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SINGULARLY PERTURBED QUADRATICALLY NONLINEAR DIRICHLET PROBLEMS

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ABSTRACT. The Dirichlet problem for singularly perturbed elliptic equations of the form $\varepsilon\Delta u = A(\mathbf{x},u)\nabla u\cdot\nabla u+\mathbf{B}(\mathbf{x},u)\cdot\nabla u+C(\mathbf{x},u)$ in $\Omega \in E^n$ is studied. Under explicit and easily checked conditions, solutions are shown to exist for ε sufficiently small and to exhibit specified asymptotic behavior as $\varepsilon \to 0$. The results are obtained using a method based on the theory of partial differential inequalities.

1. Introduction. Let Ω be an open, bounded set in Euclidean n-space E^n endowed with the standard inner product, denoted "$\cdot$". Suppose that the boundary Γ of Ω is a smooth $(n-1)$-dimensional manifold. Points in E^n are denoted $\mathbf{x}$ or $(x_1, x_2, \ldots, x_n)$. For smooth functions $u = u(\mathbf{x})$ defined on Ω, let ∇u denote the gradient of u and $\Delta u \equiv \nabla\cdot\nabla u$ denote the Laplacian operator acting on u.

In this paper we consider the singularly perturbed quadratically nonlinear Dirichlet problem

$$\begin{gathered}\varepsilon\Delta u = A(\mathbf{x},u)(\nabla u\cdot\nabla u)+\mathbf{B}(x,u)\cdot\nabla u+C(\mathbf{x},u)\quad\text{for } \mathbf{x} \text{ in } \Omega,\\ u(\mathbf{x},\varepsilon) = f(\mathbf{x})\quad\text{for } \mathbf{x} \text{ on } \Gamma.\end{gathered}\tag{1}$$

The scalar valued functions A, C, and f and the vector valued function $\mathbf{B}$ are assumed to be smooth. The parameter ε is assumed to be small and positive.

Our goal in this paper is the determination of the asymptotic behavior of solutions of problem (1) under various conditions on $A, \mathbf{B}, C$, and f. We shall be interested in solutions which exhibit rapid variation in regions of Ω whose area vanishes with ε, but which otherwise remain near certain solutions of the so-called reduced problem

$$\begin{gathered}0 = A(\mathbf{x},U)\nabla U\cdot\nabla U+\mathbf{B}(\mathbf{x},U)\cdot\nabla U+C(\mathbf{x},U)\quad\text{in } \Omega,\\ U(\mathbf{x}) = f(\mathbf{x})\quad\text{for } \mathbf{x} \text{ in } \Gamma'\subset\Gamma.\end{gathered}\tag{2}$$

These regions of rapid variation, or layers, may occur along the boundary segment $\Gamma - \Gamma'$ (boundary layer behavior) and/or along closed curves properly contained in Ω (interior layer behavior). The term "layer" is borrowed from fluid mechanics, where solutions of flow equations often satisfy simplified equations away from fixed boundaries, but exhibit rapid variation near these boundaries.

The method we use to study the asymptotic behavior of solutions of problem (1) is based on a recent and well-known theorem due to Amann [**1**] concerning partial differential inequalities. The method is discussed fully in §3, but the basic idea

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of the method is to substitute solving problem (1) with solving a pair of partial differential inequalities. The differential inequalities are solved with the help of coordinate transformations devised by Fife [**8**] and Fife and Greenlee [**9**]. The coordinate transformations allow a reduction of the partial differential inequalities to ordinary differential inequalities which are readily solved. These coordinate transformations are discussed in §2.

An important assumption we make throughout the paper is that the characteristic curves of the reduced problem (2) are nowhere tangent to curves along which we expect layer behavior. From the classical theory of characteristics (cf. [**4**]), this condition may be expressed analytically as

$$[2A(\mathbf{x}, U(\mathbf{x}))\nabla U + \mathbf{B}(\mathbf{x}, U(\mathbf{x}))] \cdot \mathbf{n}(\mathbf{x}) \neq 0 \tag{3}$$

along the appropriate curve for each reduced solution U of interest and for $\mathbf{n}(\mathbf{x})$ a normal vector to the curve. We shall actually place certain sign requirements on the characteristic quantity (3) which reflect the fact that the characteristics are "outgoing" in a sense to be made precise later.

The Dirichlet problem (1) arises in a number of applied contexts. Applications relating to reaction-diffusion systems are discussed in Ames [**2**]. Cunningham [**5**] has shown how problem (1) arises in the study of electron density variations in semiconductor diodes. Finally, a generalized version of problem (1) has been proposed by Bell, Cosner, and Bertiger [**3**] as a model of vein formation in young leaves.

A number of results concerning the asymptotic behavior of solutions of problem (1) are known for the case $A(\mathbf{x}, u) \equiv 0$. For this case, we mention the work of Howes [**13, 15, 16**] and Kelley [**17**]. Howes [**13**] has also considered the case $A \neq 0$ and $B \equiv 0$ by employing somewhat stronger conditions than we employ in this paper. Both authors use methods based on partial differential inequalities which are similar to the method we use here.

There appears to be no published work on the asymptotic behavior of solutions of the full n-dimensional problem (1), but the one-dimensional version has been studied extensively. We mention here the work of Dorr, Parter, and Shampine [**7**], and Howes [**11, 12, 14**]. It is interesting to note that the characteristic condition (3) for the n-dimensional problem corresponds in one dimension to the classical condition for linear stability of the reduced solution U. We shall make extensive use of the one-dimensional theory throughout the paper.

We conclude this introduction with a review of asymptotic terminology. We say that a function $p(\mathbf{x}, \varepsilon)$ is $O(\varepsilon^n)$ if $\lim_{\varepsilon \to 0} p(\mathbf{x}, \varepsilon)/\varepsilon^n$ exists. In other words, $p(\mathbf{x}, \varepsilon)$ is $O(\varepsilon^n)$ if p behaves like ε^n as $\varepsilon \to 0$. We say that the function $p(\mathbf{x}, \varepsilon)$ is transcendentally small if $p(\mathbf{x}, \varepsilon)$ is $O(\varepsilon^n)$ for every n. Thus, a transcendentally small term (abbreviated T.S.T.) behaves like $\exp(-\tau/\varepsilon)$ for some $\tau > 0$ as $\varepsilon \to 0$. Finally, we say that a function $p(\mathbf{x}, \varepsilon)$ approaches zero exponentially for $\mathbf{x}$ in a set X if $|p(\mathbf{x}, \varepsilon)| < K \exp(-\tau(\mathbf{x}, \varepsilon))$, where K is a positive constant and $\lim_{\varepsilon \to 0} \tau(\mathbf{x}, \varepsilon) = \infty$ for each $\mathbf{x}$ in X. In a region in which τ is bounded away from zero, a term which decays exponentially is also a transcendentally small term.

2. Coordinate transformations. In the remainder of the paper we make use of a pair of local coordinate systems first used by Fife [**8**] and Fife and Greenlee

[9]. The purpose of these local coordinate systems is, as mentioned in the Introduction, to allow the reduction of certain partial differential inequalities to ordinary differential inequalities in selected subregions of Ω.

For convenience, we shall take Ω to be a subset of E^2. We further assume that the boundary Γ of Ω is given by the equation $F(x,y) = 0$, where $\mathbf{\nabla} F \neq \mathbf{0}$ along Γ, and $\mathbf{\nabla} F$ is an outward normal to Γ. It will be clear in what follows that our results generalize to E^n for every n.

Let $\mathcal{C}$ be a smooth closed curve properly contained in Ω. We assume that $\mathcal{C}$ is given by the equation $J(x,y) = 0$, where $\mathbf{\nabla} J \neq \mathbf{0}$ along $\mathcal{C}$. The curve $\mathcal{C}$ divides Ω into two nonempty, disjoint open subsets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{C}$. For a given real number a we set $S_a = \{\mathbf{x} \text{ in } \Omega\colon |\mathbf{x} - \mathcal{C}| < a\}$. The set S_a is a band of width $2a$ surrounding the curve $\mathcal{C}$. Similarly, we set $S_a^b = \{\mathbf{x} \text{ in } \Omega\colon |\mathbf{x} - \Gamma| < a\}$. With a number d to be specified below, we set $S = S_d$, $S^b = S_d^b$.

In S, we use a coordinate system (t,s), where $t(x,y) = \pm|\mathbf{x} - \mathcal{C}|$, $t > 0$ for $\mathbf{x} = (x,y)$ in Ω_1, $t < 0$ for $\mathbf{x} = (x,y)$ in Ω_2, and where $s(x,y)$ is the arc length along $\mathcal{C}$ from some reference point to the point on $\mathcal{C}$ closest to (x,y). In S^b we use a coordinate system (r,z), where $r(x,y) = |\mathbf{x} - \Gamma|$ for $\mathbf{x} = (x,y)$ in S^b and where $z(x,y)$ is arc length along Γ to the point on Γ closest to (x,y).

The Jacobiàn of the transformation $(x,y) \to (t,s)$ is simply $x_t y_s - x_s y_t = (x_t, y_t) \cdot (y_s, -x_s)$. Now, the vector $(y_s, -x_s)$ is normal to the curve $\mathcal{C}$. Thus, a sufficient condition for the nonvanishing of the Jacobian along $\mathcal{C}$ is that $(x_t, y_t) \cdot \mathbf{\nabla} J \equiv \partial J/\partial t \neq 0$ along $\mathcal{C}$. Since $\partial J/\partial t = |\mathbf{\nabla} J|$ in this case, we see that we really only need $\mathbf{\nabla} J \neq \mathbf{0}$ along $\mathcal{C}$. By continuity, for some $d > 0$ the Jacobian does not vanish in $S = S_d$. Similarly if the boundary Γ of Ω is given by $F(x,y) = 0$, then the condition $\mathbf{\nabla} F \neq \mathbf{0}$ along Γ insures that the Jacobian of the transformation $(x,y) \to (r,z)$ does not vanish in $S^b = S_d^b$. The coordinate transformations are therefore bijective in S and S^b.

For ease of notation, we shall simply replace (x,y) by (t,s) in symbols for functions evaluated in S and by (r,z) in symbols for functions evaluated in S^b. Thus, for example, $u(t,s)$ really means $u(X(t,x), Y(t,s))$, where $x = X(t,s)$ and $y = Y(t,s)$.

We note that the curve $t(x,y)$ =constant is a closed curve everywhere parallel to the closed curve $\mathcal{C}$, while the curve $s(x,y)$ =constant is a line normal to $\mathcal{C}$ at the point s on $\mathcal{C}$ corresponding to (x,y). More precisely, we have $\mathbf{\nabla} t = -\rho \mathbf{\nabla} J$ for some real number ρ, and $\mathbf{\nabla} s \cdot \mathbf{\nabla} J = 0$. It follows, then, that $\mathbf{\nabla} t \cdot \mathbf{\nabla} s = 0$ for all (x,y) in S. Similarly, $\mathbf{\nabla} r \cdot \mathbf{\nabla} z = 0$ for all (x,y) in S^b.

In terms of the local coordinates (t,s), the Laplacian operator Δ acting on a function u may be written

$$\Delta u = [t_x^2 + t_y^2]u_{tt} + [s_x^2 + s_y^2]u_{ss} + [t_{xx} + t_{yy}]u_t + [s_{x\bar{x}} + s_{yy}]u_s,$$

where we note that the coefficient of u_{tt} does not vanish in S since the Jacobian of the transformation $(x,y) \to (t,s)$ does not vanish in S. Also in terms of (t,s) coordinates we have

$$\begin{aligned} \mathbf{\nabla} u \cdot \mathbf{\nabla} u &= (t_x^2 + t_y^2)u_t^2 + 2(t_x s_x + t_y s_y)u_t u_s + (s_x^2 + s_y^2)u_s^2 \\ &= (t_x^2 + t_y^2)u_t^2 + (s_x^2 + s_y^2)u_s^2 \end{aligned}$$

since $\mathbf{\nabla} t \cdot \mathbf{\nabla} s = 0$. Similar formulas may be written in terms of (r,z) coordinates in S^b.

3. Differential inequalities. The principal tool we use in the study of the asymptotic behavior of solutions of problem (1) is a theorem on differential inequalities due to Amann [**1**]. This theorem states that to establish the existence of a solution to an elliptic boundary value problem such as (1), it is sufficient to solve a certain pair of partial differential inequalities. Furthermore, the solutions of the partial differential inequalities provide upper and lower bounds for the solution of the boundary value problem. The following version of the theorem is written in a form directly applicable to problem (1).

THEOREM (AMANN). *Let Ω be a bounded, open set in E^2. Suppose the boundary Γ of Ω is given by $F(x,y) = 0$, where F is a function of class C^{2+k}, where $0 < k < 1$. Suppose further that the functions $A = A(x,y,u)$, $\mathbf{B} = \mathbf{B}(x,y,u) = (B_1(x,y,u),\ B_2(x,y,u))$, and $C = C(x,y,u)$ are of class C^1 on $\Omega \times E^1$, and that $f = f(x,y)$ is continuous for (x,y) on Γ. Suppose finally that there exist functions $\alpha = \alpha(x,y,\varepsilon)$ and $\beta = \beta(x,y,\varepsilon)$ (called lower and upper solutions, respectively) of class C^2 in Ω and of class C^1 on Γ which satisfy, for ε sufficiently small (say $\varepsilon \le \varepsilon_0$), the inequalities*

$$\varepsilon\Delta\alpha \ge A(x,y,\alpha)\nabla\alpha\cdot\nabla\alpha + \mathbf{B}(x,y,\alpha)\cdot\nabla\alpha + C(x,y,\alpha) \quad \text{in } \Omega,$$
$$\alpha(x,y,\varepsilon) \le f(x,y) \quad \text{on } \Gamma,$$
$$\varepsilon\Delta\beta \le A(x,y,\beta)\nabla\beta\cdot\nabla\beta + \mathbf{B}(x,y,\beta)\cdot\nabla\beta + C(x,y,\beta) \quad \text{in } \Omega,$$
$$\beta(x,y,\varepsilon) \ge f(x,y) \quad \text{on } \Gamma,$$

and

$$\alpha(x,y,\varepsilon) \le \beta(x,y,\varepsilon) \quad \text{in } \overline{\Omega}.$$

Then, there exists a solution $u = u(x,y,\varepsilon)$ of problem (1) for $\varepsilon \le \varepsilon_0$ such that $\alpha(x,y,\varepsilon) \le u(x,y,\varepsilon) \le \beta(x,y,\varepsilon)$.

This theorem remains valid if the lower and upper solutions α and β are not differentiable at points on a closed curve $\mathcal{C}$ contained in Ω, provided that these functions behave appropriately at these points. The behavior required at a point (x_0, y_0) of nondifferentiability is conveniently expressed in terms of the local coordinates (t,s) relative to $\mathcal{C}$. Suppose that the point $(0,s_0)$ is the point on $\mathcal{C}$ corresponding to (x_0,y_0). Then, we require that $D_l\alpha(0,s,\varepsilon) \le D_r\alpha(0,s_0,\varepsilon)$ and $D_l\beta(0,s_0,\varepsilon) \ge D_r\beta(0,s_0,\varepsilon)$, where D_l and D_r denote derivatives with respect to t from the negative t (Ω_2) side of $\mathcal{C}$ and the positive t (Ω_1) side of $\mathcal{C}$, respectively. The functions α and β are otherwise as in Amann's theorem. A proof of this extended version of Amann's theorem is easily adapted from a similar theorem in DeSanti [**6**]. The one-dimensional version of the extension is due to Habets and Laloy [**10**].

The version of the above theorem proved by Amann applies to a much wider class of elliptic problems than problem (1). Namely, the Laplacian operator may be replaced by any uniformly elliptic operator with twice Hölder continuously differentiable coefficients. Furthermore, the Dirichlet boundary condition may be replaced by the condition $Hu = f(x,y)$ along Γ, where H is a linear differential operator. Extensions to still wider classes of problems are discussed in Amann [**1**] and Sperb [**18**].

4. Boundary layer theory. In this section we use the method of differential inequalities described in the previous section to study the boundary layer behavior of solutions of

$$\varepsilon\Delta u = A(x,y,u)\nabla u\cdot\nabla u + \mathbf{B}(x,y,u)\cdot\nabla u + C(x,y,u) \quad \text{in } \Omega, \qquad u(x,y,\varepsilon) = f(x,y) \quad \text{on } \Gamma. \tag{4}$$

More precisely, we seek to establish the existence of a solution $u = u(x,y,\varepsilon)$ for ε sufficiently small, such that $\lim_{\varepsilon\to 0} u(x,y,\varepsilon) = U(x,y)$, where U is a certain solution of the reduced problem

$$0 = A(x,y,U)\nabla U\cdot\nabla U + \mathbf{B}(x,y,U)\cdot\nabla U + C(x,y,U) \quad \text{in } \Omega, \qquad U(x,y) = f(x,y) \quad \text{on } \Gamma' \subset \Gamma. \tag{5}$$

If all solutions of (5) are such that $U(x,y) = f(x,y)$ on all of Γ, then it is clear that there can be no boundary layer solution of problem (4). In this case, the exact solution is uniformly approximated to $O(\varepsilon)$ by the reduced solution U. This situation is rare, and typically corrections in the reduced solution must be introduced near the subset $\Gamma - \Gamma'$ of the boundary Γ.

The boundary layer behavior of the solutions of problem (4) also depends upon the nature of the characteristics of the reduced problem (5). Namely, we shall find that boundary layer behavior is possible provided that the characteristics of (5) are outgoing at each point of $\Gamma - \Gamma'$, that is, if $[2A(x,y,U(x,y))\nabla U + \mathbf{B}(x,y,U(x,y))]\cdot\nabla F > 0$ along $\Gamma - \Gamma'$.

The following theorem is our main result on the boundary layer behavior of solutions of problem (4).

THEOREM (BOUNDARY LAYER BEHAVIOR). *Assume*

(a) *the functions* $A = A(x,y,u)$, $\mathbf{B} = \mathbf{B}(x,y,u) = (B_1(x,y,u),\ B_2(x,y,u))$, *and* $C = C(x,y,u)$ *are of class* C^1 *on* $\Omega\times E^1$, *and* $f = f(x,y)$ *is continuous on* Γ;

(b) *there exists a solution* $U = U(x,y)$ *of class* C^2 *on* Ω *of reduced problem* (5);

(c) *the characteristics of the reduced problem are outgoing along* $\Gamma - \Gamma'$, *that is,*

$$[2A(x,y,U(x,y))\nabla U + \mathbf{B}(x,y,U(x,y))]\cdot\nabla F > 0$$

for (x,y) *in the set* $\Gamma - \Gamma'$;

(d) *the vector field* $[2A(x,y,U(x,y))\nabla U + \mathbf{B}(x,y,U(x,y))]$ *does not vanish for any* (x,y) *in* $\Omega\cup\Gamma - \Gamma'$;

(e) *for* (x,y) *in the set* $\Gamma - \Gamma'$, *either*

$$\int_{\eta}^{U(x,y)} A(x,y,u)\,du < 0$$

holds for all η *such that* $f(x,y) < \eta < U(x,y)$ *in the case* $f(x,y) < U(x,y)$, *or*

$$\int_{U(x,y)}^{\eta} A(x,y,u)\,du > 0$$

holds for all η *such that* $U(x,y) < \eta < f(x,y)$ *in the case* $U(x,y) < f(x,y)$.

Then, for ε *sufficiently small, say* $\varepsilon \le \varepsilon_0$, *there exists a solution* $u = u(x,y,\varepsilon)$ *of problem* (4) *which exhibits boundary layer behavior as* $\varepsilon \to 0$. *More precisely, we have the estimate*

$$u(x,y,\varepsilon) = U(x,y) + v(x,y,\varepsilon) + O(\varepsilon), \tag{6}$$

where v is a boundary layer correction function which converges to zero exponentially as $\varepsilon \to 0$ for (x,y) in Ω.

PROOF. We give the proof only for the case $f(x,y) > U(x,y)$. The proof for the remaining case is similar.

The main idea of the proof is to construct lower and upper solutions α and β as in Amann's theorem. These lower and upper solutions are simple modifications of the reduced solution $U(x,y)$ which readily allow the deduction of the estimate (6).

We note at the outset that condition (d) implies that there exists a constant vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$ such that

$$[2A(x,y,U(x,y))\nabla U + \mathbf{B}(x,y,U(x,y))] \cdot \boldsymbol{\nu} > 1$$

for all (x,y) in Ω. For notational ease, we set $h(x,y) = \nu_1 x + \nu_2 y$, so that $\boldsymbol{\nu} = \nabla h$.

We take as lower and upper solutions the following functions:

$$\alpha(x,y,\varepsilon) = U(x,y) - \varepsilon\gamma\exp[\lambda h(x,y)] \quad \text{in } \Omega,$$
$$\beta(x,y,\varepsilon) = \begin{cases} U(x,y) + \varepsilon\gamma\exp[\lambda h(x,y)] & \text{in } \Omega - S^b, \\ U(r,z) + \overline{v}(r,\varepsilon) + \varepsilon\gamma\exp[\lambda h(r,z)] & \text{in } S^b, \end{cases}$$

where λ is such that $\lambda > l$ for l an upper bound on

$$|A_u(x,y,\xi)\nabla U \cdot \nabla U + \mathbf{B}_u(x,y,\xi) \cdot \nabla U + C_u(x,y,\xi)|$$

for (x,y) in $\overline{\Omega}$ and for $U(x,y) \le \xi \le f(x,y)$, and γ is such that

$$\gamma(\lambda - l)\min_{\overline{\Omega}}\{\exp[\lambda h(x,y)]\} > M = \max_{\overline{\Omega}}\{\Delta U\}. \tag{7}$$

The function $\overline{v}$ is of the form $\overline{v}(r,\varepsilon) = w(r)\hat{v}(r,\varepsilon)$, where $w(r)$ is a C^2-cut-off function with $w(r) = 1$ for $r < d/2$, $w(r) = 0$ for $r > d$, and $0 < w(r) < 1$ for all r, and where the function $\hat{v}$ satisfies

$$\varepsilon\hat{v}_{rr} < A'(\hat{v})\hat{v}_r^2 = \left[\min_{\Gamma-\Gamma'} A(0,z,U(0,z)+\hat{v})\right]\hat{v}_r^2 \quad \text{in } S^b,$$
$$\hat{v} > 0, \quad \hat{v}_r < 0, \quad \hat{v}(0,\varepsilon) = \max_{\Gamma-\Gamma'}\{f(0,z) - U(0,z)\},$$
$$\hat{v} \to 0 \text{ exponentially as } \varepsilon \to 0 \text{ for } r > 0.$$

The existence of such a function $\hat{v}$ is a consequence of conditions (b) and (e), and is establsihed in Howes [**12**].

We note first of all that $\alpha \le \beta$ on all of $\overline{\Omega}$, and that α and β are both of class C^2 on Ω. Further, it is clear that $\alpha \le f$ and $\beta \ge f$ along the boundary segment $\Gamma - \Gamma'$.

Let us now complete the proof that α is a lower solution in the sense of Amann's theorem. From the Mean Value Theorem, we have

$$\begin{aligned}
\varepsilon\Delta\alpha &- A(x,y,\alpha)\nabla\alpha \cdot \nabla\alpha - \mathbf{B}(x,y,\alpha)\cdot\nabla\alpha - C(x,y,\alpha) \\
&= \varepsilon\Delta U + \varepsilon\gamma\exp[\lambda h(x,y)]\{\lambda[2A(x,y,U(x,y))\nabla U + \mathbf{B}(x,y,U(x,y))]\cdot\boldsymbol{\nu} \\
&\quad + [A_u(x,y,U(x,y))\nabla U \cdot \nabla U + \mathbf{B}_u(x,y,U(x,y))\cdot\nabla U + C_u(x,y,U(x,y))]\} \\
&\quad + O(\varepsilon^2) \\
&\ge -\varepsilon M + \varepsilon\gamma\exp[\lambda h(x,y)](\lambda - l) + O(\varepsilon^2) > 0
\end{aligned}$$

for ε sufficiently small by inequality (7). Thus, the function α satisfies the differential inequality for a lower solution from Amann's theorem.

We turn now to a demonstration that the function β satisfies the appropriate differential inequality of Amann's theorem for an upper solution. We do this by examining the behavior of β in S^b near the boundary Γ and then in Ω away from the boundary.

To begin, we note that since $\hat{v}$ converges to zero exponentially as $\varepsilon \to 0$, for each $\kappa > 0$ there is an ε_1 such that $-\hat{v}_r > \kappa\hat{v}$ for $\varepsilon < \varepsilon_1$ (cf. Fife [**8**]). Thus we have $-m\hat{v}_r > l\hat{v}$, where l is as above and

$$m = \left\{\min_{S^b}(2A(0,z,U(0,z))\nabla U + \mathbf{B}(0,z,U(0,z))) \cdot \nabla r\right\}$$

for ε sufficiently small. We note that $m < 0$ since

$$[2A(0,z,U(0,z))\nabla U + \mathbf{B}(0,z,U(0,z))] \cdot \nabla F > 0$$

from condition (c).

Now, for $r < d/2$ we have from the Mean Value Theorem,

$$\begin{aligned}
&\varepsilon\Delta\beta - A(x,y,\beta)\nabla\beta\cdot\nabla\beta - \mathbf{B}(x,y,\beta)\cdot\nabla\beta - C(x,y,\beta)\\
&\quad = (r_x^2 + r_y^2)[\varepsilon\hat{v}_{rr} - A(r,z,U(r,z)+\hat{v})\hat{v}_r^2]\\
&\qquad - [(2A(r,z,U(r,z))\nabla U + \mathbf{B}(r,z,U(r,z)))\cdot\nabla r]\hat{v}_r\\
&\qquad - [A_u(r,z,U(r,z)+\varphi) + \mathbf{B}_u(r,z,U(r,z)+\varphi) + C_u(r,z,U(r,z)+\varphi)]\\
&\qquad + O(\varepsilon)\\
&\quad \le (r_x^2 + r_y^2)[\varepsilon\hat{v}_{rr} - A'(\hat{v})\hat{v}_r^2] - m\hat{v}_r + l\hat{v} + O(\varepsilon) < 0
\end{aligned}$$

for ε and d sufficiently small, where $0 < \varphi \le f(0,z) - U(0,z)$. Thus, the function β satisfies the appropriate differential inequality of Amann's theorem in S^b for $r \le d/2$.

Let us now consider that portion of S^b in which $r > d/2$. Since the function $\hat{v}$ converges to zero exponentially as $\varepsilon \to 0$, $\hat{v}$ is a transcendentally small term (T.S.T.) for $r > d/2$. Expanding as before, but explicitly listing the $O(\varepsilon)$ terms, we have

$$\begin{aligned}
&\varepsilon\Delta\beta - A(x,y,\beta)\nabla\beta\cdot\nabla\beta - \mathbf{B}(x,y,\beta)\cdot\nabla\beta - C(x,y,\beta)\\
&\quad = \varepsilon\{\Delta U - \gamma\exp(\lambda h(r,z))[\lambda((2A(r,z,U(r,z))\nabla U + \mathbf{B}(r,z,U(r,z)))\cdot\boldsymbol{\nu})\\
&\qquad\quad + A_u(r,z,U(r,z)+\varphi)\nabla U\cdot\nabla U + \mathbf{B}_u(r,z,U(r,z)+\varphi)\\
&\qquad\qquad\qquad\qquad \cdot\nabla U + C_u(r,z,U(r,z)+\varphi)]\}\\
&\qquad + O(\varepsilon^2) + \text{T.S.T}\\
&\quad \le \varepsilon[M - \gamma\exp(\lambda h(r,z))](\lambda - l) + O(\varepsilon^2) + \text{T.S.T} < 0
\end{aligned}$$

by virtue of inequality (7) for ε sufficiently small. Thus, β satisfies the appropriate differential inequality of Amann's theorem in S^b for $r > d/2$.

Let us now consider the case (x,y) in $\Omega - S^b$. In this case, the function β is of the form

$$\beta(x,y,\varepsilon) = U(x,y) + \varepsilon\gamma\exp[\lambda h(x,y)].$$

Thus, the proof that β satisfies the appropriate differential inequality of Amann's theorem in $\Omega - S^b$ is essentially identical to the proof for the case $r > d/2$ in S^b. The details are omitted.

We have thus far shown that the functions α and β satisfy all the conditions of Amann's theorem for ε sufficiently small. We conclude, then, that there exists a solution $u = u(x, y, \varepsilon)$ of problem (4) such that

$$\alpha(x, y, \varepsilon) \leq u(x, y, \varepsilon) \leq \beta(x, y, \varepsilon).$$

Since α and β both converge to $U(x, y)$ as $\varepsilon \to 0$, it must be true that the solution u behaves in the same way. The estimate (6) follows immediately from the above chain of inequalities. This completes the proof of the theorem.

To illustrate the application of the theorem, let us consider the problem

$$\begin{aligned} \varepsilon \Delta u &= u_x^2 + u_y^2 + (3x+2)u_x + yu_y + u \quad \text{in } \Omega, \\ u(x, y, \varepsilon) &= 1 \quad \text{on } \Gamma, \end{aligned} \tag{8}$$

where Ω is the region inside the circle given by $F(x, y) = x^2 + y^2 - \frac{1}{2} = 0$. In this example, $A(x, y, u) = 1$, $\mathbf{B}(x, y, u) = (3x + 2, y)$, and $C(x, y, u) = u$.

The reduced equation from problem (8) is

$$U_x^2 + U_y^2 + (3x + 2)U_x + yU_y + U = 0.$$

A solution of this equation is $U(x, y) = 0$. We note that in this case the reduced solution U does not equal the boundary data anywhere on Γ, that is, $\Gamma' = \emptyset$. Thus, we anticipate boundary layer behavior along all of Γ.

Since $A(x, y, u) = 1 > 0$ for (x, y) on Γ, it is clear that the integral condition (e) is satisfied. Further, we have

$$[2A(x, y, U(x, y))\nabla U + \mathbf{B}(x, y, U(x, y))] \cdot \nabla F = 6x^2 + 4x + 2y^2 > 0$$

along Γ. We note finally that the vector field $[2A(x, y, U(x, y))\nabla U + \mathbf{B}(x, y, U(x, y))] = (3x + 2, y)$ does not vanish in Ω since $x > -\frac{2}{3}$. Thus, all the conditions of the theorem are satisfied. We conclude that for ε sufficiently small there exists a solution $u = u(x, y, \varepsilon)$ of problem (8) such that $\lim_{\varepsilon \to 0} u(x, y, \varepsilon) = U(x, y) = 0$ for (x, y) in Ω.

5. Interior layer theory. We turn now to a study of the interior layer behavior of solutions of the problem

$$\begin{aligned} \varepsilon \Delta u &= A(x, y, u)\nabla u \cdot \nabla u + \mathbf{B}(x, y, u) \cdot \nabla u + C(x, y, u) \quad \text{in } \Omega, \\ u(x, y, \varepsilon) &= f(x, y) \quad \text{on } \Gamma, \end{aligned} \tag{9}$$

in the vicinity of certain closed curves properly contained in Ω. More precisely, for a given closed curve $\mathcal{C}$ in Ω, we seek to establish the existence, for ε sufficiently small, of a solution $u = u(x, y, \varepsilon)$ such that $\lim_{\varepsilon \to 0} u(x, y, \varepsilon) = U_1(x, y)$ for (x, y) in Ω_1, and $\lim_{\varepsilon \to 0} u(x, y, \varepsilon) = U_2(x, y)$ for (x, y) in Ω_2, where we recall that Ω_1 is the region inside the curve $\mathcal{C}$ and Ω_2 is the region outside the curve. The functions U_1 and U_2 are certain solutions of the reduced equation

$$0 = A(x, y, U)\nabla U \cdot \nabla U + \mathbf{B}(x, y, U) \cdot \nabla U + C(x, y, U) \quad \text{in } \Omega, \tag{10}$$

but only U_2 is required to satisfy the boundary data on at least part of Γ, i.e., $U_2(x, y) = f(x, y)$ on $\Gamma' \subset \Gamma$. No such condition is imposed on U_1, since we expect the solution u to be near U_1 only far from the boundary.

Unlike boundary layer behavior, interior layer behavior is possible even if $U_2(x,y) = f(x,y)$ along all of Γ, i.e., if $\Gamma' = \Gamma$. In this case, we expect to see interior layer behavior and not boundary layer behavior. If, on the other hand, $\Gamma' \neq \Gamma$, we expect to see both boundary and interior layer behavior in $\overline{\Omega}$. Naturally, if $U_1(x,y) = U_2(x,y)$ for (x,y) on $\mathcal{C}$, then we expect not to see layer behavior at that point.

In analogy with the boundary layer situation, we shall require that the characteristics of the reduced equation (10) be outgoing along $\mathcal{C}$ for $U = U_1, U_2$. This means that the characteristics of the reduced equation for U_1 must point into Ω_2, while the characteristics of the reduced equation for U_2 must point into Ω_1. This geometric condition may be expressed analytically as

$$[2A(x,y,U_1(x,y))\nabla U_1 + \mathbf{B}(x,y,U_1(x,y))] \cdot \mathbf{n}(x,y) > 0$$

and

$$[2A(x,y,U_2(x,y))\nabla U_2 + \mathbf{B}(x,y,U_2(x,y))] \cdot \mathbf{n}(x,y) < 0$$

for (x,y) on $\mathcal{C}$, where $\mathbf{n}$ is an outward normal to the curve $\mathcal{C}$. We shall typically assume that the curve $\mathcal{C}$ is given by an equation $J(x,y) = 0$, where $\nabla J \neq \mathbf{0}$. Thus, an outward normal to $\mathcal{C}$ is given by ∇J if the region inside $\mathcal{C}$ is given by $\{(x,y)\colon J(x,y) < 0\}$. Similarly, the outward normal is given by $-\nabla J$ if the region inside $\mathcal{C}$ is given by $\{(x,y)\colon J(x,y) > 0\}$.

The following is our principal result concerning the interior layer behavior of solutions of problem (9). For simplicity, we assume that boundary layer behavior is absent. In remarks after the proof of the theorem, we indicate the modifications necessary to include boundary layer behavior.

THEOREM (INTERIOR LAYER BEHAVIOR). *Assume*

(a) *the functions* $A = A(x,y,u)$, $\mathbf{B} = \mathbf{B}(x,y,u) = (B_1(x,y,u), B_2(x,y,u))$, *and* $C = C(x,y,u)$ *are of class* C^1 *on* $\Omega \times E^1$, *and* $f = f(x,y)$ *is continuous on* Γ;

(b) *there exist two distinct solutions* $U_1 = U_1(x,y)$ *and* $U_2 = U_2(x,y)$ *of the reduced equation* (10) *such that* $U_2(x,y) = f(x,y)$ *for each* (x,y) *on* Γ;

(c) *there exists a smooth closed curve* $\mathcal{C}$ *given by* $J(x,y) = 0$, *where*

$$J(x,y) = \int_{U_1(x,y)}^{U_2(x,y)} A(x,y,u)\,du$$

and $\nabla J \neq \mathbf{0}$ *along* $\mathcal{C}$;

(d) *for* (x,y) *on* $\mathcal{C}$, *either*

$$\int_{\eta}^{U_2(x,y)} A(x,y,u)\,du < 0$$

holds for all η *such that* $U_1(x,y) < \eta < U_2(x,y)$ *in the case* $U_1(x,y) < U_2(x,y)$, *or*

$$\int_{U_2(x,y)}^{\eta} A(x,y,u)\,du > 0$$

holds for all η *such that* $U_2(x,y) < \eta < U_1(x,y)$ *in the case* $U_2(x,y) < U_1(x,y)$;

(e) *the characteristics of the reduced equation* (10) *are outgoing along* $\mathcal{C}$, *that is,*

$$[2A(x,y,U_1(x,y))\nabla U_1 + \mathbf{B}(x,y,U_1(x,y))] \cdot \mathbf{n}(x,y) > 0$$

and

$$[2A(x,y,U_2(x,y))\nabla U_2 + \mathbf{B}(x,y,U_2(x,y))] \cdot \mathbf{n}(x,y) < 0$$

for (x,y) *on* $\mathcal{C}$, *where* $\mathbf{n}$ *is an outward normal to* $\mathcal{C}$ ($\mathbf{n} = \nabla J$ *if* $J(x,y) < 0$ *defines the inside of* $\mathcal{C}$, *and* $\mathbf{n} = -\nabla J$ *if* $J(x,y) > 0$ *defines the inside of* $\mathcal{C}$);

(f) *the vector field* $2A(x,y,U_1(x,y))\nabla U_1 + \mathbf{B}(x,y,U_1(x,y))$ *does not vanish in* Ω_1, *and the vector field* $2A(x,y,U_2(x,y))\nabla U_2 + \mathbf{B}(x,y,U_2(x,y))$ *does not vanish on* Ω_2.

Then, for ε *sufficiently small, say* $\varepsilon \leq \varepsilon_0$, *there exists a solution* $u = u(x,y,\varepsilon)$ *of problem* (9) *which exhibits interior layer behavior along the closed curve* $\mathcal{C}$ *in* Ω. *More precisely,*

$$u(x,y,\varepsilon) = \begin{cases} U_1(x,y) + v_1(x,y,\varepsilon) + O(\varepsilon) & \text{in } \Omega_1, \\ U_2(x,y) + v_2(x,y,\varepsilon) + O(\varepsilon) & \text{in } \Omega_2, \end{cases} \tag{11}$$

where v_1 *and* v_2 *converge to zero exponentially as* $\varepsilon \to 0$ *for* (x,y) *in* Ω_1 *and* Ω_2, *respectively.*

PROOF. We consider only the case $U_2(x,y) < U_1(x,y)$ for (x,y) on $\mathcal{C}$. The proof for the case $U_2(x,y) > U_1(x,y)$ is similar.

As in the proof of the boundary layer theorem of §4, we construct lower and upper solutions. However in the present situation it is more convenient to construct these functions as in the extended version of Amann's theorem given in §2.

We define the functions α and β as follows:

$$\alpha(x,y,\varepsilon) = \begin{cases} U_2(x,y) - \varepsilon\gamma_2 \exp[\lambda_2 h_2(x,y)] & \text{in } \Omega_2, \\ U_1(t,s) + \overline{v}_1(t,\varepsilon) - \varepsilon\gamma_1 \exp[\lambda_1 h_1(x,y)] & \text{in } S \cap \Omega_1, \\ U_1(x,y) - \varepsilon\gamma_1 \exp[\lambda_1 h_1(x,y)] & \text{in } \Omega_1 - S, \end{cases}$$

$$\beta(x,y,\varepsilon) = \begin{cases} U_2(x,y) + \varepsilon\gamma_2 \exp[\lambda_2 h_2(x,y)] & \text{in } \Omega_2 - S, \\ U_2(t,s) + \overline{v}_2(t,\varepsilon) + \varepsilon\gamma_2 \exp[\lambda_2 h_2(t,s)] & \text{in } S \cap \Omega_2, \\ U_1(x,y) + \varepsilon\gamma_1 \exp[\lambda_1 h_1(x,y)] & \text{in } \Omega_1, \end{cases}$$

where γ_i, λ_i, and h_i are as in the boundary layer theorem of §4 for $U = U_i$ in Ω_i, $i = 1,2$, and where $\overline{v}_1(t,\varepsilon) = \overline{w}(t)\hat{v}_1(t,\varepsilon)$ and $\overline{v}_2(t,\varepsilon) = \overline{w}(t)\hat{v}_2(t,\varepsilon)$ for $\overline{w}$ a C^2-cut-off function such that $\overline{w}(t) = 1$ for $|t| \leq d/2$, $\overline{w}(t) = 0$ for $|t| > d$, and $0 < \overline{w} < 1$ for all t. In addition, the functions $\hat{v}_1$ and $\hat{v}_2$ satisfy

$$\varepsilon\hat{v}_{1tt} > A''(\hat{v}_1)(\hat{v}_{1t})^2 = \max_s[A(0,s,U_1(0,s) + \hat{v}_1)](\hat{v}_{1t})^2,$$

$$\hat{v}_1(0,\varepsilon) = \min_s[U_2(0,s) - U_1(0,s)], \qquad \hat{v}_1 < 0,$$

$$\hat{v}_{1t} > 0, \qquad \lim_{\varepsilon\to 0}\hat{v}_{1t}(0,\varepsilon) = \infty$$

$$\hat{v}_1 \to 0 \text{ exponentially as } \varepsilon \to 0 \text{ for } t > 0,$$

$$\varepsilon\hat{v}_{2tt} < A'''(\hat{v}_2)(\hat{v}_{2t})^2 = \min_s[A(0,s,U_2(0,s) + \hat{v}_2)](\hat{v}_{2t})^2,$$

$$\hat{v}_2(0,\varepsilon) = \max_s[U_1(0,s) - U_2(0,s)], \qquad \hat{v}_2 > 0,$$

$$\hat{v}_{2t} > 0, \qquad \lim_{\varepsilon\to 0}\hat{v}_{2t}(0,\varepsilon) = \infty$$

$$\hat{v}_2 \to 0 \text{ exponentially as } \varepsilon \to 0 \text{ for } t > 0.$$

The existence of such functions is a consequence of conditions (b), (c), and (d) and the one-dimensional theory due to Howes [**12**].

It is clear from the construction of α and β that $\alpha \leq \beta$ on $\overline{\Omega}$, and that $\alpha \leq f \leq \beta$ along Γ. The functions α and β are clearly not of class C^2 on all of Ω since these functions have discontinuous gradients across the curve $\mathcal{C}$. However, along $\mathcal{C}$ we have

$$D_l\alpha(0,s,\varepsilon) = U_{2t}(0,s) + O(\varepsilon) < U_{1t}(0,s) + \hat{v}_{1t}(0,\varepsilon) + O(\varepsilon) = D_r\alpha(0,s,\varepsilon),$$

and

$$D_l\beta(0,s,\varepsilon) = U_{2t}(0,s) + \hat{v}_{2t}(0,\varepsilon) + O(\varepsilon) > U_{1t}(0,s) + O(\varepsilon) = D_r\beta(0,s,\varepsilon)$$

for ε sufficiently small. Thus α and β satisfy the appropriate differential inequalities along $\mathcal{C}$ in the extended version of Amann's theorem.

The proof that α and β are lower and upper solutions in $\Omega - S$ is essentially identical to that part of the proof of the boundary layer theorem in §4 that deals with the region $\Omega - S^b$. Hence, the details are omitted. It remains only to verify that α and β satisfy the appropriate differential inequalities in $S - \mathcal{C}$.

We note that since $\hat{v}_1$ and $\hat{v}_2$ both converge to zero exponentially as $\varepsilon \to 0$, for each $\kappa > 0$ there is a number ε_1 such that $\hat{v}_{it} > \kappa|\hat{v}_i|$ for $i = 1,2$ and $\varepsilon < \varepsilon_1$. Thus, if

$$\begin{aligned} m_1 &= \max\{[2A(t,s,U_1(t,s))\nabla U_1 + \mathbf{B}(t,s,U_1(t,s))] \cdot \nabla t\}, \\ m_2 &= \min\{[2A(t,s,U_2(t,s))\nabla U_2 + \mathbf{B}(t,s,U_2(t,s))] \cdot \nabla t\}, \end{aligned}$$

and l is an upper bound on $|A_u(t,s,\xi)\nabla U_i \cdot \nabla U_i + \mathbf{B}_u(t,s,\xi) \cdot \nabla U_i + C_u(t,x,\xi)|$ for $i = 1,2$ and for $U_2(t,s) \leq \xi \leq U_1(t,s)$, then we have $-m_1\hat{v}_{1t} + l\hat{v}_1 > 0$ and $-m_2\hat{v}_{2t} + l\hat{v}_2 < 0$. (Note that $m_1 < 0$ and $m_2 > 0$ by condition (f) for d sufficiently small since ∇t is parallel to $-\mathbf{n}$.)

Let us consider first the function α. Expanding as in the proof of the boundary layer theorem of §4, and recalling that $\hat{v}_1 = \overline{v}_1$ for $|t| \leq d/2$, we have for such t the following:

$$\begin{aligned} &\varepsilon\Delta\alpha - A(x,y,\alpha)\nabla\alpha \cdot \nabla\alpha - \mathbf{B}(x,y,\alpha) \cdot \nabla\alpha - C(x,y,\alpha) \\ &\quad = (t_x^2 + t_y^2)[\varepsilon\hat{v}_{1tt} - A(t,s,U_1(t,s) + \hat{v}_1)\hat{v}_{1t}^2] \\ &\qquad - [(2A(t,s,U_1(t,s))\nabla U_1 + \mathbf{B}(t,s,U_1(t,s))) \cdot \nabla t]\hat{v}_{1t} \\ &\qquad - [A_u(t,s,U_1(t,s) + \theta) + \mathbf{B}_u(t,s,U_1(t,s) + \theta) + C_u(t,s,U_1(t,s) + \theta)]\hat{v}_1 \\ &\qquad\quad + O(\varepsilon) \\ &\quad > (t_x^2 + t_y^2)[\varepsilon\hat{v}_{1tt} - A''(\hat{v}_1)(\hat{v}_{1t})^2] - m_1\hat{v}_{1t} + l\hat{v}_1 > 0 \end{aligned}$$

for ε sufficiently small and for d sufficiently small, where θ is such that $U_2(t,s) - U_1(t,s) \leq \theta < 0$.

Now, for $t > d/2$, the function $\hat{v}_1$ is a transcendentally small term. Thus, the proof that α satisfies the appropriate differential inequality is essentially identical to the corresponding part of the proof of the boundary layer theorem of §4. The details are omitted.

Let us consider the function β. As above, we have for $t \geq -d/2$ the following:

$$\begin{aligned}
\varepsilon\Delta\beta &- A(x,y,\beta)\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\beta - \mathbf{B}(x,y,\beta)\cdot\boldsymbol{\nabla}\beta - C(x,y,\beta) \\
&= (t_x^2+t_y^2)[\varepsilon\hat{v}_{2tt} - A(t,s,U_2(t,s)+\hat{v}_2)\hat{v}_{2t}^2] \\
&\quad - [(2A(t,s,U_2(t,s))\boldsymbol{\nabla}U_2 + \mathbf{B}(t,s,U_2(t,s))\cdot\boldsymbol{\nabla}t]\hat{v}_{2t} \\
&\quad - [A_u(t,s,U_2(t,s)+\theta') + \mathbf{B}_u(t,s,U_2(t,s)+\theta') + C_u(t,s,U_2(t,s)+\theta')]\hat{v}_2 \\
&\quad + O(\varepsilon) \\
&\leq (t_x^2+t_y^2)[\varepsilon\hat{v}_{2tt} - A'''(\hat{v}_2)(\hat{v}_{2t})^2] - m_2\hat{v}_{2t} + l\hat{v}_2 < 0
\end{aligned}$$

for ε sufficiently small, where $0 < \theta' \leq U_1(t,s) - U_2(t,s)$. For $t < -d/2$, the term $\hat{v}_2$ is transcendentally small, and so the proof proceeds as in the boundary layer case. The details are omitted.

We have thus shown that the functions α and β satisfy all the conditions of the extended version of Amann's theorem given in §3. It follows, therefore, that there exists a solution $u = u(x,y,\varepsilon)$ of problem (9) such that

$$\alpha(x,y,\varepsilon) \leq u(x,y,\varepsilon) \leq \beta(x,y,\varepsilon).$$

Since α and β both converge to U_1 in Ω_1 and to U_2 in Ω_2, the solution u must behave in the same way. The estimate (11) follows immediately from the nature of α and β and from the above chain of inequalities.

To include boundary layer behavior in the above theorem, we need only assume that $U_2 \neq f$ along Γ and assume that the conditions of the boundary layer theorem hold for $U = U_2$, $\Omega = \Omega_2$. A proof of this follows easily from the proofs of the two main theorems, since the boundary layer correction term used in §4 is zero in the vicinity of $\mathcal{C}$ and the interior layer correction terms $\overline{v}_1$ and $\overline{v}_2$ used above are zero in the vicinity of the boundary.

We conclude this section with an illustration of the application of the interior layer theorem. Let us consider the problem

$$(12)\qquad \begin{gathered}
\varepsilon\Delta u = (r^2-u)(u_x^2+u_y^2) + (6u-7)(xu_x+yu_y) - 6(u-1)(2r^2-1) \quad \text{in } \Omega, \\
u = 2 \quad \text{on } \Gamma_1, \qquad u = 1 \quad \text{on } \Gamma_2,
\end{gathered}$$

where $r^2 = x^2+y^2$, and Ω is the annular region bounded by the circles Γ_1 and Γ_2, where Γ_1 is given by $F_1(x,y) = x^2+y^2-1 = 0$ and Γ_2 is given by $F_2(x,y) = x^2+y^2-4 = 0$. In this example, $A(x,y,u) = (r^2-u)$, $\mathbf{B}(x,y,u) = (6u-7)(x,y)$, and $C(x,y,u) = -6(u-1)(2r^2-1)$.

The reduced equation from problem (12) is

$$0 = (r^2-U)(U_x^2+U_y^2) + (6U-7)(xU_x+yU_y) - 6(U-1)(2r^2-1).$$

Two solutions of this equation are $U_1(x,y) = 1+r^2$ and $U_2(x,y) = 1$. We note that U_1 is equal to the boundary data along Γ_1, and U_2 is equal to the boundary data along Γ_2. Thus, boundary layer behavior is impossible.

Let us turn now to the interior layer conditions. We have

$$J(x,y) = \int_{1+r^2}^{1} (r^2-u)\,du = r^2(2-r^2)/2.$$

The equation $J(x,y) = 0$ defines a circle $\mathcal{C}$ of radius $\sqrt{2}$. We note that $\nabla J \neq \mathbf{0}$ along $\mathcal{C}$, and that ∇J is an inward normal to $\mathcal{C}$. Moreover, for (x,y) on $\mathcal{C}$, we have

$$\int_1^{\eta} (r^2 - u)\, du = (1-\eta)(\eta - 3)/2 > 0$$

for $1 < \eta < 1 + r^2 = 3$. Thus, the integral conditions of the interior layer theorem are satisfied.

The closed curve $\mathcal{C}$ divides Ω into two disjoint regions Ω_1 and Ω_2, with $\Omega_1 = \{(x,y)\colon 1 < x^2 + y^2 < 2\}$ and $\Omega_2 = \{(x,y)\colon 2 < x^2 + y^2 < 4\}$. It is clear that the vector field $2(r^2 - U_1)\nabla U_1 + (6U_1 - 7)(x,y) = (6r^2 - 5)(x,y)$ does not vanish in Ω_1, and the vector field $2(r^2 - U_2)\nabla U_2 + (6U_2 - 7)(x,y) = -(x,y)$ does not vanish in Ω_2. Further, with $-\nabla J$ an outward normal to $\mathcal{C}$, we have

$$[2(r^2 - U_1)\nabla U_1 + (6U_1 - 7)(x,y)] \cdot (-\nabla J) = 28 > 0 \quad \text{along } \mathcal{C},$$

and

$$[2(r^2 - U_2)\nabla U_2 + (6U_2 - 7)(x,y)] \cdot (-\nabla J) = -4 < 0 \quad \text{along } \mathcal{C}.$$

Thus, all conditions of the interior layer theorem are satisfied. We conclude that problem (12) has a solution $u = u(x,y,\varepsilon)$ such that $\lim_{\varepsilon\to 0} u(x,y,\varepsilon) = U_1(x,y) = 1 + r^2$ in Ω_1 and $\lim_{\varepsilon\to 0} u(x,y,\varepsilon) = U_2(x,y) = 1$ in Ω_2.

6. Concluding remarks. A major assumption made in this paper is that the characteristics of the reduced problem derived from the full problem are outgoing (in the proper sense) along those curves along which we expect layer behavior. Critical use is made of this assumption in the proofs of the two main theorems. However, this characteristic condition imposes rather severe limits on the nature of the solutions of the reduced equation. It is natural to wonder if the condition may be relaxed in some way. Some indication that it cannot has been given by Howes [**16**], who showed that for the quasilinear case (i.e. $A \equiv 0$), boundary layer behavior is impossible if the characteristics of the reduced equation are "incoming" along the boundary. A forthcoming paper will address this question for the full quadratically nonlinear problem.

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