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SPANIER-WHITEHEAD DUALITY IN ETALE HOMOTOPY

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ABSTRACT. We construct a (mod-l) Spanier-Whitehead dual for the etale homotopy type of any geometrically unibranched and projective variety over an algebraically closed field of arbitrary characteristic. The Thom space of the normal bundle to imbedding any compact complex manifold in a large sphere as a real submanifold provides a Spanier-Whitehead dual for the disjoint union of the manifold and a base point. Our construction generalises this to any characteristic. We also observe various consequences of the existence of a (mod-l) Spanier-Whitehead dual.

Introduction. In this paper we establish the existence of a (mod-l) Spanier-Whitehead dual for the etale homotopy type of any geometrically unibranched and projective variety over an algebraically closed field of arbitrary characteristic. This generalises the familiar construction of the Spanier-Whitehead dual for a compact complex manifold. In the forthcoming papers [**J-2** and **J-3**] we make use of this to establish a Becker-Gottlieb type transfer for proper and smooth maps of smooth quasi-projective varieties.

Recall that associated to every finite spectrum X there exists another spectrum, denoted DX, and called the Spanier-Whitehead dual of X, which is characterised by the following property. Let Σ^0 denote the sphere spectrum, while E denotes any arbitrary spectrum. Then there exists a map $\mu\colon \Sigma^0 \to X \wedge DX$ of spectra which induces isomorphisms

$$[\mu]\colon h^{-q}(X,E) \to h_q(DX,E) \quad \text{and} \quad [\tau\mu]\colon h^{-q}(DX,E) \to h_q(X,E) \qquad \text{for all } q.$$

Here τ is the map interchanging the two factors X and DX while $h_*(\ ,E)$ ($h^*(\ ,E)$) is the generalised homology (generalised cohomology, respectively) with respect to the spectrum E. (See [**Sw**, pp. 321–335] for a general reference on the familiar notion of Spanier-Whitehead duality in topology.)

If X happens to be the suspension spectrum associated to a compact closed real manifold M, there exists an explicit geometric construction of its Spanier-Whitehead dual. If α is the normal bundle to imbedding M in a large sphere as a smooth closed submanifold, then a suitable desuspension of the Thom space of this bundle forms a Spanier-Whitehead dual for M_+. We observe that this construction therefore provides a Spanier-Whitehead dual for any compact complex manifold, by merely forgetting its complex structure. This construction is generalised here for projective and geometrically unibranched varieties over any algebraically closed field.

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An alternative approach to Spanier-Whitehead duality is a construction along the lines of Spanier's original approach using function space duality. This would involve constructing a Spanier-Whitehead dual to any sheaf of finite spectra on the etale site and would bring out the connections between this type of duality and other duality results, for example Verdier duality. This will be explored in a separate paper.

Spanier-Whitehead duality (in the familiar setting of topology) provides a connection between Poincaré duality and the Gysin-Thom isomorphism. The Gysin-Thom isomorphism for the normal bundle α (as above) is equivalent to Poincaré duality for the compact manifold M. This correspondence is also carried over to our setting. The Gysin-Thom isomorphism for topological (i.e., etale) K-theory is established in [**T**]. The above correspondence therefore establishes Poincaré duality in topological (i.e., etale) K-theory. We also establish some results that enable us to obtain the Lefschetz number property for the Becker-Gottlieb type transfer constructed in [**J-2**].

In §1 we consider generalised cohomology and homology theories associated to inverse systems of pairs of simplicial sets. We require that these theories are given by spectra which have finite homotopy groups, and with only l-primary torsion for some prime l. Our main interest is in the inverse systems of simplicial sets arising as the etale topological type of algebraic varieties. The results of this section help to formalise the concept of (mod-l) Spanier-Whitehead duality.

In §2 we construct a vector bundle over the projective space $\mathbf{P}^n$ (over the algebraically closed field k) which lifts to the stable normal bundle to imbedding the complex projective space $\mathbf{P}^n$ as a compact real submanifold in a large sphere in characteristic 0. §3 introduces the Thom-Pontrjagin construction in etale homotopy. The existence of a (mod-l) Spanier-Whitehead dual for the etale homotopy type of any geometrically unibranched and projective variety is established in §4. Much of the formalism of Spanier-Whitehead duality is carried over to this setting. §5 deals with a few straightforward application.

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1. Generalised homology and cohomology of inverse systems of spaces. First of all we establish the following terminology:

(i) A *space* always means a simplicial set (which we assume again is a Kan complex).

(ii) A *spectrum* always means a simplicial spectrum in the sense of [**K**].

Let l be a fixed prime number. In this paper we consider only spectra E that satisfy the following condition:

(1.$*$) The homotopy groups $\pi_i(E)$ are finite and with only l-primary torsion for every i.

Recall from [**K**] that any spectrum is weakly equivalent to a group spectrum, namely the one obtained by applying the free-group functor. Following [**K-W**] we call a map of spectra $E \to B$ a fibration if the induced map of free-group spectra $F(E) \to F(B)$ is an epimorphism. We use this device to obtain Postnikov towers of spectra as in [**Br**, p. 455].

Let p and q be two integers with $p \leq q$. Given a spectrum E, $E[-\infty, q]$ will denote the spectrum obtained by identifying two simplices of E with the same q-dimensional faces. This provides us with the fibration $E \to E[-\infty, q]$. The fiber of this map is denoted $E[q+1, \infty]$. We let $E[p, q]$ be the spectrum $(E[p, \infty])[-\infty, q]$. We now obtain the fibration sequence of spectra:

$$E[q, r-1] \to E[p, r-1] \to E[p, q-1],$$

where $p < q < r$. Observe that $E[p, p]$ is an Eilenberg-Mac Lane spectrum with

$$\pi_p(E[p, p]) = \pi_p(E).$$

The spectra $E[p, q]$ with $q < p$ are all trivial.

We reserve the symbol Σ^k for the k-fold suspension of the sphere spectrum, and S^k will denote the k-dimensional sphere. $M_k(l)$ will denote the Moore spectrum, which is defined as the homotopy cofiber of the map $\Sigma^{k-1} \xrightarrow{\times l} \Sigma^{k-1}$. Bousefield-Kan completion will be used for both spaces and spectra. This will always be away from a prime p or at the prime 1 and will be denoted by the familiar symbol $\hat{}$.

Let S be the category of pointed spaces, while we let Sp be the category of spectra. Let E be a spectrum satisfying the finiteness and torsion conditions above. If

$$(X, Y) = \{(X^i, Y^i) | i \in I\}$$

is a pro-object made up of pairs in S, we obtain the pro-object $\{\Sigma^0(X^i/Y^i) | i \in I\}$ in Sp, abbreviated to $\Sigma X/Y$. Therefore from now on we will restrict ourselves to the categories Sp and pro-Sp. (However, many of our results are true only for spectra which are the suspension spectra associated to spaces.) The homotopy category corresponding to Sp will be denoted HSp, while the corresponding one for S will be denoted HS.

We now set up the framework of generalised homology and cohomology. A few results on generalised cohomology appear in [**Fr-1**, Chapter 16]. The desirable properties of such theories can be summarised as follows.

(i) Generalised homology should be a functor from the category pro-Sp to the category of graded abelian groups. (In particular we regard homology as a functor to groups and not pro-groups as in [**A-M**].) Generalised cohomology should be such a contravariant functor.

(ii) Both generalised homology and cohomology should be invariant under *weak equivalences.* (A map $f\colon \{X^i\} \to \{Y^j\}$ in pro-HS (pro-HSp) is called a weak equivalence if the induced map $\#f\colon \{\#X^i\} \to \{\#Y^j\}$ is an isomorphism in pro-HS (pro-HSp, respectively). Here $\#X^i$ denotes $\{\cos k_n X^i\}$.)

(iii) Under "reasonable" circumstances there should exist strongly convergent Atiyah-Hirzebruch spectral sequences.

(1.1) DEFINITION. Let $X = \{X^i | i \in I\}$ be in pro-Sp or pro-HSp. We then define generalised homology and cohomology of X with respect to a spectrum E as follows.

Generalised homology:

$$h_*(X; E) = \lim_{p \to -\infty} \varinjlim{}_I h_*(X^i; E[p, \infty]).$$

Generalised cohomology (Friedlander [**Fr-1**, p. 158]):

$$h^*(X; E) = \lim_{\infty \leftarrow p} \varinjlim{}_I h^*(X^i; E[-\infty, p]).$$

We will presently show the need for taking the direct limit over p as $p \to -\infty$ in homology (the inverse limit over p as $p \to \infty$ in cohomology, respectively).

Recall the existence of an infinite family of maps

$$\cdots \to \Sigma^{8(k-1)} M_2 \to \Sigma^{8k} M_2 \to \cdots \to \Sigma^8 M_2 \to M_2,$$

where M_2 is the $Z/2$-Moore space. (See [**A**] for details.) It is known that the maps in the above family induce isomorphism in nonconnective complex K-theory.

$\{M_2^{8k}\}$ clearly forms a pro-simplicial set. It is also clear that $\{M_2^{8k}\}$ is weakly equivalent to a point. Hence the complex K-theory of this pro-simplicial set should be isomorphic to that of a point. We observe (from our definition above) that indeed

$$h_*(\{M_2^{8k}\}; BU) = h_*(\mathrm{pt}\,; BU),$$

where we have used BU to represent the spectrum giving nonconnective complex K-theory, and pt denotes a space consisting of a single point. There are similar examples to show the need for adopting our definition of generalised cohomology.

(1.2) LEMMA. *Let E be a spectrum. For every integer n, there exists an integer m such that*

$$h_n(X; E) = h_n(X; E[-\infty, m])$$

and

$$h^n(X; E) = h^n(X; E[m, \infty])$$

for all spaces X.

PROOF. Let p be any integer. Then we observe that the spectrum $E[p, \infty]$ is $(p-1)$-connected. Now let X be a space as above. Then $X \wedge E[p, \infty]$ is also $(p-1)$-connected. Therefore $[\Sigma^n, X \wedge E[p, \infty]] = 0$ if $p > n$.

Now the fibration sequence

$$X \wedge E[p, \infty] \to X \wedge E[-\infty, \infty] \to X \wedge E[-\infty, p-1]$$

implies that

$$[\Sigma^n, X \wedge E[-\infty, \infty]] = [\Sigma^n, X \wedge E[-\infty, p-1]]$$

if $p > n$. This proves the lemma for homology. A similar argument proves the corresponding result in cohomology.

(1.3) COROLLARY. *Let E be a spectrum satisfying the finiteness and torsion conditions in* (1.$*$). *Let X be in pro-*S *or in pro-*HS. *Then we obtain the following isomorphisms:*

$$\begin{aligned} h_*(X; E) &= \lim_{p \to -\infty} h_*(X; E[p, \infty]) \\ &= \lim_{\infty \leftarrow q} \lim_{p \to -\infty} h_*(X; E[p, q]) \\ &= \lim_{p \to -\infty} \lim_{\infty \leftarrow q} h_*(X; E[p, q]), \\ h^*(X; E) &= \lim_{\infty \leftarrow p} h^*(X; E[-\infty, p]) \\ &= \lim_{q \to -\infty} \lim_{\infty \leftarrow p} h^*(X; E[q, p]) \\ &= \lim_{\infty \leftarrow p} \lim_{q \to -\infty} h^*(X; E[q, p]). \end{aligned}$$

PROOF. The proof follows immediately from the lemma.

(1.4) PROPOSITION. *Let $E' \to E \to E''$ be a fibration sequence of spectra satisfying the finiteness and torsion conditions* (1.$*$) *above. Let Z be a scheme whose etale homotopy type will be denoted by*

$$Z_{\mathrm{ht}} = \{Z^i | i \in I\}.$$

(*See* [**Fr-1**, *p.* 37].)

(i) *If Z is of finite type over an algebraically closed field k of characteristic p (where $p \neq l$), we obtain the following long exact sequence in generalised cohomology.*

$$\cdots \to h^*(Z_{\mathrm{ht}}; E') \to h^*(Z_{\mathrm{ht}}; E) \to h^*(Z_{\mathrm{ht}}; E'') \to \cdots.$$

(ii) *We obtain the following long exact sequence in generalised homology provided Z is also geometrically unibranched and connected in addition to the above hypothesis.*

$$\cdots \to h_*(Z_{\mathrm{ht}}; E') \to h_*(Z_{\mathrm{ht}}; E) \to h_*(Z_{\mathrm{ht}}; E'') \to \cdots.$$

PROOF. Let Z_{ht} be as above. Then for each Z^i, $i \in I$, we obtain the long exact sequence

$$\cdots \to h^*(Z^i; E'[-\infty, p]) \to h^*(Z^i; E[-\infty, p]) \to h^*(Z^i; E''[-\infty, p]) \to \cdots.$$

Here $p < \infty$. Taking the direct limit over I still provides us with a similar long exact sequence. The assumptions on Z ensure that the cohomology groups

$$\varinjlim_I h^*(Z^i; E[-\infty, p]) = h^*(Z_{\mathrm{ht}}; E[-\infty, p])$$

are all finite. Therefore on taking the inverse limit over p (as $p \to \infty$), we still obtain a similar long exact sequence.

For each Z^i, $i \in I$, and each $p > -\infty$ fixed, we obtain the following long exact sequence in generalised homology:

$$\cdots \to h_*(Z^i; E'[p, \infty]) \to h_*(Z^i; E[p, \infty]) \to h_*(Z^i; E''[p, \infty]) \to \cdots.$$

Taking the inverse limit over I still leaves us with a corresponding long exact sequence because of the assumptions on Z. (The assumption that Z is connected and geometrically unibranched ensures that its etale homotopy type is pro-finite—see [**A-M**, p. 124].)

We remark that it is possible to replace the etale homotopy type by the etale topological type everywhere in Proposition (1.4). To see this we first observe there is a weak equivalence between Z_{ht} and Z_{et}. (See [**Fr-1**, p. 53].) This implies the following isomorphism of pro-abelian groups:

$$h^*(Z^i; E[s, t])_{i \in \mathrm{HR}(Z)} = h^*(Z^j; E[s, t])_{j \in \mathrm{HRR}(Z)}$$

and similarly in homology where $-\infty < s < t < \infty$. Now Corollary (1.3) implies a similar isomorphism for arbitrary spectra E satisfying the conditions in (1.$*$).

Let X be in pro-Sp or pro-HSp. Let E be spectrum as in (1.$*$) and also such that $E = E[s, t]$ for some $-\infty < s < t < \infty$. We then observe the following isomorphisms:

$$h_n(X; E) = [\Sigma^n, X \wedge E]$$

and similarly

$$h^n(X; E) = [\Sigma^{-n} \wedge X, E],$$

where [,] denotes Hom in the category pro-HSp. This representation enables us to carry over many of the usual results on generalised homology and cohomology to this setting. We now have the following result.

(1.5) PROPOSITION. *Assume we have a pairing of spectra $\sigma\colon E \wedge E' \to E''$ as in* [**K-W**, *p.* 253]. *Let X and Y be in pro-*S *or in pro-*HS. *Then we obtain the following products (as well as the usual relations among them).*

(i) $\wedge\colon h_n(X;E) \otimes h_m(Y;E') \to h_{n+m}(X \wedge Y;E'')$.
(ii) $\wedge\colon h^n(X;E) \otimes h^m(Y;E') \to h^{n+m}(X \wedge Y;E'')$.
(iii) $/\colon h^p(X \wedge Y;E) \otimes h_q(Y;E') \to h^{p-q}(X;E'')$.
(iv) $\backslash\colon h^p(X;E) \otimes h_q(X \wedge Y;E') \to h_{q-p}(Y;E'')$.
(v) $\cup\colon h^p(X,A;E) \times h^q(X,B;E') \to h^{p+q}(X, A \cup B;E'')$.
(vi) $\cap\colon h^p(X,A;E) \times h_q(X, A \cup B;E') \to h_{q-p}(X,B;E'')$.

Here A, B and X are indexed by the same left-filtering category I, and $A^i \to X^i$ as well as $B^i \to X^i$ is a cofibration for each $i \in I$.

PROOF. We can assume without loss of generality that (X,A), (X,B), X and Y are indexed by the same left-filtering category I. The naturality of the corresponding products when X and Y are spaces ((X,A) and (X,B) are pairs of spaces) then provide the first four products (last two products, respectively). Moreover, in view of the remarks just above, these products have the familiar representations when the spectra involved are finite as in the remarks above. Using Corollary 1.3, we deduce all the usual properties of these products for spectra in general.

Next we briefly show the existence of strongly convergent Atiyah-Hirzebruch spectral sequences.

(1.6) THEOREM (FRIEDLANDER, SEE [**Fr-1**, P. 168]). *Let Z be a scheme of finite type over an algebraically closed field k of characteirstic p. If E is a spectrum satisfying the finiteness and torsion conditions as in* (1.$*$) *with $l \neq p$, then there is a strongly convergent Atiyah-Hirzebruch spectral sequence with*

$$E_2^{p,q} = H^p_{\mathrm{et}}(Z;\pi_{-q}E) \Rightarrow h^{p+q}(Z_{\mathrm{et}};E).$$

(1.7) THEOREM. *Let Z be a scheme as in Theorem* (1.6). *Assume, in addition, that Z is also connected and geometrically unibranched. Let E be a spectrum as above.*

Then there is a strongly convergent Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H^{\mathrm{et}}_p(Z : \pi_q E) \Rightarrow h_{p+q}(Z_{\mathrm{et}};E).$$

PROOF. The exact couple associated to the spectral sequence arises from the long exact sequence obtained in Proposition (1.4)(ii). The identification of the E^2-term is clear. (See [**Br**, p. 451] for example.) To show strong convergence it now suffices to prove the following:

> for each n, there is an integer $u(n)$ such that $h_n(Z_{\mathrm{et}};E[N,\infty]) = 0$ for all $N > u(n)$.

We claim $u(n) = n+1$ suffices. For if $N > (n+1)$ and $p = n - q$, then

$$\begin{aligned}\pi_q(E[N,\infty]) &= 0 && \text{if } q < N,\\ &= \pi_q(E) && \text{if } q \geq N.\end{aligned}$$

If $q \geq N$, then it follows that $p = n - q \leq n - N < n - n - 1 < 0$. Since our spectral sequence is right-half-plane, this shows $u(n) = n + 1$ suffices.

REMARKS. We make the following observations.

(1) One gets similar strongly convergent spectral sequences for any X in pro-HS, provided each of the spectra $\Sigma^0 \wedge X^i$ that make up the representing inverse system have finite homotopy groups in each dimension.

(2) Let X be as in Remark (1). Let E be a spectrum satisfying the finiteness and torsion conditions (1.$*$). If E has only finitely many nontrivial homotopy groups in negative dimensions, or if X has finite cohomological dimension with respect to Z/l-coefficients, then the groups $h_n(X; E)$ and $h^n(X; E)$ are finite for each n.

(3) Let $X = \{X^i\}$ and $Y = \{Y^i\}$ be as in (1). Then observe that each of the $\hat{Y^j}$ satisfies the finiteness and torsion conditions in (1.$*$), where $\hat{}$ denotes the Bousfield-Kan completion at the prime l (see [**B-K**, p. 183]). Therefore one obtains the isomorphisms

$$[X, \#Y\hat{}\,] = \varprojlim_i h^0(X; \hat{Y^i})$$

and

$$[\Sigma^0, X \wedge (Y)\hat{}\,] = \varprojlim_j h_0(X; \hat{Y^j}).$$

2. The normal bundle to imbedding the n-dimensional complex projective space in a large sphere. In this section $\mathbf{P}^n$ will denote the n-dimensional complex projective space. It admits a closed imbedding as a real submanifold in a large sphere S^{2N}. Let α denote the normal bundle to this imbedding. Let $O_{\mathbf{P}}(1)$ denote the tautological line bundle over $\mathbf{P}^n$. It is well known that

$$(n+1)O_{\mathbf{P}}(1) = \tau(\mathbf{P}^n) \oplus \varepsilon^1_{\mathbf{C}},$$

where $\tau(\mathbf{P}^n)$ is the tangent bundle to $\mathbf{P}^n$ and $\varepsilon^1_{\mathbf{C}}$ is the 1-dimensional trivial complex bundle over $\mathbf{P}^n$. (See [**M-S**, pp. 169–170 and p. 152].) Since the tangent bundle to S^{2N} is stably trivial, and

$$\tau(\mathbf{P}^n) \oplus \alpha = \tau(S^{2N})|\mathbf{P}^n,$$

we obtain the following isomorphism of *stable* real bundles:

$$\alpha = \varepsilon_{\mathbf{R}}/(n+1)O_{\mathbf{P}}(1), \tag{2.1}$$

where $\varepsilon_{\mathbf{R}}$ is a trivial real bundle of large dimension.

However this imbedding of $(n+1)O_{\mathbf{P}}(1)$ in a large trivial bundle is valid only in characteristic 0. Nevertheless we have an imbedding of the dual bundle $(n+1)O_{\mathbf{P}}(-1)$ in a large (complex) trivial bundle over $\mathbf{P}^n$ which is valid in all characteristics. We therefore obtain another bundle

$$\varepsilon_{\mathbf{C}}/(n+1)O_{\mathbf{P}}(-1), \tag{2.2}$$

where $\varepsilon_{\mathbf{C}}$ is a complex trivial bundle of large dimension.

It follows from topological K-theory that the bundles in (2.1) and (2.2) are *stably* isomorphic as real bundles. (Observe that given real vector bundles α, α' and β so that $\alpha \oplus \beta = \alpha' \oplus \beta$, all we can conclude is that α is stably isomorphic to α'. We apply this observation to the above situation by letting $\beta = (n+1)O_{\mathbf{P}}(-1) = (n+1)O_{\mathbf{P}}(1)$ and α (α') be the complementary bundles to the imbeddings of β in the trivial bundle ε as in (2.1) ((2.2), respectively).) Hence both bundles have the same Thom spaces (stably). We will call the bundle in (2.2) *the stable normal bundle to imbedding the projective space* $\mathbf{P}^n$ in a large sphere, and this will be denoted by α.

3. Thom-Pontrjagin construction in etale homotopy. In this section we provide an analogue of the familiar Thom-Pontrjagin construction. Recall that the usual form of this construction depends strongly on the tubular neighborhood theorem. Tubular neighborhoods in the etale setting have been developed in [**Cox**] and [**Fr-1**, p. 147]. We utilise the tubular neighborhoods as in [**Fr-1**] freely in this section.

Let $\mathbf{P}^n$ be the n-dimensional projective space over the algebraically closed field k of arbitrary characteristic p. Let α be the bundle over $\mathbf{P}^n$ defined in the last section. Let $\alpha_{\mathbf{C}}$ denote the corresponding bundle over the complex projective space $\mathbf{P}^n$.

We first of all obtain the (usual) *Thom-Pontrjagin collapse* map

$$S^{2N} \to T(\alpha_{\mathbf{C}}), \tag{3.1}$$

where $T(\alpha_{\mathbf{C}})$ is the Thom space of $\alpha_{\mathbf{C}}$. (See [**Sw**, p. 332] for a discussion of this. We observe in this context that $T(\alpha_{\mathbf{C}})$ could also be defined as the homotopy cofiber of the map

$$(E(\alpha_{\mathbf{C}}) - \mathbf{P}^n) \to E(\alpha_{\mathbf{C}}), \tag{3.2}$$

where $E(\alpha_{\mathbf{C}})$ is the total space of the bundle $\alpha_{\mathbf{C}}$.)

We now define Thom spaces for algebraic vector bundles as follows.

(3.3) DEFINITION. Let X be a quasi-projective variety over k, and let N be a vector bundle over X. Then we define the *Thom space of the bundle* N to be the homotopy cofiber of the map

$$(E(N) - X)_{\text{et}} \to E(N)_{\text{et}},$$

where $E(N)$ is the total space of the bundle N.

Now we make use of the comparison theorem (see [**A-M**, p. 144]) to obtain the weak equivalence

$$T(\alpha_{\mathbf{C}})\hat{} = \widehat{T(\alpha)_{\text{et}}}.$$

(We remind the reader that the above completions are as in [**B-K**] and are away from the characteristic of k.) Therefore we obtain from (3.1) a map

$$S^{2N} \to \widehat{T(\alpha)_{\text{et}}}, \tag{3.1$'$}$$

which will be denoted TP_1.

Let Z be a closed subvariety of $\mathbf{P}^n$ of dimension d. Let $t_{E|\mathbf{P}^n}$ ($t_{E|Z}$) denote the *etale tubular neighborhood* of $\mathbf{P}^n$ in $E(\alpha)$ (of Z in $E(\alpha)$, respectively). Similarly, we let $(t_{E|\mathbf{P}^n} - \mathbf{P}^n)$ $(t_{E|Z} - Z)$ denote the corresponding deleted etale tubular neighborhoods. We now use the results of Cox (see [**Cox**, Theorems 3.2 and 5.1]) to obtain the weak equivalence

$$\widehat{T(\alpha)_{\text{et}}} \approx [(t_{E|\mathbf{P}^n})_{\text{et}}/(t_{E|\mathbf{P}^n} - \mathbf{P}^n)_{\text{et}}]\hat{}. \tag{3.4}$$

We now construct a map

$$[(t_{E|\mathbf{P}^n})_{\text{et}}/(t_{E|\mathbf{P}^n} - \mathbf{P}^n)_{\text{et}}]\hat{} \to [(t_{E|Z})_{\text{et}}/(t_{E|Z} - Z)_{\text{et}}]\hat{}, \tag{3.5}$$

which will be denoted TP_2.

This 'homotopy collapse' is the analogue of the Thom-Pontrjagin collapse in our setting. We state our results in the following more general situation. Assume

$$Z \to X \to E \tag{3.6}$$

are both closed imbeddings of connected schemes of finite type over k, with dimensions d, $d+c$, and $d+c+q$, respectively.

(3.7) THEOREM. (*Taking* $X = E$, *this specialises to Proposition* 15.6 *in* [**Fr-1**].) *Assuming the situation of* (3.6), *we have the following weak equivalences*:
(i) $(t_{E|X})_{\text{et}} \approx (t_{E|Z})_{\text{et}} \vee_{(t_{E|Z}-Z)} (t_{E|X} - Z)_{\text{et}}$; *and*
(ii) $[(t_{E|Z})_{\text{et}}/(t_{E|Z} - Z)_{\text{et}}]^{\frown} \approx [(t_{E|X})_{\text{et}}/(t_{E|X} - Z)_{\text{et}}]^{\frown}$.

The right side of (i) is a homotopy pushout, while the two sides of (ii) are homotopy cofibers and $\approx$ denotes weak equivalence. We merely remark that the proof of this theorem is along the same lines as the proof of the special case $E = X$ in [**Fr-1**] and leave the details to the reader. (See [**J-T**, Chapter 2] for more details.)

Observe that (ii) provides a 'homotopy collapse'

$$[(t_{E|X})_{\text{et}}/(t_{E|X} - X)_{\text{et}}]^{\frown} \to [(t_{E|Z})_{\text{et}}/(t_{E|Z} - Z)_{\text{et}}]^{\frown}$$

(defined up to homotopy). We let TP_2 (as in (3.5)) be this map in the special case when $E = E(\alpha)$ and $X = \mathbf{P}^n$.

(3.8) PROPOSITION. *Assume, in addition to the hypothesis of* (3.7), *that* E *is a quasi-projective variety and that* X *and* Z *are complete subvarieties.*

Then the collapse map

$$TP_2\colon [(t_{E|X})_{\text{et}}/(t_{E|X} - X)_{\text{et}}]^{\frown} \to [(t_{E|X})_{\text{et}}/(t_{E|X} - Z)_{\text{et}}]^{\frown}$$

is of degree one on (*ordinary*) *cohomology with* Z/l*-coefficients* (*i.e., it sends the top cohomology class to the top cohomology class.*)

Assuming (3.7) we now provide a proof for Proposition (3.8).

PROOF. We first observe that the maps induced by

$$[(t_{E|X})_{\text{et}}/(t_{E|X} - X)_{\text{et}}]^{\frown} \to [(t_{E|X})_{\text{et}}/(t_{E|X} - Z)_{\text{et}}]^{\frown}$$

and

$$[(E_{\text{et}}/(E - X)_{\text{et}})]^{\frown} \to [(E_{\text{et}}/(E - Z)_{\text{et}})]^{\frown}$$

are identical in homology and cohomology with Z/l-coefficients.

This follows from the isomorphisms

$$\begin{aligned} H^*(((t_{E|X})_{\text{et}}/(t_{E|X} - X)_{\text{et}}); Z/l) &= H^*(((t_{E|X})_{\text{et}}, (t_{E|X} - X)_{\text{et}}); Z/l) \\ &= H^*(E_{\text{et}}, (E - X)_{\text{et}}; Z/l) \\ &= H^*(\overline{E}_{\text{et}}, (\overline{E} - X_{\text{et}}); Z/l) \end{aligned}$$

and similarly for Z. Here $\overline{E}$ denotes the completion of E; i.e., E is a Zariski open subvariety of $\overline{E}$ and $\overline{E}$ is a complete variety. The last isomorphism is obtained by excision in the etale setting and uses the fact that X is complete (see [**Mi**, p. 92]). The second isomorphism is established in [**Fr-1**, p. 152].

Now the proof is completed by using the commutative diagram

$$\begin{tikzcd} H_{2m}(\overline{E}_{\text{et}}, (\overline{E} - X)_{\text{et}}; Z/l) \arrow[r] \arrow[dr] & H_{2m}(\overline{E}_{\text{et}}, (\overline{E} - Z)_{\text{et}}; Z/l) \arrow[d] \\ & H_{2m}(\overline{E}_{\text{et}}; Z/l) \end{tikzcd}$$

where $m = d + c + q = \dim E$, the horizontal map $= TP_{2*}$, and the other maps are the usual maps in the long exact sequence for the pairs $(\overline{E}_{\text{et}}, (\overline{E} - X)_{\text{et}})$ and $(\overline{E}_{\text{et}}, (\overline{E} - Z)_{\text{et}})$.

4. (mod-l) **Spanier-Whitehead duality.** Let $\mathbf{P}^n$ be the n-dimensional projective space over k, and $E =$ the total space of the bundle α constructed earlier. Let Z be a closed subvariety of $\mathbf{P}^n$.

We first construct a duality map in this context as follows. First of all we obtain the following weak equivalence from [**Cox**, Theorems 3.2 and 5.1]:

$$\widehat{T(\alpha)_{\mathrm{et}}} \approx [(t_{E|\mathbf{P}^n})_{\mathrm{et}}/(t_{E|\mathbf{P}^n} - \mathbf{P}^n)_{\mathrm{et}}]\widehat{\ }\,.$$

Now composing TP_1 and TP_2 we get

$$\begin{aligned} TP\colon S^{2N} &\to \widehat{T(\alpha)_{\mathrm{et}}} = [(t_{E|\mathbf{P}^n})_{\mathrm{et}}/(t_{E|\mathbf{P}^n} - \mathbf{P}^n)_{\mathrm{et}}]\widehat{\ } \\ &\to [(t_{E|\mathbf{P}^n})_{\mathrm{et}}/(t_{E|\mathbf{P}^n} - Z)_{\mathrm{et}}]\widehat{\ } \approx [(t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}}]\widehat{\ }\,, \end{aligned} \tag{4.1}$$

where we let $2N = 2 \cdot (n+q)$. (Recall that q is the dimension of the vector bundle α.) Recall from Proposition (3.8) that TP is *degree one* in homology with Z/l coefficients. This will be referred to as the *Thom-Pontrjagin collapse map* from now on.

The next step is to obtain the diagonal map

$$\mathrm{diag}\colon (t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}} \to (t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}} \wedge (t_{E|Z})_{\mathrm{et}\,+}$$

which can be obtained in the usual manner (see [**Sw**, pp. 332] or [**J-1**, p. 25]).

Composing the Thom-Pontrjagin collapse and the diagonal maps we obtain the *duality map*

$$\begin{aligned} \mu\colon S^{2N} &\to [(t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}} \wedge (t_{E|Z})_{\mathrm{et}\,+}]\widehat{\ } \\ &\approx [(t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}} \wedge (Z_{\mathrm{et}\,+})]\widehat{\ }\,, \end{aligned} \tag{4.2}$$

where we made use of the weak equivalence

$$(t_{E|Z})_{\mathrm{et}\,+} \approx (Z_{\mathrm{et}\,+})$$

(see [**Fr-1**, p. 150] for this).

(4.3) Remarks. (i) From now on we will use $T(Z)_{\mathrm{et}}$ ($T(Z)_{\mathrm{ht}}$) to denote

$$[(t_{E|Z})_{\mathrm{et}}/(t_{E|Z} - Z)_{\mathrm{et}}] \qquad ([(t_{E|Z})_{\mathrm{ht}}/(t_{E|Z} - Z)_{|\mathrm{ht}}],\ \text{respectively}).$$

(ii) We proceed to show that $\widehat{T(Z)_{\mathrm{ht}}}$ is a Spanier-Whitehead dual (see (4.5)) of $\widehat{Z_{\mathrm{ht}\,+}}$ if Z is geometrically unibranched. (When Z is smooth, it will turn out that $\widehat{T(Z)_{\mathrm{ht}}} \approx$ the completed Thom space of the normal bundle to the composite imbedding $Z \to \mathbf{P}^n \to E$.) We first show that the duality map in (4.2) induces a map from the generalised homology (cohomology) of $\widehat{Z_{\mathrm{ht}\,+}}$ to the generalised cohomology (homology, respectively) of $\widehat{T(Z)_{\mathrm{ht}}}$ which is natural with respect to the representing spectrum. To establish Spanier-Whitehead duality, we need to show that this map is an isomorphism for all spectra as in (1.$*$). Use of Corollary (1.3), Proposition (1.4) and a Postnikov truncation argument (see the discussion following (1.$*$)) reduce this to the case of an Eilenberg-Mac Lane spectrum. Poincaré-Lefschetz duality shows that the map is an isomorphism in this case.

We next choose a generator for $H_{2N}(S^{2N}; Z/l)$ (which will be denoted j_{2N}) and call it the *orientation class* in $H_{2N}(S^{2N}; Z/l)$.

(4.4) PROPOSITION. *Let Z and E be as in the beginning of this section. Then slant product with the calss $\mu_*(j_{2N})$ induces isomorphisms*

$$H^p_{\text{et}}(E, E - Z; Z/l) \xrightarrow{\cong} H^{\text{et}}_{2N-p}(Z; Z/l)$$

and

$$H^p_{\text{et}}(Z; Z/l) \xrightarrow{\cong} H^{\text{et}}_{2N-p}(E, E - Z; Z/l),$$

where $2N$ is the dimension of the sphere in which $\mathbf{P}^n_C$ is imbedded.

PROOF. The proof consists of the following observations. First $TP_*(j_{2N})$ in $H^{\text{et}}_{2N}(E, E - Z; Z/l)$ is a generator because of the degree-oneness of TP.

Next we observe the isomorphism

$$H^{\text{et}}_*(\overline{E}, \overline{E} - Z; Z/l) = H^{\text{et}}_*(E, E - Z; Z/l),$$

where $\overline{E}$ is a completion of E. Observe that we can choose E to be smooth in this case by taking

$$\overline{E} = \text{Proj}(\alpha \oplus \varepsilon_1),$$

where ε_1 is the 1-dimensional trivial bundle over $\mathbf{P}^n$. Therefore by Poincaré-Lefschetz duality (see [**Fr-1**, p. 176])

$$TP_*(j_{2N}) \colon H^p_{\text{et}}(Z; Z/l) \to H^{2N-p}_{\text{et}}(E, E - Z; Z/l)$$

is an isomorphism. Finally, we make use of the familiar relationship between cap and slant products as well as the fact that $\mu = \text{diag}(TP)$. We have thereby proven the proposition.

We next formalise the concept of (*mod-l*) *Spanier-Whitehead duality*. We will assume in what follows that E denotes a spectrum satisfying the conditions in (1.$*$). Let X and Y be in pro-Sp or pro-HSp and let Σ^0 be the sphere spectrum. Observe that a map $\sigma \colon \Sigma^0 \to (X \wedge Y)^\frown$ in pro-Sp (or in pro-HSp) induces maps

$$[\sigma] \colon h^q(X; E) \to h_{-q}(Y; E)$$

and

$$[\tau \cdot \sigma] \colon h^q(Y; E) \to h_{-q}(X; E)$$

(where τ is the map interchanging the two factors X and Y).

To see this we let $\{f \colon X \to E[r,p] | r, p\}$ represent a class in $h^q(X; E)$. Then by smashing with Y we obtain maps $\{f \wedge \text{id} \colon (X \wedge Y) \to E[r,p] \wedge Y | r, p\}$ which are compatible as r and p vary. Since E satisfies the finiteness and torsion conditions mentioned earlier (in particular, E has only l-primary torsion), the spectrum $E[r,p] \wedge Y$ is already Z/l-complete. Hence the map $(f \wedge \text{id})$ factors through $(f \wedge \text{id})^\frown$. Now making use of the map σ, we obtain the map

$$\{\Sigma^0 \xrightarrow{\sigma} (X \wedge Y)^\frown \xrightarrow{(f \wedge \text{id})^\frown} E[r,p] \wedge Y | r, p\}.$$

This provides the maps denoted $[\sigma]$ and $[\tau \cdot \sigma]$.

(4.5) DEFINITION. The map σ will be called a (*mod-l*) *Spanier-Whitehead duality map* (*or* (*mod-l*) *S-duality map*) if $[\sigma]$ and $[\tau \cdot \sigma]$ are both isomorphisms for all spectra E, with $\pi_i(E)$ finite and with only l-primary torsion for each i. In this case Y will be called a (*mod-l*) *Spanier-Whitehead dual of X* (*or* (*mod-l*) *S-dual of X*).

REMARKS. (1) Observe that Proposition (4.4) shows that the map μ as in (4.2) satisfies the conditions in (4.5) with respect to the Eilenberg-Mac Lane spectrum $K(Z/l)$.

(2) A map $\sigma: \Sigma^m \to (X \wedge Y)\hat{}$ will also be called a (mod-l) S-duality map if the induced map $\Sigma^{-m}\sigma: \Sigma^0 \to (X \wedge (\Sigma^{-m}Y))\hat{}$ is a (mod-l) S-duality map.

In what follows we assume Z is a connected and geometrically unibranched projective variety and $T(Z)_{\mathrm{ht}}$ ($T(Z)_{\mathrm{et}}$) will be as in (4.3)(i). We proceed to show that the map of (4.2) is a (mod-l) S-duality map. First we state the following result.

(4.6) PROPOSITION. *Given a map of spectra $E \to E'$ (where the spectra satisfy the finiteness and torsion conditions in* (1.$*$)), *the following square commutes*:

$$\begin{array}{ccc} h^{2N-q}(T(Z)_{\mathrm{ht}}; E) & \xrightarrow{[\mu]} & h_q(Z_{\mathrm{ht}\,+}; E) \\ \downarrow & & \downarrow \\ h^{2N-q}(T(Z)_{\mathrm{ht}}; E') & \xrightarrow{[\mu]} & h_q(Z_{\mathrm{ht}\,+}; E') \end{array}$$

There is a similar commutative square made up of the maps $[\tau \cdot \mu]$. (*Here q is any integer and $2N$ is the dimension of the sphere in which Z is "imbedded"*.)

PROOF. We first observe that Corollary (1.3) enables us to restrict to the spectra $E[r,p]$ and $E'[r,p]$ with $-\infty < r < p < \infty$. In this situation Proposition (4.6) reduces to the following commutative square:

$$\begin{array}{ccc} [\Sigma^{-2N+q}T(Z)_{\mathrm{ht}}, E[r,p]] & \xrightarrow{[\mu]} & [\Sigma^q, Z_{\mathrm{ht}\,+} \wedge E[r,p]] \\ \downarrow & & \downarrow \\ [\Sigma^{-2N+q}T(Z)_{\mathrm{ht}}, E'[r,p]] & \xrightarrow{[\mu]} & [\Sigma^q, Z_{\mathrm{ht}\,+} \wedge E'[r,p]] \end{array}$$

The description of the maps $[\mu]$ above shows that this square commutes.

(4.7) THEOREM. *The duality map of* (4.2) *is a* (*mod-l*) *S-duality map.*

PROOF. We again make use of Corollary (1.3) to reduce to the case when the spectra involved are of the form $E[r,p]$, with $-\infty < r < p < \infty$. Next we consider the fibration sequence of spectra given by $E[q, r-1] \to E[p, r-1] \to E[p, q-1]$. Proposition (1.4) now provides the long exact sequences that make up the two rows of the following diagram:

$$\begin{array}{cccc} \cdots \to h^{2N-t}(T(Z)_{\mathrm{ht}}; E[q,r-1]) \to & h^{2N-t}(T(Z)_{\mathrm{ht}}; E[p,r-1]) & & \\ \downarrow & \downarrow & \to h^{2N-t}(T(Z)_{\mathrm{ht}}; E[p,q-1]) \to \cdots & \\ \cdots \to h_t(Z_{\mathrm{ht}\,+}; E[q,r-1]) \to & h_t(Z_{\mathrm{ht}\,+}; E[p,r-1]) & \downarrow & \\ & & \to h_t(Z_{\mathrm{ht}\,+}; E[p,q-1]) \to \cdots & \end{array}$$

where the vertical maps are $[\mu]$. The commutativity of this diagram follows from Proposition (4.6). There is a similar commutative diagram involving $[\tau \cdot \mu]$.

To show that $[\mu]$ is an isomorphism, we first of all recall (from Proposition (4.4)) that it is an isomorphism for Eilenberg-Mac Lane spectra of the form $E[p,p]$, where E is as in (1.$*$). It follows from the above commutative diagram (using the five lemma and induction) that $[\mu]$ is an isomorphism for all spectra of the form $E[p,q]$, $-\infty < p < q < \infty$. The general case now follows from Corollary (1.3).

REMARKS. *Formal consequences of* (*mod-l*) *Spanier-Whitehead duality.*

Let $\mu\colon \Sigma^0 \to (X \wedge DX)\hat{}$ and $\mu'\colon \Sigma^0 \to (Y \wedge DY)\hat{}$ be (mod-l) S-duality maps so that the following two conditions are satisfied:

(i) X, Y, DX and DY are objects in pro-HSp obtained by smashing objects of pro-HS with the sphere spectrum Σ^0; and

(ii) each of the spectra X^i (Y^i, DX^i and DY^i) forming the inverse system representing X (Y, DX and DY, respectively) satisfy the finiteness and torsion conditions in (1.$*$). (Observe that this implies $X \wedge DX$ and $Y \wedge DY$ are both Z/l-complete.)

Under the above hypotheses we obtain the isomorphisms

$$\begin{aligned}[X, \#Y] &= \varprojlim{}_j [X, \#Y^j] = \varprojlim{}_j h^0(X; Y^j) \\ &= \varprojlim{}_j h_0(DX; Y^j) = [\Sigma^0, DX \wedge Y],\end{aligned}$$

from Corollary (4.7) and the remarks at the end of §1, since each of the spectra Y^j satisfies the conditions in (1.$*$). Therefore we denote the above isomorphism $[X, \#Y] \to [\Sigma^0, DX \wedge Y]$ by $[\mu]$. Similarly, we obtain the isomorphism $[\tau \cdot \mu']$.

We therefore obtain an isomorphism (denoted D)

(4.8) $$D\colon [X, \#Y] \to [\Sigma^0, DX \wedge Y] \to [DX, \#DY]$$

by letting $D = [\tau \cdot \mu']^{-1} \cdot [\mu]$. We conclude this discussion by listing the following key properties of this duality isomorphism. (See [**J-T**, Chapter 4] for more details.)

(4.9) If $f\colon X \to \#Y$ is a map in pro-HSp, its dual $Df\colon DX \to \#DY$ is characterised by the commutativity of the following square in pro-HSp:

$$\begin{array}{ccc} \Sigma^0 & \xrightarrow{\mu} & XDX \\ \mu' \downarrow & & \downarrow (\#f \wedge \mathrm{id}) \\ Y \wedge DY & \xrightarrow{(\mathrm{id} \wedge \#Df)} & \#Y \wedge \#DX \end{array}$$

(4.10) Let Z be a projective geometrically unibranched variety, let $T(Z)_{\mathrm{ht}}$ be its dual as in (4.3)(i), and let μ be the duality map. Then the map

$$\begin{aligned}\sigma\colon \Sigma^{4N} &\xrightarrow{\mu \wedge \mu} [(Z_{\mathrm{ht}+} \wedge T(Z)_{\mathrm{ht}}) \wedge (Z_{\mathrm{ht}+} T(Z)_{\mathrm{ht}})]\hat{} \\ &\xrightarrow{\alpha} [(Z_{\mathrm{ht}+} \wedge T(Z)_{\mathrm{ht}}) \wedge (Z_{\mathrm{ht}+} \wedge T(Z)_{\mathrm{ht}})]\hat{},\end{aligned}$$

where α is given by $\alpha(x \wedge y \wedge x' \wedge y') = x \wedge y' \wedge x' \wedge y$ is a (mod-l) S-duality map for $[(Z_{\mathrm{ht}+} \wedge T(Z)_{\mathrm{ht}})]\hat{}$.

5. Applications. First of all we obtain a Becker-Gottlieb transfer for the following commutative triangle:

(5.1) $$\begin{array}{ccc} X & \xrightarrow{f} & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & \operatorname{Spec} k & \end{array}$$

where f is an automorphism of X, $p\colon X \to \operatorname{Spec} k$ is the structure map of X, and, moreover, X is asumed to be a smooth projective variety. This is used to prove the Lefschetz number property of the Becker-Gottlieb transfer, which is constructed in [**J-T**, Chapter 5] and [**J-3**] for proper and smooth maps $p\colon X \to Y$ of smooth quasi-projective varieties.

Let X be as above and let $(TX)_{\mathrm{ht}}$ be its (mod-l) S-dual constructed as before. Let μ' be the map

$$\Sigma^{2N} \xrightarrow{\mu} [X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}}]^{\frown} \wedge \Sigma^0 \to [X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}}]^{\frown} \wedge M_1(l^\nu),$$

where μ is the duality map for X and $\Sigma^0 \to \Sigma^0 \to M_1(l^\nu)$ is the usual cofibration sequence. Then μ' is dual to a map

$$D\mu' \colon [X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}}]^{\frown} \to \#\Sigma^{2N} M_1(l^\nu).$$

We proceed to define the S-duality transfer and establish its Lefschetz number property.

(5.2) DEFINITION. We define the *(mod-l) S-duality transfer* to be the following:

$$\Sigma^{2N} \xrightarrow{\mu} [\Sigma(X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}})]^{\frown} \xrightarrow{\mathrm{id} \wedge [(f_{+\,\mathrm{ht}},\, \mathrm{id}) \circ \mathrm{diag}]} [\Sigma(X_{\mathrm{ht}\,+} \wedge X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}})]^{\frown}$$
$$\xrightarrow{D\mu' \wedge \mathrm{id}} \#\Sigma^{2N} M_1(l) \wedge X_{\mathrm{ht}\,+}.$$

The *l-adic S-duality transfer* is the map obtained by replacing $M_1(l)$ in the above diagram by the homotopy inverse limit $\operatorname{holim} M_1(l^\nu)$.

Next we let $\overline{f}$ be the map

$$\overline{f} \colon \Sigma^{2N} \xrightarrow{\mu} (\Sigma X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}})^{\frown} \xrightarrow{(f_{\mathrm{ht}\,+} \wedge \mathrm{id})^{\frown}} (\Sigma X_{\mathrm{ht}\,+} \wedge T(X)_{\mathrm{ht}})^{\frown} \xrightarrow{D\mu'} \#\Sigma^{2N} M_1(l).$$

We will let $\overline{\overline{f}}$ denote the map obtained by replacing $M_1(l)$ in the above diagram by $\operatorname{holim} M_1(l^\nu)$. In order to prove the Lefschetz number property of these transfer maps, we next observe

$$H_{2N}(\Sigma^{2N} M_1(l^\nu); Z/l) = Z/l$$

and

$$\varprojlim{}_k H_{2N}(\Sigma^{2N} \operatorname{holim} M_1(l^\nu); Z/l^k) \otimes_{\mathbf{Z}_l} Q_l = Q_l.$$

We define the degree of $\overline{f}$ (and $\overline{\overline{f}}$) as follows. Let j_{2N} denote a generator of $H_{2N}(S^{2N}; Z/l)$ ($\varprojlim_k H_{2N}(S^{2N}; Z/l^k) \otimes_{\mathbf{Z}_l} Q_l$, respectively). Then we let

$$\overline{f}_*(j_{2N}) = (\operatorname{degree} \overline{f}) \cdot j_{2N} \quad (\overline{\overline{f}}_*(j_{2N}) = (\operatorname{degree} \overline{\overline{f}}) \cdot j_{2N}).$$

We next observe that the map $p \colon X \to \operatorname{Spec} k$ induces a map $p_{\mathrm{ht}\,+} \colon \Sigma^0 X_{\mathrm{ht}\,+} \to \Sigma^0$. Now $p_{\mathrm{ht}\,+} \cdot \mathrm{tr}(f) = \overline{f}$ and $p_{\mathrm{ht}\,+} \circ \mathrm{tr}(f) = \overline{\overline{f}}$ (in the (mod-l) and l-adic cases, respectively). Therefore the Lefschetz number property is established by showing that the Lefschetz number of f in homology with Z/l-coefficients (l-adic coefficients) $=$ (degree $\overline{f}$) ((degree $\overline{\overline{f}}$), respectively).

A computation exactly as in [**B-G**, Lemma 2] using the homotopy commutativity of the diagram

$$\begin{tikzcd}
\Sigma^{4N} \arrow[r, "\sigma"] \arrow[ddr, "\#\mu \wedge \#\mathrm{id}"'] & (\Sigma(\widehat{T(X)_{\mathrm{ht}}} \wedge \widehat{X_{\mathrm{ht}\,+}}) \wedge \Sigma(\widehat{T(X)_{\mathrm{ht}}} \widehat{X_{\mathrm{ht}\,+}}))^{\frown} \arrow[d, "\tau(\mathrm{id} \wedge D\mu')"] \\
& (\Sigma(\widehat{T(X)_{\mathrm{ht}}} \widehat{X_{\mathrm{ht}\,+}})^{\frown} \wedge \#\Sigma^{2N} M_1(l)) \arrow[d, "\#"] \\
& \#(\Sigma(\widehat{T(X)_{\mathrm{ht}}} \wedge \widehat{X_{\mathrm{ht}\,+}})^{\frown} \wedge \#\Sigma^{2N} M_1(l))
\end{tikzcd}$$

and the corresponding diagram with the Moore spectrum $M_1(l)$ replaced by the holim $M_1(l^\nu)$ establishes this. (The homotopy-commutativity of the above diagram follows from (4.9) and (4.10). See [**J-T**, pp. 56–57] for more details.)

Connections between the various duality results. We henceforth assume that E is a ring spectrum satisfying again the conditions (1.$*$). Let $i: \Sigma^0 \to E$ be the unit of the ring spectrum E. Let X be a geometrically unibranched projective variety, and let $T(X)_{\mathrm{ht}}$ be its dual with μ the duality map.

Let j_{2N} in $h_{2N}(S^{2N}; E)$ be the class represented by the suspension of the unit i. Then we observe $[\mu]_*(j_{2N}) \approx \mu \wedge i$, and the duality isomorphism $[\mu]$ is obtained by taking the slant product with this class (see [**J-T**, p. 77] for details).

(5.3) DEFINITION. Assume the above situation. Assume also that X is of dimension d and that it is "imbedded" (as in §2) in the sphere S^{2N}. Let $2c = 2N-2d$. Then a *Thom class* (*or Gysin-Thom class*) for X with respect to E is a class θ_X in $h^{2c}(T(X)_{\mathrm{ht}}; E)$ such that the maps $\bigcup \theta_X: h^q(X_{\mathrm{ht}\,+}; E) \to h^{2c+q}(T(X)_{\mathrm{ht}}; E)$ are isomorphisms for all q.

A *fundamental class for X with respect to E* is a class $[X]$ in $h_{2d}(X_{\mathrm{ht}}; E)$ such that $[X] \bigcap: h^q(X_{\mathrm{ht}}; E) \to h_{2d-q}(X_{\mathrm{ht}}; E)$ is an isomorphism for all q.

(5.4) PROPOSITION. *Let X be a projective, geometrically unibranched variety, and let $T(X)_{\mathrm{ht}}$ be its (mod-l) S-dual. Let E be a ring spectrum as above. Let j_{2N} in $h_{2N}(S^{2N}; E)$ also be as above.*

If θ_X is a Thom class for X with respect to E, then $\theta_X \backslash \mu_*(j_{2N})$ is a fundamental class. Conversely, if $[X]$ is a fundamental class for X with respect to E, then the inverse image of $[X]$ under the (mod-l) S-duality isomorphism $\backslash \mu_*(j_{2N})$ is a Thom class.

PROOF. The proof of this result follows along the usual lines as in [**Sw**, p. 334]. (See [**J-T**, p. 80] for more details.)

We conclude with an application of the above result to (mod-l^ν) topological (i.e., etale) K-theory. (mod-l^ν) etale K-theory is the generalised cohomology theory associated (as in §1) to the spectrum $BU \wedge M(l^\nu)$. Here BU is the spectrum giving complex K-theory, and $M(l^\nu)$ is a Moore spectrum. The corresponding generalised homology theory is also assumed to be obtained as in §1.

(5.5) COROLLARY. *Let X be a projective smooth variety. Then we obtain Poincaré-duality isomorphism between the (mod-l^ν) etale K-homology and (mod-l^ν) etale K-cohomology groups.*

PROOF. Let P^n be the ambient projective space in which X is imbedded as a closed subvariety. Let the normal bundle to this imbedding be δ. Let α be the vector bundle over P^n constructed in §2. We imbed Z by the zero section in the total space $E(\alpha|_X \oplus \delta)$. Let the dimension of this bundle be c. We denote the bundle $(\alpha|_X \oplus \delta)$ by Φ.

It is shown in [**J-3**] (and [**J-T**, Appendix B]) that in this case the (mod-l) S-dual of $X_{\mathrm{ht}\,+}$ is given by $\widehat{T(\Phi)}_{\mathrm{ht}}$, which is the Thom space of the vector bundle Φ. Therefore it follows from Proposition (5.10) that all we need is a Thom class θ_X in $h^{2c}(\widehat{T(\Phi)}_{\mathrm{ht}}; BU \wedge M(l^\nu))$. The existence of such a class is established in [**T**] (by letting it be the image of a class from (mod-l^ν) algebraic K-theory). The corollary follows.

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