

## VECTOR BUNDLES AND PROJECTIVE MODULES

BY

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ABSTRACT. Serre and Swan showed that the category of vector bundles over a compact space  $X$  is equivalent to the category of finitely generated projective modules over the ring of continuous functions on  $X$ . In this paper, titled after the famous paper by Swan, this result is extended to an arbitrary topological space  $X$ . Also the well-known homotopy classification of the vector bundles over compact  $X$  up to isomorphism is extended to arbitrary  $X$ . It is shown that the  $K_0$ -functor and the Witt group of the ring of continuous functions on  $X$  coincide, and they are homotopy-type invariants of  $X$ .

**1. Statement of results.** Let  $F$  be either the real numbers  $R$ , the complex numbers  $C$ , or the quaternions  $H$ . For a topological space  $X$ , let  $F^X$  denote the ring of all continuous functions  $X \rightarrow F$ .

When  $X$  is compact Hausdorff, it is well known (Swan [5]) that the category  $\mathcal{P}(F^X)$  of finitely generated projective  $F^X$ -modules is equivalent to the category of  $F$ -vector bundles over  $X$ . Later Goodearl [2] observed that the equivalence holds in the more general case of paracompact Hausdorff  $X$  if we restrict ourselves to the bundles of finite type. This restriction excludes vector bundles of unbounded dimension which cannot come from  $\mathcal{P}(F^X)$ .

The first goal of this paper is to extend this result (with an appropriate definition of finite type) to an arbitrary topological space  $X$ .

**DEFINITION.** A bundle over  $X$  (see [5]) is of *finite type* if there is a finite partition  $S$  of 1 on  $X$  (i.e. a finite set  $S$  of nonnegative continuous functions on  $X$  whose sum is 1) such that the restriction of the bundle to the set  $\{x \in X : f(x) \neq 0\}$  is trivial for each  $f$  in  $S$ .

For example, when  $X$  is compact Hausdorff, every bundle over  $X$  is of finite type. For a normal space  $X$  our definition of finite type is equivalent to the following definition [2]: there is a finite open covering  $T$  of  $X$  such that the restriction of the bundle to each  $U \in T$  is trivial.

**THEOREM 1.** *The category  $\mathcal{P}(F^X)$  is equivalent to the category of  $F$ -vector bundles of finite type over  $X$ .*

Our second theorem concerns the classification problem for the bundles of finite type over a topological space  $X$ , i.e. the problem of identification of the isomorphism classes in  $\mathcal{P}(F^X)$  with homotopy classes. Let  $G(F^n)$  be the set of all subspaces of the  $n$ -dimensional vector space  $F^n$  over  $F$ . As usual,  $G(F^n)$  is endowed with

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the topology of the disjoint union of the Grassmann manifolds  $G_m(F^n)$ , where  $0 \leq m \leq n$ . The topology on  $G(F^n)$  can be given explicitly by identifying  $G(F^n)$  with the subset  $Y_1 = \{p = p^2 = p^* \in M_n F\}$  or  $Y_2 = \{a \in M_n F : a^2 = aa^* = 1_n\}$  of the matrix ring  $M_n F$  (which has the topology of the direct product of  $n^2$  copies of  $F$ ). The following 2 subsets of  $M_n F$  are homotopically equivalent to  $G(F^n)$ , so they can replace  $G(F^n)$  in Theorem 2 below:  $Y_3 = \{e = e^2 \in M_n F\}$ ,  $Y_4 = \{a = a^* \in \mathrm{GL}_n F\}$ . The inclusion  $F^n \subset F^{n+1}$  induces the inclusion  $G(F^n) \subset G(F^{n+1})$  which corresponds to the map  $b \mapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$  on the sets  $Y_1$  and  $Y_3$  and to the map  $b \mapsto \begin{pmatrix} b & 0 \\ 0 & -1 \end{pmatrix}$  on the sets  $Y_2$  and  $Y_4$ .

**THEOREM 2.** *There is a bijection between any two of the following five sets:*

- (1) *the isomorphism classes of  $F$ -vector bundles of finite type over  $X$ ;*
- (2) *the isomorphism classes in  $\mathcal{P}(F^X)$ ;*
- (3) *the isomorphism classes in  $\mathcal{P}(F_0^X)$ , where  $F_0^X \subset F^X$  is the subring of bounded functions on  $X$ ;*
- (4) *the isomorphism classes of finitely generated projective  $F^X$ -modules with positive definite hermitian forms;*
- (5) *the inductive limit of the homotopy classes of continuous maps from  $X$  to  $G(F^n)$  as  $n \rightarrow \infty$ .*

**COROLLARY.** *The set of isomorphism classes in  $\mathcal{P}(F^X)$  is a homotopy-type invariant of  $X$ . In particular,  $K_0 F^X$ , the Grothendieck group of  $\mathcal{P}(F^X)$ , is a homotopy-type invariant of  $X$ .*

**REMARK.** The rings  $F^X$  and  $F_0^X$  in Theorem 2 can be replaced by dense subrings (see [6]). For example, if  $X$  is a manifold and  $A$  is the ring of smooth functions on  $X$ , then there is a bijection between the isomorphism classes in  $\mathcal{P}(F^X)$  and  $\mathcal{P}(A)$ . In particular,  $K_0 F^X = K_0 A$  is a homotopy-type invariant of  $X$ . Note that the algebraic  $K_0$ -functor does not take into account any topology on rings.

Our third goal is to compute the Witt group of hermitian forms over  $F^X$ . Let  $\mathcal{P}^+(F^X)$  denote the category of hermitian spaces  $(P, \Phi)$  over  $F^X$ , where  $P \in \mathcal{P}(F^X)$  and  $\Phi$  is a nonsingular hermitian form on  $P$  (nonsingular here means that the corresponding map  $P \rightarrow P^*$  is an isomorphism). The morphisms in  $\mathcal{P}^+(F^X)$  preserve scalar products. In particular, they have trivial kernels.

**THEOREM 3.** *The set of isomorphism classes in  $\mathcal{P}^+(F^X)$  is the direct square of the set of isomorphism classes of objects in  $\mathcal{P}(F^X)$ .*

**COROLLARY.** *The Grothendieck group of the category  $\mathcal{P}^+(F^X)$ , i.e. the Witt-Grothendieck group of  $F^X$ , is isomorphic to  $K_0 F^X \times K_0 F^X$ . The Witt group of  $F^X$  is isomorphic to  $K_0 F^X$ .*

Recall that the Witt group is the cokernel of the homomorphism of the Grothendieck groups induced by the hyperbolic functor  $\mathcal{P}(F^X) \rightarrow \mathcal{P}^+(F^X)$ .

## 2. Proof of Theorem 1.

**LEMMA 4.** *Every object  $P$  in  $\mathcal{P}(F^X)$  is isomorphic to the column space of a square hermitian idempotent matrix  $e = e^2 = e^*$  over  $F^X$ .*

**PROOF.** Since  $P$  is isomorphic to a direct summand of the free module  $(F^X)^n$  for some natural number  $n$ , it is isomorphic to the column space (i.e. to the image

$a(F^X)^n$ ) of an  $n$ -by- $n$  idempotent matrix  $a = a^2$  over  $F^X$ . Since  $a = a^2$ , we have  $(a^*)^2 = a^*$  and

$$(1 + (1 - 2a)(1 - 2a^*))a^* = 2aa^* = a(1 + (1 - 2a)(1 - 2a^*)),$$

where 1 stands for the identity matrix  $1_n \in M_n F^X$ . Since  $(1 - 2a)(1 - 2a^*)$  is a hermitian semipositive matrix, there is a hermitian matrix  $g = g^*$  in  $\mathrm{GL}_n F^X$  such that  $g^2 = 1 + (1 - 2a)(1 - 2a^*)$ . We set  $e = gag^{-1}$ . Then  $e^2 = ga^2g^{-1} = gag^{-1} = e$  and  $e^* = g^{-1}a^*g = gag^{-1} = e$ . So  $P$  is isomorphic to the column space of this  $e = e^2 = e^*$ . The lemma is proved.

Every object  $P$  in  $\mathcal{P}(F^X)$  gives an  $F$ -vector bundle  $\Gamma^*(P)$  over  $X$  in the usual way. Namely, the  $F$ -vector space at a point  $x$  of  $X$  is  $P$  evaluated at  $x$ ; i.e.  $P \otimes_v F^X$ , where  $v: F^X \rightarrow F$  is given by  $v(f) = f(x)$  for any continuous function  $f: X \rightarrow F$ . When  $P$  is the column space of a matrix  $e = e^2 \in M_n F^X$ , the fiber at  $x$  is just  $e(x)F^n$ . The local triviality of  $\Gamma^*(P)$  follows from the following lemma.

**LEMMA 5.** *If  $e = e^2 \in M_n F^X$ ,  $x, y \in X$ , and  $g = e(x)e(y) + (1 - e(x))(1 - e(y)) \in \mathrm{GL}_n F$ , then the matrices  $e(x)$  and  $e(y)$  over  $F$  are similar.*

**PROOF.** We have  $e(x)g = ge(y)$ . When  $g \in \mathrm{GL}_n F$ , we conclude that  $g^{-1}e(x)g = e(y)$ . The lemma is proved.

**COROLLARY 6.** *For every object  $P$  in  $\mathcal{P}(F^X)$ , the corresponding  $F$ -vector bundle  $\Gamma^*(P)$  over  $X$  is of finite type.*

**PROOF.** By Lemma 4,  $P$  is isomorphic to the column space of  $e = e^2 = e^* \in M_n F^X$  for some  $n, e$ . The set  $Y_1 = \{p = p^2 = p^* \in M_n F\}$  is compact (note that  $|p| \leq 1$  for each  $p$  in  $Y_1$ ). Therefore there is a finite partition  $S'$  of 1 on  $Y_1$  such that  $|p - q| < 1/3$  whenever  $f'(p)f'(q) \neq 0$  for some  $f'$  in  $S'$ . (Note that  $F^n$  has the usual metric  $|(z_j)|^2 = \sum z_j^* z_j$  which makes it a Banach space, so  $M_n F$  is a Banach algebra with respect to the operator norm.)

The matrix  $e$  above can be considered as a continuous map  $X \rightarrow Y_1$ . So the partition  $S'$  gives a finite partition  $S$  of 1 on  $X$ . We have  $|e(x) - e(y)| < 1/3$  whenever  $x$  and  $y$  are in the same part (i.e.  $f(x)f(y) \neq 0$  for some  $f$  in  $S$ ).

For any part  $U(f) = \{x \in X: f(x) \neq 0\}$ , where  $f \in S$ , and any  $x, y$  in  $U(f)$  we have, as in Lemma 5,  $e(x)g = ge(y)$ , where

$$g = e(x)e(y) + (1 - e(x))(1 - e(y)) = 1 - (e(y) - e(x))(1 - 2e(y)).$$

Since

$$|g - 1| = |(e(y) - e(x))(1 - 2e(y))| \leq |e(y) - e(x)| |1 - 2e(y)| < (1/3) \cdot 3 = 1,$$

we conclude that  $g \in \mathrm{GL}_n F$ . Fixing  $x \in U(f)$ , we have  $e(y) = g^{-1}e(x)g$ , where  $g$  depends continuously on  $y \in U(f)$ . Since  $F$  is a division algebra, the column space of  $e(z)$  is a finite dimensional vector space over  $F$  for each  $z$  in  $X$ . Therefore the column space of the restriction of  $e$  to  $U(f)$  is a finitely generated free  $F^{U(f)}$ -module. So the restriction of the bundle  $\Gamma^*(P)$  to  $U(f)$  is trivial. Thus, the bundle is of finite type. The corollary is proved.

**LEMMA 7.** *For every  $F$ -vector bundle  $\xi$  of finite type over  $X$ , the  $F^X$ -module  $\Gamma(\xi)$  of its global sections is finitely generated and projective (i.e.  $\Gamma(\xi) \in \mathcal{P}(F^X)$ ).*

**PROOF.** Let  $S$  be a finite partition of 1 on  $X$  such that  $\xi$  is trivial over each  $U(f) = \{x \in X: f(x) \neq 0\}$ ,  $f \in S$ . For each  $f$  in  $S$  we pick a free basis

$a_1(f), \dots, a_{n(f)}(f)$  for  $\Gamma(\xi)|_{U(f)}$ . We would like to extend these  $a_j(f)$  to global sections of the bundle. However, in general, this can be done only after a modification of  $a_j(f)$ .

We write  $a_j(f) = \sum c_{i,j}(f, g)a_i(g)$  on  $U(f) \cap U(g)$  for every  $g$  in  $S$ . It is clear that there is a continuous nonnegative function  $f'$  on  $X$  such that  $f'$  and  $f$  have the same zero set and  $c_{i,j}(f, g)f' \rightarrow 0$  on  $U(f) \cap U(g)$  whenever  $f(x) \rightarrow 0$ , for all  $i, j$  and  $g$ . Then

$$b_j(f) = \begin{cases} a_j(f)f' & \text{on } U(f), \\ 0 & \text{elsewhere on } X \end{cases}$$

is a continuous global section of the bundle. Moreover,  $\{b_j(f): 1 \leq j \leq n(f)\}$  is a basis for  $\Gamma(\xi)|_{U(f)}$ .

Let us show now that  $\{b_j(f): 1 \leq j \leq n(f), f \in S\}$  is a generating set for the  $F^X$ -module  $\Gamma(\xi)$ . Let  $s \in \Gamma(\xi)$ . We write  $s|_{U(f)} = \sum c_j(f)b_j(f)$  with  $c_j(f) \in F^{U(f)}$ . There is a modification (depending on  $s$ ) of the partition  $S$  such that the open covering  $\{U(f): f \in S\}$  of  $X$  stays the same, but for the new partition we have that  $c_i(f)f \rightarrow 0$  on  $U(f)$  whenever  $f \rightarrow 0$ . Then  $d_j(f) \in F^X$  for  $d_j(f)$  defined by

$$d_j(f) = \begin{cases} c_j(f)f & \text{on } U(f), \\ 0 & \text{elsewhere on } X. \end{cases}$$

Moreover,

$$s = \sum_{f \in S} \sum_{j=1}^{n(f)} d_j(f)b_j(f).$$

Therefore we have a surjective map from  $(F^X)^N$ , where  $N = \sum_{f \in S} n(f)$  is the number of the generators, onto  $\Gamma(\xi)$ . Using a positive definite hermitian form on  $(F^X)^N$  (say, the standard form  $(a, b) \mapsto \sum a_j^* b_j$  for  $a = (a_j)$ ,  $b = (b_j)$  in  $(F^X)^N$ ), we obtain, as in [5], that the kernel of the homomorphism  $(F^X)^N \rightarrow \Gamma(\xi)$  is a direct summand of  $(F^X)^N$ . So  $\Gamma(\xi)$  is a projective  $F^X$ -module. Thus, Lemma 7 is proved.

Putting together Lemmas 4 and 7, we conclude, as in [5] (see also [3 and 4]) that the functors  $\Gamma$  and  $\Gamma^*$  give an equivalence of the category  $\mathcal{P}(F^X)$  and the category of  $F$ -vector bundles over  $X$  of finite type.

### 3. Proof of Theorem 2.

*Isomorphism of the sets (1) and (2).* This follows from Theorem 1.

*Isomorphism of the sets (2) and (3).* By Lemma 4 every object  $P$  of  $\mathcal{P}(F^X)$  is isomorphic to an object extended from some object of  $\mathcal{P}(F_0^X)$ . Let now  $P$  and  $Q$  be objects of  $\mathcal{P}(F_0^X)$  which are isomorphic over the bigger ring  $F^X$ . We want to prove that  $P$  and  $Q$  are isomorphic in  $\mathcal{P}(F_0^X)$ .

We represent  $P$  and  $Q$  as the column spaces of matrices  $p = p^2 = p^*$  and  $q = q^2 = q^*$  in  $M_n F_0^X$ . Using the stabilization operations  $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$  (which allowed us to take the same  $n$  for  $p$  and  $q$ ), and an isomorphism of  $P$  and  $Q$  over  $F^X$ , we can assume that  $g^{-1}pg = q$  for some  $g \in \mathrm{GL}_n F^X$ .

Let  $g = g'h$  be the polar decomposition of  $g$ , i.e.  $g' = g'^*$  is hermitian,  $h = h^{*-1}$  is orthogonal (unitary),  $g'^2 = gg^*$ , and the centralizer of  $g'$  in  $M_n F^X$  is the same as that of  $gg^*$ , so  $pg' = g'p$ . Then  $h \in \mathrm{GL}_n F_0^X$  is bounded and  $h^{-1}ph = h^*pg = q$ . So  $P$  and  $Q$  are isomorphic over  $F_0^X$ .

*Isomorphism of the sets (2) and (4).* Let us show that every object  $P$  in  $\mathcal{P}(F^X)$  carries a positive definite hermitian form. By Lemma 4,  $P$  is isomorphic to the column space  $e(F^X)^n$  of a matrix  $e = e^2 = e^*$  in  $M_n F^X$ . We define a hermitian positive definite form

$$\Phi: e(F^X)^n \times e(F^X)^n \rightarrow F^X$$

by

$$\Phi((a_i), (b_i)) = \sum a_i^* b_i.$$

Note that  $(1 - e)(F^X)^n$  is the orthogonal complement to  $e(F^X)^n$ .

Let us show now that any two positive definite hermitian forms  $\Phi$  and  $\Psi$  on  $P$  are isomorphic. Following [4, Theorem 8.8], consider  $h = \Phi^{-1}\Psi \in \text{Aut}(P)$  (where  $\Phi$  and  $\Psi$  are regarded as maps  $P \rightarrow P^*$ ). Then  $h$  is selfadjoint and positive with respect to  $\Phi$  (namely,

$$\begin{aligned} \Phi(hu, v) &= (\Phi hu)v = (\Psi u)v = \Psi(u, v) = \Psi(v, u)^* \\ &= (\Phi(hv, u))^* = \Phi(u, hv)^{**} = \Phi(u, hv) \end{aligned}$$

and  $\Phi(hv, v) = \Psi(v, v) > 0$  for  $0 \neq v \in P$ ). Let  $g$  be its selfadjoint positive square root. Then  $\Psi = \Phi h = \Phi g^2 = g^* \Phi g$  (i.e.  $\Psi(u, v) = \Phi(hu, v) = \Phi(g^2 u, v) = \Phi(gu, gv)$ ), so  $\Phi$  and  $\Psi$  are isomorphic.

*Isomorphism of the sets (2) and (5).* We identify  $G(F^n)$  with  $\{p = p^2 = p^* \in M_n F\}$  and  $\text{Hom}(X, G(F^n))$  (i.e. the continuous maps  $X \rightarrow G(F^n)$ ) with  $\{e = e^2 = e^* \in M_n F^X\}$ .

By Lemma 4, every object  $P$  in  $\mathcal{P}(F^X)$  is isomorphic to the column space of such a matrix  $e$ . Let us show that the image of  $e$  in  $\varinjlim \pi(X, G(F^n))$  does not depend on the choice of  $e$  (here  $\pi(X, G(F^n))$  denotes the homotopy classes in  $\text{Hom}(X, G(F^n))$ ).

Let  $a = a^2 = a^*$  in  $M_m F^X$  and  $b = b^2 = b^*$  in  $M_n F^X$  have isomorphic column spaces. Using the stabilization maps  $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$  which do not change images in the inductive limit, we can assume that  $m = n$  and the matrices  $a$  and  $b$  are similar. That is,  $b = gag^{-1}$  with  $g$  in  $\text{GL}_n F^X$ .

Then we have, in  $M_{2n} F^X$ , that

$$\begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

By the Whitehead lemma, the matrix  $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$  is the product of elementary matrices and, hence, homotopic to the identity matrix. So  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$  are homotopic in the set  $\{e = e^2 \in M_{2n} F^X\}$ . This homotopy can be retracted to a homotopy in the subset  $\{e = e^2 = e^* \in M_{2n} F^X\}$  (using, for example, the proof of Lemma 5 with  $X$  replaced by  $X \times \{t: 0 \leq t \leq 1\}$ ). Thus,  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$  are homotopic in  $\text{Hom}(X, G(F^{2n}))$ .

So we have a well-defined map from the isomorphism classes of objects in  $\mathcal{P}(F^X)$  to  $\varinjlim \pi(X, G(F^n))$ . This map is onto, as one can see by assigning to each  $e = e^2 = e^*$  in  $M_n F^X$  its column space, which gives a map from  $\varinjlim(\text{Hom}(X, G(F^n)))$  to  $\mathcal{P}(F^X)$  (up to isomorphism in  $\mathcal{P}(F^X)$ , this map is onto by Lemma 4).

The following lemma completes our proof.

LEMMA 8. *Any two matrices in  $\text{Hom}(X, G(F^n))$  which are homotopic are similar. In particular, their column spaces are isomorphic.*

PROOF. Let  $a, b \in \text{Hom}(X, G(F^n))$  be homotopic. That is, there is a continuous map  $c: X \times \{t: 0 \leq t \leq 1\} \rightarrow G(F^n)$  such that  $c(\cdot, 0) = a$  and  $c(\cdot, 1) = b$ .

Let  $\Pi G(F^n)$  denote the set of continuous maps  $\{t: 0 \leq t \leq 1\} \rightarrow G(F^n)$ . Then  $c$  can be regarded as a continuous map  $X \rightarrow \Pi G(F^n)$ . Note that the metric on  $G(F^n)$  (induced by the norm on  $M_n F$ ) gives a metric on  $\Pi G(F^n)$ .

Since  $\Pi G(F^n)$  is a metric space, it is paracompact. Let  $S'$  be a locally finite partition of 1 on  $\Pi G(F^n)$  such that  $|p - q| < 1/9$  whenever  $p$  and  $q$  are in the same part. The map  $c$  gives a locally finite partition  $S$  of  $X$  such that  $|c(x, t) - c(y, t)| < 1/9$  for all  $t$  whenever  $x$  and  $y$  are in the same part, i.e.  $f(x)f(y) \neq 0$  for some  $f \in S$ .

For each  $f$  in  $S$  we pick a point  $x = x(f)$  in  $U(f) = \{z \in X: f(z) \neq 0\}$  and a positive number  $\varepsilon(f)$  such that  $|c(x, t) - c(x, s)| < 1/9$  whenever  $|t - s| < \varepsilon(t)$ . Then

$$\begin{aligned} |c(z, t) - c(z, s)| &\leq |c(x, t) - c(x, s)| + |c(z, t) - c(x, t)| + |c(z, s) - c(x, s)| \\ &< 1/9 + 1/9 + 1/9 = 1/3 \end{aligned}$$

for every  $z$  in  $U(f)$  whenever  $|t - s| < \varepsilon(f)$ .

We set  $\delta(z) = \sum_{f \in S} \varepsilon(f)f(z)$  for every  $z$  in  $X$ . Then  $\delta$  is a continuous positive function on  $X$  such that  $|c(z, t) - c(z, s)| < 1/3$  whenever  $|t - s| < \delta(z)$ .

For every  $z$  in  $X$  and any integer  $k \geq 0$  we set

$$t_k(z) = \min(1, k\delta(z)), \quad c_k(z) = c(z, t_k(z)).$$

Then  $c_0 = a$ ,  $c_k(z) = b(z)$  for  $k \geq 1/\delta(z)$ , and  $|c_k(z) - c_{k+1}(z)| < 1/3$  for all  $z$  and  $k$ .

We set

$$g_k = c_k c_{k+1} + (1 - c_k)(1 - c_{k+1}) \quad \text{for all } k.$$

Then (see Lemma 4 and its proof)  $c_k g_k = g_k c_{k+1}$ ,  $|1 - g_k| < 1$ ; hence  $g_k \in \text{GL}_n(F^X)$  for all  $k$ . Moreover,  $g_k(z) = 1$  for  $k \geq 1/\delta(z)$ .

Finally, we set  $G(z) = g_0(z)g_1(z)\dots$ . Then  $g \in \text{GL}_n F^X$  and  $ag = gb$ ; hence  $g^{-1}ag = b$ . Lemma 8 is proved.

Theorem 2 is proved.

REMARK. Using that  $G(F^n)$  is homotopically equivalent to  $\{h = h^* \in \text{GL}_n F^X\}$  we obtain that any two homotopic nonsingular hermitian forms over  $F^X$  are isomorphic (when  $X$  is compact, this is well known; see [4]). In [1] the converse is claimed (that every two isomorphic forms are homotopic) when  $F = C$  and  $X$  is a finite CW complex. However, this claim is false, as the following counterexample shows. Let

$$X = S^3 = \left\{ (x_i) \in R^4 : \sum x_k^2 = 1 \right\}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$g = \begin{pmatrix} x_1 + x_2 i & x_4 i - x_3 \\ x_3 + x_4 i & x_1 - x_2 i \end{pmatrix} \in \text{GL}_2 F, \quad b = g^* ag.$$

If  $a$  and  $b$  were homotopic, we would have  $b = h^* ah$  with  $h$  being the product of matrices close to 1; hence  $h \in \text{GL}_1 F^X(E_2 F^X)$ . Then  $gh^{-1} \in U(F^X) \subset$

$\mathrm{GL}_1 F^X(E_2 F^X)$ ; hence  $g \in \mathrm{GL}_1 F^X(E_2 F^X)$ . But homotopically  $g$  is the identity map  $S^3 \rightarrow S^3$  and hence cannot be passed through the embedding  $S^1 \subset S^3$  (which corresponds to the embedding  $\mathrm{GL}_1 C \subset \mathrm{GL}_2 C$ ). Of course,  $a$  and  $b$  are *stably* homotopic.

REMARK. Lemma 8 can be easily extended to an arbitrary Banach algebra  $F$  (rather than  $F = R, C$ , or  $H$ ). Namely, we can assert the following result.

**THEOREM 9.** *Let  $F$  be a Banach algebra with 1. Then for any topological space  $X$  and any  $n$ , every two homotopic maps  $X \rightarrow \{p = p^2 \in M_n F\}$  give similar matrices in  $M_n F^X$ . Therefore there is a bijection between the finitely generated projective  $F^X$ -modules up to isomorphism and  $\varinjlim \pi(X, \{p = p^2 \in M_n F\})$ . In particular, this set of isomorphism classes and, hence,  $K_0 F^X$  are homotopy-type invariants of  $X$ .*

Moreover, checking the properties of  $F$  which were actually used in the proof of Theorem 2, we see that we can replace the condition that  $F$  is a Banach algebra with 1 in Theorem 9 by the following weaker condition:  $F$  is a metrizable topological  $R$ -algebra with 1 such that the map  $z \mapsto z^{-1}$  is defined and continuous in a neighbourhood of 1.

**4. Proof of Theorem 3.** This theorem can be restated as follows. For any nonsingular hermitian form  $\Phi$  on any object  $P$  in  $\mathcal{P}(F^X)$  there is an orthogonal decomposition  $P = P_1 \oplus P_2$  such that the restriction of  $\Phi$  on  $P_1$  (resp.  $P_2$ ) is positive (resp. negative); moreover, such a decomposition is unique up to isomorphism.

For compact  $X$  this statement is Theorem 8.13 of [4]. This theorem is stated in [4] in terms of bundles, so the condition that  $X$  is compact is needed to obtain a positive definite hermitian form. This condition can be replaced by the condition that the bundle is of finite type. Therefore to prove our Theorem 3 we can just repeat the arguments of [4].

REMARK. The referee attracted my attention to the fact that some ideas and computations in this paper can be traced back to the book *Rings of operators* by I. Kaplansky (Benjamin, 1968). In particular, our Lemma 4 is contained in Theorem 26 of the book. Also the referee pointed out the paper *Partitions of unity in the theory of fibrations* by A. Dold (Ann. of Math. **78** (1963), 223–255) which is one of the first studies of fibrations using locally finite partitions of unity.

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