = \rho_1^{-1}\rho_{l+1}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \lambda_0 \begin{pmatrix} \alpha \\ 1 \end{pmatrix}.$$

Since λ_0, together with a root of unity ς, generates $\mathcal{O}_U^\times$ by (46), in view of (50), R_0 together with an element of order 4, 8 or 12 corresponding to ς, generates $Z(A)$.

(52) As a final remark, let us apply our method to find a fundamental unit of $\mathcal{O}_K$, $K = F(\sqrt{\Delta})$, where $\Delta \in \mathcal{O}$, $\Delta \ne 0, \pm 1$, Δ is square-free in $\mathcal{O}$. Put $\pi = 1 - i$ and

$$\alpha = \begin{cases} \sqrt{\Delta} & \text{if } \Delta \equiv 0 \pmod{\pi} \text{ or } \Delta \equiv \pm i \pmod 2, \\ (1+\sqrt{\Delta})/\pi & \text{if } \Delta \equiv \pm 1 + 2i \pmod 4, \\ (1+\sqrt{\Delta})/2 & \text{if } \Delta \equiv 1 \pmod 4, \\ (1+\sqrt{-\Delta})/2 & \text{if } \Delta \equiv -1 \pmod 4. \end{cases}$$

Then $\mathcal{O}_K = \langle \alpha, 1\rangle$. The proof of this is a straightforward exercise and is left to the reader. Clearly $\mathcal{O}_K$ is the coefficient ring of the module $\mathcal{O}_K$. Thus by finding the convergents of $\mathcal{O}_K$ we get a fundamental unit of $\mathcal{O}_K$ via (46).

Appendix. Reducible case.

(1) We shall summarize the results for the case when the characteristic polynomial f is reducible over F. Since the proofs are straightforward, we shall omit them. Put $f(t) = (t - e_1)(t - e_2)$, where e_1 and $e_2 \in \mathcal{O}$.

(2) Given $A \in M(f)$, we can find $R \in \mathrm{GL}_2(\mathcal{O})$ such that

$$RAR^{-1} = \begin{pmatrix} e_1 & a \\ 0 & e_2 \end{pmatrix},$$

where a is in the first quadrant (including the real axis but not the imaginary axis).

(3) Suppose $e_1 = e_2 = e$. If a and b are in the first quadrant and $\left(\begin{smallmatrix} e & a \\ 0 & e \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} e & b \\ 0 & e \end{smallmatrix}\right)$, then $a = b$.

(4) Let $A = \left(\begin{smallmatrix} e & a \\ 0 & e \end{smallmatrix}\right)$. If $a = 0$, then $Z(A) = \mathrm{GL}_2(\mathcal{O})$. If $a \neq 0$, then $Z(A)$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$.

(5) Assume $e_1 \neq e_2$. Put $e = e_1 - e_2$. Given $a \in \mathcal{O}$, we can find $R \in \mathrm{GL}_2(\mathcal{O})$ such that

$$R \begin{pmatrix} e_1 & a \\ 0 & e_2 \end{pmatrix} R^{-1} = \begin{pmatrix} e_1 & r \\ 0 & e_2 \end{pmatrix},$$

where (i) $r = 0$, (ii) $r/e = (1+i)/2$ or (iii) $0 < \mathrm{Re}(r/e) \leq \frac{1}{2}$ and $0 \leq \mathrm{Im}(r/e) < \frac{1}{2}$.

(6) If a/e and b/e are in the quarter square in the sense of (5) for r/e and $\left(\begin{smallmatrix} e_1 & a \\ 0 & e_2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} e_1 & b \\ 0 & b_2 \end{smallmatrix}\right)$, then $a = b$.

(7) Let a/e be as in (6) and $A = \left(\begin{smallmatrix} e_1 & a \\ 0 & e_2 \end{smallmatrix}\right)$. Then $Z(A)$ is generated by

(i) $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ if $a/e = 0$,

(ii) $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ if $a/e = 1/2$,

(iii) $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & i \end{smallmatrix}\right)$ if $a/e = (1+i)/2$,

(iv) $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$ otherwise.

EXAMPLE 1.

$$A = \begin{pmatrix} 16+33i & 17+67i \\ 11+14i & 15+30i \end{pmatrix}, \qquad B = \begin{pmatrix} 26+61i & 1+51i \\ 7 & 5+2i \end{pmatrix}.$$

The characteristic polynomial of A and B is $f(t) = t^2 - (31+63i)t + 1$ and its discriminant is $\Delta = -3012 + 3906i$.

$$\alpha = \phi(A) = \frac{1+3i+\sqrt{\Delta}}{2(11+14i)}, \qquad \beta = \phi(B) = \frac{21+59i+\sqrt{\Delta}}{2(7)}.$$

Computing Q_n and $\alpha_n = Q_n^{-1}\alpha_{n-1}$ for α, we get

$$\begin{aligned}
Q_1 &= I, & \alpha_1 &= \alpha, \\
Q_2 &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, & \alpha_2 &= \frac{43+53i+\sqrt{\Delta}}{2(25-17i)}, \\
Q_3 &= \begin{pmatrix} 2i & -1 \\ 1 & 0 \end{pmatrix}, & \alpha_3 &= \frac{25+47i+\sqrt{\Delta}}{2(17-4i)}, \\
Q_4 &= \begin{pmatrix} 3i & -1 \\ 1 & 0 \end{pmatrix}, & \alpha_4 &= \frac{-1+55i+\sqrt{\Delta}}{2(13-56i)},
\end{aligned}$$

$$Q_5 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_5 = \frac{-25+57i+\sqrt{\Delta}}{2(29-5i)},$$
$$Q_6 = \begin{pmatrix} -1+2i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_6 = \frac{-13+69i+\sqrt{\Delta}}{2(-5-20i)},$$
$$Q_7 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_7 = \frac{23+31i+\sqrt{\Delta}}{2(11+14i)} \equiv \alpha_1.$$

On the other hand, $\beta_1 = \beta$ and this is not $\equiv$ to any α_n. Thus $A \not\sim B$. Note that computation gives

$$Q_2 = \begin{pmatrix} 4+8i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \beta_2 = \frac{35+53i+\sqrt{\Delta}}{2(51-7i)}$$

for β and the next convergent of $\langle \beta_2, 1\rangle$ after 1 is $(1+i)\beta_2 - i$ (cf. (38) and (41)).

EXAMPLE 2.

$$A = \begin{pmatrix} 16+33i & 17+67i \\ 11+14i & 15+30i \end{pmatrix}, \qquad B = \begin{pmatrix} 72+85i & -5-29i \\ 176-7i & -41-22i \end{pmatrix}.$$

This A is the same as in Example 1 and the characteristic polynomial of B is the same as that of A.

$$\beta = \phi(B) = \frac{113+107i+\sqrt{\Delta}}{2(176-7i)}, \quad B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \beta_1 = \frac{-113-107i+\sqrt{\Delta}}{2(5+29i)}$$

for β, and we recognize that $\beta_1 \equiv \alpha_5$, in fact, $\beta = -i\alpha_5 + (-3+i)$. Thus $A \sim B$. Now compute (cf. (42))

$$A_5 = Q_1Q_2Q_3Q_4Q_5 = \begin{pmatrix} 15-i & 14+3i \\ 7-2i & 7 \end{pmatrix},$$
$$R_1 = B_1 \begin{pmatrix} -i & -3+i \\ 0 & 1 \end{pmatrix} A_5^{-1} = \begin{pmatrix} 7-2i & -15+i \\ 19-20i & -47+32i \end{pmatrix}.$$

We have $R_1AR_1^{-1} = B$. Noting $\alpha_7 = \alpha_1 + 1$, we compute (cf. (51))

$$A_7 = \begin{pmatrix} -16-33i & -1-34i \\ -11-14i & -4-16i \end{pmatrix},$$
$$R_0 = A_7 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A_1^{-1} = \begin{pmatrix} -16-33i & -17-67i \\ -11-14i & -15-30i \end{pmatrix} = -A.$$

Thus $Z(A)$ is generated by A and iI and we get all $R \in \mathrm{GL}_2(\mathcal{O})$ such that $RAR^{-1} = B$. In view of (50), the eigenvalue of A, $\lambda = (31+63i+\sqrt{\Delta})/2$, is a fundamental unit of $\mathcal{O}_U$, where $U = \langle \alpha, 1\rangle$.

EXAMPLE 3.

$$A = \begin{pmatrix} 1+4i & -5i \\ 2+4i & -3-i \end{pmatrix},$$
$$f(t) = t^2 - (-2+3i)t + (-19-3i), \qquad \Delta = 71,$$

$$Q_1 = I, \quad \alpha_1 = \alpha = \frac{4+5i+\sqrt{\Delta}}{2(2+4i)}, \quad A_1 = I.$$

$\sigma = \alpha_1 - 1$ is a convergent of U_1 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. Take $\sigma_2 = \sigma$ (cf. (38)).

$$Q_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \frac{3i+\sqrt{\Delta}}{2(-2+4i)}, \quad A_2 = Q_2.$$

$\sigma = \alpha_2$ is a convergent of U_2 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. But since $\sigma_2^{-1} = -\sigma$, $\sigma_3 \neq \sigma$.

$$Q_3 = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \frac{8+i+\sqrt{\Delta}}{2}, \quad A_3 = \begin{pmatrix} -1-i & -1 \\ -i & -1 \end{pmatrix},$$
$$Q_4 = \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_4 = \frac{8-i+\sqrt{\Delta}}{2(-2-4i)}, \quad A_4 = \begin{pmatrix} -9-8i & 1+i \\ -1-8i & 1 \end{pmatrix}.$$

$\sigma = \alpha_4 - i$ is a convergent of U_4 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. Take $\sigma_5 = \sigma$.

$$Q_5 = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_5 = \frac{-3i+\sqrt{\Delta}}{2(2-4i)}, \quad A_5 = \begin{pmatrix} 9-8i & 9+8i \\ 8 & 1+8i \end{pmatrix}.$$

$\sigma = \alpha_5$ is a convergent of U_5 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. But since $\sigma_5^{-1} = -\sigma$, $\sigma_6 \neq \sigma$,

$$Q_6 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_6 = \frac{4-5i+\sqrt{\Delta}}{2(-5i)}, \quad A_6 = \begin{pmatrix} 18 & -9+8i \\ 9+8i & -8 \end{pmatrix},$$
$$Q_7 = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_7 = \frac{6+5i+\sqrt{\Delta}}{2(-3-3i)}, \quad A_7 = \begin{pmatrix} -9+26i & -18 \\ -16+9i & -9-8i \end{pmatrix}.$$

$\sigma = \alpha_7 + 1$ is a convergent of U_7 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. Take $\sigma_8 = \sigma$.

$$Q_8 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_8 = \frac{i+\sqrt{\Delta}}{2(3-3i)}, \quad A_8 = \begin{pmatrix} -9-26i & 9-26i \\ 7-17i & 16-9i \end{pmatrix}.$$

$\sigma = \alpha_8$ is a convergent of U_8 such that $|\sigma| = 1$ and $\sigma \not\cong 1$. But since $\sigma_8^{-1} = -\sigma$, $\sigma_9 \neq \sigma$.

$$Q_9 = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_9 = \frac{6+5i+\sqrt{\Delta}}{2(-5)}, \quad A_9 = \begin{pmatrix} 35-35i & 9+26i \\ 33-2i & -7+17i \end{pmatrix},$$
$$Q_{10} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_{10} = \frac{4-5i+\sqrt{\Delta}}{2(4+2i)}, \quad A_{10} = \begin{pmatrix} -26+61i & -35+35i \\ -40+19i & -33+2i \end{pmatrix}.$$

$\sigma = \alpha_{10} + i$ is a convergent of U_{10} such that $|\sigma| = 1$ and $\sigma \not\cong 1$. Take $\sigma_{11} = \sigma$.

$$Q_{11} = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_{11} = \frac{-3i+\sqrt{\Delta}}{2(-4+2i)}, \quad A_{11} = \begin{pmatrix} 26+61i & 26-61i \\ -14+42i & 40-19i \end{pmatrix}.$$

We note that $\alpha_{11} \equiv \alpha_1$; $\alpha_{11} = -i\alpha_1 + i$. Since $|\sigma_{11}| = 1$ and $|\sigma_2| = 1$, $\sigma_2 \cong \sigma_{11}^{-1}$ (cf. (44)). Thus we take $\sigma_{10}\sigma_{11}$ as σ_{10} and take $Q_{10}Q_{11}$ as Q_{10}:

$$Q_{10} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1+i & 1 \\ -1 & -1 \end{pmatrix}, \qquad \alpha_{10} = \frac{-3i+\sqrt{\Delta}}{2(-4+2i)},$$

$$A_{10} = \begin{pmatrix} 26+61i & 26-61i \\ -14+42i & 40-19i \end{pmatrix}.$$

Since $\alpha_{10} = -i\alpha_1 + i$,

$$R_0 = A_{10}\begin{pmatrix} -i & i \\ 0 & 1 \end{pmatrix} A_1^{-1} = \begin{pmatrix} 61-26i & -35-35i \\ 42+14i & -2-33i \end{pmatrix},$$

and $Z(A)$ is generated by R_0 and iI.

EXAMPLE 4. $\Delta = 71$, $K = F(\sqrt{71})$. Since $71 \equiv -1 \pmod 4$, with $\alpha = (1+\sqrt{-71})/2$, $\mathcal{O}_K = \langle \alpha, 1\rangle$ (cf. (52)). Compute Q_n, α_n and A_n. (In this computation, we encounter convergents σ of some U_n such that $|\sigma| = 1$ and $\sigma \not\cong 1$.)

$$Q_1 = I, \qquad \alpha_1 = \alpha, \qquad A_1 = I,$$

$$Q_2 = \begin{pmatrix} 4i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \frac{-1+8i+\sqrt{-71}}{2(2-4i)}, \qquad A_2 = Q_2,$$

$$Q_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_3 = \frac{-3+\sqrt{-71}}{2(2+4i)}, \qquad A_3 = \begin{pmatrix} -1-4i & -4i \\ -1 & -1 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_4 = \frac{-5+4i+\sqrt{-71}}{2(-5i)}, \qquad A_4 = \begin{pmatrix} 4-5i & 1+4i \\ -1-i & 1 \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_5 = \frac{5+6i+\sqrt{-71}}{2(-3+3i)}, \qquad A_5 = \begin{pmatrix} -3+9i & -4+5i \\ 2+i & 1+i \end{pmatrix},$$

$$Q_6 = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_6 = \frac{1+\sqrt{-71}}{2(-3-3i)}, \qquad A_6 = \begin{pmatrix} 5+8i & 3-9i \\ 2-i & -2-i \end{pmatrix},$$

$$Q_7 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_7 = \frac{5+6i+\sqrt{-71}}{2(-5)}, \qquad A_7 = \begin{pmatrix} -2-17i & -5-8i \\ -4 & -2+i \end{pmatrix},$$

$$Q_8 = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_8 = \frac{-5+4i+\sqrt{-71}}{2(-4+2i)}, \qquad A_8 = \begin{pmatrix} -22-6i & 2+17i \\ -2+5i & 4 \end{pmatrix},$$

$$Q_9 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_9 = \frac{-3+\sqrt{-71}}{2(-4-2i)}, \qquad A_9 = \begin{pmatrix} -20+11i & 22+6i \\ 2+5i & 2-5i \end{pmatrix},$$

$$Q_{10} = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_{10} = \frac{-1+8i+\sqrt{-71}}{2(i)}, \quad A_{10} = \begin{pmatrix} 33+26i & 20-11i \\ 7-7i & -2-5i \end{pmatrix},$$

$\alpha_{10} \equiv \alpha_1$; $\alpha_{10} = -i\alpha_1 + (4+i)$. Since $\rho_1 = 1$, $\rho_{10} = (33+26i) - (7-7i)\alpha$ is a fundamental unit of $\mathcal{O}_K$ (cf. (46)).

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