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ON MAXIMAL FUNCTIONS AND POISSON-SZEGÖ INTEGRALS

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ABSTRACT. We study a class of maximal functions of Hardy-Littlewood type defined on spaces of homogeneous type and we give necessary and sufficient conditions for the corresponding maximal operators to be of weak type $(1,1)$. As a consequence we show that Poisson-Szegö integrals of L^p functions possess certain boundary limits which are not implied by Korányi's theorem.

Introduction. The classical theorem of Fatou is concerned with the existence of nontangential boundary limits of the Poisson integral of a function f defined on $\mathbf{R}^n$ or on the unit circle. Such limits exist almost everywhere if f is in $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$ (see [**SW**, Chapter II]). In [**NS**], Nagel and Stein improve Fatou's theorem by showing the existence a.e. of limits within certain approach regions which are not contained in any nontangential region. The problem is reduced to showing that a maximal operator "of Hardy-Littlewood type" associated to the approach regions is of weak type $(1,1)$.

In this paper we consider a similar question for Poisson-Szegö integrals on the unit ball of $\mathbf{C}^n$ and on the generalized half-plane (also called Siegel half-space). Here the analogue to Fatou's theorem is the theorem of Korányi [**K2**] asserting the existence a.e. of limits within the so-called "admissible regions." As is well known, these regions allow parabolic tangential approach to the boundary along certain directions (the "complex directions"). We show that it is possible to strengthen the conclusion of Korányi's theorem as to allow for some approach regions not contained in any admissible region.

The maximal operator of Hardy-Littlewood type considered in [**NS**] is defined as follows. Let Ω be a subset of $\mathbf{R}^{n+1}_+$ with $(0,0) \in \overline{\Omega}$, i.e. Ω is an approach region at $0 \in \mathbf{R}^n$. We are interested in the existence for almost every $x_0 \in \mathbf{R}^n$ of the limit

$$(*) \qquad \lim_{\substack{(x,r)\to(0,0)\\(x,r)\in\Omega}} u(x_0+x,r),$$

where u is the Poisson integral of some $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$. (We denote points of $\mathbf{R}^{n+1}_+$ by (x,r), with $x \in \mathbf{R}^n$ and $r > 0$.) If f is locally integrable on $\mathbf{R}^n$, the maximal function $M_\Omega f$ is defined by

$$M_\Omega f(x_0) = \sup_{(x,r)\in\Omega} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x_0+t)|\, dt$$

for each $x_0 \in \mathbf{R}^n$. (Here $B(x,r)$ is the ball $\{t \in \mathbf{R}^n\colon |t-x| < r\}$, and $|E|$ denotes the n-dimensional volume of the set E.) To show the existence a.e. of the limit

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$(*)$ it suffices to show that M_Ω is of weak type $(1,1)$. In [**NS**], a necessary and sufficient condition on Ω is given for M_Ω to be of weak type $(1,1)$. The condition is essentially (modulo some mild restriction on the shape of Ω) that the cross-sectional measure of Ω at height r should be bounded by a constant times r^n. It then turns out that there are approach regions Ω not contained in any nontangential cone $\Gamma_\alpha = \{(x,r) \in \mathbf{R}^{n+1}_+ : |x| < \alpha r\}$, $\alpha > 0$, but for which M_Ω is of weak type $(1,1)$, and therefore the limit $(*)$ exists for almost every $x_0 \in \mathbf{R}^n$.

The situation for Poisson-Szegö integrals on the generalized half-plane D is quite similar. The boundary of D can be identified with a Heisenberg group H_n and so we can think of D as being $H_n \times (0,\infty)$, just as $\mathbf{R}^{n+1}_+$ is identified with $\mathbf{R}^n \times (0,\infty)$. Thus a point of D has "coordinates" (x,r), with $x \in H_n$ and $r > 0$. On H_n one can define a translation-invariant metric (which determines a family of "balls") and a translation-invariant Borel measure μ. Given $\Omega \subset H_n \times (0,\infty)$ and $f \in L^1_{\text{loc}}(\mu)$, define

$$M_\Omega f(x_0) = \sup_{(x,r)\in\Omega} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(x_0 t)|\, d\mu(t)$$

for each $x_0 \in H_n$, where $B(x,r)$ is the ball of center $x \in H_n$ and radius $r > 0$. Assume that $(0,0) \in \overline{\Omega}$. As before, the existence for almost every $x_0 \in H_n$ of the limit

$$\lim_{\substack{(x,r)\to(0,0)\\(x,r)\in\Omega}} u(x_0 x, r),$$

where u is the Poisson-Szegö integral of $f \in L^p(\mu)$, $1 \le p \le \infty$, is a consequence of the weak type $(1,1)$ of M_Ω.

Thus we seek to characterize those regions $\Omega \subset H_n \times (0,\infty)$ for which M_Ω is of weak type $(1,1)$. By analogy with the result of Nagel and Stein [**NS**] (for $\Omega \subset \mathbf{R}^{n+1}_+$), in our case one would expect the condition on Ω to be essentially that the cross-sectional μ-measure of Ω at "height" r should be bounded by a constant times the μ-measure of a ball (in H_n) of radius r. This is in fact the right condition. Nevertheless, it is interesting to notice that the method of proof in [**NS**] does not work in this situation. The reason is that the proof of a modified covering lemma as given in [**NS**] for $\mathbf{R}^n$ breaks down for H_n because of the noncommutativity of the Heisenberg group.

Therefore a different approach is needed. The key idea is that although $M_\Omega f$ cannot in general be dominated *pointwise* by the Hardy-Littlewood maximal function Mf (unless Ω is nontangential, but we are interested precisely in approach regions that are not contained in nontangential cones), it is still possible to dominate the distribution function of $M_\Omega f$ by that of Mf, provided that Ω satisfies the appropriate condition (in terms of cross-sectional measure). The weak type $(1,1)$ of M_Ω is then a consequence of the weak type $(1,1)$ of M.

An additional advantage of this approach is that it can be used in a much more general situation. Both $\mathbf{R}^n$ and H_n are examples of spaces of homogeneous type (see [**CW**]). We are thus led to study maximal operators of Hardy-Littlewood type in the framework of the spaces of homogeneous type. This has the virtue of clarifying the role played by the group structure of $\mathbf{R}^n$ or H_n. It turns out that it is possible to do away with any group structure and still be able to prove results on weak type $(1,1)$ for the maximal operators under consideration. The group structure just simplifies the statement of the results. In particular, the commutativity of the group structure of $\mathbf{R}^n$ is totally irrelevant.

The paper is divided into four sections. In §1, we study generalizations of M_Ω acting on functions defined on spaces of homogeneous type X where no group structure is assumed. Since no group translations are available, it becomes necessary to assume that for each $x_0 \in X$ we are given a set $\Omega_{x_0} \subset X \times (0, \infty)$ (the "approach region at x_0"). If f is locally integrable on X, $M_\Omega f$ is defined by

$$M_\Omega f(x_0) = \sup_{(x,r)\in\Omega_{x_0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|, \qquad x_0 \in X.$$

We give necessary and sufficient conditions on the family $\{\Omega_{x_0} \colon x_0 \in X\}$ for M_Ω to be of weak type $(1,1)$. In the main result of this section, Theorem 1.5, no mention is made of the Hardy-Littlewood maximal operator M. However, the idea underlying its proof is to dominate the measure of the set where $M_\Omega f > \lambda$ $(\lambda > 0)$ by the measure of the set where $Mf > \lambda$.

In §2, we consider a more specific situation. The space of homogeneous type X is now also a group, the measure and the metric are invariant under left-translations, and M_Ω commutes with left-translations. (Of course, right-translations could be used instead.) In this setting the results from §1 take on a very simple form. In particular, we recover the result of Nagel and Stein [**NS**] as an easy consequence of Theorem 2.4. In fact, our methods can be used to give a short direct proof of that result (see [**Su**]).

In §3, we apply the results of the preceding section to the case $X = H_n$ (Heisenberg group) in order to study boundary limits of Poisson-Szegö integrals on the generalized half-plane D. We are able to show the existence a.e. of certain limits which are not implied by the theorem of Korányi [**K2**]. That is, we obtain the existence a.e. of certain "nonadmissible" limits.

The analogous question for the unit ball B of $\mathbf{C}^n$ is taken up in §4. We use the Cayley transform to transfer the results from D to B. Since admissible convergence is preserved by the Cayley transform, we obtain results of "nonadmissible" convergence for Poisson-Szegö integrals on the unit ball.

1. Maximal functions on spaces of homogeneous type. We recall the definition of space of homogeneous type [**CW**, Chapter III]:

DEFINITION. Let X be a topological space. Assume d is a pseudo-distance on X, i.e. a nonnegative function defined on $X \times X$ satisfying

(i) $d(x,x) = 0$; $d(x,y) > 0$ if $x \neq y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq K[d(x,y) + d(y,z)]$, where K is some fixed constant.

Assume further that

(a) the balls $B(x,r) = \{y \in X \colon d(x,y) < r\}$ form a basis of open neighborhoods at $x \in X$

and that μ is a Borel measure on X such that

(b) $0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$, where A is some fixed constant.

Then we call (X, d, μ) a *space of homogeneous type.*

Properties (iii) and (b) will be referred to as the "triangle inequality" and the "doubling property." Note that (b) implies that for every $C > 0$ there exists

$A_C < \infty$ such that

$$\mu(B(x, Cr)) \le A_C \mu(B(x,r)) \tag{1.1}$$

for all $x \in X$ and $r > 0$. In fact, if $C < 2^n$ then we can take $A_C = A^n$.

We define a general Hardy-Littlewood type maximal operator on a space of homogeneous type X as follows. Suppose that for each $x_0 \in X$ we are given a set $\Omega_{x_0} \subset X \times (0, \infty)$. Let Ω denote the family $\{\Omega_{x_0}\}_{x_0 \in X}$. Thus at each $x_0 \in X$, Ω determines a collection of balls, namely $\{B(x,r)\colon (x,r) \in \Omega_{x_0}\}$. For $f \in L^1_{\mathrm{loc}}(d\mu)$ (i.e. f is integrable over balls) and $x_0 \in X$ set

$$M_\Omega f(x_0) = \sup_{(x,r)\in\Omega_{x_0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|\, d\mu, \tag{1.2}$$

where $|E|$ denotes $\mu(E)$. If Ω_{x_0} is empty, set $M_\Omega f(x_0) = 0$.

EXAMPLES. (1) If $\Omega_{x_0} = \{(x,r) \in X \times (0,\infty)\colon d(x,x_0) < r\}$, then $M_\Omega f$ is the usual Hardy-Littlewood maximal function Mf defined by

$$Mf(x_0) = \sup \frac{1}{|B|} \int_B |f|\, d\mu,$$

where the supremum is taken over all the balls B containing x_0.

(2) If $\alpha > 1$ and we set $\Omega_{x_0} = \{(x,r)\colon d(x,x_0) < \alpha r\}$, then $M_\Omega f$ is a "non-tangential" Hardy-Littlewood maximal function. When $X = \mathbf{R}^n$ (so $X \times (0,\infty)$ can be identified with the upper half-space $\mathbf{R}^{n+1}_+$), the set Ω_{x_0} in this example is a nontangential cone in $\mathbf{R}^{n+1}_+$ with vertex $x_0 \in \mathbf{R}^n$ and aperture α.

(3) If $X = \mathbf{R}^n$ and $\Omega_0 \subset \mathbf{R}^{n+1}_+$ is given, set $\Omega_{x_0} = (x_0, 0) + \Omega_0$ for each $x_0 \in \mathbf{R}^n$. Then (1.2) defines the maximal function $M_\Omega f$ introduced in [**NS**, p. 87].

In all that follows it may be helpful to keep in mind the case $X = \mathbf{R}^n$ ($X \times (0,\infty)$ identified with $\mathbf{R}^{n+1}_+$).

We are interested in knowing when M_Ω is of weak type $(1,1)$. Since M_Ω is obviously bounded on $L^\infty(\mu)$, weak type $(1,1)$ implies strong type (p,p) for $p > 1$, by the Marcinkiewicz interpolation theorem.

Our condition on Ω for M_Ω to be of weak type $(1,1)$ involves the following sets.

DEFINITION. For $\alpha > 0$ and $(x,r) \in X \times (0,\infty)$ set

$$S_\alpha(x,r) = \{x_0 \in X\colon \Omega_{x_0}(r) \cap B(x,\alpha r) \ne \varnothing\}, \tag{1.3}$$

where $\Omega_{x_0}(r) = \{x \in X\colon (x,r) \in \Omega_{x_0}\}$ is the "cross-section" of Ω_{x_0} "at height r."

An alternative description of $S_\alpha(x,r)$ is as follows. Define

$$\Omega^\alpha_{x_0} = \{(x,r) \in X \times (0,\infty)\colon d(x,y) < \alpha r \text{ for some } (y,r) \in \Omega_{x_0}\}.$$

In other words,

$$\Omega^\alpha_{x_0}(r) = \bigcup_{y\in\Omega_{x_0}(r)} B(y,\alpha r) = \{x \in X\colon d(x, \Omega_{x_0}(r)) < \alpha r\}.$$

Then $x_0 \in S_\alpha(x,r)$ if and only if $(x,r) \in \Omega^\alpha_{x_0}$.

Recall that an operator T is of *weak type* (p,p), $0 < p < \infty$, if there exists a constant $C < \infty$ so that

$$|\{|Tf| > \lambda\}| \le C(\|f\|_p/\lambda)^p$$

for all $f \in L^p$ and $\lambda > 0$.

PROPOSITION 1.4. *If the maximal operator M_Ω defined by* (1.2) *is of weak type* (p,p) *for some* p, $0 < p < \infty$, *then for every* $\alpha > 0$ *there is a constant* $\beta < \infty$ *so that*

$$|S_\alpha(x,r)| \le \beta |B(x,r)|$$

for all $(x,r) \in X \times (0,\infty)$. (*Recall that* $|E|$ *stands for* $\mu(E)$.)

PROOF. Let f be the characteristic function of $B(x, K(\alpha+1)r)$, where K is the constant in the triangle inequality. Then

$$\|f\|_p^p = |B(x, K(\alpha+1)r)| \le A'|B(x,r)|,$$

where $A' = A_{K(\alpha+1)}$ (see 1.1)). If $x_0 \in S_\alpha(x,r)$, then $(y,r) \in \Omega_{x_0}$ for some $y \in B(x, \alpha r)$. Therefore by the triangle inequality, $B(y,r) \subset B(x, K(\alpha+1)r)$ and hence

$$M_\Omega f(x_0) \ge \frac{1}{|B(y,r)|} \int_{B(y,r)} |f| = 1.$$

Thus

$$|S_\alpha(x,r)| \le |\{M_\Omega f \ge 1\}| \le C\|f\|_p^p \le CA'|B(x,r)|$$

as desired.

We intend to show that the same condition on the sets $S_\alpha(x,r)$ is sufficient for M_Ω to be of weak type $(1,1)$. A key role in the proof will be played by the following result [**CW**, p. 69].

COVERING LEMMA. *Let E be a bounded subset of X, i.e. E is contained in some ball. Let $r(x)$ be a positive number for each $x \in E$. Then there is a (finite or infinite) sequence of disjoint balls $B(x_i, r(x_i))$, $x_i \in E$, such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \le 2r(x_i)$.*

The last assertion follows from the proof given in [**CW**]. In fact, x_n is chosen inductively in such a way that

$$x_n \in E_n = E \setminus \bigcup_{i=1}^{n-1} B(x_i, 4Kr(x_i))$$

and

$$r(x_n) > \tfrac{1}{2} \sup_{x \in E_n} r(x).$$

Given $x \in E$, let n be the first index so that $x \in B(x_n, 4Kr(x_n))$. Thus $x \in E_n$ and therefore $r(x_n) > r(x)/2$.

We now state and prove the main result of this section.

THEOREM 1.5. *Assume that Ω satisfies*

(i) *If $x_0 \in X$, $(x,r) \in \Omega_{x_0}$ and $s \ge r$, then $(x,s) \in \Omega_{x_0}$.*

(ii) *There are constants $\alpha > 0$ and $\beta < \infty$ so that $|S_\alpha(x,r)| \le \beta|B(x,r)|$ for all $x \in X$ and $r > 0$.*

Then M_Ω is of weak type $(1,1)$ and hence of strong type (p,p) for $1 < p \le \infty$.

PROOF. Let $f \in L^1(\mu)$. We must show that there is a constant $C < \infty$, independent of f, so that

$$|\{M_\Omega f > \lambda\}| \le C\|f\|_1/\lambda$$

for all $\lambda > 0$.

Fix $\lambda > 0$. Set

$$E_\lambda = \left\{ x \in X \colon \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| > \lambda \right\}.$$

For each $x \in E_\lambda$ let

$$r(x) = \sup \left\{ r > 0 \colon \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| > \lambda \right\}.$$

Thus for each $x \in E_\lambda$ we have $r(x) > 0$ and

$$\frac{1}{|B(x,r(x))|} \int_{B(x,r(x))} |f| \geq \lambda. \tag{1.6}$$

(We are tacitly assuming that $r(x)$ is everywhere finite. There is no loss of generality in doing so, for if $r(x) = \infty$ for some x, then $|X|^{-1} \int_X |f| \geq \lambda$ by monotone convergence. But this says $|X| \leq \|f\|_1/\lambda$, so there is nothing to prove.)

Assume first that E_λ is bounded. Apply the Covering Lemma to the balls $B(x, r(x))$ to obtain a sequence of disjoint balls $B(x_i, r_i)$, $r_i = r(x_i)$, so that

$$E_\lambda \subset \bigcup_i B(x_i, 4Kr_i).$$

We want to show that $\{M_\Omega f > \lambda\} \subset \bigcup_i S_\alpha(x_i, (4K/\alpha) r_i)$. If $M_\Omega f(x_0) > \lambda$, then

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f| > \lambda$$

for some $(x,r) \in \Omega_{x_0}$. Thus $x \in E_\lambda$ and $r \leq r(x)$. By the last part of the Covering Lemma, $x \in B(x_i, 4Kr_i)$ for some i such that $r(x) \leq 2r_i$. We can assume $\alpha \leq 2K$ in (ii) because $S_\alpha(x,r) \subset S_{\alpha'}(x,r)$ if $\alpha \leq \alpha'$. Consequently, $r \leq r(x) \leq 2r_i \leq (4K/\alpha) r_i$ and, by (i),

$$\left(x, \frac{4K}{\alpha} r_i \right) \in \Omega_{x_0}.$$

But $x \in B(x_i, \alpha(4K/\alpha) r_i)$, so we get

$$x \in \Omega_{x_0} \left(\frac{4K}{\alpha} r_i \right) \cap B\left(x_i, \alpha \frac{4K}{\alpha} r_i \right)$$

and therefore $x_0 \in S_\alpha(x_i, (4K/\alpha) r_i)$. This shows that

$$\{M_\Omega f > \lambda\} \subset \bigcup_i S_\alpha \left(x_i, \frac{4K}{\alpha} r_i \right).$$

Thus

$$\begin{aligned}
|\{M_\Omega f > \lambda\}| &\le \sum_i \left| S_\alpha\left(x_i, \frac{4K}{\alpha} r_i\right)\right| \\
&\le \beta \sum_i \left| B\left(x_i, \frac{4K}{\alpha} r_i\right)\right| && \text{(by (ii))} \\
&\le \beta A_{4K/\alpha} \sum_i |B(x_i, r_i)| && \text{(by (1.1))} \\
&\le \beta A_{4K/\alpha} \lambda^{-1} \sum_i \int_{B(x_i, r_i)} |f| && \text{(by (1.6))} \\
&\le \beta A_{4K/\alpha} \lambda^{-1} \|f\|_1
\end{aligned}$$

since the balls $B(x_i, r_i)$ are disjoint.

If E_λ is not bounded, fix $a \in X$ and $R > 0$. The above argument shows that

$$\begin{aligned}
&|\{x_0 \in X \colon M_\Omega f(x_0) > \lambda \text{ and } x \in E_\lambda \cap B(a, R) \text{ for some } (x, r) \in \Omega_{x_0}\}| \\
&\qquad \le \beta A_{4K/\alpha} \lambda^{-1} \|f\|_1
\end{aligned}$$

since we can apply the Covering Lemma to the balls $\{B(x, r(x)) \colon x \in E_\lambda \cap B(a, R)\}$. Letting $R \to \infty$ we obtain the same weak type estimate as before. The proof is complete.

EXAMPLE. Let Ω_{x_0} be as in Example (2) after the definition of $M_\Omega f$, i.e. $M_\Omega f$ is a "nontangential" Hardy-Littlewood maximal function of f. The operator M_Ω is of weak type $(1, 1)$, because the triangle inequality and the doubling property imply $M_\Omega f \le c(\alpha) M f$, where Mf is the usual Hardy-Littlewood maximal function of f. On the other hand, it is very easy to check that Ω satisfies the hypotheses of Theorem 1.5. In fact,

$$S_\alpha(x, r) \subset B(x, K(\alpha + a)r),$$

and (ii) follows by the doubling property, while (i) is obvious.

The crucial hypothesis in Theorem 1.5 is (ii). Condition (i) was included because it makes the statement of the theorem somewhat simpler. We now proceed to show that when studying the weak type properties of M_Ω, there is no real loss of generality in requiring that (i) be satisfied.

Define

$$\hat{\Omega}_{x_0} = \{(x, r) \in X \times (0, \infty) \colon (x, s) \in \Omega_{x_0} \text{ for some } s \le r\}$$

and let $\hat{S}_\alpha(x, r)$ be defined as in (1.3) but with $\hat{\Omega}_{x_0}$ in place of Ω_{x_0}. Note that $S_\alpha(x, r) \subset \hat{S}_\alpha(x, r)$ since $\Omega_{x_0} \subset \hat{\Omega}_{x_0}$. Thus the following result shows that the conclusion of Proposition 1.4 can be strengthened.

PROPOSITION 1.7. *If M_Ω is of weak type (p, p) for some p, $0 < p < \infty$, then for every $\alpha > 0$ there is $\beta < \infty$ so that*

$$|\hat{S}_\alpha(x, r)| \le \beta |B(x, r)|$$

for all $(x, r) \in X \times (0, \infty)$.

PROOF. Let f be the characteristic function of $B(x, K(\alpha + 1)r)$. As in the proof of Proposition 1.4 (now with $\hat{\Omega}$ in place of Ω), if $x_0 \in \hat{S}_\alpha(x, r)$ then $B(y, r) \subset B(x, K(\alpha + 1)r)$ for some $(y, r) \in \hat{\Omega}_{x_0}$. But then $(y, s) \in \Omega_{x_0}$ for some $s \le r$, hence

$B(y,s) \subset B(x, K(\alpha+1)r)$ and therefore

$$M_\Omega f(x_0) \geq \frac{1}{|B(y,s)|} \int_{B(y,s)} f = 1.$$

Thus

$$|\hat{S}_\alpha(x,r)| \leq |\{M_\Omega f \geq 1\}| \leq C\|f\|_p^p \leq CA_{K(\alpha+1)}|B(x,r)|$$

and the proof is complete.

We can now give a necessary and sufficient condition for M_Ω to be of weak type $(1,1)$.

THEOREM 1.8. *M_Ω is of weak type $(1,1)$ if and only if there are constants $\alpha > 0$ and $\beta < \infty$ so that*

$$(1.9) \qquad |\hat{S}_\alpha(x,r)| \leq \beta|B(x,r)| \quad \textit{for every } (x,r) \in X \times (0,\infty).$$

PROOF. The "only if" part follows from Proposition 1.7.

Conversely, if (1.9) holds then the sets $\hat{\Omega}_{x_0}$ satisfy hypothesis (ii) of Theorem 1.5. Since they also satisfy (i), $M_{\hat{\Omega}}$ is of weak type $(1,1)$. But $M_\Omega f \leq M_{\hat{\Omega}} f$ for all f, because $\Omega_{x_0} \subset \hat{\Omega}_{x_0}$ for every $x_0 \in X$. Thus M_Ω is of weak type $(1,1)$.

2. The case of a group. In this section we assume that (X, d, μ) is a space of homogeneous type and that X is a group such that the pseudo-distance d is left-invariant, that is,

$$(\text{G1}) \qquad xB(y,r) = B(xy,r) \quad \text{for all } x,y \in X,\ r > 0.$$

We let e denote the identity element of the group X.

REMARK. d is left-invariant if and only if

$$d(x,y) = \|x^{-1}y\|,$$

where $\|\cdot\|$ is a nonnegative function on X satisfying

(I) $\|x\| = 0$ if and only if $x = e$,
(II) $\|xy\| \leq K(\|x\| + \|y\|)$,
(III) $\|x^{-1}\| = \|x\|$

(see [**KV**, p. 577, Remark 1]).

Let Ω_e be a subset of $X \times (0,\infty)$. For each $x_0 \in X$ set

$$(2.1) \qquad \Omega_{x_0} = \{(x_0 x, r) \colon (x,r) \in \Omega_e\}.$$

In other words, $\Omega_{x_0}(r) = x_0 \Omega_e(r)$.

LEMMA 2.2. *If (2.1) holds, then*

$$S_\alpha(x,r) = B(x,\alpha r)[\Omega_e(r)]^{-1}.$$

In particular, $S_\alpha(x,r) = xS_\alpha(e,r)$ if (G1) also holds.

PROOF. If $x_0 \in S_\alpha(x,r)$, then $(y,r) \in \Omega_{x_0}$ for some $y \in B(x,\alpha r)$. Then $(x_0^{-1}y, r) \in \Omega_e$ and therefore

$$x_0 = y(x_0^{-1}y)^{-1} \in B(x,\alpha r)[\Omega_e(r)]^{-1}.$$

Conversely, if $x_0 = yz^{-1}$ with $y \in B(x, \alpha r)$ and $z \in \Omega_e(r)$, then $x_0 z \in \Omega_{x_0}(r)$ and $x_0 z = y \in B(x, \alpha r)$. Thus

$$\Omega_{x_0}(r) \cap B(x, \alpha r) \neq \emptyset,$$

that is, $x_0 \in S_\alpha(x, r)$. This proves the lemma.

We now make the following additional assumptions:

$$\text{(G2)} \qquad \mu \text{ is left-invariant: } \mu(xE) = \mu(E);$$

$$\text{(G3)} \qquad \mu(E^{-1}) = \mu(E).$$

Note that (G2) and (G3) imply that μ is right-invariant, although we do not need this fact. Also note that (G1) implies

$$[B(e,r)]^{-1} = B(e,r), \tag{2.3}$$

because if $x \in B(e,r)$ then $e = x^{-1}x \in B(x^{-1}e, r) = B(x^{-1}, r)$, hence $x^{-1} \in B(e,r)$.

We now show that under these assumptions Theorem 1.8 takes on a very simple form.

THEOREM 2.4. *Let* (X, d, μ) *be a space of homogeneous type. Assume* X *is also a group and* (G1), (G2), *and* (G3) *hold. Given a set* $\Omega \subset X \times (0, \infty)$ *we define*

$$M_\Omega f(x_0) = \sup_{(x,r)\in\Omega} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x_0 y)|\, d\mu(y)$$

for $f \in L^1_{\text{loc}}(d\mu)$ *and* $x_0 \in X$. (*Notice that we are writing* Ω *in place of our earlier* Ω_e. *As usual,* $|E|$ *denotes* $\mu(E)$.)

Let

$$\hat{\Omega} = \{(x,r) \in X \times (0,\infty) \colon (x,s) \in \Omega \text{ for some } s \le r\}.$$

Then M_Ω *is of weak type* $(1,1)$ *if and only if there are constants* $\alpha > 0$, $\beta < \infty$ *so that*

$$\left| \bigcup_{y \in \hat{\Omega}(r)} B(y, \alpha r) \right| \le \beta |B(e,r)| \quad \text{for all } r > 0, \tag{2.5}$$

where $\hat{\Omega}(r) = \{y \in X \colon (y,r) \in \hat{\Omega}\}$.

PROOF. Note that this M_Ω is a particular case of the operator studied in the preceding section. In fact, it is precisely the operator defined by (1.2) if the sets Ω_{x_0} are defined by (2.1) and (G1) and (G2) hold. By Lemma 2.2 applied to the sets $\hat{\Omega}_{x_0}$,

$$\begin{aligned}
|\hat{S}_\alpha(x,r)| &= |B(x,\alpha r)[\hat{\Omega}(r)]^{-1}| \\
&= |B(e,\alpha r)[\hat{\Omega}(r)]^{-1}| && \text{(by (G1) and (G2))} \\
&= |\hat{\Omega}(r)[B(e,\alpha r)]^{-1}| && \text{(by (G3))} \\
&= |\hat{\Omega}(r) B(e,\alpha r)| && \text{(by (2.3))} \\
&= \left| \bigcup_{y\in\hat{\Omega}(r)} B(y,\alpha r) \right| && \text{(by (G1))}.
\end{aligned}$$

Thus condition (1.9) becomes (2.5), since $|B(x,r)| = |B(e,r)|$. The proof is complete.

We give two examples of spaces of homogeneous type satisfying (G1)–(G3):

(A) $X = \mathbf{R}^n$, μ is Lebesgue measure, and $d(x,y) = \|x-y\|$ with $\|\cdot\|$ defined by

$$\|(x_1,\ldots,x_n)\| = \left(\sum_1^n |x_i|^{2/a_i}\right)^{1/2},$$

where $a_1 > 0, \ldots, a_n > 0$ are fixed numbers. This "nonisotropic" $\mathbf{R}^n$ (which reduces to the usual "isotropic" euclidean $\mathbf{R}^n$ when all the $a_i = 1$) is a space of homogeneous type satisfying (G1)–(G3). The "ellipsoid" $B(0,r)$ has dimension r^{a_i} along the x_i-direction.

(B) Let X be the Heisenberg group $H_n = \mathbf{C}^n \times \mathbf{R}$, whose (noncommutative) group law is given by

$$(z_1,\ldots,z_n,t)\cdot(w_1,\ldots,w_n,s) = \left(z_1+w_1,\ldots,z_n+w_n, t+s+2\operatorname{Im}\sum_1^n z_k\bar{w}_k\right)$$

or, in more compact notation,

$$(z,t)\cdot(w,s) = (z+w, t+s+2\operatorname{Im}\langle z,w\rangle)$$

(see e.g. [**FS** or **KV**]). We let μ denote Lebesgue measure (on $\mathbf{R}^{2n+1}$) and set $d(x,y) = \|x^{-1}y\|$ with

$$\|(z,t)\| = \max\left\{\sum_1^n |z_k|^2, |t|\right\}.$$

(One could also define $\|(z,t)\| = (\sum_1^n |z_k|^2 + |t|)^{1/2}$. The important point is the different homogeneity in z and in t.) The resulting space of homogeneous type also satisfies (G1)–(G3). This example will be used in the next section.

We apply Theorem 2.4 to show that there are many regions Ω not contained in any "cone"

$$\Gamma_\alpha(e,0) = \{(x,r)\in X\times(0,\infty)\colon d(x,e) < \alpha r\} \qquad (\alpha > 0)$$

for which M_Ω is of weak type $(1,1)$. In other words, Ω need not be "nontangential" in order for M_Ω to be of weak type $(1,1)$. In the remainder of this section we follow rather closely some of the arguments in [**NS**].

PROPOSITION 2.6. *Let (X,d,μ) be a space of homogeneous type satisfying (G1)–(G3). Let (x_n,r_n) be a sequence in $X\times(0,\infty)$ such that $r_{n+1}\le r_n$ and $d(x_{n+1},e)\le Cr_n$ for all n, where the constant $C<\infty$ does not depend on n. Let Ω be the union of the "cones" $\Gamma_1(x_n,r_n)$, i.e.*

$$\Omega = \{(x,r)\in X\times(0,\infty)\colon d(x,x_n) < r - r_n \text{ for some } n\}.$$

If M_Ω is defined as in Theorem 2.4, then M_Ω is of weak type $(1,1)$ and of strong type (p,p) for $1<p\le\infty$.

PROOF. First notice that $\hat{\Omega} = \Omega$. Thus by Theorem 2.4 it suffices to check (2.5) with Ω in place of $\hat{\Omega}$.

Fix $\alpha > 0$. Given $r > 0$, let N be the first index so that $r_N < r$ (if $r_n \geq r$ for all n, then $\Omega(r)$ is empty and there is nothing to prove). If $y \in \Omega(r)$, then either $d(y, x_N) < r - r_N < r$ or $d(y, x_n) < r - r_n < r$ for some $n \geq N+1$. In the latter case,

$$d(y,e) \leq K[d(y,x_n) + d(x_n,e)] < K[r + Cr_N] < K(C+1)r,$$

since $d(x_n, e) \leq Cr_{n-1} < Cr_N$ if $n \geq N+1$. Thus if $z \in B(y, \alpha r)$, then either

$$d(z, x_N) < K(\alpha+1)r \quad \text{or} \quad d(z,e) < K(\alpha + K(C+1))r.$$

Therefore we have

$$\bigcup_{y \in \Omega(r)} B(y, \alpha r) \subset B(x_N, K(\alpha+1)r) \cup B(e, K(\alpha + KC + K)r)$$

and (2.5) follows. The proof is complete.

The sequence (x_n, r_n) need not satisfy $d(x_n, e) \leq a r_n$ for all n and some fixed $a < \infty$. That is, if $r_n \searrow 0$ and

$$\limsup_{n \to \infty} \frac{d(x_n, e)}{r_n} = +\infty,$$

then Ω is not "nontangential." However, the corresponding maximal operator M_Ω is still of weak type $(1,1)$.

One case in which it is easy to construct such sequences is the following. Suppose that for every $\lambda \geq 0$ there is some $x \in X$ with $\|x\| = \lambda$, where $\|x\|$ denotes $d(x,e)$. (This condition is satisfied, in particular, by the examples considered above: isotropic and nonisotropic $\mathbf{R}^n$, and the Heisenberg group.) Let $\varphi(t)$, $0 \leq t \leq 1$, be any positive continuous function with $\varphi(0) = 0$ and

$$\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0. \tag{2.7}$$

We construct the sequence (x_n, r_n) as follows. Choose $x_1 \in X$ with $\|x_1\| = 1$ and set $r_1 = \varphi(\|x_1\|)$. Inductively choose $x_{n+1} \in X$ with $\|x_{n+1}\| = r_n$ and set

$$r_{n+1} = \min\{r_n/2, \varphi(\|x_{n+1}\|)\}.$$

Then (x_n, r_n) satisfies the hypotheses of Proposition 2.6. On the other hand, since $r_n \searrow 0$ we have $r_n = \varphi(\|x_n\|)$ for n large, because (2.7) implies $\varphi(\|x_{n+1}\|) = \varphi(r_n) < r_n/2$ for n large. But then it follows from (2.7) that $(x_n, r_n) = (x_n, \varphi(\|x_n\|))$ approaches $(e, 0)$ "tangentially"; in fact it does so with an "order of tangency" determined by φ.

3. Poisson-Szegö integrals on generalized half-planes. Let D be the generalized half-plane

$$D = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} \colon \operatorname{Im} z_{n+1} > \sum_1^n |z_k|^2 \right\}.$$

In [**K2**] the notion of admissible convergence is introduced and it is shown that Poisson-Szegö integrals of functions in $L^p(\partial D)$, $1 \leq p \leq \infty$, have admissible limits at almost every point of the boundary ∂D. We shall apply our previous results to show the existence of certain "nonadmissible" limits a.e.

Throughout the remainder of the paper, we write "Poisson" instead of "Poisson-Szegö."

We need some notation. For $z = (z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1}$ set

$$h(z) = \operatorname{Im} z_{n+1} - \sum_1^n |z_k|^2,$$

so that $D = \{z \in \mathbf{C}^{n+1} \colon h(z) > 0\}$ and $\partial D = \{z \in \mathbf{C}^{n+1} \colon h(z) = 0\}$. Let $H_n = \mathbf{C}^n \times \mathbf{R}$ be the Heisenberg group of real dimension $2n+1$ (cf. example (B) in §2). Define $\psi \colon \mathbf{C}^{n+1} \to H_n$ by

$$\psi(z) = (z_1, \dots, z_n, \operatorname{Re} z_{n+1}).$$

Then the restriction of ψ to ∂D is a bijection between ∂D and H_n.

Define a map $\Phi \colon D \to H_n \times (0, \infty)$ by $\Phi(z) = (\psi(z), h(z))$. Clearly Φ is one-to-one and onto. It can be thought of as a coordinate system on D: $\psi(z)$ is the "projection" onto ∂D (identified with H_n) and $h(z)$ is the "height" above ∂D.

The group H_n acts on D (and on ∂D) as follows. If $g = (w_1, \dots, w_n, s) \in H_n$ and $z \in D$ (or $z \in \partial D$), then

$$g \cdot z = \left(w_1 + z_1, \dots, w_n + z_n, s + z_{n+1} + 2i \sum_1^n \bar{w}_k z_k + i \sum_1^n |w_k|^2 \right).$$

It is readily checked that $h(g \cdot z) = h(z)$ and $\psi(g \cdot z) = g\psi(z)$. In other words, the group action preserves the "layers" $D(a) = \{h(z) = a\}$ for every $a \geq 0$, and if we identify $D(a)$ with H_n via ψ, then the restriction of the group action to $D(a)$ is just left translation on H_n. In particular, we can identify ∂D with H_n.

On H_n we consider the pseudo-norm

$$\|(z_1, \dots, z_n, t)\| = \max \left\{ |t|, \sum_1^n |z_k|^2 \right\}$$

and the corresponding left-invariant pseudo-distance

$$d(x, y) = \|x^{-1} y\| \qquad (x, y \in H_n).$$

We let μ denote $(2n+1)$-dimensional Lebesgue measure on H_n. Thus $\mu(B(x, r)) = c_n r^{n+1}$.

On ∂D we consider the measure β defined by

$$\int_{\partial D} f \, d\beta = \int_{H_n} f \circ \psi^{-1} \, d\mu,$$

where ψ^{-1} is the inverse of the restriction of ψ to ∂D. The Poisson kernel of D is **[K2]**

$$P(u, z) = c_{n+1} \frac{\rho(z, z)^{n+1}}{|\rho(u, z)|^{2n+2}} \qquad (u \in \partial D, z \in D),$$

where $\rho(z, w) = i(\bar{w}_{n+1} - z_{n+1}) - 2 \sum_1^n z_k \bar{w}_k$. If f is a function defined on ∂D, then its Poisson integral is given by

$$F(z) = \int_{\partial D} f(u) P(u, z) \, d\beta(u)$$

for $z \in D$, provided that the integral makes sense.

In studying boundary limits of Poisson integrals on D the main idea is to view D as $H_n \times (0, \infty)$. It turns out that Poisson integrals are dominated by maximal operators of the type considered in §2, in much the same way as Poisson integrals on the upper half-plane are controlled by the usual Hardy-Littlewood maximal operator (see e.g. [**S**, Chapter 3]). Let Ω' be a subset of D (to be thought of as an approach region at $0 \in \partial D$). For $\varsigma_0 \in \partial D$ set $x_0 = \psi(\varsigma_0)$, i.e. x_0 is the unique element of H_n such that $x_0 \cdot 0 = \varsigma_0$. Let $\Omega'_{\varsigma_0} = x_0 \cdot \Omega'$ be the "approach region at ς_0." The following lemma is then the analogue to Lemma 4 in [**NS**].

LEMMA 3.1. *Assume that $\Omega' \subset D$ has the property that if $z \in \Omega'$ and $r \geq 0$, then $z + (0, \dots, 0, ir) \in \Omega'$. Let $\Omega = \Phi(\Omega')$, so $\Omega \subset H_n \times (0, \infty)$. Let f be a function on ∂D such that its Poisson integral F is defined on D. If $\varsigma_0 \in \partial D$ and $x_0 = \psi(\varsigma_0)$, then*

$$\sup_{z \in \Omega'_{\varsigma_0}} |F(z)| \leq c(n) M_\Omega(f \circ \psi^{-1})(x_0),$$

where M_Ω is defined as in Theorem 2.4 and ψ^{-1} is the inverse of $\psi|\partial D$.

PROOF. Let $e = (0, \dots, 0, 1)$. The Poisson kernel satisfies the following estimates [**K2**]:

$$P(u, ite) \leq \frac{c}{t^{n+1}} \quad \text{for all } u \in \partial D,$$

$$P(u, ite) \leq c\frac{t^{n+1}}{\rho^{2n+2}} \quad \text{if } \|\psi(u)\| \geq \rho.$$

Since $P(g \cdot u, g \cdot z) = P(u, z)$ for all $g \in H_n$, we have

$$P(u, z) \leq \frac{c}{t^{n+1}} \quad \text{for all } u \in \partial D,$$

$$P(u, z) \leq c\frac{t^{n+1}}{\rho^{2n+2}} \quad \text{if } \delta(u, z) \geq \rho,$$

where $t = h(z)$ and $\delta(u, z) = \|\psi(u)^{-1}\psi(z)\| = d(\psi(u), \psi(z))$.

Fix $z \in \Omega'_{\varsigma_0}$ and let $t = h(z)$. Then

$$\begin{aligned} |F(z)| &\leq \int_{\delta(u,z)<t} |f(u)| P(u, z)\, d\beta(u) \\ &\quad + \sum_{k=0}^{\infty} \int_{2^k t \leq \delta(u,z) < 2^{k+1} t} |f(u)| P(u, z)\, d\beta(u) \\ &= \mathrm{I} + \sum_0^{\infty} \mathrm{II}_k. \end{aligned}$$

By the above estimates and the definition of β,

$$\begin{aligned} \mathrm{I} &\leq \frac{c}{t^{n+1}} \int_{\delta(u,z)<t} |f(u)|\, d\beta(u) \\ &= \frac{c}{t^{n+1}} \int_{d(x,\psi(z))<t} |f(\psi^{-1}(x)|\, d\mu(x) \\ &= \frac{c'}{|B(\psi(z), t)|} \int_{B(\psi(z),t)} |f \circ \psi^{-1}|\, d\mu. \end{aligned}$$

(As usual, $|E|$ means $\mu(E)$.) But $z \in \Omega'_{\varsigma_0}$ implies $(\psi(z), t) = \Phi(z) \in \Omega_{x_0}$ because

$$\begin{aligned}\Phi(\Omega'_{\varsigma_0}) &= \Phi(x_0 \cdot \Omega') = (\psi(x_0 \cdot \Omega'), h(x_0 \cdot \Omega')) \\ &= (x_0 \psi(\Omega'), h(\Omega')) = \Omega_{x_0}.\end{aligned}$$

Thus

$$\mathrm{I} \leq c' M_\Omega (f \circ \psi^{-1})(x_0).$$

On the other hand,

$$\begin{aligned}\mathrm{II}_k &\leq \frac{ct^{n+1}}{(2^k t)^{2n+2}} \int_{\delta(u,z) \leq 2^{k+1} t} |f(u)| \, d\beta(u) \\ &= c' 2^{(1-k)(n+1)} \frac{1}{|B(\psi(z), 2^{k+1}t)|} \int_{B(\psi(z), 2^{k+1}t)} |f \circ \psi^{-1}| \, d\mu \\ &\leq 2^{n+1} c' (2^{-n-1})^k M_\Omega (f \circ \psi^{-1})(x_0)\end{aligned}$$

since $(\psi(z), 2^{k+1}t) = \Phi(z + i(2^{k+1} - 1)te) \in \Phi(\Omega'_{\varsigma_0}) = \Omega_{x_0}$. Therefore

$$\sum_0^\infty \mathrm{II}_k \leq 2^{n+1} c' \sum_{k=0}^\infty (2^{-n-1})^k M_\Omega (f \circ \psi^{-1})(x_0)$$

and the lemma follows.

REMARK. We have stated Lemma 3.1 for Poisson integrals, but a similar result holds in a more general situation. Assume that $K \in L^1 \cap L^\infty(H_n)$ is of the form $K(x) = \varphi(\|x\|)$, where φ is a nonnegative nonincreasing function on $[0, \infty)$. For $\lambda > 0$ and $x \in H_n$ set

$$K_\lambda(x) = \lambda^{-n-1} K(\delta_{\lambda^{-1}}(x)),$$

where for $r > 0$ the dilation δ_r is defined by

$$\delta_r(z_1, \ldots, z_n, t) = (r^{1/2} z_1, \ldots, r^{1/2} z_n, rt).$$

If f is a function on ∂D and F is defined on D by the convolution

$$F(z) = \int_{H_n} (f \circ \psi^{-1})(x) K_{h(z)}(x^{-1} \psi(z)) \, d\mu(x) \qquad (z \in D), \tag{3.2}$$

then the conclusion of the lemma holds with $c(n)$ replaced by a constant depending on K. The proof is similar to that of Lemma 4 in [**NS**]. Notice that (3.2) defines the Poisson integral of f when $K(x) = P(\psi^{-1}(x), ie)$, as an easy computation shows. In this case we do not have $K(x) = \varphi(\|x\|)$. However,

$$K(x) \leq \mathrm{const}(1 + \|x\|^2)^{-n-1},$$

which is enough to obtain the same conclusion, since only size estimates are involved.

The preceding lemma shows that: if $\Omega' \subset D$ contains $z + (0, \ldots, 0, ir)$ for all $r \geq 0$ whenever it contains z, and if $\Omega = \Phi(\Omega')$ is such that the maximal operator M_Ω is of weak type $(1,1)$ on H_n, then the maximal operator $\mathcal{M}_{\Omega'}$ defined by

$$\mathcal{M}_{\Omega'} f(\varsigma_0) = \sup_{z \in \Omega'_{\varsigma_0}} |F(z)|,$$

where F is the Poisson integral of f, is also of weak type $(1,1)$. In fact, if $\mathcal{M}_{\Omega'}f(\varsigma_0) > \lambda > 0$, then $M_\Omega(f\circ\psi^{-1})(\psi(\varsigma_0)) > c(n)^{-1}\lambda$, by the lemma. Thus

$$\beta(\{\mathcal{M}_{\Omega'}f > \lambda\}) = \mu(\psi(\{\mathcal{M}_{\Omega'}f > \lambda\})) \le \mu(\{M_\Omega(f\circ\psi^{-1}) > c(n)^{-1}\lambda\})$$
$$\le C_\Omega \frac{\|f\circ\psi^{-1}\|_{L^1(\mu)}}{c(n)^{-1}\lambda} = C_\Omega c(n)\frac{\|f\|_{L^1(\beta)}}{\lambda}.$$

The Korányi regions can be defined by

$$\Gamma_\alpha(0) = \{z\in D\colon \|\psi(z)\| < \alpha h(z)\}$$

and

$$\Gamma_\alpha(g\cdot 0) = g\cdot\Gamma_\alpha(0) \qquad (g\in H_n).$$

It is obvious that $\Phi(\Gamma_\alpha(\varsigma_0)) = \Gamma_\alpha(\psi(\varsigma_0),0)$, where the "cone" $\Gamma_\alpha(x_0,r)\subset H_n\times(0,\infty)$ is defined by

$$\Gamma_\alpha(x_0,r) = \{(y,s)\in H_n\times(0,\infty)\colon d(y,x_0) < \alpha(s-r)\}.$$

Thus Korányi's theorem on admissible convergence of Poisson integrals [**K2**] is now a consequence of Lemma 3.1, since we know that the maximal operator M_{Ω_α}, $\Omega_\alpha = \Gamma_\alpha(0,0)$, is of weak type $(1,1)$ (see the example after Theorem 1.5).

On the other hand, let $\Omega\subset H_n\times(0,\infty)$ be a set of the type considered in Proposition 2.6 and not contained in $\Gamma_\alpha(0,0)$ for any $\alpha > 0$ (see remarks after Proposition 2.6). Then $\Omega' = \Phi^{-1}(\Omega)$ is not contained in any Korányi region $\Gamma_\alpha(0)$. However it follows from Proposition 2.6 and Lemma 3.1 that $\mathcal{M}_{\Omega'}$ is of weak type $(1,1)$. Thus we obtain convergence within Ω'_ς for almost all $\varsigma\in\partial D$ for Poisson integrals of functions in $L^p(\partial D)$, $1\le p\le\infty$. That is, if $f\in L^p(\partial D)$, $1\le p\le\infty$, and F is its Poisson integral, then

$$\lim_{\substack{z\to\varsigma\\ z\in\Omega'_\varsigma}} F(z) = f(\varsigma)$$

for almost every $\varsigma\in\partial D$. Nevertheless, this convergence is not admissible in the sense of [**K2**].

4. Poisson-Szegö integrals on the unit ball of $\mathbf{C}^n$. A similar result holds for B, the unit ball of $\mathbf{C}^{n+1}$. The generalized Cayley transform $\mathcal{C}$ carries B onto D and, as remarked in [**K2**], admissible convergence in B is equivalent via $\mathcal{C}$ to admissible convergence in D. Let $g\in L^p(\partial B)$, $1\le p\le\infty$, and set $f = g\circ\mathcal{C}^{-1}$, so f is defined on ∂D. It follows from [**K1**] that $f\in L^p(P_{ie}\,d\beta)$, where P_{ie} is the Poisson (i.e. Poisson-Szegö) kernel of D at $ie = (0,\dots,0,i)$, and that $P[f]\circ\mathcal{C} = \mathcal{P}[g]$, where $P[\cdot]$ and $\mathcal{P}[\cdot]$ denote Poisson integrals on D and on B, respectively. We shall show that if $f\in L^p(P_{ie}\,d\beta)$, $1\le p\le\infty$, then $P[f]$ has limits within Ω'_ς for almost every $\varsigma\in\partial D$, provided that Ω' is an approach region (i.e. $0\in\overline{\Omega'}$) for which M_Ω ($\Omega = \Phi(\Omega')$) is of weak type $(1,1)$. Since Ω' need not be admissible, we can then conclude that if $g\in L^p(\partial B)$, $1\le p\le\infty$, then its Poisson integral $\mathcal{P}[g]$ has certain "nonadmissible" limits at a.e. point of ∂B.

To complete the argument, let $f\in L^p(P_{ie}\,d\beta)$, $1\le p\le\infty$. Note that $P[f]$ is then defined on D since the Poisson kernel at $z\in D$ satisfies $P_z\le c(z)P_{ie}$, where $c(z)$ is some positive number depending on z. For $R > 0$ define

$$f_1(\varsigma) = \begin{cases} f(\varsigma) & \text{if } \|\psi(\varsigma)\|_{H_n} < R,\\ 0 & \text{otherwise,}\end{cases}$$

and $f_2 = f - f_1$. If $\|\psi(\varsigma)\| < R$, then clearly

$$\lim_{\substack{z\to\varsigma\\ z\in\Omega'_\varsigma}} P[f_2](z) = 0$$

because ς is not in the support of f_2. On the other hand, $f_1 \in L^p(\beta)$ since P_{ie} is bounded below on compact subsets of ∂D. Therefore if Ω' is as before,

$$\lim_{\substack{z\to\varsigma\\ z\in\Omega'_\varsigma}} P[f_1](z) = f_1(\varsigma) \quad \text{a.e.}$$

It follows that

$$\lim_{\substack{z\to\varsigma\\ z\in\Omega'_\varsigma}} P[f](z) = f(\varsigma) \tag{4.1}$$

for a.e. $\varsigma \in \partial D$ with $\|\psi(\varsigma)\| < R$. Letting $R \to \infty$ we obtain (4.1) for a.e. $\varsigma \in \partial D$.

We finish by mentioning the following result of Hakim and Sibony [**HS**]. Let $\alpha > 1$ and let $h\colon (0,1] \to [\alpha, +\infty)$ be a nonincreasing function with

$$\lim_{x\to 0^+} h(x) = +\infty. \tag{4.2}$$

For each $\varsigma \in \partial B$ define an approach region

$$\begin{aligned} D(\varsigma) = D_{\alpha,h}(\varsigma) = \{z \in B\colon &|1 - \langle z, \varsigma\rangle| \le \alpha(1 - |\langle z, \varsigma\rangle|) \\ &\text{and } |1 - \langle z, \varsigma\rangle| \le h(|1 - \langle z, \varsigma\rangle|)\cdot(1 - |z|)\}. \end{aligned} \tag{4.3}$$

Then there exists a bounded holomorphic function u on B such that for a.e. $\varsigma \in \partial B$ the limit

$$\lim_{\substack{z\to\varsigma\\ z\in D(\varsigma)}} u(z)$$

fails to exist. Recall that the Korányi regions ("admissible regions" in [**K2**]) are given by

$$D_\alpha(\varsigma) = \{z \in B\colon |1 - \langle z, \varsigma\rangle| \le \alpha(1 - |z|)\}.$$

Because of (4.2) the approach regions $D_{\alpha,h}(\varsigma)$ are wider than the Korányi regions $D_\alpha(\varsigma)$, but only in the complex directions ($D_{\alpha,h}(\varsigma)$ is nontangential in the special real direction). Thus the result of Hakim and Sibony shows that as far as the existence of boundary limits a.e. is concerned, the regions $D_\alpha(\varsigma)$ are best possible among approach regions defined by inequalities as in (4.3), even if we restrict our attention to bounded holomorphic functions. Nonetheless, as our results show, some boundary limits do exist which are not a consequence of Korányi's theorem.

The situation just described may be compared to the one-dimensional case. By Fatou's theorem, nontangential limits at a.e. point of the unit circle T exist for Poisson integrals of functions in $L^1(T)$. On the other hand, let C_0 be a curve in the unit disk U approaching the point 1 tangentially, and let $C_\theta = e^{i\theta}\cdot C_0$ be the result of rotating C_0 around the origin by the angle θ ($0 < \theta < 2\pi$). Thus C_θ approaches $e^{i\theta}$ tangentially. Littlewood [**L**] (see also Zygmund [**Z**]) showed that there is a bounded holomorphic function u in U such that for almost every θ the limit

$$\lim_{\substack{z\to e^{i\theta}\\ z\in C_\theta}} u(z)$$

fails to exist. However, it is shown in [**NS**] that some limits which are not nontangential do exist a.e. for Poisson integrals of functions in $L^1(T)$. The point, of course, is that an approach region which is not nontangential (i.e. not contained in any nontangential angle) need not be tangential.

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