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ON STARSHAPED REARRANGEMENT AND APPLICATIONS[1]

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ABSTRACT. A radial symmetrization technique is investigated and new properties are proven. The method transforms functions u into new functions u^* with starshaped level sets and is therefore called starshaped rearrangement. This rearrangement is in general not equimeasurable, a circumstance with some surprising consequences. The method is then applied to certain variational and free boundary problems and yields new results on the geometrical properties of solutions to these problems. In particular, the Lipschitz continuity of free boundaries can now be easily obtained in a new fashion.

Introduction. In [**26**] G. Szegö introduced the concept of radial symmetrization. This is a nonequimeasurable rearrangement which transforms functions $u:\Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^2$ with starshaped level sets $\Omega_c := \{x \in \overline{\Omega} | u(x) \geq c\}$ into functions u^* with symmetric starshaped level sets. Later [**2, 3, 15, 16, 22**] his approach was generalized to transform functions with level sets containing the origin into starshaped and not necessarily symmetric level sets and to higher dimensions. We call a set $D \subset \mathbf{R}^n$ *starshaped with respect to* x_0 if $x \in D$ implies $tx + (1-t)x_0 \in D$ for every $t \in [0,1]$. We call a set D *starshaped* if it is starshaped with respect to the origin. Therefore a more appropriate name for this rearrangement might be "starshaped rearrangement," in particular, since the notion of radial symmetrization can be easily confused with circular or spherical symmetrization or spherically symmetric rearrangement.

One of the standard results in the theory of rearrangements is that the capacity of a condenser decreases under rearrangement; i.e.,

$$\int_\Omega |\nabla u(x)|^p \, dx \geq \int_{\Omega^*} |\nabla u^*(x)|^p \, dx \quad \text{for } p \geq 1,$$

where $u \equiv 1$ on Ω_1, $u = 0$ on $\partial\Omega$ and $U_\varepsilon(0) \subset \Omega_1 \subset \Omega$, $\varepsilon > 0$, $0 \leq u \leq 1$ in Ω.

This feature can be found for starshaped rearrangement, too [**2, 3, 15, 16, 22**] but only at the sacrifice of defining different rearrangements $u^{*(p)}$ for different values of p. Corollary 1.4 states

$$\int_\Omega |\nabla u(x)|^p \geq \int_\Omega |\nabla u^{*(p)}(x)|^p \, dx \quad \text{for } p \geq 1, \tag{0.1}$$

provided $\Omega \subset \mathbf{R}^n$ is a bounded starshaped domain containing $U_\varepsilon(0)$ and $u:\overline{\Omega} \to [0,1]$, $u \in C^{0,1}(\overline{\Omega})$, $u \equiv 1$ in $U_\varepsilon(0)$ and $u = 0$ on $\partial\Omega$.

Another convenient feature of other rearrangements is their equimeasurability. A rearrangement $u \to u^*$ is called equimeasurable if the n-dimensional Lebesgue

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measure $m_n(\Omega_c)$ coincides with $m_n(\Omega_c^*)$ for all $c \in \operatorname{range} u$, where Ω_c^* is the corresponding level set of u^*. Among other things, equimeasurability implies Cavalieri's principle

$$\int_\Omega F(u(x))\,dx = \int_{\Omega^*} F(u^*(x))\,dx \tag{0.2}$$

for every Borel measurable function F.

Unfortunately u and $u^{*(p)}$ (for $p \geq 1$) are not equimeasurable, although u and $u^{*(0)}$ are. Since property (0.2) appears to be desirable for potential applications, one might ask if, in addition to (0.1), the inequality

$$\int_\Omega |\nabla u(x)|^p\,dx \geq \int_\Omega |\nabla u^{*(0)}(x)|^p\,dx \tag{0.3}$$

holds under the same conditions as (0.1).

As a first result we give a counterexample to (0.3). The proof is based on a surprising feature of this kind of rearrangement. Unlike Schwarz and Steiner symmetrization [**23**], the "surface" or perimeter of a level set can increase under rearrangement.

As a second result we discuss the strict inequality in (0.1). If u is smooth in a sense defined below, the equality sign in (0.1) implies $u = u^{*(p)}$. This is relevant because it has applications to variational problems with multiple solutions.

A third result is the fact that for nonequimeasurable rearrangements, equality (0.2) can be replaced by an inequality.

Therefore starshaped rearrangement can be applied to variational functionals of type

$$J(v) = \int_\Omega \{|\nabla v(x)|^p + F(v)\}\,dx \tag{0.4}$$

or of type

$$J(v) = \int_\Omega \{|\nabla v(x)|^2 + \chi_{\{v>0\}}\}\,dx. \tag{0.5}$$

Our results provide new statements about the geometry of level sets of solutions, e.g., Lipschitz continuity of the free boundary in a "jet problem" in $\mathbf{R}^n$ [**1**]. They also give new proofs of some already known results.

Other ways to prove starshapedness of level sets of a function u are usually based on maximum principles and require fairly strong regularity properties of u, which are unrealistic for our applications. We refer to [**11–13, 19, 24**] for those and similar questions.

Throughout the paper we use the common notation $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$ for Sobolev spaces of functions [**9, 18**]. $C(D)$ denotes functions that are continuous in D. If D is closed, they are continuous on D. $C^{k,\alpha}(D)$ are functions with derivatives up to order k and with α-Hölder continuous kth derivative, $0 \leq \alpha \leq 1$. A boundary $\partial\Omega$ of a domain Ω in $\mathbf{R}^n$ is said to be of class $C^{k,\alpha}$ if it can be described locally by a $C^{k,\alpha}$ function of $n-1$ variables. $C^{0,1}$ boundaries $\partial\Omega$ are called Lipschitz continuous. As usual $\mathbf{R}^+$ denotes $(0,\infty)$, $\mathbf{R}_0^+ = [0,\infty)$, $\mathbf{N} = \{1,2,3,\ldots\}$, $\mathbf{N}_0 = \{0,1,2,3,\ldots\}$ and ε and δ are sufficiently small positive real numbers. Unless otherwise indicated, all occurring integrals are understood to be Lebesgue integrals. This paper is a revised and abbreviated version of [**14**].

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I. Starshaped rearrangement. A common principle of symmetrization or rearrangement techniques is to replace a level set $\Omega_c := \{x \in \overline{\Omega} | u(x) \geq c\}$ by another one Ω_c^* which has the desired properties of symmetry or, in our case, starshapedness. The function u^* can then be constructed from its level sets Ω_c^* just like a three-dimensional mountain can be constructed from a map that shows all of its level lines.

For our purposes we shall define the rearrangement $u^{*(p)}$ of a function u under the following general assumptions.

(1.1) $\Omega \subset \mathbf{R}^n$ is a bounded starshaped domain containing $U_\varepsilon(0)$.

(1.2) $u := \overline{\Omega} \to [0,1]$, $u \in C^{0,1}(\overline{\Omega})$, $u \equiv 1$ in $U_\varepsilon(0)$ and $u = 0$ on $\partial\Omega$.

Therefore each nonempty level set of u will contain $U_\varepsilon(0)$. Let $D \subset \mathbf{R}^n$ be compact and $U_\varepsilon(0) \subset D$. To define $D^{*(p)}$, we shall use n-dimensional polar coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ of a point $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, which are defined by the relations

$$\begin{aligned} &|x| = r, \quad x_1 = r\cos\theta_1, \ldots, \\ &x_k = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{k-1}\cos\theta_k \quad \text{for } k = 2, \ldots, n-1, \\ &x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1}, \end{aligned} \tag{1.3}$$

where $0 \leq \theta_k \leq \pi$ for $k = 1, \ldots, n-2$ and $-\pi \leq \theta_{n-1} \leq \pi$. For typographical reasons let θ denote the vector of the angular coordinates $(\theta_1, \ldots, \theta_{n-1})$ and T the $(n-1)$-dimensional hypercube $[0,\pi]^{n-2} \times [-\pi, \pi]$. A point $x \in \mathbf{R}^n$ has the representation $x = (r, \theta)$, where $\theta \in T$ and $r \in \mathbf{R}_0^+$. The ray $\{x \in \mathbf{R}^n | x = (r,\theta),\ \theta = \theta',\ r \geq \varepsilon\}$ will be denoted by $L(\theta')$ and $D(\theta') = D \cap L(\theta')$.

Let $g: \mathbf{R}^+ \to \mathbf{R}$ be a positive and continuous function and let G be its primitive. We define

$$h(\theta) := \int_{D(\theta)} g(r)\,dr + G(\varepsilon), \tag{1.4}$$

$$R(\theta) := G^{-1}(h(\theta)), \tag{1.5}$$

where the integral in (1.4) is understood to be a Lebesgue integral. Notice that $R(\theta)$ does not depend on ε. For compact sets D containing $U_\varepsilon(0)$ we define

$$D^* := \{x \in \mathbf{R}^n | 0 \leq |x| \leq R(\theta),\ \theta \in T\}.$$

If D is empty, then by definition D^* is empty.

This construction transforms given sets D into starshaped ones, since the sets D^* are starshaped with respect to zero. In particular, let us from now on consider a special class of rearrangements, namely those induced by the family of metrics

$$g(r) = r^{\beta - 1} \quad \text{with } \beta \in \mathbf{R}. \tag{1.6}$$

In addition to the assumption $U_\varepsilon(0) \subset D$ and D compact, let us now assume that the rays $\theta =$ constant intersect ∂D in a finite (odd) number $2m+1$ of points

$$0 < \varepsilon \le r_1(\theta) < r_2(\theta) < \cdots < r_{2m+1}(\theta) \quad \text{with } m \in \mathbf{N}_0.$$

Then for $\beta \neq 0$

$$\begin{aligned} h(\theta) &= \frac{1}{\beta}(r_1^\beta - r_2^\beta + \cdots + r_{2m+1}^\beta), \\ R(\theta) &= (r_1^\beta - r_2^\beta + \cdots + r_{2m+1}^\beta)^{1/\beta}, \end{aligned} \tag{1.7}$$

while for $\beta = 0$

$$\begin{aligned} h(\theta) &= \log r_1 - \log r_2 + \cdots + \log r_{2m+1}, \\ R(\theta) &= \frac{r_1 r_3 r_5 \cdots r_{2m+1}}{r_2 r_4 \cdots r_{2m}}. \end{aligned} \tag{1.8}$$

The rearrangement of D under the metric $g(r) = r^{n-p-1}$ will be denoted by $D^{*(p)}$ from now on, where $p \in \mathbf{R}$, $n \in \mathbf{N}$. A simple calculation shows

$$\int_D dx = \int_{D^{*(0)}} dx \tag{1.9}$$

for compact sets D containing $U_\varepsilon(0)$. Now we are in a position to define the (decreasing) *starshaped rearrangement* $u^{*(p)}$ of a function u satisfying (1.1), (1.2) as follows:

$$u^{*(p)}(x) := \sup\{c \in [0,1] | x \in \Omega_c^{*(p)}\} \quad \text{for } x \in \overline{\Omega}^* \; (= \overline{\Omega}). \tag{1.10}$$

Then it is well known [**2, 3, 15, 16, 22**] that for every $p \ge 1$

$$\Omega_c^{*(p)} = \{x \in \overline{\Omega} | u^{*(p)}(x) \ge c\} \quad \text{for each } c \in [0,1], \tag{1.11}$$

so that the level sets of $u^{*(p)}$ are starshaped and

$$u^{*(p)} \text{is uniformly Lipschitz continuous on } \overline{\Omega}. \tag{1.12}$$

A new result is the following.

THEOREM 1.1. *There exist a domain* $\Omega \subset \mathbf{R}^2$ *and a function* $v \in W_0^{1,1}(\Omega)$ *satisfying* (1.1), (1,2), *for which* (0,1) *fails*; *i.e.*,

$$\int_\Omega |\nabla v^{*(0)}(x)|\, dx > \int_\Omega |\nabla v(x)|\, dx \tag{1.13}$$

holds.

The proof is based on H. Grabmüller's "long nose" (Lemma 1.2) and on Federer's coarea formula.

LEMMA 1.2. *There exists a compact domain* $D \subset U_\varepsilon(0)$ *in* $\mathbf{R}^2$ *such that the perimeter of* D *is shorter than the perimeter of* $D^{*(0)}$.

To prove Lemma 1.2 we simplify a construction in [**10**]. Let D be the union of an ε-neighborhood of zero with $0 < \varepsilon \ll 1$ and of the angular sector $\{x = (r,\theta) \in \mathbf{R}^2 | 1 \le r \le 2, |\theta| \le \varphi\}$ where $\varphi \in (0,\pi)$ is determined below. The perimeter of D

has length $|\partial D| = 2+3\varphi+2\pi\varepsilon$; $D^{*(0)} = \{x = (r,\theta) \in \mathbf{R}^2 | 0 \le r \le \sqrt{3}+O(\varepsilon),\ |\theta| \le \varphi\} \cup U_\varepsilon(0)$ and the perimeter of $D^{*(0)}$ has length $|\partial D^{*(0)}| = 2\sqrt{3} + \sqrt{3}\varphi + O(\varepsilon)$. Apparently $|\partial D^{*(0)}| > |\partial D|$ for sufficiently small φ, which proves the lemma.

To prove Theorem 1.1 let $\Omega \subset \mathbf{R}^2$ be a ball with radius 3 and center in the origin and let D be the set constructed in the proof of Lemma 1.2. D will be the support of a function $v\colon \Omega \to [0,1]$ defined by

$$v(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega}\backslash D, \\ 2/\varepsilon \cdot d(x,\partial D) & \text{if } x \in D \text{ and } d(x,\partial D) \le \varepsilon/2, \\ 1 & \text{if } x \in D \text{ and } d(x,\partial D) > \varepsilon/2. \end{cases} \tag{1.14}$$

If ε and φ are sufficiently small, we have the strict inequality $|\partial\Omega_c| < |\partial\Omega_c^{*(0)}|$ for every $c \in (0,1)$, according to Lemma 1.2. But Federer's coarea formula states [**5**]

$$\int_\Omega |\nabla v(x)|\, dx = \int_{\mathbf{R}} P(t)\, dt,$$

where $P(t) =$ perimeter of $\{x \in \Omega | w(x) > t\}$. This implies (1.13).

To formulate the next theorem, we introduce some definitions. We call a function $u\colon \Omega \to [0,1]$ *simple* if u satisfies (1.1), (1.2) and if u is piecewise linear in Ω. We call a function $u\colon \Omega \to [0,1]$ *smooth* if u satisfies (1.1), (1.2) and if u has the properties

(i) For almost every $\theta \in T$ and every $c \in (0,1)$ the set of points $\{(r,\theta) \in \Omega | u(r,\theta) = c\}$ is finite.

(ii) For almost every $\theta \in T$ the set of points $\{(r,\theta) \in \Omega | (\partial u/\partial r)(r,\theta) = 0$ and $u(r,\theta) \in (0,1)\}$ is finite.

THEOREM 1.3. *Let u be simple or smooth and $p \ge 1$ and let $H(\theta,t)\colon T\times[0,1] \to \mathbf{R}^+$ be continuous. Then the inequality*

$$\int_\Omega H(\theta, u^{*(p)})|\nabla u^{*(p)}(x)|^p\, dx \le \int_\Omega H(\theta,u)|\nabla u(x)|^p\, dx \tag{1.15}$$

holds, and for $p > 1$ equality holds only if $u = u^{*(p)}$.

Inequality (1.15) was derived in [**2, 3**] under the additional assumption $H \equiv 1$. A careful inspection of their proof reveals the extension given here and, in particular, the strictness of this inequality. The reader who wants to check this should observe that in the notation of [**3**], one has to set $p(\rho) = \rho + \varepsilon^{n-p}$ for $n \ne \alpha$ and $p(\rho) \equiv 1$ for $n = \alpha$.

COROLLARY 1.4. *Let u satisfy (1.1), (1.2). Then the following holds for $p \ge 1$:*

$$\int_\Omega |\nabla u^{*(p)}(x)|^p\, dx \le \int_\Omega |\nabla u(x)|^p\, dx. \tag{1.16}$$

This follows with more or less standard approximation arguments and using a trick from [**6**]. One approximates u by simple functions u_n and shows that $\{u_n^{*(p)}\}_{n\in\mathbf{N}}$ has a subsequence converging to $u^{*(p)}$ in $W_0^{1,p}(\Omega)$. At the suggestion of the referee the details are left to the reader.

Next we intend to find a substitute for (0.2). To this end we want to relate $u^{*(0)}$ to $u^{*(p)}$ for $p \ge 1$ or $\Omega_c^{*(0)}$ to $\Omega_c^{*(p)}$.

LEMMA 1.5. *Let $m \in \mathbf{N}_0$ and $0 < \varepsilon \le r_1 < r_2 < \cdots < r_{2m+1}$, $r_j \in \mathbf{R}$, let $2 \le n \in \mathbf{N}$ and $1 \le p < \infty$, $p \ne n$. Then the following inequalities hold:*

$$(r_1^{n-p} - r_2^{n-p} + \cdots + r_{2m+1}^{n-p})^{1/(n-p)} \le (r_1^n - r_2^n + \cdots + r_{2m+1}^n)^{1/n}, \tag{1.17}$$

$$\frac{r_1 \cdot r_3 \cdots r_{2m+1}}{r_2 \cdot r_4 \cdots r_{2m}} \le (r_1^n - r_2^n + \cdots + r_{2m+1}^n)^{1/n}. \tag{1.18}$$

If we introduce the notation $y_j = r_j^n$, $\alpha = (n-p)/n$, inequality (1.17) is equivalent to

$$F(y_1, \ldots, y_{2m+1}) := (y_1^\alpha - y_2^\alpha + \cdots + y_{2m+1}^\alpha)^{1/\alpha} - y_1 + y_2 - \cdots - y_{2m+1} \le 0. \tag{1.19}$$

For $n > p$ or $\alpha > 0$ inequality (1.19) is recorded in [**20**], and the proof is reduced to Weinberger's inequality [**17**, p. 112]. For $p > n$ or $\alpha < 0$ one can either modify the proof of Payne and Weinstein in an obvious fashion or apply Theorem 5 on p. 112 in [**17**].

Finally l'Hospital's rule implies that (1.18) follows from (1.17). This was kindly pointed out by the referee. Another proof of (1.18) would follow along the lines of [**20**].

THEOREM 1.6. *Let u satisfy* (1.1), (1.2), *and $p \ge 1$. Then the following inequalities hold:*

$$u^{*(0)}(x) \ge u^{*(p)}(x) \quad in\ \Omega, \tag{1.20}$$

and

$$\int_\Omega F(u(x))\,dx \ge \int_\Omega F(u^{*(p)}(x))\,dx \tag{1.21}$$

for $F: [0,1] \to \mathbf{R}$ monotone nondecreasing and Borel measurable.

It suffices to prove this theorem for simple functions; the rest follows from an approximation argument. The level sets of simple functions are polyhedral, so for almost every $\theta \in T$ the boundary of a level set is intersected only finitely often, an odd number of times. Now we recall (1.7), (1.8) and Lemma 1.5 to see that

$$\Omega_c^{*(0)} \supset \Omega_c^{*(p)} \quad \text{modulo a nullset}, \tag{1.22}$$

so that (1.20) holds a.e. in Ω, and by continuity of $u^{*(0)}$ and $u^{*(p)}$ (1.20) holds everywhere in Ω. Now (1.21) follows from (1.20), (1.9) and (0.2), since

$$\int_\Omega F(u(x))\,dx = \int_\Omega F(u^{*(0)}(x))\,dx \ge \int_\Omega F(u^{*(p)}(x))\,dx.$$

To conclude our results on starshaped rearrangement, we give a counterexample to another type of inequality,

$$\int_\Omega u(x)v(x)\,dx \le \int_\Omega u^*(x)v^*(x)\,dx \tag{1.23}$$

which does hold for Schwarz, Steiner or any other equimeasurable rearrangement.

COROLLARY 1.7. *There exist a domain $\Omega \subset \mathbf{R}^2$ and functions u and v satisfying* (1.1), (1.2) *such that for $p \geq 1$ inequality* (1.23) *fails. Moreover, for these functions even the following inequality holds*:

$$\int_\Omega u^{*(p)}(x)v(x)\,dx > \int_\Omega u^{*(p)}(x)v^{*(p)}(x)\,dx. \tag{1.24}$$

To prove this corollary we extend the definition of starshaped rearrangement to characteristic functions of compact sets containing $U_\varepsilon(0)$. Then we set $\tilde{u}(x) = \tilde{u}^{*(p)}(x) = \chi_{B_{3(0)}}(x)$, where $B_3(0)$ is the ball with radius 3 and center in the origin. We set $\tilde{v}(x) = \chi_R(x)$ where $R = B_\varepsilon(0) \cup (B_2(0)\backslash B_1(0))$ and verify

$$\int_\Omega \tilde{u}^{*(p)}(x)\tilde{v}(x)\,dx \geq \int_\Omega u^{*(p)}(x)\tilde{v}^{*(p)}(x)\,dx$$

by computation, using (1.7) and (1.8). Now (1.24) follows after mollification of $\tilde{v}$ and $\tilde{u}$.

II. Applications.

EXAMPLE 1. *Capacitory problems.* Let $\Omega_1 \subset \mathbf{R}^n$ be starshaped with respect to the origin and contain $U_\varepsilon(0)$. Let $\Omega_1 \Subset \Omega_0 \subset \mathbf{R}^n$, where Ω_0 is bounded and starshaped with respect to zero. Let $u \in C^{0,1}(\overline{\Omega}_0)$, $0 \leq u \leq 1$, be a solution to the variational problem, $p \geq 1$,

$$\text{(V1)} \quad \operatorname*{Min}_{v\in A_1} J_1(v) := \int_\Omega \left\{\frac{1}{p}|v(x)|^p + F(v(x))\right\}dx,$$

where $A_1 := \{v \in W_0^{1,p}(\Omega_0) | v \equiv 1 \text{ on } \Omega_1\}$.

COROLLARY 2.1. *Suppose $F\colon [0,1] \to \mathbf{R}$ is monotone nondecreasing and continuous and that at least one of the conditions* (i), (ii) *or* (iii) *holds*:

(i) *Problem* (V1) *has a unique solution.*
(ii) *Problem* (V1) *has only smooth solutions.*
(iii) *If the problem has two different solutions u and w, then either* $\operatorname{supp} u \supset \operatorname{supp} w$ *and $u > w$ in* $\operatorname{supp} u\backslash\overline{\Omega}_1$, *or* $\operatorname{supp} w \supset \operatorname{supp} u$ *and $w > u$ in* $\operatorname{supp} w\backslash\overline{\Omega}_1$. *Furthermore, let F be nonconstant.*

Then every solution u of problem (V1) *has starshaped level sets.*

Under assumptions (i) or (ii) the proof follows from Theorem 1.3, Theorem 1.6 and from the observation $J_1(u^{*(p)}) = J_1(u)$. Under assumption (iii) we note that if u is a solution, then so is $u^{*(p)}$ and set $w = u^{*(p)}$ to reach a contradiction. Therefore, $u = u^{*(p)}$.

If F is differentiable with derivative $f\colon [0,1] \to \mathbf{R}$, then solutions u of the variational problem (V1) are weak solutions of the degenerate elliptic boundary value problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in } \Omega, \tag{2.1}$$

$$u = 1 \quad \text{on } \partial\Omega_1, \qquad u = 0 \quad \text{on } \partial\Omega_0. \tag{2.2}$$

If F is convex with subdifferential $f = \partial F$, then u is the unique weak solution of the differential inclusion

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \in f(u) \quad \text{in } \Omega \tag{2.3}$$

under boundary conditions (2.2).

Previous versions of Corollary 2.1 for special p and F (mainly $F \equiv 0$, $p = 2$) are known [**2, 3, 11, 13, 15, 16, 22**]. The novelty is that the starshapedness can now be derived without monotonicity assumptions on f and without differentiability assumptions on f or u.

Finally we want to indicate a case in which assumption (iii) of Corollary 2.1 is satisfied. In the paper of A. Friedman and D. Phillips [**9**] one can find that (iii) holds, provided

$$p = 2, \qquad f(t) = \begin{cases} t^\alpha f_0(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad \text{for some } \alpha \in (0,1) \tag{2.4}$$

and $m \leq f_0(t) \leq M$, $0 < m \leq M < \infty$.

EXAMPLE 2. *Exterior free boundary problems in the context of reaction-diffusion.* Let Ω_1 be given as before and let $u\colon \mathbf{R}^n \to [0,1]$ be a Lipschitz continuous solution of the variational problem

$$\text{(V2)} \quad \operatorname*{Min}_{v \in A_2} J_2(v) := \int_\Omega \left\{ \frac{1}{p} |\nabla v(x)|^p \, dx + \lambda^2 F(v(x)) \right\} dx,$$

$$\text{where } A_2 := \{ v \in W^{1,2}_{\text{loc}}(\mathbf{R}^n) | v \equiv 1 \text{ on } \overline{\Omega}_1 \},$$

where $F\colon \mathbf{R} \to \mathbf{R}_0^+$ is a convex, monotone nondecreasing function with $F(0) = 0$, and where $\lambda \in \mathbf{R}^+$ is given. Furthermore, suppose that

$$\text{the support of } u \text{ is bounded.} \tag{2.5}$$

A sufficient condition for (2.5) is

$$\int_0^1 [F(t)]^{-1/p} \, dt < \infty, \tag{2.6}$$

as was kindly pointed out to me by J. I. Diaz [**4**]. As λ tends to infinity, the support of u shrinks to a boundary layer; see [**7, 9**]. As a consequence of Corollary 2.1 the support of u and all level sets will be starshaped with respect to zero. A further consequence is

COROLLARY 2.2. *Suppose, in addition to the above assumptions on Ω_1 and F, that*

(2.7) *Ω_1 is starshaped with respect to each point y in a small nonempty closed neighborhood $U_\delta(0)$ of the origin.*

Then all the level sets Ω_c of u for $c \in (0,1)$ and the support of u have Lipschitz-continuous boundary.

For the proof we can construct a nonempty cone C with vertex on $\partial\Omega_c$ such that $\partial\Omega_c \cap C$ consists only of the vertex. Notice that (2.7) holds if Ω_1 is convex.

EXAMPLE 3. *An exterior boundary value problem arising in potential flow.* Let $u\colon \mathbf{R}^n \to \mathbf{R}$ be a solution of the variational problem

$$\text{(V3)} \quad \operatorname*{Min}_{v \in A_3} J_3(v) := \int_{\mathbf{R}^n \setminus \Omega_1} \{ |\nabla v(x)|^2 + \lambda^2 \chi_{\{v>0\}}(x) \} \, dx,$$

$$\text{where } A_3 := \{ v \in W^{1,2}_{\text{loc}}(\mathbf{R}^n) | v \equiv 1 \text{ on } \overline{\Omega}_1 \}.$$

Here χ_S denotes the characteristic function of a set S. The following results were derived in the pioneering paper [**1**] of H. W. Alt and L. Caffarelli. Let $\partial\Omega_1$ be sufficiently smooth and Ω_1 bounded. Then there exists a solution $u \in A_3$ to problem (V3). Any solution of (V3) has bounded support and belongs to $C^{0,1}(\mathbf{R}^n)$. Furthermore, u is harmonic in the set $\{x \in \mathbf{R}^n \backslash \overline{\Omega}_1 | u(x) > 0\}$, and $0 \leq u(x) \leq 1$ in $\mathbf{R}^n$. Formally u is a solution to the free boundary problem

$$\Delta u = 0 \quad \text{on } \{u > 0\} \backslash \overline{\Omega}_1, \tag{2.8}$$

$$u = 1 \quad \text{on } \partial\Omega_1, \tag{2.9}$$

$$u = 0 \quad \text{and} \quad |\partial u / \partial n| = \lambda \quad \text{on } \partial\{u > 0\}. \tag{2.10}$$

Notice that in particular the last condition has to be interpreted in a generalized sense, since the free boundary $\partial\{u > 0\}$ might not be sufficiently smooth to define a normal vector field on it. Assuming a certain flatness condition, the free boundary is known to be of class $C^{1,\alpha}$ and even analytic. This flatness condition can be verified for the plane case $n = 2$. If the free boundary satisfies the interior sphere condition, one can easily prove uniqueness [**27, 28**]; see also [**25**].

For dimensions $n \geq 3$ one has examples of solutions to (2.8), (2.9), (2.10) with singular free boundary.

COROLLARY 2.4. *Suppose that u is any solution of problem* (V4), *that Ω_1 is bounded and satisfies* (2.7). *Then all the level sets of u are starshaped and the free boundary $\partial\{u > 0\}$ is Lipschitz continuous.*

For the proof we can modify an idea in [**9**, §1] and show that different solutions u and v of (V4) are nested in the sense of Corollary 2.1(iii). Then we can proceed as in the proof of Corollary 2.1.

In two dimensions the starshapedness of $\{u > 0\}$ was derived in [**28**] by a different method.

An extension of this application to functionals of type

$$J(v) = \int_{\mathbf{R}^n \backslash \overline{\Omega}_1} \{|\nabla v(x)|^2 + v^\alpha(x) \chi_{\{v>0\}}(x)\}\, dx, \qquad \alpha \in (0,1),$$

is possible. Such a functional was studied in [**21**].

EXAMPLE 4. *An obstacle problem.* Let $\Omega_0 \subset \mathbf{R}^n$ be starshaped with respect to zero and let $\partial\Omega_0$ be sufficiently smooth. Let $\psi \in C^{1,1}(\overline{\Omega}_0)$ be given with $\psi < 0$ on $\partial\Omega_0$, $\psi(0) = \max\{\psi(x) | x \in \overline{\Omega}_0\}$, and suppose that all the level sets of ψ are starshaped with respect to zero. Let u be a solution of the variational problem

$$\text{(V4)} \qquad \operatorname*{Min}_{v \in A_4} J_1(v), \quad \text{where } A_4 := \{v \in W^{1,2}(\Omega_0) | v \geq \psi \text{ a.e. in } \Omega_0\}.$$

Then under the assumptions of Corollary 2.1 the function u has starshaped level sets. This can be deduced by cutting off ψ at "height" $\psi(0) - \delta$ and by the limiting process $\delta \to 0^+$ [**13**]. For another proof under the stronger assumptions $x \cdot \nabla\psi < 0$ in $\Omega_0 \backslash \{0\}$ and F convex, we refer to [**11**].

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