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ON INDUCTIVE LIMITS OF MATRIX ALGEBRAS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Let $\mathfrak{A}$ be a UHF algebra and $\mathscr{A}(\mathbf{D})$ the disk algebra. If $\mathfrak{A} = [\bigcup_{n \geqslant 1} \mathfrak{A}_n]^-$ and α is a product-type automorphism of $\mathfrak{A}$ which leaves each $\mathfrak{A}_n$ invariant, then α defines an embedding

$$\mathfrak{A}_n \otimes \mathscr{A}(\mathbf{D}) \overset{\iota_n}{\hookrightarrow} \mathfrak{A}_{n+1} \otimes \mathscr{A}(\mathbf{D}).$$

The inductive limit of this system is a Banach algebra whose maximal ideal space is closely related to that of the disk algebra if the Connes spectrum $\Gamma(\alpha)$ is finite.

I

Let $\mathfrak{A}$ be a UHF-algebra, and let $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ be a nested sequence of subalgebras of $\mathfrak{A}$, each of which is isomorphic to a complex matrix algebra, such that $\mathfrak{A} = (\bigcup_{n \geqslant 1} \mathfrak{A}_n)^-$. Let $\mathscr{A}(\mathbf{D})$ be the disk algebra; that is, $\mathscr{A}(\mathbf{D})$ is the commutative Banach algebra of functions that are continuous on the closed unit disk $\mathbf{D}^-$ and holomorphic in the interior. In this note we consider Banach algebras $\mathscr{C}$ of the following sort: $\mathscr{C}$ possesses a nested sequence $\{\mathscr{C}_n\}_{n=1}^{\infty}$ of subalgebras, $\mathscr{C}_n$ is isomorphic to $\mathfrak{A}_n \otimes \mathscr{A}(\mathbf{D})$ and $\mathscr{C}$ is the closure of the union of the $\mathscr{C}_n$'s. To be more precise, let α be a "product-type" automorphism of the UHF-algebra $\mathfrak{A}$. Denote by $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ the closed subalgebra of the C^*-crossed product $\mathbf{Z} \times_\alpha \mathfrak{A}$ which is generated by the nonnegative powers of α. Such algebras have been called semicrossed products [**1**, **9**]. The class of Banach algebras treated here is semicrossed products of UHF-algebras by product-type actions. That Banach algebras $\mathscr{C}$ of this kind possess a nested sequence of subalgebras $\{\mathscr{C}_n\}_{n=1}^{\infty}$ with $\mathscr{C}_n \cong \mathfrak{A}_n \otimes A(\mathbf{D})$ and $\bigcup \mathscr{C}_n$ dense in $\mathscr{C}$, is proved below (cf. Proposition II.5, Lemma IV.4).

It is not surprising that there is a close connection between ideals in the disk algebra and ideals in $\mathscr{C} = \mathbf{Z}^+ \times_\alpha \mathfrak{A}$. Our main result states that if the Connes spectrum $\Gamma(\alpha)$ is a finite subgroup of the unit circle, then there is a natural one-to-one correspondence between the strong structure space (or maximal ideal space) $\mathscr{M}$ of $\mathscr{C}$ and $\mathbf{D}^-/\Gamma(\alpha)$ ($\mathbf{D}^-$ the closed unit disk); in fact, the correspondence is even a homeomorphism for the respective hull-kernel topologies (Theorem 4.13). This work reveals a phenomenon which illustrates how greatly the ideal theory of such algebras can differ from that of $\mathscr{C}^*$-algebras. Recall that if $\mathscr{C}$ is a unital

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C^*-algebra and $\{\mathscr{C}_n\}_{n=1}^{\infty}$ a nested sequence of unital C^*-subalgebras with dense union, and if $J \subset \mathscr{C}$ is any (closed, two-sided) ideal, then $J = [\bigcup_{n\geqslant 1} J \cap \mathscr{C}_n]^-$. (This was proved in **[2]** in the context of AF-algebras, though Brattelli later observed it holds more generally **[3]**.) But if $\mathscr{C}$ is a Banach algebra of the type $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$, and $\mathscr{C} = [\bigcup \mathscr{C}_n]^-$, where $\{\mathscr{C}_n\}_{n=1}^{\infty}$ is a nested sequence of unital subalgebras, then for an (closed, two-sided) ideal $J \subset \mathscr{C}$ it may or may not be the case that J equals $[\bigcup_{n\geqslant 1}(J \cap \mathscr{C}_n)]^-$. If this holds, we will say J is *ascending* with respect to $\{\mathscr{C}_n\}_{n=1}^{\infty}$. For example, it can happen that only a proper subset of the maximal ideal space $\mathscr{M}$ consists of ascending ideals; if $\mathscr{M}_a \subset \mathscr{M}$ denotes those maximal ideals which are ascending with respect to $\{\mathscr{C}_n\}_{n=1}^{\infty}$, then "typically" we will have $\mathscr{M}_a$ corresponds to $\partial\mathbf{D} \cup \{0\}/\Gamma(\alpha)$. Power **[10]** has studied nest algebras where every ideal is ascending (which he calls inductive).

The paper is organized as follows: In §II the framework of semicrossed products is introduced, along with some remarks on inductive limits. §III pertains to spectral theory of automorphisms; the one new result in this section, which apparently does not follow immediately from the general theory, is Proposition III.10. The Main Theorem, describing the strong structure space of semicrossed products of UHF-algebras by product-type actions, appears in §IV.

II

II.1. We begin by reviewing some standard facts concerning inductive limits of Banach algebras which will be needed later. The proofs (which are omitted) are essentially as in **[11**, 1.23.2], only with "star isomorphism" replaced by "isometric isomorphism".

Let $\{\mathscr{C}_n\}_{n=1}^{\infty}$ be a sequence of Banach algebras with identity, and $\iota_n\colon \mathscr{C}_n \to \mathscr{C}_{n+1}$ an isometric embedding, with $\iota_n(1) = 1$, $n = 1, 2, \ldots$. Then there is a unital Banach algebra $\mathscr{C}$ and isometric embeddings $\Phi_n\colon \mathscr{C}_n \to \mathscr{C}$, with $\Phi_n(1) = 1$, such that the diagram

$$\begin{tikzcd} \mathscr{C}_{n+1} \arrow[dr, "\Phi_{n+1}"] & \\ \mathscr{C}_n \arrow[u, "\iota_n"] \arrow[r, "\Phi_n"'] & \mathscr{C} \end{tikzcd}$$

commutes. If $\tilde{\mathscr{C}}_n$ is the subalgebra $\Phi_n(\mathscr{C}_n) \subset \mathscr{C}$, then $\{\tilde{\mathscr{C}}_n\}_{n=1}^{\infty}$ is a nested sequence of subalgebras whose union is norm dense in $\mathscr{C}$. $\mathscr{C}$ is the inductive limit of $\{\mathscr{C}_n; \iota_n\}$.

Conversely, let $\mathscr{D}$ be a Banach algebra with identity which contains a sequence $\{\mathscr{D}_n\}_{n=1}^{\infty}$ of subalgebras with identity which satisfies

(i) $\mathscr{D}$ is the norm closure of $\bigcup_n \mathscr{D}_n$;

(ii) there is an isomorphic isomorphism Ψ_n of $\mathscr{C}_n$ onto $\mathscr{D}_n \subset \mathscr{D}$ such that the diagram

$$\begin{tikzcd} \mathscr{C}_n \arrow[d, "\iota_n"'] \arrow[dr, "\Psi_n"] & \\ \mathscr{C}_{n+1} \arrow[r, "\Psi_{n+1}"] & \mathscr{D} \end{tikzcd}$$

commutes. Then, there exists an isometric isomorphism θ of $\mathscr{C}$ onto $\mathscr{D}$ such that $\theta(\tilde{\mathscr{C}}_n) = \mathscr{D}_n$ and $\theta|_{\mathscr{C}_n} = \Psi_n\Phi_n^{-1}$.

II.2. COROLLARY. *Let $\mathscr{C}$ be a Banach algebra with identity, and let $\{\mathscr{C}_n\}_{n=1}^{\infty}$ be a nested sequence of closed subalgebras containing the identity. Suppose that $\mathscr{C}$ is the norm closure of $\bigcup_{n\geq 1}\mathscr{C}_n$, and let ι_n: $\mathscr{C}_n \to \mathscr{C}_{n+1}$ be the identity map. Then $\mathscr{C}$ is isometrically isomorphic to the inductive limit of $\{\mathscr{C}_n; \iota_n\}$.*

II.3. *Notation.* If $\mathscr{C} = \varinjlim \{\mathscr{C}_n; \iota_n\}$, we will not distinguish between $\mathscr{C}$ and the subalgebra $\tilde{\mathscr{C}}_n = \Phi_n(\mathscr{C}_n) \subset \mathscr{C}$.

II.4. *Semicrossed products.* Let α be an automorphism of a (separable) C^*-algebra $\mathfrak{A}$, and let $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ be the subalgebra of $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$ consisting of functions F: $\mathbf{Z} \to \mathfrak{A}$ supported on the nonnegative integers. Thus, $\|F\|_1 = \sum_{n\geq 0}\|F(n)\|$, and

$$(FG)(m) = \sum_{n\geq 0} F(n)\alpha^n(G(m-n)) = \sum_{n=0}^{m} F(n)\alpha^n(G(m-n)).$$

Denote by $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ the enveloping Banach algebra of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ with respect to the class of contractive Hilbert space representations. It follows from **[9]** that the norm of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ is also given by

$$\|F\| = \sup\|(U\times\rho)(F)\|, \qquad F\in\mathbf{Z}^+\times_\alpha\mathfrak{A},$$

where the supremum is taken over all covariant representations (U,ρ) of $(\mathfrak{A},\alpha)$; in fact, there is a faithful covariant representation for which equality holds. This shows that $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ may be considered as a closed, nonselfadjoint subalgebra of the C^*-crossed product, $\mathbf{Z}\times_\alpha\mathfrak{A}$. $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ will be called the semicrossed product of $\mathfrak{A}$ with α.

If $K(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ is the subalgebra of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ of functions with finite support, $K(\mathbf{Z}^+, \mathfrak{A}, \alpha) \subset \mathbf{Z}^+\times_\alpha\mathfrak{A}$ is dense.

II.5. Let $\mathfrak{A}$ be a C^*-algebra with 1 and $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ a nested sequence of subalgebras of $\mathfrak{A}$ such that $1\in\mathfrak{A}_n$, $n = 1, 2, \ldots$. Let $\alpha\in\operatorname{Aut}(\mathfrak{A})$ leave each subalgebra $\mathfrak{A}_n$ invariant, and set $\alpha_n = \alpha|\mathfrak{A}_n$. $K(\mathbf{Z}^+, \mathfrak{A}_n, \alpha_n)$ can be regarded in an obvious way as a subalgebra of $K(\mathbf{Z}^+, \mathfrak{A}_{n+1}, \alpha_{n+1})$. This gives rise to an isomorphic embedding ι_n: $\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n \to \mathbf{Z}^+\times_{\alpha_{n+1}}\mathfrak{A}_{n+1}$.

PROPOSITION. $\mathbf{Z}^+\times_\alpha\mathfrak{A} = \varinjlim\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n; \iota_n\}$.

PROOF. Let π be a faithful representation of $\mathfrak{A}$ and u a unitary (on some Hilbert space) such that (π, u) is a covariant representation of $(\mathfrak{A}, \alpha)$. Then $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ is the completion of $K(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ in the norm $\|x\| = \|(u\times\pi)(x)\|$. If $\pi_n = \pi|\mathfrak{A}_n$, then π_n is faithful representation of $\mathfrak{A}_n$, and for $x\in K(\mathbf{Z}^+, \mathfrak{A}_n, \alpha_n)$, $\|x\| = \|(u\times\pi_n)(x)\|$. Now the dense subalgebra $\mathfrak{A}_n$ is α-invariant, and $K(\mathbf{Z}^+, \bigcup\mathfrak{A}_n, \alpha) \subset K(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ is dense in the semicrossed product norm, and hence is dense in $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ as well.

On the other hand, we may regard the algebras $\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$ as a nested sequence of subalgebras of the inductive limit, $\varinjlim\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n; \iota_n\}$, whose union is dense in the inductive limit norm (cf. II.3). Now since $K(\mathbf{Z}^+, \mathfrak{A}_n, \alpha_n)$ is dense in $\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$,

$$\bigcup_n K(\mathbf{Z}^+, \mathfrak{A}_n, \alpha_n) = K\Big(\mathbf{Z}^+, \bigcup_n \mathfrak{A}_n, \alpha_n\Big)$$

is a dense subalgebra of both $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ and $\varinjlim\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n; \iota_n\}$. Thus the proof will be complete if we can show that the two norms agree on this subalgebra. So let $x \in \bigcup_n K(\mathbf{Z}^+, \mathfrak{A}_n, \alpha_n)$. Then $x \in K(\mathbf{Z}^+, \mathfrak{A}_N, \alpha_N)$ for some N. Viewing x as an element of the semicrossed product $\mathbf{Z}^+\times_\alpha\mathfrak{A}$, $\|x\| = \|(u \times \pi)(x)\|$. Also, $(u \times \pi)(x) = (u \times \pi_N)(x)$. But $\|(u \times \pi_N)(x)\|$ is the norm of x, viewed as an element of $\mathbf{Z}^+\times_{\alpha_N}\mathfrak{A}_N$, which coincides with the inductive limit norm of x, since $\mathbf{Z}^+\times_{\alpha_N}\mathfrak{A}_N$ is isometrically embedded in the inductive limit, $\varinjlim\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n; \iota_n\}$.

II.6. COROLLARY. $\mathbf{Z}^+\times_\alpha\mathfrak{A} = [\bigcup_{n\geqslant 1}\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n]^-$.

III

III.1 For each positive integer n, let $[n]$ be a positive integer, and $M^{(n)}$ the $[n]\times[n]$ complex matrices. Let $\mathfrak{A}$ be the UHF-algebra $\otimes_{n\geqslant 1}M^{(n)}$. So if $\mathfrak{A}_n = \otimes_{k=1}^n M_{(n)}$, $\mathfrak{A} = [\bigcup_{n\geqslant 1}\mathfrak{A}_n]^-$, where the embedding $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$ given by $x \xrightarrow{j_n} x \otimes I^{(n)}$, where $I^{(n)}$ is the identity in $M^{(n)}$.

For each $n \geqslant 1$ let $v^{(n)}$ be a unitary element of $M^{(n)}$, and $e_{ij}^{(n)}$ $(1 \leqslant i, j \leqslant [n])$ a full set of matrix units for $M^{(n)}$ such that $e_{ii}^{(n)}$, $i = 1,\ldots,[n]$, are minimal, mutually orthogonal selfadjoint projections with respect to which $v^{(n)}$ has spectral decomposition

$$v^{(n)} = \sum_{i=1}^{[n]} \exp\left(2\pi\sqrt{-1}\,\gamma_i^{(n)}\right)e_{ii}^{(n)}.$$

We will assume that $-1/2 \leqslant \gamma_i^{(n)} < 1/2$ for all i, and that $\gamma_1^{(n)} = 0$. (Since our interest is in the action $\mathrm{Ad}(v^{(n)})$, the assumption $\gamma_1^{(n)} = 0$ is just a convenient normalization.) Set

$$v^{(n)}(t) = \sum_{i=1}^{[n]} \exp\left(2\pi\sqrt{-1}\,\gamma_i^{(n)}t\right)e_{ii}^{(n)}, \qquad t \in \mathbf{R},$$

$$v_{(n)}(t) = \bigotimes_{k=1}^{n} v^{(k)}(t), \qquad \alpha^{(n)}(t) = \mathrm{Ad}\left(v^{(n)}(t)\right),$$

$$\alpha_{(n)}(t) = \bigotimes_{k=1}^{n} \alpha^{(k)}(t), \quad \text{and} \quad \alpha(t) = \bigotimes_{n=1}^{\infty} \alpha^{n}(t),$$

the corresponding strongly continuous one-parameter group of automorphisms. α is called a product-type automorphism of $\mathfrak{A}$. (The parameter t will usually be omitted when $t = 1$.) In case the sequence $\{v_{(n)}(t)\}$ converges in norm, call $v(t)$ the limit; if $v(t)$ exists for all t, it is a norm continuous group of unitaries in $\mathfrak{A}$.

Let

$$\Omega_{n,m} = \left\{\lambda \in \mathbf{R}\colon \lambda = \sum_{k=n+1}^{m} \varepsilon_k\gamma_{j_k}^{(k)}, 1 \leqslant j_k \leqslant [k], \varepsilon_k \in \{0,1\}\right\}$$

$(0 \leqslant n < m)$. Set $\Omega_n = \Omega_{0,n}$, $\Omega_n' = \bigcup_{m=n+1}^{\infty}\Omega_{n,m}$, $\Omega = \Omega_0'$, and $\Omega_\infty = \bigcap_{n=1}^{\infty}\overline{\Omega}_n'$. Next, set $\Gamma_{n,m}$ (resp., Γ_n, Γ_n', Γ) equal to $\{\exp(\sqrt{-1}\,\lambda)\colon \lambda \in \Omega_{n,m}$ (resp., $\Omega_n, \Omega_n', \Omega)\}$. Finally, set $\Gamma_\infty = \bigcap_{n=1}^{\infty}\overline{\Gamma}_n'$.

III.2. LEMMA. $\Gamma_\infty \Gamma_\infty^{-1} = \bigcap_{n=1}^\infty [\Gamma_n'(\Gamma_n')^{-1}]^-$.

PROOF. Clearly, $\Gamma_\infty \Gamma_\infty^{-1} \subset \bigcap_{n=1}^\infty [\Gamma_n'(\Gamma_n')^{-1}]^-$. Conversely, let $c \in \bigcap_{n=1}^\infty [\Gamma_n'(\Gamma_n')^{-1}]^- = \bigcap_{n=1} \overline{\Gamma}_n'(\overline{\Gamma}_n')^{-1}$. Then there exist $\{a_n\}$, $\{b_n\} \subset \Gamma_n'$, $n = 1, 2, \ldots$, such that $c = a_n b_n^{-1}$. Passing to a subsequence, we may assume that $\{a_n\}$, $\{b_n\}$ are both convergent. Let $a = \lim a_n$, $b = \lim b_n$. Since $\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subset \Gamma_n'$ for all n, it follows that $a \in \Gamma_\infty$; also $b \in \Gamma_\infty$. Hence $c = ab^{-1} \in \Gamma_\infty \Gamma_\infty^{-1}$.

III.3. LEMMA. *Γ_∞ is a closed subsemigroup of the unit circle.*

PROOF. Let $a, b \in \Gamma_\infty$. Then, given $\varepsilon > 0$ and a positive integer n, there exists $a' \in \Gamma_n'$ such that $|a - a'| < \varepsilon/2$. By definition of Γ_n', $a' \in \Gamma_{n,m}$ for some $m > n$. Again, there exists $b \in \Gamma_m'$ such that $|b - b'| < \varepsilon/2$. Now a', $b' \in \Gamma_n'$, and

$$|ab - a'b'| \leqslant |ab - a'b| + |a'b - a'b'| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $ab \in \overline{\Gamma}_n'$; since this holds for each $n \geqslant 1$, $ab \in \Gamma_\infty$.

III.4. COROLLARY. *Γ_∞ is either a finite subgroup of the unit circle* $\mathbf{T}$, *or else* $\Gamma_\infty = \mathbf{T}$.

PROOF. These are the only closed subsemigroups of $\mathbf{T}$.

III.5. COROLLARY. $\Gamma_\infty = \Gamma_\infty \Gamma_\infty^{-1} = \bigcap_{n=1}^\infty [\Gamma_n'(\Gamma_n')^{-1}]^-$.

PROOF. This follows from III.2 and III.4.

III.6. Next we recall some facts concerning the infinitesimal generators of the one-parameter group $\alpha(t)$. Let

$$\mathscr{A}x = -\sqrt{-1} \lim_{t \to 0} \frac{\alpha(t)(x) - x}{t};$$

the domain of $\mathscr{A}$ consists of all $x \in \mathfrak{A}$ for which the limit exists. If σ, σ_p, and σ_c denote the spectrum, the point spectrum, and continuous spectrum, respectively, then

(a) $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A}) \cup \sigma_c(\mathscr{A}) \subset \mathbf{R}$, and $\sigma_c(\mathscr{A}) = \sigma_p(\mathscr{A})^- \setminus \sigma_p(\mathscr{A})$;

(b) $\sigma_p(\mathscr{A}) = \Omega - \Omega = \{\lambda - \lambda' \colon \lambda, \lambda' \in \Omega\}$,

(c) for each $\lambda \in \sigma_p(\mathscr{A})$, $\ker(\mathscr{A} - \lambda I)$ is infinite dimensional, and

$$\ker(\mathscr{A} - \lambda I) = \left\{ \bigcup_{n \geqslant 1} \ker[(\mathscr{A} - \lambda I) \mid \mathfrak{A}_n] \right\}^-;$$

(d) $\mathscr{A}$ is bounded iff $\sum_{n=1}^\infty \gamma^{(k)} < \infty$, where $\gamma^{(k)} = \max_{i \leqslant i, j \leqslant [k]} |\gamma_i^{(k)} - \gamma_j^{(k)}|$. If $\mathscr{A}$ is bounded, $\|\mathscr{A}\| = \sum_{k=1}^\infty \gamma^{(k)} = r_\sigma(\mathscr{A})$, the spectral radius (cf. [**8**]).

III.7. (a) $\sigma(\alpha(t)) = \sigma_p(\alpha(t)) \cup \sigma_c(\alpha(t)) \subset \mathbf{T}$, and $\sigma_c(\alpha(t)) = \sigma_p(\alpha(t)) \setminus \sigma_p(\alpha(t))$ (similarly for $v(t)$);

(b) $\sigma_p(\alpha(t)) = \{\exp(\sqrt{-1}\, \lambda t \colon \lambda \in \Omega - \Omega\}$;

(c) $\{\alpha(t)\}$ is norm continuous iff $\{v(t)\}$ exists in $\mathfrak{A}$ for all t iff $\sum_{R=1}^\infty \gamma^{(k)} < \infty$, where $\gamma^{(k)} = \max_{1 \leqslant i \leqslant [k]} |\gamma_i^{(k)}|$.

We sketch the proof. It follows from general principles (e.g., [**6**, Theorem 2.2.2] that $\sigma(\alpha(t)) \supset \exp(\sqrt{-1}\, t\sigma(\mathscr{A}))$. However, a calculation like that in [**8**, p. 182] yields the inverse inclusion as well. This, together with $\sigma_p(\alpha(t)) = \exp(\sqrt{-1}\, t\sigma_p(\mathscr{A}))$ [**6**, Corollaries 2.2.3 and 2.2.5], implies the results.

III.8. COROLLARY. *Let $\alpha = \alpha(1)$. Then $\sigma(\alpha) = \overline{\Gamma}\overline{\Gamma}^{-1}$, which coincides with the Arveson spectum of α. (Here α is the automorphism $\alpha(1)$, not the one-parameter group.)*

PROOF. This follows from III.7 and [7, Corollary 8.1.11].

III.9. COROLLARY. *The following are equivalent*:
(a) $\{\alpha(t)\}$ *is norm continuous and hence unitarily implemented by* $\{v(t)\} \subset \mathfrak{A}$;
(b) $\Omega_\infty = (0)$.

PROOF. $\Omega_\infty = (0)$ iff $\tau_n \to 0$, where $\tau_n = \sup\{|\tau|: \tau \in \Omega'_n\}$. Let

$$\gamma^{(n)} = \max_{1 \leqslant i \leqslant [n]} \left|\gamma_i^{(n)}\right|.$$

Then $\tau_n \leqslant \sum_{k=n+1}^{\infty}\gamma^{(k)} \leqslant 2\tau_n$. Thus, $\tau_n \to 0$ is equivalent to the convergence of $\sum_{n=1}^{\infty}\gamma^{(n)}$, which in turn is equivalent to the norm continuity of $\{\alpha(t)\}$ by III.7(c).

III.10. PROPOSITION. *If Γ_∞ is a finite group, there exists an automorphism β of $\mathfrak{A}$ with $\sigma(\beta) = \Gamma_\infty$ and a unitary $u \in \mathfrak{A}$ such that $\alpha = (\mathrm{Ad}\, u) \circ \beta = \beta \circ (\mathrm{Ad}\, u)$.*

PROOF. *Let* $\Gamma_\infty = \{\exp(\sqrt{-1}\, 2\pi k\lambda_0): k = 0, 1, \ldots, p-1\}$ $(\lambda_0 = 1/p)$. *Define* $\tau_k^{(n)} \in [-\lambda_0/2, \lambda_0/2)$ by $\tau_k^{(n)} = \lambda_k^{(n)} \pmod{\lambda_0}$, and set

$$\tau^{(n)} = \max_{1 \leqslant k \leqslant [n]} \left\{\left|\tau_k^{(n)}\right|, n = 1, 2, \ldots\right\}.$$

We claim that $\sum_{n=1}^{\infty}\tau^{(n)} < \infty$. Suppose, to the contrary, that $\sum_{n=1}^{\infty}\tau^{(n)} = +\infty$, and let $\mathbf{P} = \{n$: there exists j, $1 \leqslant j \leqslant [n]$, such that $\tau_i^{(n)} = \tau^{(n)}\}$. Now $\tau^{(n)} \to 0$ as $n \to \infty$; if not, there is a subsequence $\tau^{(n_k)}$ and $\varepsilon > 0$ with $\tau^{(n_k)} \geqslant \varepsilon$. But then there is a further subsequence $\{n_{k_i}\}_{i=1}^{\infty}$ and $m_i \in \{1, \ldots, [n_{k_i}]\}$ such that $\{\tau_{m_i}^{n_{k_i}}\}$ converges to some value $\lambda \in [-\lambda_0/2, \lambda_0/2]$. But in that case $\lambda \in \Omega_\infty$, so $\exp(\sqrt{-1}\, 2\pi\lambda) \in \Gamma_\infty$, a contradiction. Next, either $\sum_{n \in \mathbf{P}}\tau^{(n)} = +\infty$ or else $\sum_{n \in \mathbf{N}\setminus\mathbf{P}}\tau^{(n)} = +\infty$ (or both). Say, without loss of generality, $\sum_{n \in \mathbf{P}}\tau^{(n)} = +\infty$. Then Ω'_N is dense in $\mathbf{R}^+$ for every N. Indeed, let $\lambda > 0$ and $N \in \mathbf{Z}^+$ be given. Let $n_1 = \inf\{n: n \in \mathbf{P}, n > N, \tau^{(n)} < \lambda\}$. Suppose $n_1, \ldots, n_k$ have been chosen. Let

$$n_{k+1} = \inf\left\{n: n \in \mathbf{P}, n > n_k, \tau^{(n_1)} + \cdots + \tau^{(n_k)} + \tau^{(n)} < \lambda\right\}.$$

The conditions $\lim_{n\to\infty}\tau^{(n)} = 0$ and $\sum_{n \in \mathbf{P}, n > N}\tau^{(n)} = +\infty$ imply that given $\varepsilon > 0$ there exists $k \in \mathbf{Z}^+$ such that $\tau^{(n_1)} + \cdots + \tau^{(n_k)} \in (\lambda - \varepsilon, \lambda)$. Thus Ω'_n is dense in $\mathbf{R}^+$, and so $\Omega_\infty = \bigcap_{n=1}^{\infty}\overline{\Omega}'_n \supset \mathbf{R}^+$. But then $\Gamma_\infty = \mathbf{T}$. This proves the claim.

Set $u^{(n)} = \sum_{k=1}^{[n]}\exp(\sqrt{-1}\, 2\pi\tau_k^{(n)})e_{kk}^{(n)}$, and $u_n = \otimes_{k=1}^{n} u^{(n)}$. By II.7(c) and $\sum_{n=1}^{\infty}\tau^{(n)} < \infty$, $\{u_n\}$ converges to a unitary u in $\mathfrak{A}$. Set $\beta = \mathrm{Ad}(u^{-1}) \circ \alpha$. By construction, the spectrum of β is Γ_∞, and $\mathrm{Ad}(u^{-1})$ commutes with α.

III.11. Recall that the Connes spectrum $\Gamma(\alpha)$ is defined to be $\bigcap_{\mathscr{B}}\sigma(\alpha|\mathscr{B})$, where the intersection ranges over all α-invariant, hereditary subalgebras $\mathscr{B} \subset \mathfrak{A}$. Let $\mathscr{B}_n$ be the smallest hereditary subalgebra of $\mathfrak{A}$ containing $e_{ii}^{(1)} \otimes \cdots \otimes e_{ii}^{(n)}$. $\mathscr{B}_n$ is a UHF-algebra invariant under α, and $\alpha|\mathscr{B}_n$ is a product-type automorphism. So from Corollary III.8 the spectrum of α restricted to $\mathscr{B}_n$ is $\overline{\Gamma}'_n(\overline{\Gamma}'_n)^{-1}$. Thus the Connes spectrum $\Gamma(\alpha) \subset \bigcap_{n=1}^{\infty}\Gamma'_n(\overline{\Gamma}'_n)^{-1} = \Gamma_\infty$.

COROLLARY. $\Gamma_\infty = \Gamma(\alpha)$.

PROOF. From $\alpha = \mathrm{Ad}(u) \circ \beta$ and $\beta^p = 1$, $p = |\Gamma_\infty|$, it follows that $\alpha(p) = \mathrm{Ad}(u^p)$ is unitarily implemented. On the other hand, $\alpha(n)$ is not unitarily implemented unless $\beta(n)$ is, which will not be the case unless $n \in \mathbf{Z}p$, as can be seen from [3, Lemma 3.5]. It follows that $\Gamma_\infty^\perp = \Gamma(\alpha)^\perp$, and hence $\Gamma_\infty = \Gamma(\alpha)$.

IV

IV.1. Unless stated otherwise, an ideal in a Banach algebra is a closed, two-sided ideal.

DEFINITION. Let $\mathscr{B}$ be a Banach algebra, and suppose $\mathscr{B}$ contains a nested sequence of closed subalgebras $\{\mathscr{B}_n\}_{n=1}^\infty$ such that $\mathscr{B} = [\bigcup_{n=1}^\infty \mathscr{B}_n]^-$. An ideal $J \subset \mathscr{B}$ is called *ascending with* respect to the sequence $\{\mathscr{B}_n\}$ if

$$J = \left[\bigcup_{n=1}^{\infty} (J \cap \mathscr{B}_n)\right]^- .$$

If the sequence $\{\mathscr{B}_n\}$ is understood, we will simply say that J is *ascending*.

IV.2. Next we state the analogue for semicrossed products of a fact for C^*-crossed products. Let α, β be automorphisms of a C^*-algebra $\mathscr{B}$ with identity, and suppose there is a unitary $u \in \mathscr{B}$ such that $\alpha(B) = \mathrm{Ad}(u) \circ \beta(B)$, $B \in \mathscr{B}$. Then $\mathbf{Z}^+ \times_\alpha \mathscr{B} \cong \mathbf{Z}^+ \times_\beta \mathscr{B}$, and the (isometric) isomorphism is given on the dense subalgebra $K(\mathbf{Z}^+, \mathscr{B}, \alpha)$ by

$$\sum_{n \geqslant 0} \delta^\alpha \otimes B_n \to \sum_{n \geqslant 0} \delta_n^\beta \otimes B_n \alpha^{n-1}(u) \cdots \alpha(u) u$$

($\sum_{n \geqslant 0} \delta_n^\alpha \otimes B_n \in \mathbf{K}(\mathbf{Z}^+, B, \alpha)$ is the function F: $\mathbf{Z}^+ \to \mathscr{B}$ such that $F(n) = B_n$) [1, III.4].

IV.3. We will adhere to the notation of §III. Fix a product-type automorphism α of the UHF-algebra $\mathfrak{A} = (\bigcup_{n \geqslant 1} \mathfrak{A}_n)^-$. If $\alpha_n = \alpha | \mathfrak{A}_n$, by IV.2, $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ is isometrically isomorphic with $\mathbf{Z}^+ \times_{\mathrm{id}} \mathfrak{A}_n = \mathfrak{A}_n \otimes A(\mathbf{D})$. If π_n is any faithful representation of $\mathfrak{A}_n$ on a Hilbert space $\mathscr{H}_n$ and L is the representation of $A(\mathbf{D})$ by multiplication on the classical Hardy space H^2, the semicrossed product norm of $x \in \mathfrak{A}_n \otimes A(\mathbf{D})$ is the norm of $(\pi_n \otimes L)(x)$ acting on $\mathscr{H}_n \otimes H^2$. (Although $\mathfrak{A}_n \otimes A(\mathbf{D})$ is semisimple, and hence has a unique norm topology, it is important that a particular norm be specified when taking inductive limits.) Next we define a sequence of isometric embeddings ι_n: $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A} \to \mathbf{Z}^+ \times_{\alpha_{n+1}} \mathfrak{A}_{n+1}$; the lemma which follows is essentially [3, Theorem 2.1].

IV.4. LEMMA. *Define Λ_n as the set of functions a from $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots\}$ such that $a(i) \in \{1, 2, \ldots, [i]\}$ for all $i = 1, 2, \ldots, n$. Then $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ is linearly spanned by elements $f_{ab}^{(n)}(g)$, $a, b \in \Lambda_n$, $g \in A(\mathbf{D})$. These elements satisfy*

$$f_{ab}^{(n)}(g) f_{cd}^{(n)}(h) = \delta_{bc} f_{ad}^{(n)}(gh)$$

for all $a, b, c, d \in \Lambda_n$; $g, h \in A(\mathbf{D})$. The embedding $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n \xrightarrow{\iota_n} \mathbf{Z}^+ \times_{\alpha_{n+1}} \mathfrak{A}_{n+1}$ is given by the relation

$$f_{ab}^{(n)}(g) = \sum_{k=1}^{[n+1]} f_{a_k b_k}^{(n+1)}(g_{\gamma_k(n+1)}),$$

where $a_k, b_k \in \Lambda_{n+1}$ *are both defined by*

$$a_k(i) = \begin{cases} a(i) & \text{if } 1 \leqslant i \leqslant n, \\ k, & \text{if } i = n+1, \end{cases}$$

and $g_\gamma(z) = g(\exp(-2\pi\sqrt{-1}\,\gamma)z)$, $z \in \overline{\mathbf{D}}$. *An isomorphism* $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A} \to \mathfrak{A}_n \otimes A(\mathbf{D})$ *is given by* $f_{ab}^{(n)}(g) \to e_{ab}^{(n)} \otimes g$. (*By* $e_{ab}^{(n)}$ *we mean* $e_{a(1)b(1)}^{(n)} \otimes \cdots \otimes e_{a(n)b(n)}^{(n)}$.)

IV.5. NOTATION. We may now form the inductive limit of $\{\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n; \iota_n\}$. By Proposition II.5, the inductive limit is $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$. We will view $\{\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n\}$ as a nested sequence of subalgebras of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$, whose union is dense (cf. II.3, II.6). By an ascending ideal we mean an ideal that is ascending with respect to the sequence $\{\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n\}$.

IV.6. Let $\mathcal{J}$ be a nonzero closed ideal in the disk algebra $A(\mathbf{D})$, and let K be the intersection of the zeroes of the functions in $\mathcal{J}$ on $\mathbf{T} = \partial\mathbf{D}$, the unit circle. Let F be the greatest common divisor of the inner parts of the nonzero functions in $\mathcal{J}$. Then $\mathcal{J}$ is precisely the set of functions of the form Fg, where g ranges over the functions in $A(\mathbf{D})$ which vanish on K [**4**, p. 85].

Now let $J \subset \mathbf{Z}^+ \times_\alpha \mathfrak{A}$ be an ascending ideal (cf. IV.5), and set $J_n = J \cap \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$. The isomorphism $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n \to \mathfrak{A}_n \otimes A(\mathbf{D})$ maps J_n to an ideal in $\mathfrak{A}_n \otimes A(\mathbf{D})$; but ideals here are of the form span $\{e_{ab}^{(n)} \otimes h\colon h \in \mathcal{J}_n\}$ where $\mathcal{J}_n \subset A(\mathbf{D})$ is an ideal. Thus J_n is linearly spanned by elements $f_{ab}^{(n)}(g)$, where $a, b \in \Lambda_n$ and g belongs to some ideal $\mathcal{J}_n \subset A(\mathbf{D})$. As above, $\mathcal{J}_n$ corresponds to a compact set K_n of $\mathbf{T}$ (having Lebesgue measure zero) and an inner function F_n. Let $a, b \in \Lambda_n$ and $g \in A(\mathbf{D})$ vanish on K_n. By Lemma IV.4 we have

$$f_{ab}^{(n)}(F_n g) = \sum_{k=1}^{[n+1]} f_{a_k b_k}^{(n+1)}\left((F_n)_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}}\right).$$

In particular,

$$f_{cd}^{(n+1)}\left((F_n)_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}}\right) \in J_{n+1}$$

for all $c, d \in \Lambda_{n+1}$, $k \in \{1, 2, \ldots, [n+1]\}$. But then $f_{cd}^{(n+1)}((F_n)_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}})$ must be of the form $f_{cd}^{(n+1)}(F_{n+1}h)$, where $h \in A(\mathbf{D})$ vanishes on K_{n+1}. As F_{n+1} is the greatest common divisor of the inner parts of all nonzero functions in $\mathcal{J}_{n+1}$, F_{n+1} must divide the inner part of $(F_n)_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}}$, and hence, as $(F_n)_{\gamma_k(n+1)}$ is the greatest common divisor of the inner parts of all functions of that form, it follows that F_{n+1} divides $(F_n)_{\gamma_k^{(n+1)}}$, or, $(F_{n+1})_{-\gamma_k^{(n+1)}}$ divides F_n, $1 \leqslant k \leqslant [n+1]$. In fact, we claim that F_n is the least common multiple of $\{(F_{n+1})_{-\gamma_k^{(n+1)}}\colon 1 \leqslant k \leqslant [n+1]\}$. For let F_n' be the least common multiple. If $g \in A(\mathbf{D})$ is such that $f_{cd}^{(n+1)}((F_n')_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}}) \in J_{n+1}$, $1 \leqslant k \leqslant [n+1]$, then the condition $J_n = J_{n+1} \cap \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ implies

$$f_{ab}^{(n)}(F_n' g) = \sum_{k=1}^{[n+1]} f_{a_k b_k}^{(n+1)}\left((F_n')_{\gamma_k^{(n+1)}} g_{\gamma_k^{(n+1)}}\right) \in J_n.$$

By definition of F_n', F_n' divides F_n; but the characterization of F_n implies F_n divides F_n', so F_n, F_n' agree up to a unit.

Let $L_n = \{\gamma\colon \gamma \in [-\frac{1}{2}, \frac{1}{2})$ and $\exp(2\pi\sqrt{-1}\,\gamma) \in K_n)\}$.

LEMMA. $F_n = \operatorname{lcm}\{(F_{n+1})_{-\gamma_k^{(n+1)}}\colon k = 1,\dots,[n+1]\}$,

$$L_n = \bigcup_{k=1}^{[n+1]} \left(L_{n+1} - \gamma_k^{(n+1)}\right) \pmod 1$$

and

$$K_n = \bigcup_{k=1}^{[n+1]} \exp\left(-2\pi\sqrt{-1}\,\gamma_k^{(n+1)}\right) K_{n+1}.$$

PROOF. The first statement has been proved, and the second follows from [**3**, Corollary 2.2] by taking complements.

IV.7. Since each inner function is uniquely decomposable as the product of a singular inner function and a Blaschke product, we can write $F_n = S_n B_n$. Here B_n is a Blaschke product, and S_n has the form

$$S_n(\mathbf{z}) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu_n(\theta)\right],$$

where μ_n is a positive singular measure supported on $L_n \subset [\frac{1}{2}, \frac{1}{2})$. Let $\mathbf{Z}_n$ be the set of zeroes of B_n in the interior $\mathbf{D}$ of the unit disk. From IV.6,

$$B_n = \operatorname{lcm}\left\{(B_{n+1})_{-\gamma_k^{(n+1)}}\colon k = 1,\dots,[n+1]\right\}.$$

This implies

$$(*) \qquad \mathbf{Z}_n = \bigcup_{k=1}^{[n=1]} \exp\left(-2\pi\sqrt{-1}\,\gamma_k^{(n+1)}\right)\mathbf{Z}_{n+1}.$$

LEMMA. *If Γ has infinite cardinality, then either each Blaschke product B_n is void, or else each B_n has a zero (of some positive order) at the origin.*

PROOF. If $0 < r < 1$, $n = 1,2,\dots$, set $\mathbf{Z}_{n,r} = \mathbf{Z}_n \cap \{z \in \mathbf{C}\colon |z| = r\}$. We will show that $\mathbf{Z}_{n,r}$ is empty. Observe that $(*)$ holds with $\mathbf{Z}_{n,r}$, $\mathbf{Z}_{n+1,r}$ in place of $\mathbf{Z}_n$, $\mathbf{Z}_{n+1}$. Hence, by $(*)$, if $\mathbf{Z}_{n,r}$ in nonvoid for some n, it is nonvoid for all n, in which case, by compactness, $\mathbf{Z}_{\infty,r} \equiv \bigcap_{n=1}^{\infty} \mathbf{Z}_{n,r}$ is nonvoid. Repeated application of $(*)$ implies, for $m > n$,

$$\mathbf{Z}_{n,r} = \bigcup_{\tau \in \Gamma_{n,m}} \tau \mathbf{Z}_{m,r},$$

and hence $\mathbf{Z}_{n,r} \supset \bigcup_{\tau \in \Gamma_{n,m}} \tau \mathbf{Z}_{\infty,r}$. Since $m > n$ is arbitrary $\mathbf{Z}_{n,r} \supset \bigcup_{\tau \in \Gamma_n'} \tau \mathbf{Z}_{\infty,r}$. If $\mathbf{z}_0 \in \mathbf{Z}_{\infty,r}$, $\mathbf{Z}_{n,r}$ contains $\{\tau \mathbf{z}_0\colon \tau \in \Gamma_n'\}$. If Γ is infinite, so is Γ_n'. But then the Blaschke product B_n would have infinitely many zeroes on the circle $|z| = |z_0| = r$. That is impossible, and so $\mathbf{Z}_{n,r}$, $\mathbf{Z}_{\infty,r}$ are void.

IV.8. LEMMA. *Let $\{K_n\}_{n=1}^{\infty}$ be as in* IV.6, *and set $K_\infty = \bigcap_{n=1}^{\infty} K_n$. Then $K_n = [(\Gamma_n')^{-1} K_\infty]^-$.*

PROOF. Repeated application of Lemma IV.6 yields $K_n = (\Gamma_{n,m})^{-1} K_m$ $(m > n)$. So $K_n \supset (\Gamma_{n,m})^{-1} K_\infty$ for all $m > n$, and hence $K_n \supset (\Gamma_n')^{-1} K_\infty$. Since K_n is closed, $K_n \supset [(\Gamma_n')^{-1} K_\infty]^-$. For the opposite inclusion, let $\tau \in K_n$ and $\tau_m \in K_m$, $\tau_m' \in \Gamma_{n,m}$ such that $\tau = \tau_m (\tau_m')^{-1}$. Passing to a subsequence, we may assume that $\{\tau_m\}$, $\{\tau_m'\}$ are convergent. If $\tau_\infty = \lim_m \tau_m$, $\tau_\infty \in K_\infty$. Since $\Gamma_{n,m} \subset \Gamma_n'$ for all $m > n$, $\{\tau_m'\} \subset \Gamma_n'$, and if $\tau_\infty' = \lim \tau_m'$, $\tau_\infty' \in (\Gamma_n')^-$. Then $\tau = \tau_\infty (\tau_\infty')^{-1}$.

IV.9. COROLLARY. *There is a one-to-one correspondence between the ascending ideals $J \subset \mathbf{Z}^+ \times_\alpha \mathfrak{A}$ and triples $(\{B_n\}_{n=1}^\infty, \{K_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty)$ satisfying*:

(i) $\{B_n\}_{n=1}^\infty$ *is a sequence of Blaschke products with*

$$B_n = \operatorname{lcm}\{(B_{n+1})_{-\gamma_k^{(n+1)}}, 1 \leqslant k \leqslant [n+1]\};$$

(ii) $\{K_n\}_{n=1}^\infty$ *is a sequence of compact subsets of* $\mathbf{T}$ *of Lebesgue measure zero with* $K_{n+1} \subset K_n$ *and* $K_n = [K_\infty(\Gamma_n')^{-1}]^-$ *or* K_n *void, where* $K_\infty = \bigcap_{n=1}^\infty K_n$;
and

(iii) $\{\mu_n\}_{n=1}$ *is a sequence of singular measures on* $[-\frac{1}{2}, \frac{1}{2})$, *with* μ_n *supported on* L_n (*cf*. IV.6), *and* $\mu_n = \operatorname{lcm}\{(\mu_{n+1})_{\gamma_k^{(n+1)}}), 1 \leqslant k \leqslant [n+1]\}$. (*The translate* ν_γ *of the measure* γ *on* $[-\frac{1}{2}, \frac{1}{2}]$ *is given by* $\int f(\theta)\, d\nu_\gamma = \int f(\theta - \gamma)\, d\nu$.)

PROOF. In IV.6, IV.7 and IV.8 it is shown that each ascending ideal $J \subset \mathbf{Z}^+ \times_\alpha \mathfrak{A}$ gives rise to a triple $(\{B_n\}_{n=1}^\infty, \{K_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty)$ satisfying (i)–(iii). Conversely, any such triple satisfying the stated conditions determines a sequence $\{J_n\}_{n=1}^\infty$ of ideals. $J_n \subset \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ and $J_n = J_{n+1} \cap \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$, $n = 1, 2, \ldots$. Then $J = (\bigcup_{n=1}^\infty J_n)^-$ is an ascending ideal in $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$.

IV.10. Let J be an ascending ideal corresponding to the triple $(\{B_n\}, \{K_n\}, \{\mu_n\})$; now K_n is invariant under $\Gamma_\infty = \Gamma(\alpha)$ and the zeroes Z_n of B_n are invariant under $\Gamma(\alpha)$.

Suppose now $\Gamma(\alpha)$ is finite; say $|\Gamma(\alpha)| = p$. Then J will be maximal in the set of all ascending ideals iff it is of one of the following forms:

(i) $\mu_n = 0$ for all n, $L_\infty = \bigcap L_n = \bigcup_{j=0}^{p-1}\{\gamma_0 + j/p\} \pmod 1$ for some $\gamma_0 \in [-\frac{1}{2}, \frac{1}{2})$, and the Blaschke products B_n are all void; or

(ii) $L_\infty = L_n = \varnothing$ and $\mu_n = 0$ for all n, and the zeroes of B_n are of multiplicity one with Z_∞, the common zeroes of the B_n, of the form $Z_\infty = \bigcup_{\omega \in \Gamma(\alpha)} \omega\lambda_0$, for some $\lambda_0 \in \mathbf{D}$.

If J is of type (i), write $J = M_{\lambda_0}$, where $\lambda_0 = \exp(2\pi\sqrt{-1}\,\gamma_0) \in \mathbf{T} = \partial\mathbf{D}$; also write $J = M_{\lambda_0}$ if J is of type (ii).

Of course, just because an ideal J is maximal among the ascending ideals does not autmatically imply that it is maximal among all (proper, closed, two-sided) ideals of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$. Let $\mathscr{M}$ denote the maximal ideals of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$, and $\mathscr{M}_a$ denote the subset of $\mathscr{M}$ consisting of ascending ideals (with respect to the sequence) $\{\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n\}$). As we have seen in Lemma IV.7, if the spectrum of α is infinite, there is no ascending ideal of the form $M_{\lambda_0}, \lambda_0 \in \mathbf{D}$, $\lambda_0 \neq 0$. Later we will show that $\mathscr{M}_a$ consists precisely of those M_{λ_0}'s ($\lambda_0 \in \mathbf{D}$) for which an ascending ideal M_{λ_0} exists. At the moment we consider the case where α has finite spectrum.

PROPOSITION. *Assume* $\sigma(\alpha) = \Gamma(\alpha)$ *is finite. Let* $\lambda_0 \in \mathbf{D}$, *then* M_{λ_0} *is a maximal ideal in* $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$.

PROOF. We need to show that the quotient $\mathbf{Z}^+ \times_\alpha \mathfrak{A}/M_{\lambda_0}$ is simple. The cases $\lambda_0 = 0$, $\lambda_0 \neq 0$ will be dealt with separately; $\mathbf{Z}^+ \times_\alpha \mathfrak{A}/M_0 \cong \mathfrak{A}$, whereas if $\lambda_0 \neq 0$, $\mathbf{Z}^+ \times_\alpha \mathfrak{A}/M_{\lambda_0}$ is a simple [AF]-algebra (which is not necessarily UHF if $p = |\Gamma(\alpha)| > 1$). In each case we will construct a representation π_{λ_0} with kernel M_{λ_0}.

Fix a nontrivial (and hence faithful) representation ρ of the UHF-algebra $\mathfrak{A}$ on some Hilbert space. The formula

$$\pi_0^{(n)}\big(f_{ab}^{(n)}(g)\big) = g(0)\rho\big(e_{ab}^{(n)}\big) \qquad \big(g \in A(\mathbf{D}),\ a, b \in \Lambda_n\big)$$

defines a contractive representation on $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ $(n \geq 1)$ such that

$$\pi_0^{(n+1)} | \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n = \pi_0^{(n)}.$$

Also, $\ker \pi_0^{(n)} = M_0 \cap \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$. This allows us to define a representation π_0 on $\bigcup_{n \geq 1} \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$, and since π_0 is contractive it extends uniquely to a representation π_0 of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$. Since the image of π_0 is precisely $\rho(\mathfrak{A})$, the kernel of π_0, M_0, is a maximal ideal in $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$.

Now let $0 \neq \lambda_0 \in \mathbf{D}^-$, and let $\rho_0, \ldots, \rho_{p-1}$ be representations of $\mathfrak{A}$ which are unitarily equivalent to ρ. Let $\omega \in T$ be a generator of $\Gamma(\alpha)$ (so $\Gamma(\alpha) = \{1, \omega, \ldots, \omega^{p-1}\}$), and define a representation $\pi_{\lambda 0}^{(n)}$ of $\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n$ by

$$\pi_{\lambda_0}^{(n)}\big(f_{ab}^{(n)}(g)\big) = \sum_{j=0}^{p-1} g\big(\omega^j \lambda_0\big)\rho_j\big(e_{ab}^{(n)}\big) \in \bigoplus_{j=0}^{p-1} \rho_j(\mathfrak{A}_n),$$

$(g \in A(\mathbf{D}),\ a, b \in \Lambda_n)$. Since the assumption $\sigma(\alpha) = \Gamma(\alpha)$ implies that the constants $\gamma_j^{(n)} \in \{0, 1/p, \ldots, (p-1)/p\}$, write $\gamma_j^{(n)} = k_j^{(n)}/p$; so $k_j^{(n)}$ is an integer in $\{0, \ldots, p-1\}$. We want to embed $\oplus_{j=0}^{p-1} \rho_j(\mathfrak{A}_n)$ into $\oplus_{j=0}^{p-1} \rho_j(\mathfrak{A}_{n+1})$ in such a way that the diagram

$$\begin{array}{ccc} \mathbf{Z}^+ \times_{\alpha_{n+1}} \mathfrak{A}_{n+1} & \overset{\pi_{\lambda_0}^{(n+1)}}{\rightarrow} & \bigoplus_{j=0}^{p-1} \rho_j(\mathfrak{A}_{n+1}) \\ \uparrow \iota_n & & \uparrow \\ \mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n & \overset{\pi_{\lambda_0}^{(n)}}{\rightarrow} & \bigoplus_{j=0}^{p-1} \rho_j(\mathfrak{A}_n) \end{array}$$

commutes. Now the formula

$$f_{ab}^{(n)}(g) = \sum_{i=1}^{[n+1]} f_{a_i b_i}^{(n+1)}\big(g_{\gamma_i}^{(n+1)}\big)$$

(notation as in Lemma IV.4) implies

$$\pi_{\lambda_0}^{(n)}\big(f_{ab}^{(n)}(g)\big) = \sum_{i=1}^{[n+1]} \pi_{\lambda_0}^{(n+1)}\Big(f_{a_i b_i}^{(n+1)}\big(g_{\gamma_i^{(n+1)}}\big)\Big);$$

i.e.,

$$\sum_{j=0}^{p-1} g\big(\omega^j \lambda_0\big)\rho_j\big(e_{ab}^{(n)}\big) = \sum_{j=0}^{p-1} \sum_{i=1}^{[n+1]} g\Big(\omega^{j + k_i^{(n+1)}} \lambda_0\Big)\rho_j\Big(e_{a_i b_i}^{(n+1)}\Big).$$

Comparing coefficients,

$$\rho_k\big(e_{ab}^{(n)}\big) = \sum_{i=1}^{[n+1]} \rho_{k - k_i^{(n+1)}}\Big(e_{a_i b_i}^{(n+1)}\Big)$$

(where $k - k_i^{(n+1)}$ is to be understood mod p). In other words, the factor $\rho_k(\mathfrak{A}_n)$ is partially embedded in $\rho_{k-k_i^{(n+1)}}(\mathfrak{A}_{n+1})$, $i = 1, \ldots, [n+1]$. By definition of Γ_∞ (which is $\Gamma(\alpha)$), given n and $k, k' \in \mathbf{Z}_p$ there exists $m > n$ and $k_{i_j}^{(j)}$, $j = n+1, \ldots, n, i_j \in \{1, \ldots, [j]\}$, such that

$$k' = k + \sum_{j=n+1}^{m} k_{i_j}^{(j)} \pmod{p}.$$

Thus, $\rho_k(\mathfrak{A}_n)$ is partially embedded in $\rho'_k(\mathfrak{A}_m)$. It follows by [**2**, Corollary 3.5] that the [AF]-algebra which is the direct limit of $\oplus_{j=0}^{p-1}\rho_j(\mathfrak{A}_n)$ with respect to these embeddings is simple. If π_{λ_0} is the representation of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ defined by the sequence $\{\pi_{\lambda_0}^{(n)}\}$, then $\ker \pi_{\lambda_0} = M_{\lambda_0}$, and so M_{λ_0} is indeed maximal in $\mathbf{Z}^+\times_\alpha\mathfrak{A}$.

IV.11. LEMMA. *Suppose* $-\frac{1}{2} \leqslant \gamma_k^{(n)}, \gamma_k'^{(n)} < \frac{1}{2}$, $1 \leqslant k \leqslant [n]$, $n = 1, 2, \ldots,$ *and suppose that* $\gamma_k^{(n)} = \gamma_k'^{(n)}$ *except for at most finitely many pairs* (k, n). *Let* α, α' *be the corresponding automorphisms of* $\mathfrak{A}$ (*cf.* III.1). *Then there is an isomorphism* Φ: $\mathbf{Z}^+\times_\alpha\mathfrak{A} \to \mathbf{Z}^+\times_{\alpha'}\mathfrak{A}$ *which maps the ascending ideals of* $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ *bijectively to those of* $\mathbf{Z}^+\times_\alpha\mathfrak{A}$.

PROOF. By hypothesis, there exists an integer N such that $\gamma_k^{(n)} = \gamma_k'^{(n)}$, $1 \leqslant k \leqslant [n]$, $n \geqslant N$. Let Φ_n: $\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n \to \mathbf{Z}^+\times_{\alpha'_n}A_n$ be the isomorphism $\Phi(f_{ab}^{(n)}(g)) = f_{ab}'^{(n)}(g)$, where $a, b \in \Lambda_n$, $g \in A(\mathbf{D})$ (notation as in IV.4). Then, for $n \geqslant N$, the diagram

$$\begin{array}{ccc} \mathbf{Z}^+\times_{\alpha_n} A_n & \xrightarrow{\Phi_n} & \mathbf{Z}^+\times_{\alpha'_n} A_n \\ \downarrow \iota_n & & \downarrow \iota'_n \\ \mathbf{Z}^+\times_{\alpha_{n+1}} A_{n+1} & \xrightarrow{\Phi^{n+1}} & \mathbf{Z}^+\times_{\alpha'_{n+1}} A_{n+1} \end{array}$$

commutes. The sequence $\{\Phi_n\}$ defines an isometric isomorphism Φ: $\bigcup_n \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n \to \bigcup_n \mathbf{Z}^+\times_{\alpha'_n}\mathfrak{A}_n$, which extends uniquely to an isomorphism Φ: $\mathbf{Z}^+\times_\alpha\mathfrak{A} \to \mathbf{Z}^+\times_{\alpha'}\mathfrak{A}$, and, by its definition, maps ascending ideals of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ bijectively to those of $\mathbf{Z}^+\times_{\alpha'}\mathfrak{A}$.

IV.12. Let M be the set of maximal ideals of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$, and $\mathcal{M}_a \subset \mathcal{M}$ be the subset of ascending maximal ideals.

PROPOSITION. *Assume the Connes spectrum* $\Gamma(\alpha)$ *is finite and that* $\sigma(\alpha) = \Gamma(\alpha)$. *Then every maximal ideal* M *of* $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ *is ascending.*

PROOF. Let $p = |\Gamma(\alpha)|$, and set $A_p(\mathbf{D}) \subset A(\mathbf{D})$ the subalgebra of p-periodic functions, i.e. $f(\omega z) = f(z)$, $\omega = \exp(2\pi\sqrt{-1}/p)$. Let Ψ_n: $\mathfrak{A}_n \otimes A(\mathbf{D}) \to \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$ be the isomorphism $\Psi_n(e_{ab}^{(n)} \otimes g) = f_{ab}^{(n)}(g)$, $n = 1, 2, \ldots$. Then the diagram

$$\begin{array}{ccccc} \mathfrak{A}_n \otimes A_p(\mathbf{D}) & \to & \mathfrak{A}_n \otimes A(\mathbf{D}) & \xrightarrow{\Psi_n} & Z^+\times_{\alpha_n}\mathfrak{A}_n \\ \downarrow j_n \otimes \mathrm{id} & & & & \downarrow \iota_n \\ \mathfrak{A}_{n+1} \otimes A_p(\mathbf{D}) & \to & \mathfrak{A}_{n+1} \otimes A(\mathbf{D}) & \xrightarrow{\Psi_{n+1}} & \mathbf{Z}^+\times_{\alpha_{n+1}}\mathfrak{A}_{n+1} \end{array}$$

commutes. This is because ι_n only involves translations by powers of $\omega = \exp(2\pi\sqrt{-1}/p)$ (cf. Lemma IV.4). The sequence $\{\Psi_n\}$ thus defines a map Ψ: $\mathfrak{A} \otimes A_p(\mathbf{D}) \to \mathbf{Z}^+\times_\alpha\mathfrak{A}$. Ψ is an isomorphism whose range is a proper subalgebra of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ (if $p > 1$).

Next we define "translation" on $\mathbf{Z}^+\times_\alpha\mathfrak{A}$. Set $Q_\zeta^{(n)}(f_{ab}^{(n)}(g)) = f_{ab}^{(n)}(h)$, where $h(z) = g(\zeta z)$ ($\zeta \in \mathbf{T}$, $g \in A(\mathbf{D})$, and $a, b \in \Lambda_n$). Since

$$\begin{array}{ccc} \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n & \xrightarrow{Q_\zeta^{(n)}} & \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n \\ \downarrow \iota_n & & \downarrow \iota_n \\ \mathbf{Z}^+\times_{\alpha_{n+1}}\mathfrak{A}_{n+1} & \xrightarrow{Q_\zeta^{(n+1)}} & \mathbf{Z}^+\times_{\alpha_{n+1}}\mathfrak{A}_{n+1} \end{array}$$

commutes, and $\{Q_\zeta^{(n)}\}$ is isometric, the sequence $\{Q_\zeta^{(n)}\}$ defines an operator Q_ζ on $\mathbf{Z}^+\times_\alpha\mathfrak{A}$, whose restriction to $\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$ is $Q_\zeta^{(n)}$.

We want to define a (linear) projection P from $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ onto the subalgebra $\Psi(\mathfrak{A} \otimes A_p(\mathbf{D})) \subset Z^+\times_\alpha\mathfrak{A}$. Set

$$P(F) = \frac{1}{p}\sum_{p=0}^{p-1} Q_\omega^k(F), \qquad F \in \mathbf{Z}^+\times_\alpha\mathfrak{A}.$$

It is easy to check that P is a projection whose range is just $\Psi(A \otimes A_p(\mathbf{D}))$. Now let M be a (proper) maximal ideal in $\mathbf{Z}^+\times_\alpha\mathfrak{A}$, and let $I(M) = M \cap Q_\omega(M) \cap \cdots \cap Q_\omega^{p-1}(M)$. Suppose $I(M) \subset M_0 = \ker\pi_0$. Since the product $MQ_\omega(M)\cdots Q_\omega^{P-1}(M) \subset I(M)$, and since maximal ideals are prime, we conclude $Q_\omega^k(M) \subset M_0$ for some k, $0 \leqslant k \leqslant p-1$. But then $M = Q_{\omega^{-1}}^k(Q_\omega^k(M)) \subset Q_{\omega^{-1}}^k(M_0) = M_0$, since M_0 is translation invariant. By maximality, $M = M_0$, so in this case M is ascending.

Thus we may assume there exists $F \in I(M)$, $F \notin M_0$. Then $P(F) \in I(M)$. But as $\pi_0(F) = \pi_0(P(F))$ and $\pi_0(F) \neq 0$, we have $\pi_0(P(F)) \neq 0$, or $P(F) \notin M_0$. Thus $I(M) \cap \Psi(\mathfrak{A} \otimes A_p(\mathbf{D})) \neq (0)$.

Let $J_n = M \cap \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$, and $J = (\bigcup J_n)^-$. Since every ideal of $\Psi(\mathfrak{A} \otimes A_p(\mathbf{D}))$ is ascending (with respect to the sequence $\Psi(\mathfrak{A}_n \otimes A_p(\mathbf{D})) \subset \mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n$), $J \cap \Psi(\mathfrak{A}_n \otimes A_p(\mathbf{D}))$ is ascending, and in particular $J_n \neq (0)$, $n = 1, 2, \ldots$.

As in IV.9, $\{J_n\}_{n=1}^\infty$ corresponds to a triple $(\{B_n\}, \{L_n\}, \{\mu_n\})$. (It is more convenient to work with L_n than K_n.) We finish the proof by showing that if a maximal ideal has nontrivial intersection with an ascending ideal, then the maximal ideal is ascending. By Lemma IV.8, $L_n = \bigcup_{k=0}^{p-1} L_\infty + k/p \pmod 1$, where $L_\infty = \bigcap L_n = L_n$ for all n. Set

$$L_n^{(1,1)} = \bigcup_{k=0}^{p-1} L_\infty \cap \left[0, \frac{1}{2}\right] + \frac{k}{p} \pmod 1.$$

If $\mathbf{Z}_\infty$ is the intersection of the zeroes $\mathbf{Z}_n$ of B_n, then $\mathbf{Z}_n = \bigcup_{k=0}^{p-1} \omega^k \mathbf{Z}_\infty$ for all n by IV.7, where $\omega = \exp(2\pi\sqrt{-1}/p)$. Let

$$\mathbf{Z}_n^{(1,1)} = \bigcup_{k=0}^{p-1} \omega^k[\mathbf{Z}_\infty \cap \{0 \leqslant \arg z \leqslant \pi\}],$$

and

$$\mathbf{Z}_n^{(1,2)} = \bigcup_{k=0}^{p-1} \omega^k [\mathbf{Z}_\infty \cap \{\pi \leqslant \arg z \leqslant 2\pi\}],$$

and $B_n^{(1,1)}$, $B_n^{(1,2)}$ be Blaschke products with $\mathbf{Z}_n^{(1,1)}$, $\mathbf{Z}_n^{(1,2)}$ as zeroes. Let $\mu_n^{(1,1)}$, $\mu_n^{(1,2)}$ be singular measures supported on $L_n^{(1,1)}$, $L_n^{(1,2)}$, respectively, such that $\mu_n^{(1,1)} + \mu_n^{(1,2)} = \mu_n$. Let $J_J^{(1,1)}$, $J^{(1,2)}$, be the ideals in $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ which correspond to the triples $(\{B_n^{(1,1,)}\}, \{L_n^{(1,1)}\}, \{\mu_n^{(1,1)}\{), (\{B_n^{(1,2,)}\}, \{L_n^{(1,2)}\}, \{\mu_n^{(1,2)}\})$. Now $J^{(1,1)}J^{(1,2)} \subset J \subset M$, and since M is maximal, either $J^{(1,1)}$ or $J^{(1,2)}$ is contained in M. Let $J^{(1)}$ be the one which is contained in M. (If both are contained in M, pick one arbitrarily.) Suppose $J^{(k)} \subset M$ has been defined in such a way that $L_\infty^{(k)} = \bigcup_{j=0}^{p-1}(I^{(k)} + j/p) \pmod 1$ where length of $I^{(k)}$ is 2^{-k}, and $\mathbf{Z}_\infty^{(k)} \subset \bigcup_{j=0}^{p-1} \omega^j S^{(k)}$, where $S^{(k)} \subset \mathbf{D}$ is a sector of width $2\pi/2k$. Let $I^{(k+1,1)}$, $I^{(k+1,2)}$ be compact intervals of length 2^{-k} in $I^{(k)}$ whose union is $I^{(k)}$, and set

$$L_\infty^{(k+1,i)} = \bigcup_{j=0}^{p-1}\left(I^{(k+1,i)} + \frac{j}{p}\right) \pmod 1, \qquad i = 1,2.$$

Also set

$$\mathbf{Z}_\infty^{(k+1,i)} = \bigcup_{j=0}^{p-1} \omega^j \big(S^{(k+1,i)} \cap \mathbf{Z}_\infty^{(k)}\big), \qquad i = 1,2,$$

and $B_n^{(k+1,1)}$ the Blashke product with $\mathbf{Z}_\infty^{(k+1,i)}$ as zeroes, $n = 1,2,\ldots$. If $\mu_n^{(k+1,1)}$ are singular measures on $L_n^{(k+1,1)}, L_n^{(k+1,2)}$, respectively, with $\mu_n^{(k)} = \mu_n^{(k+1,1)} + \mu_n^{(k+1,2)}$, and if $J^{(k+1,i)}$ $(i = 1,2)$ are the ideals in $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ corresponding to the triples $(\{L_n^{(k+1,1)}\}$, $\{B_n^{(k+1,i)}\}$, $\{\mu_n^{(k+1,i)}\})$, then $J^{(k+1,1)}J^{(k+1,2)} \subset J^{(k)} \subset M$, so either $J^{(k+1,1)}$ or $J^{(k+1,2)} \subset M$. Call $J^{(k+1)}$ the one that is contained in M. Thus the chain $\{J^{(k)}\}_{k=1}^\infty$ is contained in M, and so is $(\bigcup_k J^{(k)})^-$. Changing notation, $(\bigcup_k J^{(k)})^-$ is an ascending ideal corresponding to the triple $(\{B_n\}, \{L_n\}, \{\mu_n\})$, where $L_n = L_\infty = \bigcup_{j=0}^{p-1}\{\lambda_0 + j/p\} \pmod 1$ for some $\lambda_0 \in [0,1)$, or else $L_n = L_\infty = \varnothing$, if $\mathbf{Z}_n$ are the zeroes of B_n, $\mathbf{Z}_n = \mathbf{Z}_\infty \subset \bigcup_{j=0}^{p-1} \omega^j R$, where R is a ray emanating from the origin in $\mathbf{D}$ (possibly $\mathbf{Z}_n = \varnothing$), and $\mu_n = 0$ or else μ_n is supported on L_n. Now an argument very similar to the one just given shows that M contains an ideal M' corresponding to a triple $(\{B_n\}$, $\{L_n\}$, $\{\mu_n\})$ where $\mu_n = 0$ and either $L_n = \bigcup_{j=0}^{p-1}\{\lambda_0 + j/p\} \pmod 1$ and $\mathbf{Z}_n = \varnothing$, or else $L_n = \varnothing$ and $\mathbf{Z}_n = \bigcup_{j=0}^{p-1}\{\omega^j \lambda_0\}$ for some $\lambda_0 \in \mathbf{D}$. But such an M' is maximal, so $M = M'$ is an ascending ideal.

IV.13. Suppose that $\sigma(\alpha) = \Gamma(\alpha)$ is finite. By IV.10 and IV.12 the maximal ideals and maximal ascending ideals coincide ($\mathscr{M} = \mathscr{M}_a$), and every $M \in \mathscr{M}$ is of the form M_{λ_0}, for some $\lambda_0 \in \mathbf{D}$. Recall M_{λ_0} is the closed linear span of $f_{ab}^{(n)}(g)$, where $a, b \in \Lambda_n$ $(n = 1,2,\ldots)$ and $g \in A(\mathbf{D})$ vanishes at λ_0, $\lambda_0\omega \cdots \lambda_0\omega^{p-1}$. ($\omega$ is a generator of $\Gamma(\alpha)$.) Clearly, $M_{\lambda_0} = M_{\omega\lambda_0^0} = \cdots = M_{\omega^{p-1}\lambda_0}$. Thus, there is a one-to-one correspondence between the space $\mathscr{M}$ of maximal ideals of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ and

$\mathbf{D}/\Gamma(\alpha)$. Call φ the mapping $M \to \mathbf{D}/\Gamma(\alpha)$, $\varphi(M_{\lambda_0}) = \lambda_0\Gamma(\alpha)$. Then φ is a homeomorphism if $\mathscr{M}$ is given the hull-kernel topology and $\mathbf{D}$ has the hull-kernel topology from the disk algebra. Indeed, if $\mathscr{S} \subset \mathscr{M}$,

$$\begin{aligned}\bar{\mathscr{S}} &= \Big\{ J \in \mathscr{M} \colon J \supset \bigcap_{M \in \mathscr{S}} M \Big\} \\ &= \{ J \cap \mathscr{M} \colon J \text{ is the closed linear span of } \\ &\qquad f_{ab}^{(n)}(g),\ a, b \in \Lambda_n\ (n = 1, 2, \dots) \text{ and } g \text{ belongs} \\ &\qquad \text{to a maximal ideal } \mathscr{J} \text{ in } A(\mathbf{D}),\ \mathscr{J} \supset \ker(\varphi(S)) \}.\end{aligned}$$

In other words, $\varphi(\bar{\mathscr{S}}) = \varphi(\mathscr{S})^{-}$.

For sake of concreteness, let us recall the description of the hull-kernel topology on $\overline{\mathbf{D}}$. The closed sets $V \subset \overline{\mathbf{D}}$ are of the following form:

(i) $V \cap \mathbf{D}$ is either finite or countable; if $V \cap \mathbf{D}$ is countable, say $\{\lambda_1, \lambda_2, \dots\}$, then $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$.

(ii) $V \subset \partial\mathbf{D}$ is a closed subset (in the usual topology of the circle) of Lebesgue measure zero, and contains every accumulation point of $V \cap \mathbf{D}$ [**4**, p. 89].

Let us still assume $\Gamma(\alpha)$ finite, but relax the condition $\sigma(\alpha) = \Gamma(\alpha)$. How does the situation change? Not surprisingly, the answer depends on how much $\sigma(\alpha)$ differs from $\Gamma(\alpha)$. If $\sigma(\alpha)\setminus\Gamma(\alpha)$ is finite (equivalently, $\sigma(\alpha)$ is finite), then there is no change; Lemma 4.11 together with what we have said about implies $\mathscr{M}_a = \mathscr{M} \cong \mathbf{D}/\Gamma(\alpha)$.

Suppose, on the other hand, $\Gamma(\alpha)$ is finite but that $\sigma(\alpha)$ $(= \overline{\Gamma\Gamma^{-1}})$ has infinite cardinality. By Proposition III.10, $\alpha = (\mathrm{Ad}\, u)\beta$, where $u \in \mathfrak{A}$ is unitary, and β is a product-type automorphism of $\mathfrak{A}$ with respect to the same masa, and $\sigma(\beta) = \Gamma(\alpha) = \Gamma(\beta)$. Then $\mathbf{Z}^+\times_\alpha\mathfrak{A}$, $\mathbf{Z}^+\times_\beta\mathfrak{A}$ are isomorphic, and hence have the same maximal ideals, $\mathscr{M} \cong \mathbf{D}/\Gamma(\alpha)$. The isomorphism, most easily described on the dense subalgebras $K(\mathbf{Z}^+, \mathfrak{A}, \beta) \overset{\Psi}{\to} \mathbf{K}(Z^+, A, \alpha)$ by $\Psi(F) = F'$, where $F'(m) = F(m)u^{*m}$, does not map the sequence of subalgebras $\{\mathbf{Z}^+\times_{\beta_n}\mathfrak{A}_n\}$ onto $\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n\}$, and consequently ascending ideals in $Z^+\times_\beta\mathfrak{A}$ may not necessarily be mapped onto ascending ideals in $\mathbf{Z}^+\times_\alpha\mathfrak{A}$. (It should be noted that every maximal ideal of $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ will be ascending with respect to the sequence of subalgebras which is the image of $\{\mathbf{Z}^+\times_{\beta_n}\mathfrak{A}_n\}$ under ψ, but may not be ascending with respect to $\{\mathbf{Z}^+\times_{\alpha_n}\mathfrak{A}_n\}$.) Indeed, if the spectrum of α is infinite (equivalently, Γ is infinite), then by Lemma IV.7 $\psi(M_{\lambda_0})$ is not an ascending maximal ideal in $\mathbf{Z}^+\times_\alpha\mathfrak{A}$ for any $\lambda_0 \in \mathbf{D} \cup \{0\}$. It should be observed here that if $J \subset \mathbf{Z}^+\times_\beta\mathfrak{A}$ is an ideal corresponding to the triple $(\{B_n\}, \{L_n\}, \{\mu_n\})$, *and if* $\Psi(J) \subset \mathbf{Z}^+\times_\beta\mathfrak{A}$ is also ascending, hence corresponding to some triple $(\{B_n'\}, \{L_n'\}, \{\mu_n'\})$, $\bigcap_{n=1}^{\infty} L_n = \bigcap_{n=1}^{\infty} L_n'$ and the set of common zeroes of $\{B_n\}$ coincides with that of $\{B_n'\}$ (cf. Lemmas IV.7, IV.8). Moreover, if Γ is not nowhere dense, then by Lemma IV.8 L_n contains an interval, but since the only functions in the disk algebra (unlike $C(\mathbf{T})$) which vanish on a set of positive measure are zero, it follows that the maximal ideal $\Psi(M_{\lambda_0}) \subset \mathbf{Z}^+\times_\alpha\mathfrak{A}$ $(\lambda_0 \in \mathbf{T})$ is not ascending.

These facts are summarized in the following

THEOREM. *Assume that the Connes spectrum* $\Gamma(\alpha)$ *is finite. Then the space* $\mathcal{M}$ *of (closed, two-sided) maximal ideals of* $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ *is homeomorphic with* $\overline{\mathbf{D}}/\Gamma(\alpha)$, *where* $\overline{\mathbf{D}}$ *carries the hull-kernel topology of the disk algebra. Let* $\mathcal{M}_a$ *denote the subset of* $\mathcal{M}$ *of ideals which are ascending (with respect to the sequence* $\{\mathbf{Z}^+ \times_{\alpha_n} \mathfrak{A}_n\}$):

(i) *If the spectrum of* $\alpha, \sigma(\alpha) = \overline{\Gamma}\overline{\Gamma}^{-1}$, *is finite*, $\mathcal{M}_a = \mathcal{M}$;

(ii) *If* Γ *is infinite but nowhere dense, then* $M_a \cong \mathbf{T} \cup \{0\}/\Gamma(\alpha)$ *under the same homeomorphism which maps* $\mathcal{M}$ *to* $\mathbf{D}/\Gamma(\alpha)$;

(iii) *If* $\overline{\Gamma}$ *contains an interval, then* $\mathcal{M}_a$ *consists of a single point* $\{M_0\}$.

IV.14. REMARK. We have not treated the case $\Gamma(\alpha)$ infinite (in which case $\Gamma(\alpha) = \mathbf{T}$), but we suspect that here $\mathcal{M}$ consists of a single maximal ideal $\{M_0\}$. We can prove, however, that if the strong radical (i.e., the intersection of the maximal ideals) is nonzero, then $\mathcal{M} = \{M_0\}$. Since the result is not definitive, the proof is omitted, but it uses the same sort of "Fejér kernel" argument as in §IV of [**9**].

IV.15. REMARK. The ideas of 4.13 can also be used to characterize the maximal ideal space $\mathcal{M}$ of the C^*-crossed product $\mathbf{Z} \times_\alpha \mathfrak{A}$. Indeed, if $\Gamma(\alpha)$ is finite, $\mathcal{M} = \mathbf{T}/\Gamma(\alpha)$, where $\mathbf{T}$ has the Euclidean topology. (Of course if $\Gamma(\alpha) = \mathbf{T}$, $\mathbf{Z} \times_\alpha \mathfrak{A}$ is simple by [**3**, Corollary 3.8].)

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