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NECESSITY CONDITIONS FOR L^p MULTIPLIERS WITH POWER WEIGHTS

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ABSTRACT. It is shown that if multiplier operators are bounded on L^p with weight $|x|^\alpha$ for all functions in the space $\mathscr{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not including 0 and all multiplier functions in a standard Hörmander type multiplier class, then α must satisfy certain inequalities. This is a sequel to a previous paper in which conditions on α that were almost the same were shown to be sufficient for the norm inequality to hold.

1. Introduction. This paper is concerned with conditions that a real number α must satisfy if

$$\int_{-\infty}^{\infty} |(m\hat{f})^\vee(x)|^p |x|^\alpha \, dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx \tag{1.1}$$

for all m in a standard Hörmander type multiplier class and all f in $\mathscr{S}_{0,0}$, the Schwartz functions whose Fourier transforms have compact support not including 0. This is a sequel to [5] in which sufficient conditions were obtained on α for (1.1) to hold.

As in [5] we use the multiplier classes $M(s, \lambda)$; for λ a positive integer and $1 \leqslant s \leqslant \infty$ the class $M(s, \lambda)$ consists of all m such that

$$B(m, s, \lambda) = \|m\|_\infty + \sup_{r>0} r^{\lambda - 1/s} \left[\int_{r<|t|<2r} |m^{(\lambda)}(t)|^s \, dt \right]^{1/s} < \infty.$$

For the definition of $M(s, \lambda)$ and $B(m, s, \lambda)$ with λ not an integer, see §2. As shown in §7 of [5], these spaces are two sided versions of the multiplier classes $S(s, \lambda)$ in [2] and $\mathrm{WBV}_{s,\lambda}$ in [3].

The following was proved in [5]; it is Theorem (1.2) of [5].

THEOREM (1.2). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\lambda > \max(1/s, |1/p - 1/2|)$ *or* $\lambda = s = 1$, *m is in* $M(s, \lambda)$,

$$\max\left(-1, -p\lambda, -1 + p\left(-\lambda + \tfrac{1}{2}\right)\right) < \alpha < \min\left(p\lambda, -1 + p\left(\lambda + \tfrac{1}{2}\right), -1 + p\left(\lambda + 1 - \tfrac{1}{s}\right)\right)$$

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and $(\alpha + 1)/p$ *is not an integer, then for* f *in* $\mathscr{S}_{0,0}$

$$\int_{-\infty}^{\infty} \left|(m\hat{f})^{\vee}(x)\right|^p |x|^\alpha \, dx \leqslant CB(m,s,\lambda)^p \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx, \tag{1.3}$$

where C *is independent of* m *and* f.

The conditions on α in Theorem (1.2) seem peculiar, both because they are complicated and because taking s larger than the minimum of 2 and $p' = p/(p-1)$ does not increase the range of α. The main result of this paper is the fact that, except possibly for the strictness of the inequalities, these conditions are also necessary. The result is as follows.

THEOREM (1.4). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\lambda \geqslant 1/s$ *and* (1.1) *holds for all* m *in* $M(s,\lambda)$ *and* f *in* $\mathscr{S}_{0,0}$, *then* $\alpha > -1$,

$$\max\left(-p\lambda, -1 + p\left(-\lambda + \tfrac{1}{2}\right)\right) \leqslant \alpha \leqslant \min\left(p\lambda, -1 + p\left(\lambda + \tfrac{1}{2}\right), -1 + p\left(\lambda + 1 - \tfrac{1}{s}\right)\right)$$

and $(\alpha + 1)/p$ *is not an integer.*

In at least some cases, the end values of the inequalities for α are included in the values for which (1.1) holds; a theorem of this type is given in §6 of [**5**]. It should also be remarked that the conclusion that $(\alpha + 1)/p$ is not an integer can be proved from much weaker hypotheses. In fact, as shown in Theorem (1.2), p. 624 of [**6**], if (1.1) holds for all f in $\mathscr{S}_{0,0}$ for some m that is not a constant almost everywhere, then it follows that $(\alpha + 1)/p$ is not an integer. The proof given here uses the stronger hypothesis and, as a result, is much shorter than the proof in [**6**].

This paper consists of the proof of Theorem (1.4). §2 contains certain facts needed in later sections primarily concerning fractional derivatives and the multiplier classes $M(s,\lambda)$. These were proved in [**5**] and are quoted here for convenience. §3 gives some general procedures for generating functions in the classes $M(s,\lambda)$. These are used in §4 to produce specific examples of functions in $M(s,\lambda)$ that prove the upper bounds for α in Theorem (1.4). In §5 the facts that $\alpha > -1$ and $(\alpha + 1)/p$ is not an integer are proved directly while the lower bounds on α are obtained from the upper bounds by a duality argument.

The following definitions and notations will be used throughout this paper. Given real α and $p \geqslant 1$, we define

$$\|f\|_{p,\alpha} = \left[\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx\right]^{1/p}$$

and L_α^p as the set of f with $\|f\|_{p,\alpha} < \infty$; as usual, α may be omitted if it is 0. The space $\mathscr{S}_{0,0}$ will be as defined above. For integrable f we define the Fourier transform $\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt} \, dt$ and the inverse Fourier transform

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{ixt} \, dt.$$

For general locally integrable f, we define $\hat{f}$ to be the function that satisfies $\int_{-\infty}^{\infty} \hat{f}(x)\phi(x)\,dv = \int_{-\infty}^{\infty} f(x)\hat{\phi}(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists. The inverse Fourier transform $\check{f}$ for locally integrable functions is defined analogously. Similarly, the weak derivative of a function f on $(-\infty, \infty)$ is the function f' such that $\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists.

Throughout this paper C will denote constants not necessarily the same at each occurrence. If $g(x)$ is an expression in x, $[g(x)]^\wedge$ will denote the Fourier transform at the point x. For a number p with $1 \leqslant p \leqslant \infty$, p' will denote $p/(p-1)$. In addition to the expression $\operatorname{int}(x)$ for the greatest integer less than or equal to x, the traditional $[x]$ will also be used when not ambiguous.

2. Definitions and basic results. For convenience, we list here the definitions and theorems from [**5**] that will be needed to prove Theorem (1.4). These are various properties of the multiplier classes $M(s, \lambda)$ and fractional derivatives plus one density theorem. For proofs and further discussion, see [**5**].

We define the operator D^λ by $D^\lambda g(x) = [\check{g}(x)x^\lambda]^\wedge$, where x^λ is taken to be $|x|^\lambda e^{-i\pi\lambda}$ for $x < 0$ and the Fourier transforms are as defined in §1. To define the multiplier classes, choose a function $\psi(x)$ in C^∞ with support in $1/2 < |x| < 2$ such that $\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1$ for $x \neq 0$. Given a function $m(x)$, define $m_j(x) = m(x)\psi(2^{-j}x)$. Then for $1 \leqslant s \leqslant \infty$ and $\lambda \geqslant 0$, the multiplier class $M(s, \lambda)$ is the set of functions m such that $D^\lambda m_j$ is a locally integrable function for every j and

$$B(m, s, \lambda) = \|m\|_\infty + \sup_j 2^{j(\lambda - 1/s)} \left\| D^\lambda m_j(x) \right\|_s < \infty. \tag{2.1}$$

We will need the following two results concerning the classes $M(s, \lambda)$. The first shows that the classes are independent of the choice of the function ψ; the second is an imbedding theorem. They are Theorem (2.25) and Theorem (2.12) of [**5**].

THEOREM (2.2). *If m is in $M(s, \lambda)$, $1 \leqslant s \leqslant \infty$, $\lambda > 0$, and ϕ has $[\lambda + 1]$ bounded derivatives and support in $1/2 \leqslant |x| \leqslant 2$, then $D^\lambda[m(x)\phi(2^{-j}x)]$ is a locally integrable function and*

$$\left\| D^\lambda \left[m(x)\phi(2^{-j}x) \right] \right\|_s \leqslant CA(\phi) 2^{j(-\lambda + 1/s)} B(m, s, \lambda),$$

where C is independent of m and ϕ and $A(\phi) = \sup_{0 \leqslant k \leqslant [\lambda+1]} \|\phi^{(k)}\|_\infty$.

THEOREM (2.3). *If $1 \leqslant s \leqslant \infty$, $1 \leqslant t \leqslant \infty$, $0 \leqslant \alpha \leqslant \lambda$, m is in $M(s, \lambda)$ and one of the following holds:*

(i) $\alpha - 1/t \leqslant \lambda - 1/s$, $s > 1$ *and* $t < \infty$,

(ii) $\alpha - 1/t \leqslant \lambda - 1/s$, $s = 1$ *and* $t = \infty$,

(iii) $\alpha - 1/t < \lambda - 1/s$,

then m is in $M(t, \alpha)$ and $B(m, t, \alpha) \leqslant CB(m, s, \lambda)$ with C independent of m.

The following facts about the fractional derivative operator will also be needed. These are respectively Lemma (2.6), Lemma (2.15) and Lemma (2.18) of [**5**].

LEMMA (2.4). *If $f(x)$ is integrable, $f(x) = 0$ for x not in a compact interval I, $\lambda > -1$ and $D^\lambda f(x)$ is a locally integrable function, then there is a C, depending only on λ, such that $D^\lambda f(x) = 0$ for almost every x to the right of I and*

$$D^\lambda f(x) = C\int_I \frac{f(t)}{(t-x)^{\lambda+1}}\,dt$$

for almost every x to the left of I.

LEMMA (2.5). *If $0 < \lambda < 1$, f is in L^1 and either the function*

$$G^\lambda f(x) = \frac{d}{dx}\int_x^\infty \frac{f(t)}{(t-x)^\lambda}\,dt,$$

where the derivative is taken in the weak sense, or $D^\lambda f(x)$ is a locally integrable function, then the other is a locally integrable function and there is a nonzero constant C such that $D^\lambda f(x) = CG^\lambda f(x)$.

LEMMA (2.6). *If $0 \leqslant \lambda < 1$, $1 \leqslant s \leqslant \infty$, f is integrable, $D^\lambda f$ is a locally integrable function, ϕ is differentiable with $\|\phi'\|_\infty < \infty$ and ϕ has support in a finite interval I, then $D^\lambda(\phi f)$ is a locally integrable function and*

$$\|D^\lambda(\phi f)\|_s \leqslant C\big(\|\phi\|_\infty + |I|\,\|\phi'\|_\infty\big)\big(\|\chi_I D^\lambda f\|_s + |I|^{-\lambda}\|f\|_s\big),$$

where C is independent of f and ϕ.

Finally, we shall need the following density result; it is Theorem (1.5) of [**5**]. The space Q_k in Theorem (2.7) for $k \geqslant 0$ is the set of functions f in $L^2 \cap L^1_k$ such that $\int_{-\infty}^\infty f(x)x^j\,dx = 0$ for $0 \leqslant j \leqslant k$; the space Q_{-1} is L^2.

THEOREM (2.7). *If $1 < p < \infty$, $\alpha > -1$, k is an integer, $k \geqslant -2 + (\alpha+1)/p$, m is bounded, and*

$$\|(m\hat{f})^\vee\|_{p,\alpha} < C\|f\|_{p,\alpha} \tag{2.8}$$

for all f in $\mathscr{S}_{0,0}$, then (2.8) is true for all f in $Q_k \cap L^p_\alpha$ with the same C.

3. Lemmas for Theorem (1.4). To prove Theorem (1.4), we need a way to construct examples of functions in $M(s,\lambda)$, and we will need a modified form of (1.3) for these functions. The results needed are Lemmas (3.1), (3.5), (3.14) and (3.22).

LEMMA (3.1). *If $0 < \lambda \leqslant 1$, $|m(x)| \leqslant A/(1+|x|^\lambda)$ and $|m'(x)| \leqslant A|x|^{-\lambda}$, then m is in $M(\infty,\lambda)$ and $B(m,\infty,\lambda) \leqslant CA$, where C depends only on λ.*

For $\lambda = 1$ this is immediate; therefore, assume that $0 < \lambda < 1$. It is sufficient to show that

$$\frac{d}{dx}\int_0^\infty \frac{m(x+t)\psi(2^{-j}(x+t))}{t^\lambda}\,dt \tag{3.2}$$

has absolute value bounded by $CA2^{-j\lambda}$ for almost every x since Lemma (2.5) will then imply that $D^\lambda m_j(x)$ equals a constant independent of j times (3.2), and $|D^\lambda m_j(x)| \leqslant CA2^{-j\lambda}$ is what is needed.

To estimate (3.2) for $2^{j-2} < |x| < 2^{j+2}$, $j > 3$, write (3.2) as the sum of

$$\frac{d}{dx}\int_0^1 \frac{m(x+t)\psi(2^{-j}(x+t))}{t^\lambda}\,dt \tag{3.3}$$

and

$$\frac{d}{dx}\int_{x+1}^\infty \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda}\,dt. \tag{3.4}$$

The absolute value of (3.3) is bounded by

$$\int_0^1 \frac{|m'(x+t)\psi(2^{-j}(x+t))| + |2^{-j}m(x+t)\psi'(2^{-j}(x+t))|}{t^\lambda}\,dt.$$

This has the bound

$$C\int_0^1\left[\frac{A}{|x+t|^\lambda t^\lambda} + \frac{2^{-j}A}{t^\lambda}\right]dt \leqslant CA|x|^{-\lambda} + C2^{-j}A \leqslant CA2^{-j\lambda}.$$

The absolute value of (3.4) is bounded by

$$|m(x+1)\psi(2^{-j}(x+1))| + C\int_{x+1}^\infty \frac{|m(t)\psi(2^{-j}t)|}{(t-x)^{\lambda+1}}\,dt$$

which is bounded by

$$CA2^{-j\lambda} + CA2^{-j\lambda}\int_{x+1}^\infty (t-x)^{-\lambda-1}\,dt \leqslant C2^{-j\lambda}.$$

We next estimate (3.2) for $j > 3$ and $|x| < 2^{j-2}$ or $|x| > 2^{j+2}$. If $|x| < 2^{j-2}$, the integrand in (3.2) is 0 for t not in $[2^{j-1} - x, 2^{j+1} - x]$ and a change of variables shows that the absolute value of (3.2) is

$$\left|\frac{d}{dx}\int_{2^{j-1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda}\,dt\right| = C\left|\int_{2^{j-1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^{\lambda+1}}\,dt\right|.$$

Since $|m(t)| \leqslant A$ and $|\psi(2^{-j}t)| \leqslant C$, we obtain the upper bound $CA2^{-j\lambda}$.

If $j > 3$ and $x > 2^{j+2}$, the integral in (3.2) is 0 and the estimate is trivial. If $j > 3$ and $x < 2^{j+2}$, the integrand in (3.2) is 0 for t not in $[-2^{j+1} - x, 2^{j+1} - x]$. A change of variables shows that (3.2) has absolute value bounded by

$$\left|\frac{d}{dx}\int_{-2^{j+1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda}\,dt\right| \leqslant C\int_{-2^{j+1}}^{2^{j+1}} \frac{|m(t)\psi(2^{-j}t)|}{|x|^{\lambda+1}}\,dt.$$

The facts that $|m(t)| \leqslant A$ and $|\psi(2^{-j}t)| \leqslant C$ then show that $CA2^{-j\lambda}$ is also an upper bound for this case.

To estimate the absolute value of (3.2) for $j \leqslant 3$, perform the differentiation to get the bound

$$\int_0^\infty \frac{|m'(x+t)\psi(2^{-j}(x+t))| + |m(x+t)2^{-j}\psi'(2^{-j}(x+t))|}{t^\lambda}\,dt.$$

This is less than or equal to

$$CA\int_{2^{j-1}\leqslant|x+t|\leqslant 2^{j+1}} \frac{2^{-j\lambda}+2^{-j}}{|t|^{\lambda}}\,dt \leqslant C2^{-j\lambda}.$$

This completes the proof of Lemma (3.1).

LEMMA (3.5). *For $0<\lambda<1$, assume that $|m(\lambda,x)|\leqslant C(1+|x|)^{-\lambda}$ for all x and $|\partial m(\lambda,x)/\partial x|\leqslant C|x|^{-\lambda}$ for $|x|\leqslant 1$, where C is independent of x, and assume that $m(\lambda,x)$ is in $M(\infty,\lambda)$. For $\lambda>1$ and not an integer, define $m(\lambda,x)$ recursively by*

$$m(\lambda,x)=\int_{-\infty}^{x} g(\lambda-1,t)\,dt,$$

where

$$g(\lambda-1,t)=\frac{m(\lambda-1,t)-\theta(t)m(\lambda-1,0)}{t} - \frac{\phi(t)}{t}\int_{-\infty}^{\infty}\frac{m(\lambda-1,u)-\theta(u)m(\lambda-1,0)}{u}\,du,$$

where θ and ϕ are in C^{∞} with support in $[-1,1]$, θ is even, $\theta(0)=1$, $\phi(0)=0$ and $\int_{-1}^{1}(\phi(t)/t)\,dt=1$. Then $m(\lambda,x)$ is in $M(\infty,\lambda)$ for $\lambda>1$ if λ is not an integer.

Note that although a version of this theorem is true for integer values of λ, a modification would be needed since $|m(1,x)|\leqslant C(1+|x|)^{-1}$ for all x and $|\partial m(1,x)/\partial x|\leqslant C|x|^{-1}$ for $|x|\leqslant 1$ do not imply the integrability of $g(1,t)$ on $(-\infty,\infty)$. Since a version valid for integer values is not needed in the proof of Theorem (1.4), we will not give one here.

To prove the existence of $m(\lambda,x)$ for $\lambda>1$, we will show inductively that

$$|m(\lambda,x)|\leqslant C(1+|x|)^{-\lambda+[\lambda]} \tag{3.6}$$

and

$$\left|\frac{\partial}{\partial x}m(\lambda,x)\right|\leqslant C|x|^{-\lambda+[\lambda]},\qquad |x|<1. \tag{3.7}$$

Inequalities (3.6) and (3.7) are hypothesized for $0<\lambda<1$. Now assume that they hold with λ replaced by $\lambda-1$ for some $\lambda>1$. Then for $|t|<1$, since $-(\lambda-1)+[\lambda-1]=-\lambda+[\lambda]$,

$$|m(\lambda-1,t)-m(\lambda-1,0)|\leqslant\int_0^{|t|} C|x|^{-\lambda+[\lambda]}\,dx\leqslant C|t|^{1-\lambda+[\lambda]}.$$

This, the fact that $|\theta(t)-1|\leqslant C|t|$ and (3.6) for $\lambda-1$ imply that

$$\left|\frac{m(\lambda-1,t)-\theta(t)m(\lambda-1,0)}{t}\right|\leqslant\begin{cases} C|t|^{-\lambda+[\lambda]}, & |t|<1,\\ C|t|^{-\lambda-1+[\lambda]}, & |t|>1.\end{cases} \tag{3.8}$$

Therefore, $g(\lambda-1,t)$ exists and is integrable on $(-\infty,\infty)$. The estimates in (3.8) also directly imply (3.6) for $-\infty<x<1$ and (3.7). From the definition of $g(\lambda-1,t)$ we see that $\int_{-\infty}^{\infty}g(\lambda-1,t)\,dt=0$. Therefore $m(\lambda,x)=-\int_x^{\infty}g(\lambda-1,t)\,dt$, and (3.8) implies (3.6) for $x\geqslant 1$. This completes the induction. Note that in particular we have $\|m(\lambda,x)\|_{\infty}<\infty$ for all $\lambda>0$, λ not an integer.

We will now show inductively that $\|D^\lambda m_j(\lambda, x)\|_\infty \leqslant C2^{-j\lambda}$ for $\lambda > 1$ and not an integer, where C is independent of j. Therefore, fix λ, let $k = [\lambda]$ and assume that $m(\lambda - i, x)$ is in $M(\infty, \lambda - i)$ for $1 \leqslant i \leqslant k$. By the definition of $m(\lambda, x)$, $D^\lambda m_j(\lambda, x)$ is the sum of

$$2^{-j}D^{\lambda-1}\left[m(\lambda - 1, x)\frac{\psi(2^{-j}x)}{2^{-j}x}\right], \tag{3.9}$$

$$2^{-j}D^{\lambda-1}\left[-m(\lambda - 1, 0)\theta(x)\frac{\psi(2^{-j}x)}{2^{-j}x}\right], \tag{3.10}$$

$$-2^{-j}D^{\lambda-1}\left[\phi(x)\frac{\psi(2^{-j}x)}{2^{-j}x}\int_{-\infty}^{\infty}\frac{[m(\lambda - 1, u) - \theta(u)m(\lambda - 1, 0)]}{u}\,du\right] \tag{3.11}$$

and

$$2^{-j}D^{\lambda-1}[m(\lambda, x)\psi'(2^{-j}x)]. \tag{3.12}$$

Since $m(\lambda - 1, x)$ is in $M(\infty, \lambda - 1)$, Theorem (2.2) with $\phi(x)$ there taken to be $\psi(x)/x$ implies that (3.9) is bounded by $CB(m(\lambda - 1, x), \infty, \lambda - 1)2^{-j\lambda} \leqslant C2^{-j\lambda}$ as desired. For (3.10) and (3.11) use the fact that $\theta(x)$ and $\phi(x)$ are in $M(\infty, i)$ for all integers i. Then use Theorem (2.3) to deduce that θ and ϕ are in $M(\infty, \lambda - 1)$ and use Theorem (2.2). For (3.12) differentiate again to obtain a sum of terms like (3.9)–(3.12) with the initial 2^{-j} replaced by 2^{-2j}, $D^{\lambda-1}$ by $D^{\lambda-2}$, and ψ replaced by ψ'. These are treated in the same way; this process continues until the term

$$2^{-jk}D^{\lambda-k}[m(\lambda, x)\psi^{(k)}(2^{-j}x)] \tag{3.13}$$

is reached. Now since $|m(\lambda - 1, x)| \leqslant C$, the definition of $m(\lambda, x)$ implies $|\partial m(\lambda, x)/\partial x| \leqslant C/|x|$ and, therefore, $m(\lambda, x)$ is in $M(\infty, 1)$. Theorem (2.3) implies that $m(\lambda, x)$ is in $M(\infty, \lambda - k)$ and Theorem (2.2) shows that (3.13) also is bounded by $C2^{-j\lambda}$. This completes the proof of Lemma (3.5).

LEMMA (3.14). *If $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, k is an integer, $k \geqslant 0$, $k < \lambda < k + 1$, $-1 + pk < \alpha \leqslant -1 + p(\lambda + 1)$, $\alpha \neq -1 + p(k + 1)$,* (1.3) *holds for all f in $\mathscr{S}_{0,0}$ and all m in $M(s, \lambda)$ and $m(\lambda, x)$ is a set of functions as described in Lemma* (3.5), *then there is a constant C such that*

$$\int_{-\infty}^{\infty} \left|[m(\lambda - k, x)\hat{g}(x)]^\vee\right|^p |x|^{\alpha - kp}\,dx \leqslant C\int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha\,dx \tag{3.15}$$

for all g with compact support, k bounded derivatives and $\int_{-\infty}^{\infty} g(x)\,dx = 0$.

Since (1.3) holds for all m in $M(s, \lambda)$, it holds in particular for $m(\lambda, x)$ by Lemma (3.5) and Theorem (2.3). Now since $-1 + pk < \alpha < -1 + p(k + 2)$ and $\alpha \neq -1 + p(k + 1)$, we have $j = \operatorname{int}((\alpha + 1)/p)$ equal to k or $k + 1$, $G(x) = x^j$ is a polynomial, $V(x) = |x|^{\alpha - jp}$ is in A_p and $W(x) = |x|^\alpha = |G(x)|^p V(x)$. Applying Theorem (2.7) to (1.3) then shows that (1.3) holds for any f in $Q_k \cap L^p_\alpha$ if $m(x) = m(\lambda, x)$. In particular,

$$\int_{-\infty}^{\infty} \left|\left(m(\lambda, x)[g^{(k)}(x)]^\wedge\right)^\vee\right|^p |x|^\alpha\,dx \leqslant C\int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha\,dx \tag{3.16}$$

for any g of the type described by the lemma.

If $k = 0$, we are done. If $k > 0$, the left side of (3.16) is bounded below by

$$\int_{-\infty}^{\infty} \left| \left(\frac{\partial}{\partial x} m(\lambda, x)[g^{(k)}(x)]^{\wedge} \right)^{\vee} \right|^{p} |x|^{\alpha - p}\, dx \tag{3.17}$$

minus

$$C\int_{-\infty}^{\infty} \left| \left(m(\lambda, x)[xg^{(k)}(x)]^{\wedge} \right)^{\vee} \right|^{p} |x|^{\alpha - p}\, dx. \tag{3.18}$$

Now since $-1 + pk < \alpha \leqslant -1 + p(1 + \lambda)$ and $\alpha \neq -1 + p(k + 1)$, $(\alpha - p + 1)/p$ is not an integer and $\alpha - p \leqslant -1 + p\lambda < -1 + p(1 + \lambda - 1/t)$, where $t = \min(2, p', \infty)$. Furthermore, since $\int_{-\infty}^{\infty} g(x)\, dx = 0$, we have

$$\int_{-\infty}^{\infty} [xg^{(k)}(x)]x^{j}\, dx = 0 \quad \text{for } 0 \leqslant j \leqslant k - 1.$$

Therefore, Theorem (2.7) in conjunction with Theorem (1.2) implies that (3.18) is bounded above by the right side of (3.15) since $\operatorname{int}((\alpha - p + 1)/p) \leqslant \lambda < k + 1$. This implies that (3.17) is also bounded above by the right side of (3.15).

Using the definition of $m(\lambda, x)$, we find that (3.17) is bounded below by

$$\int_{-\infty}^{\infty} \left| \left(m(\lambda - 1, x)[g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^{p} |x|^{\alpha - p}\, dx \tag{3.19}$$

minus

$$C\int_{-\infty}^{\infty} \left| \left(\theta(x)[g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^{p} |x|^{\alpha - p}\, dx \tag{3.20}$$

and

$$C\int_{-\infty}^{\infty} \left| \left(\phi(x)[g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^{p} |x|^{\alpha - p}\, dx. \tag{3.21}$$

Since θ and ϕ are in $M(\infty, j)$ for any integer j and $(\alpha - p + 1)/p$ is not an integer, Theorem (1.2) and Theorem (2.7) show that (3.20) and (3.21) are bounded by

$$C\int_{-\infty}^{\infty} |g^{(k-1)}(x)|^{p} |x|^{\alpha - p}\, dx,$$

and this is bounded by the right side of (3.15) by Hardy's inequality. Therefore, (3.19) is also bounded by the right side of (3.15). Repeating this procedure k times completes the proof of lemma (3.14).

To estimate one of the counterexamples, we will need the following lemma.

LEMMA (3.22). *If* $-2 < b < 1$ *and* $\alpha = 0$ *or* 1, *then*

$$f(x) = \left[|x|^{b} \sin(x^{2})(\operatorname{sgn} x)^{\alpha} \right]^{\wedge}$$

is a function and

$$|f(x)| \leqslant C(1 + |x|)^{\max(-1, b)}. \tag{3.23}$$

To prove this we will use the following lemma which is a simple consequence of the Leibniz alternating series theorem.

LEMMA (3.24). *If $g(x) > 0$ and is monotone on $a \leqslant x \leqslant b$ and d is any real number, then*

$$\left|\int_a^b \sin(x+d)g(x)\,dx\right| \leqslant \pi\int_s^{s+1} g(x)\,dx,$$

where $s = b-1$ if g is increasing and $s = a$ if g is decreasing.

The following corollary of Lemma (3.24) will also be used.

LEMMA (3.25). *If $g(x) > 0$ and is monotone on $a \leqslant x \leqslant b$ and d is any real number, then*

$$\left|\int_a^b \sin(x+d)g(x)\,dx\right| \leqslant \pi g(s),$$

where $s = b$ if g is increasing and $s = a$ if g is decreasing.

Lemma (3.22) will be proved by showing that

$$\lim_{N\to\infty} 2\int_0^N i^\alpha t^b \sin(t^2)\cos\left(xt + \frac{\alpha\pi}{2}\right)dt \tag{3.26}$$

exists for every x and

$$\left|\int_0^N t^b \sin(t^2)\cos\left(xt + \frac{\alpha\pi}{2}\right)dt\right| \leqslant C(1+|x|)^{\max(-1,b)}, \tag{3.27}$$

where C is independent of N and x. This is sufficient since (3.26) and (3.27) imply that (3.26) equals $f(x)$ and (3.27) then shows that the absolute value of (3.26) has the bound asserted for $|f(x)|$. By symmetry, we need only prove these facts for $x \geqslant 0$.

To show that (3.26) exists for $x \geqslant 0$, we will use the identity

$$2\sin(t^2)\cos\left(xt + \frac{\alpha\pi}{2}\right) = \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right) + \sin\left(t^2 + xt + \frac{\alpha\pi}{2}\right). \tag{3.28}$$

If $x < M < N$, the change of variables $t - x/2 = \sqrt{u}$ shows that

$$\left|\int_M^N t^b \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right)dt\right| = \left|\int_{(M-x/2)^2}^{(N-x/2)^2} \frac{\left(\sqrt{u}+x/2\right)^b}{2\sqrt{u}}\sin\left(u - \frac{x^2}{4} - \frac{\alpha\pi}{2}\right)\right| du,$$

and since $b < 1$, Lemma (3.25) shows this is bounded by CM^{b-1}. Therefore,

$$\lim_{N\to\infty}\int_0^N t^b \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right)dt$$

exists by the Cauchy convergence criterion. The proof of the existence of the other limit obtained by substituting (3.28) in (3.26) is similar.

To prove (3.27), observe that since $b > -2$, $t^b\sin(t^2)$ has a positive derivative at 0. Let r be the smaller of $\frac{1}{4}$ and the least positive t for which $t^b\sin(t^2)$ has derivative 0. Then r depends only on b, and $t^b\sin(t^2)$ is increasing on $[0, r]$. We will now show that for $x > 1/r$ and $N > x/(1-b)$

$$\left|\int_0^r t^b \sin(t^2)\cos\left(xt + \frac{\alpha\pi}{2}\right)dt\right| \leqslant Cx^{-1}, \tag{3.29}$$

$$\left|\int_r^{x/2} t^b \sin\left[\left(\frac{x}{2} - t\right)^2 + d\right] dt\right| \leqslant Cx^b + Cx^{-1}, \tag{3.30}$$

$$\left|\int_{x/2}^N t^b \sin\left[\left(t - \frac{x}{2}\right)^2 + d\right] dt\right| \leqslant Cx^b \tag{3.31}$$

and

$$\left|\int_r^N t^b \sin\left[\left(t + \frac{x}{2}\right)^2 + d\right] dt\right| \leqslant Cx^{b-1} + Cx^{-1}, \tag{3.32}$$

where $d = \pm\alpha\pi/2 - x^2/4$ and C is independent of N and x. This is enough to prove (3.23) for $x > 1/r$ since the left side of (3.27) is bounded by the sum of the left sides of (3.29)–(3.32).

To prove (3.29) for $x > 1/r$, make the change of variables $xt = u$ in the left side and use Lemma (3.25) to get the bound

$$\frac{\pi}{x}\left(\frac{rx}{x}\right)^b \sin\left[\left(\frac{rx}{x}\right)^2\right],$$

and (3.29) follows.

To prove (3.30) for $x > 1/r$, make the change of variables $t = -\sqrt{u} + x/2$ in the left side to get the bound

$$\left|\frac{1}{2}\int_0^\beta g(u)\sin(u + d)\, du\right|,$$

where $g(u) = u^{-1/2}(-\sqrt{u} + x/2)^b$ and $\beta = (x/2 - r)^2$. Now if $g(u)$ is decreasing on $[0, \beta]$, let $s = \beta$; otherwise let s be the unique point in $[0, \beta]$ where $g(u)$ has a local minimum. By Lemma (3.24),

$$\left|\int_0^s g(u)\sin(u + d)\, du\right| \leqslant \pi\int_0^1 g(u)\, du$$

and this is bounded by Cx^b since $x \geqslant 4$. If $s < \beta$, Lemma (3.25) shows that

$$\left|\int_s^\beta g(u)\sin(u + d)\, du\right| \leqslant \pi g(\beta) = \frac{\pi r^b}{x/2 - r},$$

which is bounded by Cx^{-1}. This completes the proof of (3.30).

To prove (3.31) for $x > 1/r$, substitute $t - x/2 = \sqrt{u}$ to get

$$\left|\frac{1}{2}\int_0^{(N-x/2)^2}\left(\frac{x}{2} + \sqrt{u}\right)^b u^{-1/2}\sin(u + d)\, du\right|.$$

Since $b < 1$, $(x/2 + \sqrt{u})^b u^{-1/2}$ is decreasing for $u > 0$ and Lemma (3.24) completes the proof.

For (3.32) with $x > 1/r$, substitute $t + x/2 = \sqrt{u}$ to get $|\int_\beta^\gamma g(u)\sin(u + d)\, du|$, where $\beta = (r + x/2)^2$, $\gamma = (N + x/2)^2$ and $g(u) = \frac{1}{2}u^{-1/2}(-x/2 + \sqrt{u})^b$. If $\beta < s = [x/2(1 - b)]^2$, then $b > 0$ and g is increasing on $[\beta, s]$ and decreasing on $[s, \gamma]$. Apply Lemma (3.25) to each part; both parts have the bound $C|x|^{b-1}$. If $\beta \geqslant s$, g is decreasing on $[\beta, \gamma]$ and Lemma (3.25) can be applied directly to get the bound $C|x|^{-1}$. This completes the proof of Lemma (3.22) for $x > 1/r$.

To prove (3.27) for $0 \leqslant x < 1/r$, it is sufficient because of (3.28) to show that for $a = 2[r(1-b)]^{-1}$ and $N > a$ that

$$\left|\int_0^a t^b \sin(t^2)\cos\left(xt + \frac{\alpha\pi}{2}\right) dt\right| \leqslant C, \tag{3.33}$$

$$\left|\int_a^N t^b \sin\left[\left(\frac{x}{2} - t\right)^2 + d\right] dt\right| \leqslant C, \tag{3.34}$$

and

$$\left|\int_a^N t^b \sin\left[\left(\frac{x}{2} + t\right)^2 + d\right] dt\right| \leqslant C, \tag{3.35}$$

where C depends only on b.

Inequality (3.33) follows from the fact that the integrand is bounded. In (3.34) and (3.35), make the respective substitutions $t - x/2 = \sqrt{u}$, $t + x/2 = \sqrt{u}$ and apply Lemma (3.25). This completes the proof of Lemma (3.22).

4. Examples: proof of the upper bounds in Theorem (1.4). The proof that (1.3) for all f in $\mathscr{S}_{0,0}$ and m in $M(s,\lambda)$ implies the asserted upper bound for α will be done in three parts. We will first prove $\alpha \leqslant -1 + p(1 + \lambda - 1/s)$, next $\alpha \leqslant p\lambda$ and third $\alpha \leqslant -1 + p(\lambda + \frac{1}{2})$. The proof that $(\alpha + 1)/p$ cannot be an integer and the proof of the lower bounds are given in §5.

To prove that $\alpha \leqslant -1 + p(1 + \lambda - 1/s)$, let $\phi(x)$ be in C^∞ with $\phi(x) = 1$ for $|x| \leqslant \frac{1}{2}$, $\phi(x) = 0$ for $|x| \geqslant 1$ and $0 \leqslant \phi(x) \leqslant 1$ for all x. Let

$$m(x) = \sum_{k=1}^{\infty} 8^{k(1/s-\lambda)}\phi(x - 8^k),$$

and for $k \geqslant 2$ let

$$\hat{f}_k(x) = \phi(x8^{1-k} - 8).$$

Since $\lambda \geqslant 1/s$, $\|m\|_\infty \leqslant 1$. Because of Theorem (2.2), we may assume that the function ψ used in the definition of $B(m,s,\lambda)$ has $\psi(x) = 1$ for $\frac{3}{4} \leqslant |x| \leqslant \frac{5}{4}$. Then $m_j(x) \equiv 0$ for j not of the form $3k$ and

$$m_{3k}(x) = 2^{3k(1/s-\lambda)}\phi(x - 2^{3k}).$$

From this,

$$\|D^\lambda m_{3k}(x)\|_s = 2^{3k(1/s-\lambda)}\|(D^\lambda\phi)(x - 2^{3k})\|_s.$$

Therefore, for all $j \geqslant 1$,

$$\|D^\lambda m_j\|_s \leqslant 2^{j(1/s-\lambda)}\|D^\lambda\phi(x)\|_s.$$

Now $D^\lambda\phi$ is bounded; therefore, Lemma (2.4) implies that $\|D^\lambda\phi\|_s < \infty$. This and the definition then show that m is in $M(s,\lambda)$.

Now it is immediate that f_k is in $\mathscr{S}_{0,0}$. Since $\hat{f}_k$ has support in $[8^k - 8^{k-1}, 8^k + 8^{k-1}]$ and is 1 in $[8^k - 8^{k-2}, 8^k + 8^{k-2}]$, we have

$$m(x)\hat{f}_k(x) = 8^{k(1/s-\lambda)}\phi(x - 8^k)$$

and

$$\left|[m(x)\hat{f}_k(x)]^\vee\right| = 8^{k(1/s-\lambda)}|\check{\phi}(x)|.$$

Therefore,

$$\int_{-\infty}^{\infty} \left|[m(x)\hat{f}_k(x)]^{\vee}\right|^p |x|^\alpha dx = 8^{kp(1/s-\lambda)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx. \tag{4.1}$$

Similarly,

$$|f_k(x)| = 8^{k-1}|\check{\phi}(8^{k-1}x)|$$

and

$$\int_{-\infty}^{\infty} |f_k(x)|^p |x|^\alpha dx = 8^{(k-1)(p-\alpha-1)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx. \tag{4.2}$$

Using (4.1) and (4.2) in (1.3) then shows that

$$8^{kp(1/s-\lambda)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx \leqslant C 8^{(k-1)(p-\alpha-1)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx$$

for all $k \geqslant 0$, and this implies $\alpha \leqslant -1 + p(1 + \lambda - 1/s)$.

To prove that $\alpha \leqslant p\lambda$ for λ not an integer, fix λ, s, p and α and define

$$m(\lambda, x) = e^{-ix}/(1 + x^2)^{\lambda/2}$$

for $0 < \lambda < 1$. Note that $m(\lambda, x)$ satisfies the hypothesis of Lemma (3.5), and define $m(\lambda, x)$ for $\lambda > 1$ and not an integer as was done in Lemma (3.5). We may assume that $\alpha \leqslant -1 + p(1 + \lambda - 1/s) \leqslant -1 + p(1 + \lambda)$ by the first proof of this section. We may also assume $\alpha > -1 + pk$, where $k = [\lambda]$, since otherwise $\alpha \leqslant -1 + pk < p\lambda$ and there is nothing to prove. As mentioned before, we also assume $(\alpha + 1)/p$ is not an integer. Now let ϕ be a C^∞ function with support in $[1, 2]$ and $\phi(x) > 0$ on $(1, 2)$ and define

$$g_n(x) = 2\phi(-2nx) - \phi(-nx).$$

Then Lemma (3.14) implies that (3.15) holds for this $m(\lambda - k, x)$ and g_n. Now

$$\int_{-\infty}^{\infty} |g_n^{(k)}(x)|^p |x|^\alpha dx = Cn^{kp-\alpha-1}. \tag{4.3}$$

Also, since $[m(\lambda - k, x)\hat{g}_n(x)]^{\vee} = \check{m}(\lambda - k, x) * g_n(x)$,

$$[m(\lambda - k, x)\hat{g}_n(x)]^{\vee} = \frac{1}{n}\int_1^2 \left[\check{m}\left(\lambda - k, x + \frac{y}{2n}\right) - \check{m}\left(\lambda - k, x + \frac{y}{n}\right)\right]\phi(y)\, dy.$$

By [7, p. 132], since $0 < \lambda - k < 1$,

$$\check{m}(\lambda - k, x) = A|x - 1|^{\lambda-k-1} + o\left(|x - 1|^{\lambda-k-1}\right),$$

where $A \neq 0$. Since $1 \leqslant y \leqslant 2$, for $1 + 1/n \leqslant x \leqslant 1 + 2/n$ we have

$$|x - 1 + y/2n| \leqslant \tfrac{5}{6}|x - 1 + y/n|.$$

It follows that

$$|x - 1 + y/2n|^{\lambda-k-1} - |x - 1 + y/n|^{\lambda-k-1} > C|x - 1 + y/2n|^{\lambda-k-1}$$

and, therefore,

$$|\check{m}(\lambda - k, x + y/2n) - \check{m}(\lambda - k, x + y/n)| > Cn^{k+1-\lambda} \quad \text{for large } n.$$

Thus, for n large and $1 + 1/n \leqslant x \leqslant 1 + 2/n$, we have

$$\left|\left[m(\lambda - k, x)\hat{g}_n(x)\right]^\vee\right| \geqslant Cn^{k-\lambda}$$

with $C > 0$. Therefore, for n large,

$$\int_{-\infty}^{\infty} \left|\left[m(\lambda - k, x)\hat{g}_n(x)\right]^\vee\right|^p |x|^{\alpha - kp}\, dx \geqslant Cn^{p(k-\lambda)-1}. \tag{4.4}$$

Using (4.3) and (4.4) in (3.15) then shows that

$$Cn^{p(k-\lambda)-1} \leqslant Cn^{kp-\alpha-1}$$

for sufficiently large n. We conclude that $\alpha \leqslant p\lambda$ for λ not an integer.

If λ is an integer, choose a sequence $\{\lambda_n\}$ with $\lambda < \lambda_n < \lambda + 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$. If (1.3) holds for all f in $\mathscr{S}_{0,0}$ and m in $M(s, \lambda)$, then by Theorem (2.3) it also holds for all f in $\mathscr{S}_{0,0}$ and m in $M(s, \lambda_n)$. By the part already proved, $\alpha \leqslant \lambda_n p$. Since this is true for all n, we obtain $\alpha \leqslant \lambda p$.

Next we shall prove that $\alpha \leqslant -1 + p(\lambda + \frac{1}{2})$. As in the proof that $\alpha \leqslant p\lambda$, we need only prove this for λ not an integer. As before, we may also assume $\alpha \leqslant -1 + p(1 + \lambda)$. We will also assume that $\alpha > -1 + p(\lambda + \frac{1}{2})$; from this we will derive a contradiction and conclude that $\alpha \leqslant -1 + p(\lambda + \frac{1}{2})$.

For $0 < \lambda < 1$, define

$$m(\lambda, x) = \left[|x|^{-2\lambda}\sin(x^2)\right]^\wedge.$$

We will use Lemma (3.14) with $k = [\lambda]$. Since $\alpha > -1 + p(\lambda + \frac{1}{2})$, we have $\alpha > -1 + pk$. We must verify that $m(\lambda, x)$ satisfies the hypotheses of Lemma (3.5). By Lemma (3.22), $|m(\lambda, x)| \leqslant C(1 + |x|)^{\max(-1,-2\lambda)}$. Since $\max(-1, -2\lambda) \leqslant -\lambda$, this implies that $|m(\lambda, x)| \leqslant C(1 + |x|)^{-\lambda}$. Similarly, since $\partial m(\lambda, x)/\partial x = [|x|^{1-2\lambda}\sin(x^2)\operatorname{sgn} x]^\wedge$, Lemma (3.22) implies

$$\left|\frac{\partial}{\partial x} m(\lambda, x)\right| \leqslant C(1 + |x|)^{1-2\lambda}, \tag{4.5}$$

which is bounded for $|x| < 1$.

To show that $m(\lambda, x)$ is in $M(\infty, \lambda)$, start with the fact that $D^\lambda m(\lambda, x) = [x^\lambda |x|^{-2\lambda}\sin(x^2)]^\wedge$. By taking a linear combination of the functions in Lemma (3.22) with $b = -\lambda$, we see that

$$|D^\lambda m(\lambda, x)| \leqslant C(1 + |x|)^{-\lambda} \tag{4.6}$$

for all x. By Lemma (2.6) then with I equal to $[2^{j-1}, 2^{j+1}]$ or $[-2^{j+1}, -2^{j-1}]$, $\phi(x) = \chi_I(x)\psi(2^{-j}x)$ and $f(x) = m(\lambda, x)$ we get

$$\left\|D^\lambda \chi_I(x) m_j(\lambda, x)\right\|_\infty \leqslant C\left[\left\|\chi_I(x) D^\lambda m(\lambda, x)\right\|_\infty + 2^{-j\lambda}\left\|m(\lambda, x)\right\|_\infty\right].$$

These inequalities combined with (4.6) and the boundedness of $m(\lambda, x)$ show that $m(\lambda, x)$ is in $M(\infty, \lambda)$. Therefore, $m(\lambda, x)$ satisfies the hypothesis of Lemma (3.5).

We will now define the function $g(x)$ to be used in Lemma (3.14). Let $\phi(x)$ be a C^∞ function with support in $[0, 1]$ which is positive in $(0, 1)$. We will consider two cases. If $\alpha < -1 + p(k + 1)$, define

$$g(x) = \phi(x) - n\phi(nx). \tag{4.7}$$

The inequality $\alpha < -1 + p(k+1)$ implies

$$\int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha \, dx \leqslant Cn^{(k+1)p-\alpha-1}. \tag{4.8}$$

The rest of the proof will consist of showing that for n sufficiently large,

$$\int_{-\infty}^{\infty} \left|[m(\lambda - k, x)\hat{g}(x)]^\vee\right|^p |x|^{\alpha - kp} \, dx \geqslant Cn^y, \tag{4.9}$$

where $y = (k - 2\lambda)p + \alpha + 1$ and C is independent of n. Combining (3.15), (4.8) and (4.9) shows that $(k - 2\lambda)p + \alpha + 1 \leqslant (k+1)p - \alpha - 1$, and this contradicts the assumption $\alpha > -1 + p(\lambda + \frac{1}{2})$.

To prove (4.9), start with the fact that for $x > 1$

$$[m(\lambda - k, x)\hat{g}(x)]^\vee = C\int_{-\infty}^{\infty} g(t)|x - t|^{2k-2\lambda} \sin(x - t)^2 \, dt.$$

Since $\int_{-\infty}^{\infty} g(t)\, dt$ is 0, the right side equals

$$C\int_{-\infty}^{\infty} g(t)\Big[|x - t|^{2k-2\lambda}\sin(x - t)^2 - |x|^{2k-2\lambda}\sin x^2\Big] \, dt. \tag{4.10}$$

Then by the definition of g, $[m(\lambda - k, x)\hat{g}(x)]^\vee$ equals a constant times the sum of

$$q(x) = \int_0^1 \phi(t)|x - t|^{2k-2\lambda}\sin(x - t)^2 \, dt,$$

$$r(x) = -|x|^{2k-2\lambda}\sin x^2 \int_0^1 \phi(t)\, dt$$

and

$$s(x) = -n\int_0^{1/n} \phi(nt)\Big[|x - t|^{2k-2\lambda}\sin(x - t)^2 - |x|^{2k-2\lambda}\sin x^2\Big] \, dt.$$

We shall show that if $0 < a < 1$ and $n > 3/a$, then

$$\int_2^{an} |q(x)|^p x^{\alpha - kp} \, dx \leqslant C_1, \tag{4.11}$$

$$\int_2^{an} |r(x)|^p x^{\alpha - kp} \, dx \geqslant C_2(an)^y \tag{4.12}$$

and

$$\int_2^{an} |s(x)|^p x^{\alpha - kp} \, dx \leqslant C_3 a^p (an)^y, \tag{4.13}$$

where C_1, C_2 and C_3 are independent of n and a. By choosing a so that $C_3 a^p < \frac{1}{2}C_2$, it is clear that these imply (4.9) provided $y > 0$. To show that $y > 0$, compare the end terms of the inequalities

$$-1 + p(\lambda + \tfrac{1}{2}) < \alpha < -1 + p(k+1)$$

to see that $k > \lambda - \frac{1}{2}$. This lower bound for k and the fact that $\alpha > -1 + p(\lambda + \frac{1}{2})$ imply $y > 0$.

To prove (4.11), integrate the definition of $q(x)$ by parts to get

$$q(x) = \int_0^1 \phi'(t)\int_0^t |x - u|^{2k-2\lambda}\sin(x - u)^2 \, du\, dt.$$

Assume that $x > 2$ and make the change of variables $x - u = \sqrt{w}$ in the inner integral to get

$$q(x) = \frac{1}{2}\int_0^1 \phi'(t)\int_{(x-t)^2}^{x^2} w^{k-\lambda-1/2}\sin w\, dw\, dt.$$

Now apply Lemma (3.25) to the inner integral; this shows that $|q(x)| \leqslant Cx^{2k-2\lambda-1}$, and (4.11) follows since $\alpha < -1 + p(k+1)$ implies $p(2k - 2\lambda - 1) + \alpha - kp < -1$.

Inequality (4.12) is immediate. To prove (4.13), use the facts that for $0 \leqslant t \leqslant 1/n$ and $x > 2$,

$$\left|\left[(x-t)^{2k-2\lambda} - x^{2k-2\lambda}\right]\sin(x-t)^2\right| \leqslant Ctx^{2k-2\lambda-1} \tag{4.14}$$

and

$$\left|x^{2k-2\lambda}\left[\sin(x-t)^2 - \sin x^2\right]\right| \leqslant Ctx^{2k-2\lambda+1}. \tag{4.15}$$

These show that $|s(x)| \leqslant (c/n)x^{2k-2\lambda+1}$ for $x > 2$, and (4.13) follows. This completes the proof that $\alpha \leqslant -1 + p(\lambda + \frac{1}{2})$ for the case $\alpha < -1 + p(k+1)$.

For the case $\alpha > -1 + p(k+1)$, let

$$g(x) = n\phi(nx) - n^2\phi(n^2x).$$

This condition on α implies that (4.8) also holds for this g and we will prove (4.9). To do this, use the fact that $[m(\lambda - k, x)\hat{g}(x)]^\vee$ equals (4.10) to write it as the sum of constants times

$$Q(x) = \int_0^{1/n} n\phi(nt)\left[|x-t|^{2k-2\lambda} - |x|^{2k-2\lambda}\right]\sin(x-t)^2\, dt,$$

$$R(x) = |x|^{2k-2\lambda}\int_0^{1/n} n\phi(nt)\left[\sin(x-t)^2 - \sin x^2\right] dt$$

and

$$S(x) = \int_0^{1/n^2} n^2\phi(n^2t)\left[|x-t|^{2k-2\lambda}\sin(x-t)^2 - |x|^{2k-2\lambda}\sin x^2\right] dt.$$

The inequalities to be proved are

$$\left[\int_2^{an} |Q(x)|^p x^{\alpha-kp}\, dx\right]^{1/p} \leqslant \frac{C_1}{n}, \tag{4.16}$$

$$\left[\int_2^{an} |R(x)|^p x^{\alpha-kp}\, dx\right]^{1/p} \geqslant \frac{C_2}{n}(an)^{k-2\lambda+1+(\alpha+1)/p} \tag{4.17}$$

and

$$\left[\int_2^{an} |S(x)|^p x^{\alpha-kp}\, dx\right]^{1/p} \leqslant \frac{C_3}{n^2}(an)^{k-2\lambda+1+(\alpha+1)/p} \tag{4.18}$$

for $0 < a < 1$ and $n > 3/a$; as in the last case these are sufficient to prove (4.9) since $\alpha > -1 + p(k+1)$ implies $k - 2\lambda + 1 + (\alpha+1)/p > 0$.

For (4.16), use (4.14) to show that $|Q(x)| \leqslant (C/n)x^{2k-2\lambda-1}$ for $x > 2$. This and the fact that $\alpha < -1 + p(\lambda + 1)$ imply (4.16). For (4.17), use a trigonometric identity to show that

$$R(x) = -|x|^{2k-2\lambda}\int_0^{1/n} n\phi(nt)2\cos\left(x^2 - xt + \frac{t^2}{2}\right)\sin\left(xt - \frac{t^2}{2}\right) dt.$$

Now if $0 \leqslant t \leqslant 1/n$ and $1 \leqslant x \leqslant n$, then $\sin(xt - t^2/2) \geqslant Cxt$, and if we also have $2j\pi \leqslant x^2 \leqslant (2j + \frac{1}{3})\pi$, then $\cos(x^2 - xt + t^2/2) \geqslant C > 0$. Therefore, if $j \geqslant 1$ and $\sqrt{2j\pi} \leqslant x \leqslant \sqrt{(2j + \frac{1}{3})\pi} \leqslant n$, we have $|R(x)| \geqslant (C/n)x^{2k-2\lambda+1}$. Inequality (4.17) follows from this. For (4.18), use (4.14) and (4.15) to show that $|S(x)| \leqslant (C/n^2)x^{2k-2\lambda+1}$ for $|x| \geqslant 2$. This completes the proof that $\alpha \leqslant -1 + \lambda(p + \frac{1}{2})$ and completes the proof of the upper bounds in Theorem (1.4).

5. Completion of the proof of Theorem (1.4). This will be done in three parts. The proof that $\alpha > -1$ is done directly. This result and a duality argument based on the upper bounds proved in §4 then prove that $\alpha \geqslant \max(-p\lambda, -1 + p(-\lambda + \frac{1}{2}))$. Finally, we show that $(\alpha + 1)/p$ cannot be a positive integer.

To prove $\alpha > -1$, we will show for every integer k that $-kp < \alpha \leqslant -1$ is impossible. To do this, fix k and α and choose g in C^∞ with support in $[1, 2]$ such that $\int_1^2 g(x)\,dx = 1$ and $\int_1^2 x^j g(x)\,dx = 0$ for $1 \leqslant j \leqslant k$; the existence of such a g is shown in Lemma 2.6, p. 182, of [**1**]. Define $f(x) = \check{g}(x) - \check{g}(-x)$. Then $f(x)$ is in $\mathscr{S}_{0,0}$ and $f^{(j)}(0) = 0$ for $0 \leqslant j \leqslant k$. Therefore,

$$\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha\,dx < \infty. \tag{5.1}$$

Since (1.3) is assumed to hold for f in $\mathscr{S}_{0,0}$ and all m in $M(s, \lambda)$, it holds in particular for this $f(x)$ and $m(x) = i \operatorname{sgn} x$. Therefore, (1.3) and (5.1) imply that

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p |x|^\alpha\,dx < \infty, \tag{5.2}$$

where

$$\tilde{f}(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{|t| > \varepsilon} \frac{f(x - t)}{t}\,dt$$

is the Hilbert transform of f.

We will now show that (5.2) cannot hold. To do this, define

$$G(x) = \int_{-\infty}^{x} [g(t) - g(-t)]\,dt.$$

Since g has support in $[1, 2]$ and $g(t) - g(-t)$ has integral 0, $G(x)$ has support in $[-2, 2]$. Therefore, G is integrable and

$$x\check{G}(x) = i[\check{g}(x) - \check{g}(-x)] = if(x).$$

From this we see that

$$G(0) = i\int_{-\infty}^{\infty} \frac{f(t)}{t}\,dt$$

and

$$\tilde{f}(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(-t)}{t}\,dt = \frac{i}{\pi}G(0) = \frac{-i}{\pi}.$$

Since $\tilde{f}$ is in C^∞ and $\alpha \leqslant -1$, this shows that $\int_{-\infty}^{\infty} |\tilde{f}(x)|^p |x|^\alpha\,dx = \infty$ and contradicts (5.2). This completes the proof that $\alpha > -1$.

To prove that $\alpha \geqslant \max(-p\lambda, -1 + p(-\lambda + \frac{1}{2}))$, we may assume by the previous part that $\alpha > -1$. We may also assume $\alpha < p - 1$ since if $\alpha \geqslant p - 1$ there is nothing to prove. Now since (1.3) holds for all f in $\mathscr{S}_{0,0}$, Theorem (2.7) shows that (1.3) holds for all f in $L^2 \cap L^p_\alpha$. Now if g is any function in $\mathscr{S}_{0,0}$,

$$\left[\int_{-\infty}^{\infty} |[m(x)\hat{g}(x)]^\vee|^{p'}|x|^{-\alpha/(p-1)}\,dx\right]^{1/p'} = \sup_{f\in A}\left|\int_{-\infty}^{\infty} [m(x)\hat{g}(x)]^\vee f(x)\,dx\right|, \tag{5.3}$$

where A is the set of all f in L^2 with $\int_{-\infty}^{\infty} |f(x)|^p|x|^\alpha\,dx \leqslant 1$. Since f and g are in L^2, the right side equals

$$\sup_{f\in A}\left|\int_{-\infty}^{\infty} g(x)[m(-x)\hat{f}(x)]^\vee\,dx\right|.$$

Now use Hölder's inequality and (1.3) to conclude that

$$\int_{-\infty}^{\infty} |[m(x)\hat{g}(x)]^\vee|^{p'}|x|^{-\alpha/(p-1)}\,dx \leqslant C\int_{-\infty}^{\infty} |g(x)|^{p'}|x|^{-\alpha/(p-1)}\,dx.$$

Since this is true for all g in $\mathscr{S}_{0,0}$ and m in $M(s,\lambda)$, the proof in §4 shows that

$$-\alpha/(p-1) \leqslant \min[p'\lambda, -1 + p'(\lambda + \tfrac{1}{2})].$$

Multiplying by $1 - p$ then gives the asserted inequality.

Finally, we show that $(\alpha + 1)/p$ can not be a positive integer. Since $m(x) = \operatorname{sgn} x$ is in $M(s,\lambda)$ for all $\lambda > 0$ and s satisfying $1 \leqslant s \leqslant \infty$, it is sufficient to show that (1.3) cannot hold for this m if $(\alpha + 1)/p$ is a positive integer. To do this, fix p satisfying $1 < p < \infty$ and a positive integer k. Let $\alpha = -1 + pk$ and $m(x) = \operatorname{sgn} x$ and assume that (1.3) holds for all f in $\mathscr{S}_{0,0}$. We then have for all f in $\mathscr{S}_{0,0}$

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p|x|^\alpha\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p|x|^\alpha\,dx. \tag{5.4}$$

By Theorem (2.7), (5.4) will also hold for all f in $L^p_\alpha \cap Q_{k-2}$.

Now choose $f(x)$ in $L^\infty \cap Q_{k-2}$ with support in $[0,1]$ such that

$$\int_{-\infty}^{\infty} x^{k-1}f(x)\,dx = 1; \tag{5.5}$$

such an f exists by Lemma 2.6, p. 182, of [**1**]. Now since f is in Q_{k-2}, we see that

$$\int_{-\infty}^{\infty} \frac{x^{k-1} - t^{k-1}}{x - t} f(t)\,dt = 0;$$

therefore, for $x > 2$

$$x^{k-1}\tilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^{k-1}f(t)}{x - t}\,dt.$$

This and (5.5) show that for x large we have $|\tilde{f}(x)| \geqslant x^{-k}/2\pi$ and that, therefore, the left side of (5.4) is infinite. Since the right side of (5.4) is finite, we have a contradiction. This completes the proof that $(\alpha + 1)/p$ is not a positive integer.

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