

IRREDUCIBILITY OF MODULI SPACES OF CYCLIC UNRAMIFIED COVERS OF GENUS g CURVES

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ABSTRACT. Let $(C_1, \dots, C_r, G) = (\mathbf{C}, G)$ be an r -tuple consisting of a transitive subgroup G of S_m and r conjugacy classes C_1, \dots, C_r of G . We consider the concept of the moduli space $\mathcal{H}(\mathbf{C}, G)$ of compact Riemann surface covers of the Riemann sphere of *Nielsen class* (\mathbf{C}, G) . The irreducibility of $\mathcal{H}(\mathbf{C}, G)$ is equivalent to the transitivity of a specific permutation representation of the *Hurwitz monodromy group* (§1), but there are few general tools to decide questions about this representation. Theorem 2 gives a class of examples of (\mathbf{C}, G) for which $\mathcal{H}(\mathbf{C}, G)$ is irreducible. As an immediate corollary this gives an elementary proof and generalization of the irreducibility of the moduli space of cyclic unramified covers of genus g curves (for which Deligne and Mumford [DM, Theorem 5.15] applied Teichmüller theory and Dehn's theorem). This contrasts with the examples of (\mathbf{C}, G) in [BFr] for which $\mathcal{H}(\mathbf{C}, G)$ is reducible. These kinds of questions combined with the study of the existence of rational subvarieties of $\mathcal{H}(\mathbf{C}, G)$ have application to the realization of a group G as the Galois group of a regular extension of $\mathbb{Q}(t)$ [Fr3, §4].

1. Introduction to the fundamental moduli spaces. The most well-known moduli spaces of compact Riemann surfaces are the moduli spaces, denoted \mathcal{M}_g , of compact Riemann surfaces of genus $g \geq 1$ (in the case $g = 0$, \mathcal{M}_g can be taken to be a point). Each point of \mathcal{M}_g corresponds to exactly one isomorphism class of surfaces of genus g . Furthermore, \mathcal{M}_g is a complex analytic set (actually, algebraic) with the following key property. Let $\Phi: \mathcal{X} \rightarrow \mathcal{P}$ be a family of compact Riemann surfaces of genus g . Here that will mean that \mathcal{X} and \mathcal{P} are compact analytic sets, that Φ is a complex analytic map, and that for each point $\mathfrak{p} \in \mathcal{P}$ the set $\{x \in \mathcal{X} \mid \Phi(x) = \mathfrak{p}\} = \mathcal{X}_{\mathfrak{p}}$, the *fiber over* \mathfrak{p} , naturally inherits the structure of a compact Riemann surface of genus g . Then the natural map,

$$(1.1) \quad \Phi: \mathcal{P} \rightarrow \mathcal{M}_g,$$

defined by $\mathfrak{p} \rightarrow [\mathcal{X}_{\mathfrak{p}}]$ (the isomorphism class of $\mathcal{X}_{\mathfrak{p}}$) is complex analytic. A succinct story, with references, on the *irreducibility of* \mathcal{M}_g appears in [Fu].

Deligne and Mumford [DM, Theorem 5.15] prove the irreducibility of spaces ${}_n\mathcal{M}_g$, $n \geq 1$, $g \geq 2$, that generalize the classical moduli spaces, \mathcal{C}_n , of elliptic curves with level n structure. The irreducibility of \mathcal{C}_n follows from the identification of it

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with the quotient of the complex upper half plane by the action of

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{(n)} \right\}.$$

In the [DM] generalization, Teichmüller theory [W] and Dehn's theorem allow for a presentation of ${}_n\mathcal{M}_g$ as a quotient of a ball. These heavy tools limit the possibility of immediate generalization. This we give in a framework, considerably more elementary than that of [DM], that follows the classical tradition of [Hu].

For the sake of simplicity, but still allowing for fair comparison with [DM, Theorem 5.15] we generalize (Theorem 3) the proof of the irreducibility of ${}_o\mathcal{C}_{n,g}$, the moduli space of cyclic unramified covers of degree n of genus g curves. This corollary of [DM, Theorem 5.15] generalizes the irreducibility of the curves ${}_o\mathcal{C}_n$ that are classically identified with the quotient of the upper half plane by the group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \mid c \equiv 0 \pmod{(n)} \right\}$. In the special case $g = 4$ and 5 , $n = 2$, this is an essential ingredient of the results of [B] on the number of components of the space of singular theta divisors of dimensions 4 and 5. Following a precise description of the spaces with which we shall deal, this section concludes with a paragraph of exposition on direct and general motivation for such irreducibility results through [Fr2 and Th], connecting them to the classical inverse Galois group problem over \mathbb{Q} .

Riemann's existence theorem allows us to use combinatorial techniques in our analysis of moduli spaces. Each compact Riemann surface X can be presented as a cover $\varphi: X \rightarrow \mathbb{P}^1$ of the projective line. Let z_1, \dots, z_r be a list of the distinct points of \mathbb{P}^1 over which φ is ramified, and let $m(\varphi) = m$ denote the degree of φ . For a given surface X , it can be difficult to describe the possible values of r and m . But, there is a one-one correspondence between the elements of the following two sets [Fr1, §1]:

(1.2) (a) the quotient of $\{\sigma = (\sigma(1), \dots, \sigma(r)) \in (S_m)^r \mid \sigma(1)\sigma(2)\cdots\sigma(r) = 1 \text{ and } \langle \sigma(1), \dots, \sigma(r) \rangle = G(\sigma) \text{ is a transitive subgroup of } S_m\}$ by the relation that equivalences σ and $\gamma^{-1} \cdot \sigma \cdot \gamma = (\gamma^{-1} \cdot \sigma(1) \cdot \gamma, \dots, \gamma^{-1} \cdot \sigma(r) \cdot \gamma)$ for each $\gamma \in S_m$; and

(b) the quotient of $\{\varphi': X' \rightarrow \mathbb{P}^1 \text{ of connected covers of degree } m \text{ with branch locus in } \{z_1, \dots, z_r\}\}$ by the relation that equivalences $\varphi': X' \rightarrow \mathbb{P}^1$ and $\varphi' \circ \psi: X'' \rightarrow \mathbb{P}^1$ for $\psi: X'' \rightarrow X'$ an isomorphism.

Such a correspondence, however, depends on additional data, and cannot be regarded as functional.

Let $(C_1, \dots, C_r, G) = (\mathbf{C}, G)$ be an r -tuple consisting of a transitive subgroup G of S_m and r conjugacy classes C_1, \dots, C_r of G . Denote the set $\{\text{equivalence classes of } \sigma \in (S_m)^r \mid \text{such that } G(\sigma) = G \text{ and there exists } \beta \in S_r \text{ with } \sigma(\beta(i)) \in C_i, i = 1, \dots, r\}$ by $\mathrm{Ni}(\mathbf{C}, G)$, the *Nielsen class of* (\mathbf{C}, G) . We assume, from here on, that (\mathbf{C}, G) is so chosen that $\mathrm{Ni}(\mathbf{C}, G)$ is nonempty.

We now list $r - 1$ operators Q_1, \dots, Q_{r-1} that naturally act as permutations of the elements of $\mathrm{Ni}(\mathbf{C}, G)$ by a right-hand action. Indeed, Q_i maps the equivalence class of $\sigma = (\sigma(1), \dots, \sigma(r))$ to the equivalence class of

$$(1.3) \quad (\sigma)Q_i = (\sigma(1), \dots, \sigma(i-1), \sigma(i) \cdot \sigma(i+1) \cdot \sigma(i)^{-1}, \sigma(i), \dots, \sigma(r)), \\ i = 1, \dots, r-1.$$

Our discussion continues with a brief review from [BFr, pp. 89–95]. Identify \mathbf{P}^r with the quotient of the nonzero polynomials in x of degree at most r ,

$$\left\{ \sum_{j=0}^r a_j \cdot x^j \neq 0 \mid a_j \in \mathbf{C}, j = 0, \dots, r \right\},$$

by the relation that equivalences $\sum_{i=0}^r a_i \cdot x^i$ and $\sum_{i=0}^r a \cdot a_i \cdot x^i$ for $a \in \mathbf{C} - \{0\}$.

Consider the natural map—the *Noether cover*—

$$(1.4) \quad \Phi_r: (\mathbf{P}^1)^r \rightarrow \mathbf{P}^r$$

that maps $(z_1, \dots, z_r) \in (\mathbf{P}^1)^r$ to the equivalence class of $\prod_{j=1}^r (x - z_j)$ with the proviso that the factor $x - z_j$ is replaced by 1 if $z_j = \infty$. Let Δ_r be the subset of $(\mathbf{P}^1)^r$ consisting of points with two or more equal coordinates, and let D_r , the *discriminant locus of the Noether cover*, be the image of Δ_r under Φ_r . For $\mathbf{a}^0 \in \mathbf{P}^r - D_r$, the fundamental group, $\pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$, is the quotient of the free group generated by elements Q_1, \dots, Q_{r-1} by the following list of relations [FaBu]:

$$(1.5) \quad \begin{aligned} (a) \quad & Q_i \cdot Q_j = Q_j \cdot Q_i, \quad |i - j| \geq 2, \quad 1 \leq i, j \leq r - 1; \\ (b) \quad & Q_i \cdot Q_{i+1} \cdot Q_i = Q_{i+1} \cdot Q_i \cdot Q_{i+1}, \quad 1 \leq i \leq r - 1; \\ (c) \quad & Q_1 \cdots Q_{r-2} \cdot (Q_{r-1})^2 \cdot Q_{r-2} \cdots Q_1 = 1. \end{aligned}$$

From (1.5) the action given by (1.3) gives a permutation representation of $\pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ on the set $\text{Ni}(\mathbf{C}, G)$. Let Br_1, \dots, Br_t be the distinct orbits of this action. Covering space theory associates to each Br_i an equivalence class of unramified covers

$$(1.6) \quad \mathcal{H}(Br_i) \rightarrow \mathbf{P}^r - D_r, \quad i = 1, \dots, t.$$

Define the (absolute) Hurwitz space $\mathcal{H}(\mathbf{C}, G)$ of $\text{Ni}(\mathbf{C}, G)$ to be the disjoint union of the spaces $\mathcal{H}(Br_i)$, $i = 1, \dots, t$. In [BFr, p. 104] (or [Fr1, §4] without the use of (1.5)) it is shown that $\mathcal{H}(\mathbf{C}, G)$ is a (coarse) moduli space for covers of Nielsen type $\text{Ni}(\mathbf{C}, G)$ (i.e., covers $\varphi: X \rightarrow \mathbf{P}^1$ for which the σ given by (1.2)(a) is in $\text{Ni}(\mathbf{C}, G)$). Then $\mathcal{H}(\mathbf{C}, G)$ is irreducible if and only if $t = 1$. Denote t by $\text{Hur}(\mathbf{C}, G)$, the *Hurwitz number* of (\mathbf{C}, G) .

Theorem 2 of this paper shows that $\text{Hur}(\mathbf{C}, G) = 1$ in the following case. Let S_m act on $(\mathbf{Z}/(n))^m$ by permutation of the coordinates. Denote the semidirect product of S_m and $(\mathbf{Z}/(n))^m$ by $(\mathbf{Z}/(n))^m \times^s S_m = \overline{G}$. Indicate elements of \overline{G} by $(\alpha_1, \dots, \alpha_m; \sigma) = (\alpha; \sigma)$, $\alpha_k \in \mathbf{Z}/(n)$, $k = 1, \dots, m$ and $\sigma \in S_m$. Let G be the subgroup of \overline{G} consisting of $(\alpha; \sigma)$ such that $\alpha_1 + \dots + \alpha_m = 0$. Clearly G is normal in \overline{G} and \overline{G} may be regarded as a subgroup of $S_{m \cdot n}$. Then $\text{Hur}(\mathbf{C}, G) = 1$ if $C_1 = C_2 = \dots = C_r$ are the conjugacy class of $(\mathbf{0}; (1 \ 2))$, $r \geq 4$ is an even integer and $m \geq 3$. The evenness of r assures that $\text{Ni}(\mathbf{C}, G)$ is nonempty. Theorem 3 is a corollary, based on general principles, of Theorem 2.

The main theorem of [Fr1, §5] shows that under very mild group theoretic conditions on (\mathbf{C}, G) , the space $\mathcal{H}(\mathbf{C}, G)$ parametrizes a family of covers $\{\varphi_{\mathfrak{p}}: X_{\mathfrak{p}} \rightarrow \mathbf{P}^1 \mid \mathfrak{p} \in \mathcal{H}(\mathbf{C}, G)\}$ where the family, the map from the family to $\mathcal{H}(\mathbf{C}, G)$ and $\mathcal{H}(\mathbf{C}, G)$ are all algebraic sets defined over some cyclotomic field—in the case that $\text{Hur}(\mathbf{C}, G) = 1$. It even gives the precise cyclotomic field K in question. Little, however, is known in the case that $\text{Hur}(\mathbf{C}, G)$ exceeds 1, except that this

can happen [BFr, §3]. If, furthermore, $\mathcal{H}(\mathbf{C}, G)$ contains a K -rational subvariety (even K -unirationality often suffices, as [Fr3, §4] explains), the K -rational points of this variety parametrize a family of curves $f(x, y) = 0$ defined over K for which $K(x, y)/K(x)$ is a regular Galois extension with group G . This is all sufficiently combinatorial to suggest a program for finding \mathbf{C} , given G , so as to get the cyclotomic field in question to be \mathbb{Q} . Thompson [Th] has stated such in the case that $r = 3$ (where $\mathcal{H}(\mathbf{C}, G)$ is covered by $(\mathbb{P}^1)^3 - \Delta_3$, and is always \mathbb{Q} -rational). This continues with work of Feit [Fe], Matzat [Ma] and Walter [Wa].

Since it is unlikely that a general technique will carry the program through with just the case $r = 3$, [Fr3, Theorem 4.2] states a condition that has produced non-trivial examples with $\mathcal{H}(\mathbf{C}, G)$ a rational variety for $r > 3$. It suggests a program that adds additional conditions to \mathbf{C} to assure the rationality (and, when appropriate, \mathbb{Q} -rationality) of $\mathcal{H}(\mathbf{C}, G)$. Even in the case that $r = 4$, there are pairs (\mathbf{C}, G) with $\mathcal{H}(\mathbf{C}, G)$ nonunirational (e.g., [Fr2, Theorem 3.3] gives an example where $\mathcal{H}(\mathbf{C}, G)$ maps surjectively to the modular curve ${}_o\mathcal{C}_n$; its genus exceeds o for n suitably large, and therefore a well-known generalization of Luroth's theorem shows that $\mathcal{H}(\mathbf{C}, G)$ is nonunirational). The argument of §3 of this paper, combined with [HM], shows that for (\mathbf{C}, G) given in Theorem 2 with r suitably large, investigation of $\mathcal{H}(\mathbf{C}, G)$ is not amenable to any present day techniques that generalize the use of unirationality.

2. The group theory of moduli spaces of cyclic covers. Let $\varphi: X \rightarrow \mathbb{P}^1$ be a cover of degree m for which there are at least $m - 1$ points of X over each point of \mathbb{P}^1 . If σ corresponds to this cover by (1.2)(a), then $\sigma(i)$ is a transposition, $i = 1, \dots, r$. Such a cover is called *simple*. We are interested in the following situation. Let

$$(2.1) \quad X' \xrightarrow{\psi} X \xrightarrow{\varphi} \mathbb{P}^1$$

be a sequence of covers of compact (connected) Riemann surfaces with these properties: the genus of X is g , φ is a simple cover of degree m ; and ψ is an unramified Galois cover with group $\mathbb{Z}/(n)$. Our first theorem computes the Nielsen class of the cover $\varphi \circ \psi: X' \rightarrow \mathbb{P}^1$.

Let G be the subgroup of $\bar{G} = (\mathbb{Z}/(n))^m \times S_m$ given in §1. The *Galois closure* of $\varphi \circ \psi: X' \rightarrow \mathbb{P}^1$ is a Galois cover $\hat{\varphi}: \hat{X} \rightarrow \mathbb{P}^1$ of smallest possible degree such that there exists a sequence of covers

$$(2.2) \quad \hat{X} \xrightarrow{\hat{\psi}} X' \xrightarrow{\varphi \circ \psi} \mathbb{P}^1$$

with $(\varphi \circ \psi) \circ \hat{\psi} = \hat{\varphi}$. Up to equivalence the Galois closure is unique.

THEOREM 1. *Suppose that $m \geq 3$ in the above notation. Then the Galois group of the Galois closure of $\varphi \circ \psi: X' \rightarrow \mathbb{P}^1$ given by (2.1) is isomorphic to G . If a correspondence given by (1.2) is set up, then this cover corresponds to $\sigma' = (\sigma(1)', \dots, \sigma(r)')$ where*

$$\sigma(i)' = \left(0, \dots, 0, \underset{\substack{\uparrow \\ j\text{th pos.}}}{\alpha}, 0, \dots, 0, -\underset{\substack{\uparrow \\ k\text{th pos.}}}{\alpha}, 0, \dots, 0; \sigma(i) \right)$$

with $\sigma(i) = (j\ k) \in S_m$ and $\alpha \in \mathbb{Z}/(n)$ (j, k and α dependent on i), $i = 1, \dots, r$, $\sigma(1)' \cdots \sigma(r)' = 1$ and $G(\sigma') = G$. In particular, $r \geq 2m$, and the cover is in the Nielsen class $\text{Ni}(\mathbb{C}, G)$ with $C_1 = C_2 = \cdots = C_r$, where C_1 is the conjugacy class of $\{0; (1\ 2)\}$.

PROOF. The second of the three parts of the proof includes some notation for manipulation within the group \overline{G} to which we will refer later.

PART A. *The Galois group of $\hat{\varphi}: \hat{X} \rightarrow \mathbb{P}^1$.* There is a notational simplification if we compute using the function fields of the Riemann surfaces. Let $\mathbb{C}(X)$ (resp., $\mathbb{C}(X')$, $\mathbb{C}(\hat{X})$) be the field of meromorphic functions on X (resp., X' , \hat{X}). Also, let $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(z)$ for some indeterminate z . Then (the primitive element theorem), $\mathbb{C}(X) = \mathbb{C}(z, x)$ for some $x \in \mathbb{C}(X)$. Let $x = x_1, \dots, x_m$ be the conjugates of x over $\mathbb{C}(z)$. Since $\mathbb{C}(X')/\mathbb{C}(X)$ is a cyclic extension with group $\mathbb{Z}/(n)$, we may choose $y = y_1 \in \mathbb{C}(X)$ so that $\mathbb{C}(X') = \mathbb{C}(z, x_1, y_1^{1/n})$. Thus, $\mathbb{C}(\hat{X}) = \mathbb{C}(z, x_1, y_1^{1/n}, \dots, x_m, y_m^{1/n})$ with y_1, \dots, y_m the conjugates of y_1 over $\mathbb{C}(z)$.

Let ζ_n be a primitive n th root of 1. The conjugates of $y_1^{1/n}$ over $\mathbb{C}(z)$ are exactly $\zeta_n^\alpha \cdot y_j^{1/n}$, $j = 1, \dots, m$, $\alpha \in \mathbb{Z}/(n)$. Let $\tau \in G(\mathbb{C}(\hat{X})/\mathbb{C}(z))$. Associate to τ the element $F(\tau) \in \overline{G}$ by the following formula: if τ maps $(x_j, \zeta_n^\alpha \cdot y_j^{1/n})$ to $(x_k, \zeta_n^\beta \cdot y_k^{1/n})$, then

$$(2.3) \quad F(\tau) = \left(\begin{array}{c} \cdots, \beta - \alpha, \dots; \sigma \\ \uparrow \\ \text{jth pos.} \end{array} \right) \quad \text{where } \sigma(j) = k, \quad j = 1, \dots, m.$$

Check that F is a group homomorphism that embeds $G(\mathbb{C}(\hat{X})/\mathbb{C}(z))$ into \overline{G} . Let $D(\varphi)$ be the set of branch points of the cover $\varphi: X \rightarrow \mathbb{P}^1$.

The correspondence of (1.2) arises by choosing a suitable set $\mathcal{L}_1, \dots, \mathcal{L}_r$ of closed paths on $\mathbb{P}^1 - D(\varphi)$, all based at $z_0 \in \mathbb{P}^1 - D(\varphi)$, so that the homotopy classes of these paths generate the fundamental group $\pi_1(\mathbb{P}^1 - D(\varphi), z_0)$. Then the cover $\varphi: X \rightarrow \mathbb{P}^1$ corresponds to $(\sigma(1), \dots, \sigma(r))$, where $\sigma(i)$ gives the effect of analytically continuing the functions x_1, \dots, x_m around the path \mathcal{L}_i . In more detail, express x_1, \dots, x_m as power series in a neighborhood of z_0 . Then analytically continue each around \mathcal{L}_i to get a permutation, $\sigma(i)$, of these power series expressions, $i = 1, \dots, r$.

Since $X' \rightarrow X$ is unramified, the paths $\mathcal{L}_1, \dots, \mathcal{L}_r$ suffice to compute σ' for the cover $\varphi \circ \psi: X' \rightarrow \mathbb{P}^1$, and $\sigma(i)'$ is of the same order as $\sigma(i)$, $i = 1, \dots, r$. Because $\varphi: X \rightarrow \mathbb{P}^1$ is a simple branched cover, $\sigma(i)' = (\alpha_1, \dots, \alpha_m; \sigma(i)')$ is of order 2, and as an element in $S_{m \cdot n}$ it consists of n disjoint 2-cycles. For example, if $\sigma(i) = (j\ k)$, then a suitable notation would have

$$\sigma(i)' = (j \cdot n + 1\ k \cdot n + u_1)(j \cdot n + 2\ k \cdot n + u_2) \cdots ((j + 1) \cdot n\ k \cdot n + u_n)$$

where u_1, \dots, u_n is a permutation of $1, 2, \dots, n$ that is determined by u_1 , $i = 1, \dots, r$.

PART B. *Notation within the group \overline{G} .* In the notation of Part A we can write $\sigma(i)'$ as $(\alpha_1, \dots, \alpha_m; (j\ k))$ with $\alpha_j = u_1 - 1 = \alpha$, $\alpha_k = -\alpha$ and $\alpha_l = 0$ for $l \neq j, k$. For future computations designate this element by $(\alpha_{jk}; (j\ k))$. More generally, write $(\alpha_{jk}; \sigma)$ for σ any element of S_m , where α_{jk} denotes the first part of $\sigma(i)'$.

Let $\text{pr}: \overline{G} \rightarrow S_m$ denote the natural projection onto S_m . Thus $G(\mathbb{C}(\hat{X})/\mathbb{C}(P^1)) =$

$G(\sigma')$ is a subgroup H of \overline{G} with the following properties:

- (2.4) (a) H is generated by elements of the form $(\alpha_{jk}; (j k))$;
 (b) $\text{pr}(H) = G(\sigma)$; and
 (c) $H \cap ((\mathbb{Z}/(n))^m \times 1)$ projects surjectively onto any factor of $(\mathbb{Z}/(n))^m$.

Property (2.4)(a) implies that H is contained in G . Since $G(\sigma)$ is a transitive subgroup of S_m generated by 2-cycles, it is well known that $G(\sigma) = S_m$. The conclusion that $H = G$ follows easily if we show that H contains $(\alpha_{12}; 1)$ for each $\alpha \in \mathbb{Z}/(n)$. Indeed, this gives $(\alpha_{1k}; 1) \in H$, $k = 2, \dots, m$, and therefore $(-\alpha_2 - \dots - \alpha_m, \alpha_2, \dots, \alpha_m; 1) \in H$ for each $\alpha_2, \dots, \alpha_m \in \mathbb{Z}/(n)$. Suppose that $\tau = (\beta_1, \dots, \beta_m; \sigma) \in \overline{G}$. Explicitly compute the conjugate of $(\alpha_{jk}; (j k))$ by this element as

$$\begin{aligned} \tau \cdot (\alpha_{jk}; (j k)) \cdot \tau^{-1} &= (\beta_1, \dots, \beta_m; \sigma) \cdot (\alpha_{jk}; (j k)) \cdot (-\beta_{\sigma(1)}, \dots, -\beta_{\sigma(m)}; \sigma^{-1}) \\ &= ((\alpha + \beta_{\sigma(j)} - \beta_{\sigma(k)})_{\sigma(j)\sigma(k)}; (\sigma(j) \sigma(k))). \end{aligned}$$

PART C. Conclusion of the proof. Consider all conjugates of elements of $\{\sigma(1)', \dots, \sigma(r)'\}$ (by elements of H) to elements of the form $(\alpha_{12}; (1 2))$. Since $G(\sigma) = S_m$, (2.5) gives at least one for each $\sigma(i)'$, $i = 1, \dots, r$. Denote the collection of first coordinates so obtained by A . From $(\alpha'_{12}; (1 2)) \cdot (\alpha_{12}; (1 2)) = ((\alpha' - \alpha)_{12}; 1)$ and (2.4)(c) deduce that H contains $(\alpha_{12}; 1)$ for each $\alpha \in \mathbb{Z}/(n)$. This concludes the proof that $G(\sigma') = G$.

We are done if we show that the conjugacy class of $(\alpha_{ij}; (i j))$ contains $(0; (1 2))$. This uses that $m \geq 3$. Choose $\sigma \in S_m$ so that $\sigma(j) = 1$, $\sigma(k) = 2$ and choose $\beta_1 = -\alpha$, $\beta_2 = 0$, $\beta_3 = \alpha$ and $0 = \beta_4 = \dots = \beta_m$. Now apply (2.5). \square

Identify \mathbb{Z}/n with the group generated by $(1 2 \dots n)$ in S_n . This identification is compatible with the Galois theory of Theorem 1. Then the normalizer of G in $(S_n)^m \times^s S_m$ is $(N_n)^m \times^s S_m$, where N_n is the normalizer of $\langle (1 2 \dots n) \rangle$ in S_n . Clearly N_n is the semidirect product $\mathbb{Z}/(n) \times^s (\mathbb{Z}(n))^*$ of $\mathbb{Z}/(n)$ and the invertible elements of $\mathbb{Z}/(n)$. These groups too, may be regarded as subgroups of $S_{m \cdot n}$.

DEFINITION 1. Call a sequence of the type given by (2.1) a *simple by cyclic sequence of type* (m, r, n) .

EXAMPLE 1. *The case $m = 2$.* This case was excluded by Theorem 1. The proof, up to the point of showing that the Galois group is G , still holds. But, if n is even, then an application of (2.5) shows that $(0_{12}; (1 2))$ and $(1_{12}; (1 2)) = (1, -1; (1 2))$ are in distinct conjugacy classes of G . \square

3. Irreducibility of spaces of simple by cyclic sequences. From Theorem 1 we may identify the space of simple by cyclic sequences of type (m, r, n) , $m \geq 3$, with the covers $\gamma': X' \rightarrow \mathbb{P}^1$ of Nielsen type $\text{Ni}(\mathbf{C}, G)$, where $\deg(\gamma') = m \cdot n$ and G and \mathbf{C} are given in the statement of the theorem. Here is a typical representative of a class in $\text{Ni}(\mathbf{C}, G)$:

$$(3.1) \quad \sigma' = ((\mathbf{0}; (1 3)), (\mathbf{0}; (1 3)), \dots, (\mathbf{0}; (1 m)), (\mathbf{0}; (1 m)); (1_{12}; (1 2)), (1_{12}; (1 2)), (\mathbf{0}; (1 2)), \dots, (\mathbf{0}; (1 2)), (\mathbf{0}; (1 2)), (\mathbf{0}; (1 2))).$$

In words, the first $2(m-2)$ entries generate $\mathbf{0} \times S_{m-1}$, where S_{m-1} is the subgroup of S_m that fixes 2; the next two entries are both $(1_{12}; 1 2) = (1, -1, 0, \dots, 0; (1 2))$; and the final $r - 2 \cdot (m - 1)$ entries are repetitions of $(\mathbf{0}; (1 2))$.

From §1 the irreducibility of the space of simple by cyclic sequences of type (m, r, n) or, equivalently, of the space $\mathcal{X}(\mathbf{C}, G)$ follows if for $\sigma'' \in \text{Ni}(\mathbf{C}, G)$ we show the existence of $\tau \in (N_n)^m \times^s S_m$ (end of §2) and $Q \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ such that

$$(3.2) \quad (\tau \cdot \sigma'' \cdot \tau^{-1})Q = \sigma'.$$

The special case with $n = 1$ has been a part of many papers [Fu], and, in the main, it goes back to Clebsch [C]. We state it here, but, for completeness, include a brief proof in an appendix. Note again that r is of necessity even in the next result so that $\text{Ni}(\mathbf{B}, S_m)$ is nonempty.

PROPOSITION 1. *The space $\mathcal{X}(\mathbf{B}, S_m)$ is irreducible, where $\mathbf{B} = (B_1, \dots, B_r)$ and $B_1 = \dots = B_r$ with B_1 the conjugacy class of $(1\ 2)$ in S_m .*

Following the next three lemmas we state the main theorem.

LEMMA 1. *Denote the element*

$$(0, \dots, 0, \underset{\substack{\uparrow \\ \text{*k*th pos.}}}{(v, u)}, 0, \dots, 0; \sigma)$$

with $\sigma \in S_m$ and $(v, u) \in (\mathbf{Z}/(n)) \times^s \mathbf{Z}/(n)^*$ by $((v, u)_k; \sigma)$. By generalization of (2.5), $((v, u)_k; 1) \cdot (\alpha'_{ij}; (i\ j)) \cdot ((v, u)_k; 1)^{-1}$ is equal to the following expression:

$$(3.3) \quad \begin{aligned} & \text{(a) } ((u \cdot \alpha' + v)_{ij}; (i\ j)) \text{ if } k = i; \\ & \text{(b) } ((u^{-1} \cdot \alpha' - u^{-1} \cdot v)_{ij}; (i\ j)) \text{ if } k = j; \text{ or} \\ & \text{(c) } (\alpha'_{ij}; (i\ j)) \text{ if } k \neq i, j. \end{aligned}$$

PROOF. This follows from the natural action of N_n on $\mathbf{Z}/(n)$ (as at the end of §2, $(v, u) \in N_n$ maps $\alpha' \in \mathbf{Z}/(n)$ to $u \cdot \alpha' + v$). \square

LEMMA 2. *Let $\sigma'_i = (c_{12}^{(i)}; (1\ 2)) \in G$, $i = 1, 2, \dots, r'$. Assume that $\sigma'_1 \cdots \sigma'_{r'} = (\mathbf{0}; 1)$. Then $\sum_{i=1}^{r'} (-1)^i \cdot c^{(i)} = 0$. Assume further that $n = p \cdot n_1$, where p is a prime, and if $n_1 > 1$, then*

$$(3.4) \quad c^{(1)} \equiv c^{(2)} \equiv 1 \pmod{n_1} \quad \text{and} \quad c^{(j)} \equiv 0 \pmod{n_1}, \quad j = 3, \dots, r'.$$

Then there exists $Q \in \pi_1(\mathbf{P}^{r'} - D_{r'}, \mathbf{a}^0)$ such that $(\sigma')Q = \sigma''$ with $\sigma''_i = (d_{12}^{(i)}; (1\ 2))$, $i = 1, \dots, r'$, with these properties:

$$(3.5) \quad \begin{aligned} & \text{(a) } d^{(1)} \equiv d^{(2)} \pmod{n} \text{ and } d^{(j)} \equiv 0 \pmod{n}, \quad j = 3, \dots, r', \text{ if} \\ & \quad n_1 > 1; \text{ and} \\ & \text{(b) there exists } t \geq 0 \text{ such that } d^{(1)} \equiv d^{(2)} \equiv \dots \equiv d^{(t)} \pmod{p} \\ & \quad \text{and } d^{(j)} \equiv 0 \pmod{p}, \quad j = t + 1, \dots, r', \text{ if } n_1 = 1. \end{aligned}$$

PROOF. For $u \geq 1$ we first compute the effect of $(Q_u)^m$ on σ' . The u th and $(u + 1)$ th entries of $(\sigma')Q_u$, are, respectively, $((2 \cdot c^{(u)} - c^{(u+1)})_{12}; (1\ 2))$ and $(c_{12}^{(u)}; (1\ 2))$; the u th and $(u + 1)$ th entries of $(\sigma')Q_u^2$ are $((3 \cdot c^{(u)} - 2 \cdot c^{(u+1)})_{12}; (1\ 2))$ and $((2 \cdot c^{(u)} - c^{(u+1)})_{12}; (1\ 2))$, \dots ; and the u th and $(u + 1)$ th entries of $(\sigma')(Q_u)^m$ are

$$(3.6) \quad \begin{aligned} & ((m \cdot (c^{(u)} - c^{(u+1)}) + c^{(u)})_{12}; (1\ 2)) \quad \text{and} \\ & (((m - 1) \cdot (c^{(u)} - c^{(u+1)}) + c^{(u)})_{12}; (1\ 2)). \end{aligned}$$

Use $\langle c \rangle$ to denote the (additive) subgroup of $\mathbf{Z}/(n)$ generated by c . After an application of an element Q' of $\pi_1(\mathbf{P}^{r'} - D_{r'}, \mathbf{a}^0)$ to σ' we may assume that there is an integer t for which $c^{(j)} \equiv 0 \pmod{(n)}$ for $j \geq t + 1$. Furthermore, assume that Q' has been chosen so that t is as small as possible. In particular, $c^{(1)}, \dots, c^{(t)}$ are not congruent to $0 \pmod{(n)}$. From this point on we will work with elements of $\pi_1(\mathbf{P}^{r'} - D_{r'}, \mathbf{a}^0)$ that affect only the coordinate entries $1, \dots, t$.

First assume that $n_1 > 1$. Suppose that $t > 2$. Then apply (3.6) to the case $u = 2$. Since $c^{(2)} - c^{(3)}$ is a unit $\pmod{(n)}$, we may choose m so that $m \cdot (c^{(u)} - c^{(u+1)}) + c^{(u)} \equiv 0 \pmod{(n)}$. Furthermore, there exists an element $Q'' \in \pi_1(\mathbf{P}^{r'} - D_{r'}, \mathbf{a}^0)$ that moves only the coordinate entries $2, \dots, t$, and which moves the second coordinate entry, otherwise unchanged, to the t th coordinate. Thus, the last $r' - t + 1$ coordinate entries of $(Q_2)^m \circ Q''$ applied to σ' are of the form $(\mathbf{0}; (1 \ 2))$, contrary to our assumption about t . This concludes the proof of (3.5)(a) under the assumption that $n_1 > 1$. Now assume that $n_1 = 1$ and that p is a prime.

Assume that there exists $i < t$ such that $d^{(i)} \not\equiv d^{(i+1)} \pmod{(p)}$. Then $d^{(i)} - d^{(i+1)}$ is a unit $\pmod{(p)}$. The same argument as in the preceding paragraph then applies with $i = u$. This gives (3.5)(b) and the lemma. \square

LEMMA 3 [BFr, LEMMA 3.8]. *Let $\sigma \in (S_m)^{r'}$ with $G(\sigma)$ transitive and $\sigma(1) \cdots \sigma(r') = 1$. Let $\tau \in G(\sigma)$. Then there exists $Q \in \pi_1(\mathbf{P}^{r'} - D_{r'}, \mathbf{a}^0)$ such that $\tau^{-1} \cdot \sigma \cdot \tau = (\sigma)Q$.*

THEOREM 2. *Let $\text{Ni}(\mathbf{C}, G)$ be the Nielsen class which contains the equivalence class represented by σ' of (3.1). Then $\text{Hur}(\mathbf{C}, G) = 1$. In particular, the space of equivalence classes of simple by cyclic sequences of type (m, r, n) , with even $r \geq 2m$ and $m \geq 3$, is irreducible.*

PROOF. As discussed above, we must establish (3.2). From Proposition 1, there exist $Q' \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ and $\tau_1 \in \mathbf{0} \times S_m$ such that

$$(3.7) \quad (\tau_1 \cdot \sigma'' \cdot \tau_1^{-1})Q' = ((\alpha_{13}^{(3)}; (1 \ 3)), (\beta_{13}^{(3)}; (1 \ 3)), \dots, (\alpha_{1m}^{(m)}; (1 \ m)), (\beta_{1m}^{(m)}; (1 \ m)); \\ (\gamma_{12}^{(1)}; (1 \ 2)), \dots, (\gamma_{12}^{(r-2 \cdot (m-2))}; (1 \ 2))).$$

Write out that the product of the entries of (3.7) is $(\mathbf{0}, 1)$. The first coordinate gives these expressions in order:

$$(3.8) \quad \begin{aligned} \text{(a)} \quad & \alpha^{(3)} - \beta^{(3)} + \alpha^{(4)} - \beta^{(4)} + \dots + \alpha^{(m)} - \beta^{(m)} \\ & + \sum_{j=1}^{r-2 \cdot (m-2)} (-1)^{j-1} \cdot \gamma^{(j)} \equiv 0 \pmod{(n)}; \\ \text{(b)} \quad & \sum_{j=1}^{r-2 \cdot (m-2)} (-1)^j \cdot \gamma^{(j)} \equiv 0 \pmod{(n)}; \text{ and} \\ \text{(c)} \quad & \alpha^{(k)} - \beta^{(k)} \equiv 0 \pmod{(n)}, \quad k = 3, \dots, m. \end{aligned}$$

With no loss therefore assume that

$$(3.9) \quad \sigma'' = ((\alpha_{13}^{(3)}; (1 \ 3)), (\alpha_{13}^{(3)}; (1 \ 3)), \dots, (\alpha_{1m}^{(m)}; (1 \ m)), (\alpha_{1m}^{(m)}; (1 \ m)); \\ (\gamma_{12}^{(1)}; (1 \ 2)), \dots, (\gamma_{12}^{(r-2 \cdot (m-2))}; (1 \ 2))) \\ \text{with } \sum_j (-1)^j \cdot \gamma^{(j)} \equiv 0 \pmod{(n)}.$$

For simplicity of notation, denote $r - 2 \cdot (m - 2)$ by r' throughout the remainder. The rest of the proof divides into four parts.

PART A. *Conjugation by elements of \overline{G} .* Apply Lemma 1 in the case that $(v, u)_k = (-\alpha^{(k)}, 0)_k$, which we denote just by $(-\alpha^{(k)})_k$. Therefore if we conjugate (3.9) by the product of $((-\alpha^{(j)})_j; 1)$, $j = 3, \dots, m$, and by $((-\gamma^{(r')})_2; 1)$, we may assume that σ'' is

$$(3.10) \quad ((\mathbf{0}; (1 \ 3)), (\mathbf{0}; (1 \ 3)), \dots, (\mathbf{0}; (1 \ m)), (\mathbf{0}; (1 \ m)); (\gamma_{12}^{(1)}; (1 \ 2)), \dots, \\ (\gamma_{12}^{(r'-1)}; (1 \ 2)), (\mathbf{0}; (1 \ 2))), \quad \text{with } \gamma^{(1)} - \gamma^{(2)} + \dots + (-1)^{r'} \cdot \gamma^{(r'-1)} \equiv 0 \pmod{n}.$$

Also, the conditions of (2.4) imply that $\gamma^{(1)}, \dots, \gamma^{(r'-1)}$ generate $\mathbf{Z}/(n)$. For the moment we assume that the conclusion of the theorem holds if n is a prime.

PART B. *Induction on n .* Assume that n is not a prime and write n as $p \cdot n_1$ with $n_1 > 1$. By the induction assumption, the conclusion of the theorem holds for n_1 . Reduce the entries of (3.10) $\pmod{n_1}$ to conclude that there exists $Q^{(3)} \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ such that the last r' entries of $Q^{(3)}$ applied to σ'' (given by (3.10)) satisfy hypothesis (3.4). Thus Lemma 2 gives an element of $\pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ that acts only on the last r' coordinates of $(\sigma'')Q^{(3)}$ to give σ' , except for the possibility that the $(2m-4)+1$ and $(2m-4)+2$ entries are both $(c; (1 \ 2))$. In this case apply Lemma 1 by conjugating (σ'', Q_3) by $((0, c^{-1})_2; 1)$. This concludes the theorem if n is not a prime.

PART C. *The case that $n = p$ is a prime.* Again apply Lemma 2, but this time under the assumption that $n_1 = 1$. Thus, according to (3.5)(b), we may assume that

$$(3.11) \quad \gamma^{(1)} \equiv \gamma^{(2)} \equiv \dots \equiv \gamma^{(t)} \pmod{p} \quad \text{and} \quad \gamma^{(j)} \equiv 0 \pmod{p}, \quad j = t+1, \dots, r'.$$

Note that since $\gamma^{(1)} - \gamma^{(2)} + \dots + (-1)^{t-1} \cdot \gamma^{(t)} \equiv 0 \pmod{p}$, t must be even. Let $m' = 2 \cdot (m-2)$. Apply $Q_{m'} \circ Q_{m'+1} \circ \dots \circ Q_{m'+t}$ to (3.10) to get

$$(3.12)(a) \quad (\dots, (\mathbf{0}; (1 \ m)), (-\gamma_{2m}^{(1)}; (2 \ m)), \dots, (-\gamma_{2m}^{(1)}; (2 \ m)), (\mathbf{0}; (1 \ m)), (\mathbf{0}; (1 \ 2)), \dots),$$

where the first $(\mathbf{0}; (1 \ m))$ is in the $m' - 1$ position and the second is in the $m' + t$ position: then apply conjugation by $(-\gamma_m^{(1)}; 1)$ (as in the notation of Part A) to get (3.12)(b)

$$(\dots, (\gamma_{1m}^{(1)}; (1 \ m)), (\mathbf{0}; (2 \ m)), \dots, (\mathbf{0}; (2 \ m)), (\gamma_{1m}^{(1)}; (1 \ m)), (\mathbf{0}; (1 \ 2)), \dots);$$

and finally apply $Q^{(4)} \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ that moves the two coordinate entries of the form $(\gamma_{1m}^{(1)}; (1 \ m))$ out to the positions $r-1$ and r and leaves all other entries of the form $(\mathbf{0}; (i \ j))$. As in Part B, Lemma 1 allows us to assume $\gamma^{(1)} = 1$. Lemma 3 allows us to apply $Q^{(3)} \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ to achieve the effect of conjugation by $(2 \ m)$. Therefore assume that σ'' has these properties:

$$(3.13) \quad (a) \ \sigma(i)'' \text{ is of the form } (\mathbf{0}; (j \ k)) \text{ (with } j \text{ and } k \text{ dependent on } i), \\ i = 1, \dots, r-2; \\ (b) \ \text{the second entries in } \sigma(1)'', \dots, \sigma(r-2)'' \text{ generate } S_m; \text{ and} \\ (c) \ \sigma(r-1)'' = \sigma(r)'' = (1_{12}; (1 \ 2)), \text{ and therefore } \sigma(1)'' \dots \\ \sigma(r-2)'' = (\mathbf{0}; 1).$$

PART D. *Application of Proposition 1.* Apply Proposition 1 to $\sigma(1)''$, ..., $\sigma(r-2)''$ to find $Q^{(6)} \in \pi_1(\mathbf{P}^{r-2} - D_{r-2}, \mathbf{a}^0)$ and $\gamma \in S_m$ such that

$$(3.14) \quad (\gamma^{-1} \cdot (\sigma(1)'', \dots, \sigma(r-2)'') \cdot \gamma) Q^{(6)} \\ = ((\mathbf{0}; (1 \ 3)), (\mathbf{0}; (1 \ 3)), \dots, (\mathbf{0}; (1 \ m)), (\mathbf{0}; (1 \ m)), (\mathbf{0}; (1 \ 2)), \dots, (\mathbf{0}; (1 \ 2))).$$

Indeed, Lemma 3 allows us to assume that $\gamma = 1$. With the natural interpretation of $Q^{(6)}$ in $\pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ it is now an easy matter to find $Q^{(7)}$ and apply it to $(\sigma'')Q^{(6)}$, with σ'' given by (3.13), to get σ' . This concludes the proof of the theorem. \square

Let ${}_o\mathcal{C}_{n,g}$ be the moduli space of cyclic unramified covers of genus g curves as discussed in §1. There is a natural map from the space $\mathcal{H}(\mathbf{C}, G)$ of simple by cyclic sequences of type (m, r, n) : the point $\mathfrak{p} \in \mathcal{H}(\mathbf{C}, G)$ represented by the sequence $X' \xrightarrow{\psi} X \xrightarrow{\varphi} \mathbf{P}^1$ of (2.1) goes to the point of ${}_o\mathcal{C}_{n,g}$ that is represented by the cover $X' \xrightarrow{\psi} X$. From the moduli property this map is complex analytic. It is an old argument, repeated, say, in [Fr1, §1], that if $m \geq 2g - 1$, every Riemann surface of genus g can be presented as a simple cover of \mathbf{P}^1 of degree m . Thus, in this case, the map from $\mathcal{H}(\mathbf{C}, G)$ to ${}_o\mathcal{C}_{n,g}$ is surjective. Connectness of the manifold $\mathcal{H}(\mathbf{C}, G)$ (and of the complement in it of each finite type analytic subset of codimension 1) from Theorem 2 therefore gives the following:

THEOREM 3. *The moduli space ${}_o\mathcal{C}_{n,g}$ of cyclic unramified covers of genus g curves is irreducible.*

For a given positive integer m , $m(g) = \lceil (g+3)/2 \rceil$ is the smallest integer m for which every curve X of genus g has a covering map $\varphi: X \rightarrow \mathbf{P}^1$ of degree m [KL]. Actually, if m is suitably large compared to g , then the technique of Theorem 3 shows that the irreducibility of the space $\mathcal{H}(\mathbf{C}, G)$ follows from [DM, Theorem 5.15]. But Theorem 3 does not give Theorem 2 in the case that $m < \lceil (g+3)/2 \rceil$.

Appendix—Proof of Proposition 1. As in the proof of Theorem 2, the proof of Proposition 1 amounts to showing that if $\sigma' \in \text{Ni}(\mathbf{B}, S_m)$ (with r even and of necessity $\geq 2 \cdot (m-1)$), then there exists $\tau \in S_m$ and $Q \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ such that

$$(A.1) \quad (\tau \cdot \sigma' \cdot \tau^{-1})Q = \sigma = ((1 \ m), (1 \ m), (1 \ m-1), (1 \ m-1), \dots, \\ (1 \ 3), (1 \ 3), (1 \ 2), \dots, (1 \ 2)).$$

Our choice of σ is for the sake of efficiency of proof, rather than for it to match the choices in Theorem 2. Furthermore, Lemma 3 allows us to take $\tau = 1$ and even to conjugate by an element of S_m whenever it is desirable.

First note that we can find $Q^{(1)} \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ so that $(\sigma')Q^{(1)} = ((1 \ j_1), (1 \ j_2), \dots, (1 \ j_t), \sigma(t+1)'', \dots, \sigma(r)'') = \sigma''$, where none of $\sigma(t+1)'', \dots, \sigma(r)''$ contain the integer 1. If the integers j_1, \dots, j_t are all distinct, then the product of the first t coordinate entries of $(\sigma')Q^{(1)}$ is $(1 \ j_1 \ j_2 \cdots j_t)$. It is thus clearly impossible for the products of all coordinate entries $(\sigma')Q^{(1)}$ to be 1.

Without loss we may therefore move the two identical cycles containing 1 together at the beginning to assume that $j_1 = j_2$. There are two possibilities for the group \mathcal{H} generated by $\sigma(3)'' , \dots , \sigma(r)''$:

- (A.2) (a) $\mathcal{H} = S_m$; or
 (b) \mathcal{H} is the subgroup of S_m that fixes either 1 or j_1 .

In case (A.2)(a) we assume that $j_1 = 2$. Transfer the first two coordinate entries, unchanged, down to the right-hand side to assume that

$$\sigma'' = (\sigma(1)'', \dots, \sigma(r-2)'', (1\ 2), (1\ 2)).$$

This is now set up for an induction on r : find $Q^{(2)} \in \pi_1(\mathbf{P}^{r-2} - D_{r-2}, \mathbf{a}^0)$ such that $(\sigma(1)'', \dots, \sigma(r-2)'')Q^{(2)}$ is (A.1) with two fewer (1 2) terms on the right-hand side. With an interpretation of $Q^{(2)} \in \pi_1(\mathbf{P}^r - D_r, \mathbf{a}^0)$ (as in Part D of the proof of Theorem 2) we are done if (A.2)(a) holds.

If (A.2)(b) holds, assume with no loss that $j_1 = m$ and that \mathcal{H} acts as S_{m-1} on $\{1, 2, \dots, m-1\}$: $\sigma'' = ((1\ m), (1\ m), \sigma(3)'', \dots, \sigma(r)'')$. Again we are set up for an induction on r (with m changed to $m-1$): find $Q^{(3)} \in \pi_1(\mathbf{P}^{r-2} - D_{r-2}, \mathbf{a}^0)$ such that $(\sigma(3)'', \dots, \sigma(r)'')Q^{(3)}$ is (A.1) with the first two terms on the left side missing. Conclude as in case (A.1)(a).

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