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STOPPING TIMES AND Γ-CONVERGENCE

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ABSTRACT. The equation $\partial u/\partial t = \Delta u - \mu u$ represents diffusion with killing. The strength of the killing is described by the measure μ, which is not assumed to be finite or even σ-finite (to illustrate the effect of infinite values for μ, it may be noted that the diffusion is completely absorbed on any set A such that $\mu(B) = \infty$ for every nonpolar subset B of A). In order to give rigorous mathematical meaning to this general diffusion equation with killing, one may interpret the solution u as arising from a variational problem, via the resolvent, or one may construct a semigroup probabilistically, using a multiplicative functional. Both constructions are carried out here, shown to be consistent, and applied to the study of the diffusion equation, as well as to the study of the related Dirichlet problem for the equation $\Delta u - \mu u = 0$. The class of diffusions studied here is closed with respect to limits when the domain is allowed to vary. Two appropriate forms of convergence are considered, the first being γ-convergence of the measures μ, which is defined in terms of the variational problem, and the second being stable convergence in distribution of the multiplicative functionals associated with the measures μ. These two forms of convergence are shown to be equivalent.

1. Let D be an open set in R^d, $d \geq 2$. Let $\mathcal{M}_0$ be the class of nonnegative measures, not necessarily σ-finite, which do not charge polar sets. For each μ in $\mathcal{M}_0$, we wish to consider two problems:

Problem 1. Find the solution u of the *μ-Dirichlet problem* on D with data g on ∂D, that is:

$$-\Delta u + \mu u = 0 \quad \text{on } D, \tag{1.1}$$

$$u = g \quad \text{on } \partial D. \tag{1.2}$$

For brevity, we will say that a solution of (1.1) is *μ-harmonic* on D. (This usage is not related to the notion of h-harmonic functions, as defined, for example, in [**12**, VIII.1].)

Problem 2. Find the solution $v\colon (0,\infty) \times R^d \to R$ of the *μ-diffusion equation* with initial data equal to some measure ν on R^d, that is:

$$\partial v/\partial t = \Delta v - \mu v \quad \text{on } (0,\infty) \times R^d, \tag{1.3}$$

$$\lim_{t \downarrow 0} v(t,\cdot) = \nu \quad \text{in distribution sense.} \tag{1.4}$$

We could generalize Problem 2 to the case that v satisfies a boundary condition

$$v(t,\cdot) = 0 \quad \text{on } D^c \text{ for all } t, \tag{1.5}$$

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but this problem is really included in the previous form of Problem 2 for an appropriate choice of μ, as we shall see.

Naturally, it is necessary to give a precise meaning to equations (1.1) and (1.3) when μ is a general measure. When μ has a density in the Kato class [**1**], not necessarily positive, the usual probabilistic Feynman-Kac method [**1**] can be applied to solve Problems 1 and 2, and in present case, in which μ is required to be positive, the same approach is easily extended to general μ (§4). However, the partial differential equations (1.1) and (1.3) no longer hold in this general case, in the usual distribution sense [**8**]. One can of course declare the functions appearing in the Feynman-Kac formulae to be solutions, but this is now a definition rather than a theorem, since the equations have not been given a meaning independent of the solution method. However, even for general μ, an interpretation for (1.1) was given in [**8** and **9**], by defining u as the solution of a variational problem (§2). The measure μ in this case represents a penalization on the solution. It was shown in [**8**] that this general form of Problem 1 provides an appropriate framework for studying limits of solutions of Dirichlet problems in varying domains with "holes" (cf. [**7**, **16**–**19**]). Equation (1.3) can also be interpreted variationally, in terms of the resolvent family associated with $-\Delta + \mu$. In the present paper we will consider this formulation, and at the same time develop the probabilistic interpretation for both Problems 1 and 2. The solution of (1.3) will be defined (§4) using an appropriate *multiplicative functional* $M(\mu)$ associated with μ for each μ in $\mathcal{M}_0$. A multiplicative functional is a special type of *randomized stopping time* (§3). We will show (§4) that this functional gives the same semigroup as the variational approach to (1.3), and (§6) that the usual probabilistic formula (6.11) gives the solution to the variational form of Problem 1. The proof that the two methods of solutions are consistent is based on the general connection between γ-convergence and stable convergence described below. We will also (Theorem 6.2 and Lemma 6.3) give two criteria for the *Dirichlet regularity* of a point for a μ-harmonic function.

It should be noted that, when measures μ which are not Radon are used, two distinct measures μ_1 and μ_2 can induce the same variational solutions for Problems 1 and 2. Thus we define (§2) two measures μ_1 and μ_2 to be *equivalent* if this is the case. Because of the variational formulation of these problems, we may express the equivalence of μ_1 and μ_2 more briefly by requiring that

$$\int u^2\,d\mu_1 = \int u^2\,d\mu_2, \tag{1.6}$$

for all functions u in $H^1(R^d)$.

We will show (Lemma 4.1) that μ_1 and μ_2 are equivalent if and only if $\mu_1(V) = \mu_2(V)$ for every finely open set V in R^d.

The study of limits of solutions of Problem 1 in varying domains [**8**] referred to earlier, was carried out using the notion of γ-*convergence* of measures (§2). In particular, it was shown in [**8**] that the space of measures is compact with respect to γ-convergence, and that finite measures with smooth densities are dense in $\mathcal{M}_0$ with respect to γ-convergence. A similar analysis can be carried out for Problem 2, using the resolvent family, and we shall show this in §4. At the same time, we will develop the connection between γ-convergence of measures and *stable convergence* of stopping times. This latter convergence was applied in [**4** and **5**] to study the limits of diffusions in varying regions with holes, using, in particular, the fact that the space of stopping times is compact with respect to stable convergence [**3**]. In the present paper we will show (Theorem 3.2) that the space of multiplicative functionals is compact with respect to stable convergence. Furthermore, we show (Theorem 4.2) that a sequence (μ_n) γ-converges to μ if and only if the associated sequence of multiplicative functionals $M(\mu_n)$ converges stably to $M(\mu)$. We thus obtain a probabilistic interpretation of γ-convergence of measures. Whereas in the analytical theory of γ-convergence, the convergence is defined in terms of functionals on L^2-spaces, the probabilistic notion is expressed as a weak convergence for associated probability measures on an appropriate sample space.

In §3 we develop some general facts concerning stable convergence (Lemma 3.1, Theorem 3.1). As one application, in Theorem 3.3 we show the relation between stable convergence of multiplicative functionals and stong resolvent convergence for the associated semigroups. This enables us to show the correspondence between γ-convergence and stable convergence, since, in Theorem 2.1, γ-convergence is also characterized in terms of resolvent convergence.

2. In this section we shall approach Problems 1 and 2 of §1 analytically, from the variational standpoint. The results we state without proof are for the most part proved in [**8** and **9**], and we shall follow the terminology of those papers. We will give a precise meaning to the weak inhomogeneous boundary value problem (2.1), (2.2), and thence to the resolvent operator. The resolvent operator would provide an indirect route to the definition of diffusion with a general killing measure, i.e. to Problem 2, but we will use a more direct probabilistic approach in §§3 and 4 to accomplish the same task. After proving some convergence and regularity results for solutions of (2.1), (2.2), and in particular for resolvents, we introduce the idea of γ-convergence of measures (Definition 2.7). As we show in §4, this form of convergence is entirely parallel to the stable convergence defined probabilistically in §3, so we will develop many of our later convergence results in terms of stable convergence. The link between the two forms of convergence is made through the convergence of the resolvent operator.

Before proceeding we note some terminology. A *measure* will mean as usual a countably additive set function, taking values in $[0, \infty]$, and so not necessarily finite. In particular a measure is *nonnegative* unless explicitly stated to be a signed measure. We may at times refer to a measure as nonnegative for emphasis. (As a convenient brief notation, we will denote by $f\mu$ the measure ν such that $d\nu = f\,d\mu$, and we will write $\mathbf{1}_C$ for the indicator of a set C.) A set of classical capacity zero is a *polar* set, and a property that is true except on a polar set will be said to hold *quasi everywhere* or q.e. $L^2(D, \mu)$ denotes the measurable functions on D which are square-integrable with respect to μ on D. Let m denote Lebesgue measure on R^d. When integrating we will also denote $m(dx)$ simply by dx. We will write $L^2(D, m)$ as $L^2(D)$, and sometimes write $\|\cdot\|_{L^2(D)}$ as $\|\cdot\|$ when the meaning is clear (we will also use $\|\psi\|$ in later sections to denote the total variation norm of a signed measure ψ). We will often consider functions in the Sobolev space $H^1(D)$ (cf. [**19**]), where D is an open set in R^d, by which is meant the space of functions in $L^2(D)$ with distributional first derivatives in $L^2(D)$. $H^1(D)$ is a Banach space with norm $\|\cdot\|_{H^1(D)}$ given by

$$\|f\|_{H^1(D)} = (\|f\|^2_{L^2(D)} + \|\nabla f\|^2_{L^2(D)})^{1/2}.$$

$H^1(D)$ is closed under finite lattice operations (cf. [**19**, Appendix A of II]). Functions in $H^1(D)$ are given quasi everywhere. More precisely, for any function $v \in H^1(D)$, $\lim_{r\downarrow 0} \int_{B_r(x)} v(y)\,dy/m(B_r(x))$ exists and is finite for quasi every x in D. Here $B_r(x)$ denotes the open ball with center x and radius r. We will adopt the following convention concerning the pointwise values of a function v in $H^1(D)$: for every $x \in D$ we will always require that

$$\liminf_{r\downarrow 0} \int_{B_r(x)} \frac{v(y)\,dy}{m(B_r(x))} \le v(x) \le \limsup_{r\downarrow 0} \int_{B_r(x)} \frac{v(y)\,dy}{m(B_r(x))}.$$

With this convention the pointwise value $v(x)$ is determined quasi everywhere in D, and the function v is quasi continuous in D, i.e. for any $\varepsilon > 0$ there exists an open set U of capacity less than ε such that the restriction of v to $D - U$ is continuous. We denote by $H_0^1(D)$ the closure in $H^1(D)$ of the smooth functions with compact support in D. Intuitively this is the class of functions in $H^1(D)$ that vanish at the boundary.

There is a close connection between $H^1(D)$ and the space of charges with finite self-energy familiar from classical potential theory. Let G be the classical potential operator on R^d defined in [**12**, 1.I.5], so that $G\mu$ is the Newtonian potential of μ if $d = 3$, and $G\mu$ is the logarithmic potential of μ if $d = 2$. More generally, if D is any Green region [**12**, 1.II.13] let G^D be the classical Green potential operator on D [**12**, 1.VII.1], and let $[\mu, \nu]_D \equiv \int G^D\mu\,d\nu$ denote the corresponding energy inner product [**12**, 1.XIII.3]. It is a simple matter to show that if ψ is a bounded signed measure on D, with $[|\psi|, |\psi|]_D < \infty$, then $[\psi, \psi]_D = \int |\nabla(G^D\psi)|^2 dm$, and if $G^D\psi$ is also in $L^2(D)$, then $G^D\psi \in H_0^1(D)$ (cf. also [**20**, 1.4 and VI.1]).

We now consider an inhomogeneous version of Problem 1. Let D be an open set in R^d, μ a member of the class $\mathcal{M}_0$ defined in §1, so that μ is a measure that does not charge polar sets but may be infinite on nonpolar sets. Let $f \in L^2(D)$, $g \in H^1(D)$. We consider a solution u of

$$-\Delta u + \mu u = f \quad \text{in } D, \tag{2.1}$$

$$u = g \quad \text{on } \partial D. \tag{2.2}$$

In order to interpret these equations rigorously, we make the following definitions.

DEFINITION 2.1. A function $u \in H^1_{\mathrm{loc}}(D) \cap L^2_{\mathrm{loc}}(D, \mu)$ will be called a *local weak solution* of (2.1) if

$$\int_D \nabla u \cdot \nabla v\,dx + \int_D uv\,d\mu = \int_D fv\,dx \tag{2.3}$$

for every $v \in H^1(D) \cap L^2(D, \mu)$ with support v compact in D. When $f = 0$ we will also say that u is *μ-harmonic* on D. A local weak solution u of (2.1) will be called a *weak solution* of the boundary problem (2.1), (2.2) if

$$u - g \in H_0^1(D). \tag{2.4}$$

(Of course, (2.4) implies that $u \in H^1(D)$.)

We note that unless μ is a Radon measure (that is, $\mu(K) < \infty$ for every compact set K), the weak solutions just defined are *not* solutions in the distribution sense in D (see [**8**, Remark 3.9]). However if $\mu \in \mathcal{M}_0$ is Radon, it is proved in [**8**]

(Proposition 3.8) that u is a local weak solution of (2.1), according to Definition 2.1, if and only if

$$u \in H^1_{\mathrm{loc}}(D) \cap L^2_{\mathrm{loc}}(D, \mu) \tag{2.5}$$

and u is a solution of equation (2.1) in the sense of distributions, that is

$$\int_D \nabla u \cdot \nabla \varphi \, dx + \int_D u\varphi \, d\mu = \int_D f\varphi \, dx \quad \text{for every } \varphi \in C_0^\infty(D). \tag{2.6}$$

The solutions of (2.1), (2.2) can be characterized in variational terms as follows.

PROPOSITION 2.1. *Let D be any open set in R^d. Let $f \in L^2(D)$ be given and let $g \in H^1(D)$ be given such that there exists some $w \in H^1(D) \cap L^2(D, \mu)$ with $w - g \in H_0^1(D)$. Then u is a weak solution of* (2.1), (2.2) *if and only if u is the* (*unique*) *minimum point of the functional*

$$F(v) = \int_D |\nabla v|^2 dx + \int_D v^2 d\mu - 2\int_D fv \, dx \tag{2.7}$$

on the set $\{v\colon v \in H^1(D), v - g \in H_0^1(D)\}$. Moreover, $u \in H^1(D) \cap L^2(D, \mu)$ and condition (2.3) *holds for every $v \in H_0^1(D) \cap L^2(D, \mu)$. Furthermore, if D is bounded, such a solution u exists for arbitrary $\mu \in \mathcal{M}_0$. If D is unbounded, u exists for every $\mu \in \mathcal{M}_0$, such that $\mu \geq \lambda m$, where m denotes Lebesgue measure in R^d and λ is any positive constant.*

For D bounded, the proof can be found in [**9**, Theorem 2.4] and Proposition 2.5. The same proof can be adapted to the case D unbounded.

Note that in (2.7) the integral $\int_D v^2 d\mu$ is well defined, because $v \in H^1(D)$ can be specified up to sets of capacity zero and these sets have μ measure zero.

Let us introduce a special class of measures $\mu \in \mathcal{M}_0$, corresponding to homogeneous Dirichlet conditions on Borel sets of R^d.

DEFINITION 2.2. For any Borel set E let ∞_E denote the measure which is $+\infty$ on all nonpolar Borel subsets of E, and 0 on every Borel subset of E^c and on every polar set.

The boundary problem (2.1), (2.2), with $g \equiv 0$ on ∂D, can be formulated as an equation of the form (2.1) in R^d, provided we replace the measure μ with the measure $\mu + \infty_E$, with $E = D^c$.

PROPOSITION 2.2. *Let D be an open set in R^d, $f \in L^2(D)$, $\mu \in \mathcal{M}_0$. Then u is a weak solution of the boundary problem*

$$-\Delta u + \mu u = f \quad \text{in } D, \tag{2.8}$$

$$u = 0 \quad \text{on } \partial D \tag{2.9}$$

(*in particular, $u \in H_0^1(D)$*), *if and only if $u = U|_D$, where U is a weak solution of the equation*

$$-\Delta U + (\mu + \infty_E)U = f \quad \text{in } R^d, \text{with } E = D^c. \tag{2.10}$$

PROOF. Let us first recall that $V \in H^1(R^d)$ and $V = 0$ q.e. on $E = D^c$ implies that $V|_D \in H_0^1(D)$; see e.g. J. Deny [**11**], L. Hedberg [**16**].

Let u be a solution of (2.8), (2.9). By Proposition 2.1, $u \in H_0^1(D) \cap L^2(D, \mu)$ and (2.3) holds for every $v \in H_0^1(D) \cap L^2(D, \mu)$. Let $U = u$ in D, $U = 0$ in $E = D^c$.

Then $U \in H^1(R^d) \cap L^2(R^d, \mu + \infty_E)$. Now let $V \in H^1(R^d) \cap L^2(R^d, \mu + \infty_E)$ with compact support; since $V = 0$ q.e. on E, we have $V|_D \in H_0^1(D)$, and therefore

$$\int_{R^d} \nabla U \nabla V \, dx + \int_{R^d} UV \, d(\mu + \infty_E) = \int_{R^d} fV \, dx,$$

and hence U is a weak solution of (2.10).

Now let U be a solution of (2.10) and let $u = U|_D$. Since $U \in L^2(R^d, \mu + \infty_E)$ we have $U = 0$ q.e. on E, thus $u \in H_0^1(D)$ and u is a weak solution of (2.8), so the proposition is proved.

In view of Proposition 2.1, with every $\mu \in \mathcal{M}_0$ we associate the family of *resolvent operators* $R_\lambda^D(\mu) = (-\Delta + \mu + \lambda m)^{-1}$, D open $\subset R^d, \lambda \in R^+$, as in the following.

DEFINITION 2.3. Let D be bounded open and $\lambda \geq 0$ or D unbounded open and $\lambda > 0$. Then the operator $R_\lambda^D(\mu)\colon L^2(D) \to L^2(D)$ is defined to be the mapping that associates with every $f \in L^2(D)$ the (unique) weak solution $u \in H_0^1(D) \cap L^2(D, \mu) \subset L^2(D)$ of the problem

$$-\Delta u + (\mu + \lambda m)u = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where m denotes the Lebesgue measure on R^d.

By Proposition 2.1, the linear operators $R_\lambda^D(\mu)$ are well defined and continuous with

$$\|R_\lambda^D(\mu)\| \leq (\lambda + \lambda_1(\mu, D))^{-1}, \tag{2.11}$$

where

$$\lambda_1(\mu, D) = \inf \left[\left(\int_D |\nabla v|^2 dx + \int_D v^2 d\mu \right) \Big/ \int_D v^2 dx \right]$$

(note that $\lambda_1(\mu, D) > 0$ if D is bounded).

This is easily proved by taking $v = u$ in (2.3), where μ has been replaced by $\mu + \lambda m$, giving

$$\int_D |\nabla u|^2 dx + \int_D u^2 d\mu + \lambda \int_D u^2 dx \leq \|f\|_{L^2(D)} \|u\|_{L^2(D)},$$

hence

$$[\lambda + \lambda_1(\mu, D)] \int_D u^2 dx \leq \|f\|_{L^2(D)} \|u\|_{L^2(D)},$$

which implies (2.11).

Let us also remark that the range of $R_\lambda^D(\mu)$ is dense in $H_0^1(D) \cap L^2(D, \mu)$ with respect to the norm

$$\left[\int_D |\nabla u|^2 dx + \int_D u^2 d\mu + \int_D u^2 dx \right]^{1/2}.$$

If not, there exists $v \in H_0^1(D) \cap L^2(D, \mu)$, $v \neq 0$ such that

$$\int_D \nabla u \cdot \nabla v \, dx + \int_D uv \, d\mu + \lambda \int_D uv \, dx = 0$$

for every $u = R_\lambda^D(\mu) f$ and every $f \in L^2(D)$. Taking Definition 2.3 into account, this implies $\int_D fv \, dx = 0$ for every $f \in L^2(D)$, giving a contradiction.

For every $f \in L^2(R^d)$ and every open subset D of R^d we now define $R_\lambda^D(\mu)f$ to be $R_\lambda^D(\mu)$ applied to the restriction of f to D. We will then define $R_\lambda^D(\mu)f$ to be zero outside D, so that $R_\lambda^D(\mu)f$ is defined on all of R^d when convenient.

The following *comparison principle* holds, for local weak solutions of equation (2.1) (see [**9**, Theorem 2.10]):

PROPOSITION 2.3. *Let $\mu_1, \mu_2 \in \mathcal{M}_0$ and let u_1, u_2 be local weak solutions of the equations*

$$-\Delta u_1 + \mu_1 u_1 = f_1 \quad \text{in } D, \qquad -\Delta u_2 + \mu_2 u_2 = f_2 \quad \text{in } D,$$

where D is an open set in R^d. If $\mu_1 \le \mu_2$ as measures on D, $0 \le f_2 \le f_1$ on D, and $0 \le u_2 \le u_1$ on ∂D, then $0 \le u_2 \le u_1$ quasi everywhere in D.

We recall that for $u, v \in H^1_{\mathrm{loc}}(D)$ we say that $u \le v$ on ∂D if and only if $(v - u) \wedge 0 \in H^1_0(D)$.

COROLLARY. *Let D_1 and D_2 be open sets in R^d, $D_1 \subset D_2 \subset D$, $\mu \in \mathcal{M}_0$, $f \in L_2(D)$, $f \ge 0$. Then $R_\lambda^{D_1}(\mu)f \le R_\lambda^{D_2}(\mu)f$ quasi everywhere on R^d, provided $\lambda \ge 0$ and D is bounded or $\lambda > 0$ and D is unbounded.*

Resolvents on unbounded regions can be approximated by resolvents on bounded domains, according to

LEMMA 2.1. *Let D_n, D be open sets in R^d, with $D_n \uparrow D$. Let $\mu \in \mathcal{M}_0$, $\lambda > 0$, $f \in L_2(R^d)$. Then $R_\lambda^{D_n}(\mu)f \to R_\lambda^D(\mu)f$ in $L^2(R^d)$ as $n \to \infty$.*

PROOF. By (2.11), the operators $R_\lambda^{D_n}(\mu)$ are uniformly bounded in n from $L^2(R^d)$ into $L^2(R^d)$. Therefore, it suffices to prove the lemma for every f in a dense subset of $L^2(R^d)$. We assume that f has compact support in some D_{n_0}. We may also assume that $f \ge 0$ in D. Let $u_n = R_\lambda^{D_n}(\mu)f$ and $u = R_\lambda^D(\mu)f$. Since $f \ge 0$, u_n converges monotonically upward to a limit w in D. Clearly $u_n \in H^1_0(D_n)$ for each $n \ge n_0$, and hence by (2.7),

$$\begin{aligned}\int_{D_n} |\nabla u_{n_0}|^2 dx + \int_{D_n} u_{n_0}^2 d(\mu + \lambda m) - 2\int_{D_n} f u_{n_0} dx \\ \ge \int_{D_n} |\nabla u_n|^2 dx + \int_{D_n} u_n^2 d(\mu + \lambda m) - 2\int_{D_n} f u_n dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_D |\nabla_{n_0}|^2 dx + \int_D u_{n_0}^2 d(\mu + \lambda m) + 2\int_{D_n} f u_n dx \\ \ge \int_D |\nabla u_n|^2 dx + \lambda \int_D u_n^2 dx.\end{aligned}$$

Thus $\|u_n\|_{H^1(D)}$ is bounded and hence, since $u_n \uparrow w$, $u_n \to w$ weakly in $H^1(D)$.

Let $v \in H^1(D) \cap L^2(D, \mu)$ and let the support of v be compact in D. Let us suppose $v \ge 0$. Let n_1 be such that support $v \subset D_{n_1}$. Then, by (2.3) for each $n \ge \max(n_0, n_1)$,

$$\int_D \nabla u_n \cdot \nabla v \, dx + \int_D u_n v \, d(\mu + \lambda m) = \int_D f v \, dx.$$

By weak convergence in the first term and monotone convergence in the second term, we obtain

$$\int_D \nabla w \cdot \nabla v \, dx + \int_D wv \, d(\mu + \lambda m) = \int_D fv \, dx.$$

Since w is a weak limit of elements in $H_0^1(D)$, then $w \in H_0^1(D)$. Since $w \leq u$, $w \in L^2(D, \mu)$. Hence, by the uniqueness part of Proposition 2.1, $u = w$. This proves Lemma 2.1.

REMARK 2.1. It follows in particular from Lemma 2.1 that if $f \geq 0$ a.e. in R^d and if the functions $R_\lambda^{D_n}(\mu)f \in H^1(R^d)$ are pointwise defined in R^d according to the convention mentioned above, then $R_\lambda^{D_n}(\mu)f \uparrow R_\lambda^D(\mu)f$ q.e. in R^d.

With each $\mu \in \mathcal{M}_0$ and each open set D in R^d we associate the following functional $F_\mu^D(V)$ defined on $L^2_{\text{loc}}(D)$ by setting

$$F_\mu^D(v) = \int_D |\nabla v|^2 dx + \int_D v^2 d\mu \quad \text{if } v \in H_0^1(D), \tag{2.12}$$

$$F_\mu^D(v) = +\infty \quad \text{if } v \in L^2(D), \text{ but } v \text{ not in } H_0^1(D). \tag{2.13}$$

If $D = R^d$, we denote the corresponding functional by F_μ.

Since μ does not charge polar sets, the functional F_μ^D is lower semicontinuous in $L^2(D)$.

By Proposition 2.1, knowledge of F_μ^D is sufficient to determine the solution of the μ-Dirichlet problem. Clearly two different measures μ_1, μ_2 may give rise to the same functional. This leads to

DEFINITION 2.4. Two measures $\mu_1, \mu_2 \in \mathcal{M}_0$ are *equivalent*, in which case we write $\mu_1 \sim \mu_2$, if $F_{\mu_1}^D(v) = F_{u_2}^D(v)$ for every open set $D \subset R^d$, and every $v \in L^2(D)$.

Obviously, $\mu_1 \sim \mu_2$ if and only if

$$\int_{R^d} v^2 d\mu_1 = \int_{R^d} v^2 \, d\mu_2 \quad \text{for every } v \in H^1(R^d). \tag{2.14}$$

We will see in Lemma 4.1 that two measures are equivalent if and only if they agree on all finely open sets.

We need the following result from [**8**, Lemma 4.5]:

PROPOSITION 2.4. *For every $\mu \in \mathcal{M}_0$ there exists a nonnegative Radon measure ν, with $\nu \in H^{-1}(R^d)$, and a nonnegative Borel function $q\colon R^d \to [0, \infty]$, such that $\mu \sim q\nu$ in the sense of Definition* 2.4.

We recall that a Radon measure ν on R^d belongs to the space $H^{-1}(R^d)$ (the dual space of $H^1(R^d)$) if there exists a constant $c > 0$ such that

$$\left| \int_{R^d} \varphi \, d\nu \right| \leq c \|\varphi\|_{H^1(R^d)} \tag{2.15}$$

for every $\varphi \in C_0^\infty(R^d)$. We also recall that if ν is a nonnegative Radon measure on R^d which belongs to $H^{-1}(R^d)$, then $\nu \in \mathcal{M}_0$ and (2.15) holds for every $\varphi \in H^1(R^d)$. We define $H^{-1}(D)$ similarly for any D. If D is a Green region, it is easy to see that any signed measure ψ with finite energy is in $H^{-1}(D)$, and a sequence ψ_n which converges in energy norm converges in $H^{-1}(D)$.

We will denote by $\mathcal{M}_1$ the space of measures of the form $q\nu$, with $\nu \in H^{-1}(R^d)$, and q a nonnegative Borel function from R^d to $[0, \infty]$. Proposition 2.4 can now be expressed by saying that for each $\mu \in \mathcal{M}_0$ there exists $\mu_1 \in \mathcal{M}_1$ with $\mu_1 \sim \mu$.

DEFINITION 2.5. Let G^+ denote the operator obtained using the positive part of the classical potential. That is, $G^+ = G$ for $d > 2$, and G^+ contains the singular part of the logarithmic kernel when $d = 2$. Let $\mathcal{M}_2$ denote the space of finite measures μ on R^d such that $G^+\mu$ is bounded and continuous on R^d. It is well known that $G^+\mu$ is continuous if and only if

$$\lim_{\varepsilon \to 0} \sup_{x \in B} \int_{\{|x-y|<\varepsilon\}} k^+(|x-y|)\, \mu(dy) = 0 \tag{2.16}$$

for every bounded set B in R^d, where k denotes the classical potential kernel (see, for instance, [**1**]). It follows easily that if $\mu \in \mathcal{M}_2$ and ν is a measure with $0 \le \nu \le \mu$, then $\nu \in \mathcal{M}_2$.

We may strengthen Proposition 2.4 somewhat:

PROPOSITION 2.5. *For every $\mu \in \mathcal{M}_0$ there exists a measure $\psi \in \mathcal{M}_2$, and a nonnegative Borel function $h\colon R^d \to [0, \infty]$, such that $\mu \sim h\psi$ in the sense of Definition* 2.4.

PROOF. Clearly we may assume μ has support in a bounded ball B. For a measure ν with support in B, $G^+\nu$ will be bounded and continuous on R^d if $G\nu$ is bounded and continuous on B. By Proposition 2.4 there exists a nonnegative Radon measure $\nu \in H^{-1}(R^d)$, and a nonnegative Borel function $q\colon R^d \to [0, \infty]$, such that $\mu \sim q\nu$. Replacing ν by $q_1\nu$, where q_1 is an appropriate Borel function, we may assume that ν is finite supported on B. Since ν is finite, $G\nu$ is finite except on at most a set of capacity zero. Thus $G\nu < \infty$, ν-a.e. It follows from [**12**, 1.V.9], that there exists a sequence $\nu_n \in \mathcal{M}_2$ with $\nu = \sum_{n=1}^{\infty} \nu_n$. By choosing positive constants c_n sufficiently small, we have $\psi \equiv \sum_{n=1}^{\infty} c_n \nu_n$ such that $\psi \in \mathcal{M}_2$. Since $\nu \ll \psi$, the proposition follows.

Measures in $\mathcal{M}_2$ will play an important role in what follows. The next proposition gives one useful property of these measures.

PROPOSITION 2.6. *Let $\mu \in \mathcal{M}_2$, let D be open in R^d, $u \in H^1_{\mathrm{loc}}(D)$, u μ-harmonic on D. Then there exists w, continuous on D, such that $u = w$ quasi everywhere.*

PROOF. Let $x \in D$. Let B be an open ball containing x with compact closure in D, with diameter less than 1. Let $w^{\pm}$ denote the solution of the μ-Dirichlet problem on B with boundary data $u^{\pm}$. Let $v^{\pm}$ denote the solution of the ordinary Dirichlet problem on B with boundary data $u^{\pm}$. $w^{\pm}, v^{\pm}$ exist by Proposition 2.1. Clearly $w^{\pm} - v^{\pm} \in H^1_0(B)$. Hence $w^{\pm} = v^{\pm}$ on ∂B, so by Proposition 2.3 $w^{\pm} \le v^{\pm}$ q.e. on B. By uniqueness for the μ-Dirichlet problem (Proposition 2.1), $u = w^+ - w^-$ on B. Hence $|u| \le v^+ + v^-$ q.e. on B, and hence u is locally bounded on D.

Since $\Delta u = \mu u$ in distribution sense in B (see [**8**, Proposition 3.8]), we have

$$u = G^+(u^+|_B\mu) + G^+(u^-|_B\mu) + h$$

where h is harmonic in B. Since u^+ and u^- are bounded in B, the measures $u^+|_B\mu$ and $u^-|_B\mu$ belong to $\mathcal{M}_2$, therefore the corresponding potentials are continuous, hence u is continuous on B.

PROPOSITION 2.7. *Let D be a bounded open set in R^d, $\mu_n, \mu \in \mathcal{M}_0$ with $\mu_n \uparrow \mu$ as $n \to \infty, g \in H^1(D) \cap L^2(D,\mu)$, $g \geq 0$, $f \in L^2(D)$, $f \geq 0$. Let u_n, u denote the solutions, in the sense of Definition 2.1, of*

$$-\Delta u + \mu_n u_n = f \quad \text{in } D, \qquad u_n = g \quad \text{on } \partial D,$$
$$-\Delta u + \mu u = f \quad \text{in } D, \qquad u = g \quad \text{on } \partial D.$$

Then $u_n \downarrow u$ quasi everywhere as $n \to \infty$, and $\|u_n - u\|_{H^1(D)} \to 0$ as $n \to \infty$.

PROOF. By Proposition 2.3, $u_n \downarrow$ and $u_n \geq u \geq 0$ for all n. Let $w = \lim_{n\to\infty} u_n$. By (2.12), (2.13), and Proposition 2.1, for every $n \geq m$ we have

$$\begin{aligned} F_\mu^D(u) - 2\int_D fu\,dx \geq F_{\mu_n}^D(u) - 2\int_D fu\,dx \geq F_{\mu_n}^D(u_n) - 2\int_D fu_n\,dx \\ \geq F_{\mu_m}^D(u_n) - 2\int_D fu_n\,dx. \end{aligned} \tag{2.17}$$

Hence

$$\int_D |\nabla u|^2 dx + \int_D u^2 d\mu + 2\int_D fu_1 dx \geq \int_D |\nabla u_n|^2 dx.$$

Thus $\|u_n\|_{H^1(D)}$ is bounded, so $u_n \to w$ weakly in $H^1(D)$, and also $u_n - g \to w - g$ weakly in $H^1(D)$. Since $u_n - g \in H_0^1(D)$, $w - g \in H_0^1(D)$.

By lower semicontinuity of $F_{\mu_m}^D$ we obtain from (2.17)

$$\begin{aligned} F\mu_m^D(w) - 2\int_D fw\,dx &\leq \liminf_{n\to\infty}\left[F_{\mu_m}^D(u_n) - 2\int_D fu_n dx\right] \\ &\leq \liminf_{n\to\infty}\left[F_{\mu_n}^D(u_n) - 2\int_D fu_n dx\right] \leq \limsup_{n\to\infty}\left[F_{\mu_n}^D(u_n) - 2\int_D fu_n dx\right] \\ &\leq F_\mu^D(u) - 2\int_D fu\,dx. \end{aligned}$$

By taking the limit as $m \to \infty$ we obtain

$$F_\mu^D(w) - 2\int_D fw\,dx \geq F_\mu^D(u) - 2\int_D fu\,dx. \tag{2.18}$$

By Proposition 2.1 u is the unique minimum point of the functional

$$F_\mu^D(v) - 2\int_D fv\,dx$$

on the set $\{v : v \in H^1(D), v - g \in H_0^1(D)\}$. Since $w - g \in H_0^1(D)$, from (2.18) we obtain $w = u$ and

$$F_\mu^D(u) = \lim_{n\to\infty} F_{\mu_n}^D(u_n). \tag{2.19}$$

Since $\mu_n \uparrow \mu$ and $u_n \downarrow u$ we have

$$\int_D u^2 d\mu \leq \lim_{n\to\infty}\int_D u_n^2\,d\mu_n.$$

Therefore from (2.19) we obtain

$$\int_D |\nabla u|^2 dx = \lim_{n\to\infty}\int_D |\nabla u_n|^2 dx. \tag{2.20}$$

Since (u_n) converges to u weakly in $H^1(D)$ and $u_n - g \in H_0^1(D)$, (2.20) implies that (u_n) converges to u strongly in $H^1(D)$, so Proposition 2.7 is proved.

We now introduce a variational convergence for sequences (μ_n) in $\mathcal{M}_0$, using the notion of Γ-convergence for functionals in the calculus of variations (see [**10**] and also [**2**], where such convergence is called epi-convergence). Γ-convergence in turn is based on the abstract Kuratowski convergence, which can be formulated in an arbitrary metric space X as follows:

DEFINITION 2.6. Let (F_n) be a sequence of functions from X into $\overline{\mathbf{R}}$, and let F be a function from X into $\overline{\mathbf{R}}$. We say that (F_n) Γ-converges to F in X if the following conditions are satisfied:

(a) for every $u \in X$ and for every sequence (u_n) converging to u in X

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(b) for every $u \in X$ there exists a sequence (u_n) converging to u in X such that

$$F(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

We now define the γ-convergence of a sequence (u_n) in $\mathcal{M}_0$ in terms of the Γ-convergence of the corresponding functionals F_{u_n} defined on the space $L^2(R^d)$ by equations (2.12) and (2.13) with $D = R^d$.

DEFINITION 2.7. We say that a sequence (μ_n) in $\mathcal{M}_0$ γ-converges to the measure $\mu \in \mathcal{M}_0$ if the sequence of functionals (F_{u_n}) Γ-converges to the functional F_μ in $L^2(R^d)$ as in Definition 2.6.

The following proposition shows that our definition of γ-convergence is equivalent to Definition 4.8 of [**8**].

PROPOSITION 2.8. *Let (μ_n) be a sequence in $\mathcal{M}_0$ and let $\mu \in \mathcal{M}_0$. The following conditions are equivalent:*

(a) *(μ_n) γ-converges to μ;*

(b) *$(F^D_{\mu_n})$ Γ-converges to F^D_μ in $L^2(D)$ for every open set D in R^d;*

(c) *$(F^D_{\mu_n})$ Γ-converges to F^D_μ in $L^2(D)$ for every bounded open set D in R^d.*

PROOF. For every open set D in R^d we consider the following functionals on $L^2(D)$:

$$F^D_+(u) = \inf\left\{\limsup_{n\to\infty} F^D_{\mu_n}(u_n) \colon u_n \to u \text{ in } L^2(D)\right\}, \tag{2.21}$$

$$F^D_-(u) = \inf\left\{\liminf_{n\to\infty} F^D_{\mu_n}(u_n) \colon u_n \to u \text{ in } L^2(D)\right\}. \tag{2.22}$$

If $D = R^d$, we denote the corresponding functionals by F_+ and F_-.

It is easy to see (by a diagonal argument) that the infima in (2.21) and (2.22) are achieved by suitable sequences and that F^D_+ and F^D_- are lower semicontinuous on $L^2(D)$ (see [**10**, Proposition 1.8]). Moreover, F^D_μ is the Γ-limit in $L^2(D)$ of the sequence $(F^D_{\mu_n})$ if and only if $F^D_+ = F^D_-$ and both equal F^D_μ on $L^2(D)$.

Since $F^D_- \leq F^D_+$ on $L^2(D)$ and $F^D_-(u) = F^D_+(u) = F^D_\mu(u)$ if u is not in $H^1_0(D)$, it follows that the Γ-convergence of $F^D_{\mu_n}$ to F^D_μ is equivalent to the inequalities $F^D_+ \leq F^D_\mu \leq F^D_-$ on $H^1_0(D)$.

Let us prove that (a) implies (b). Assume (a), which is equivalent to $F_\mu = F_- = F_+$ on $L^2(R^d)$.

Let D be open in R^d. Let us prove that

$$F_\mu^D \le F_-^D \quad \text{on } H_0^1(D). \tag{2.23}$$

Let $u \in H_0^1(D)$ with $F_-^D(u) < \infty$. By (2.22) there exists a sequence (u_n) converging to u in $L^2(D)$ such that $F_-^D(u) = \liminf_{n\to\infty} F_{\mu_n}^D(u_n)$. We may assume $u_n \in H_0^1(D)$ for every n. If we extend u_n to R^d by putting $u_n = 0$ outside D, we have that $u_n \in H^1(R^d)$ and $F_{\mu_n}^D(u_n) = F_{\mu_n}(u_n)$, therefore

$$F_-^D(u) = \liminf_{n\to\infty} F_{\mu_n}(u_n) \ge F_-(u) = F_\mu(u)$$

and (2.23) is proved.

Let us prove that

$$F_+^D \le F_\mu^D \quad \text{on } H_0^1(D). \tag{2.24}$$

Let $u \in H_0^1(D)$ with compact support in d and such that $F_\mu^D(u) < \infty$. We extend u to R^d by putting $u = 0$ outside D, so that $u \in H^1(R^d)$ and $F_+(u) = F_\mu(u) = F_\mu^D(u)$. By (2.21) there exists a sequence (u_n) converging to u in $L^2(R^d)$ such that

$$F_\mu^D(u) = F_+(u) = \limsup_{n\to\infty} F_{\mu_n}(u_n) < \infty.$$

Therefore $u_n \in H^1(R^d)$ for n large enough and

$$\limsup_{n\to\infty} \int_{R^d} |\nabla u_n|^2 dx \le \limsup_{n\to\infty} F_{\mu_n}(u_n) < \infty,$$

so that (u_n) converges to u weakly in $H^1(R^d)$.

Let $\varphi \in \mathbf{C}_0^\infty(D)$ with $0 \le \varphi \le 1$ and $\varphi = 1$ on the support of u. Then $\varphi u_n \in H_0^1(D)$ and, by Rellich's theorem, the sequence (φu_n) converges to u in $L^2(D)$, hence

$$\begin{aligned} F_+^D(u) &\le \limsup_{n\to\infty} F_{\mu_n}^D(\varphi u_n) \\ &= \limsup_{n\to\infty} \left\{ \int_D [\varphi^2 |\nabla u_n|^2 + 2\varphi u_n \nabla\varphi \nabla u_n + u_n^2 |\nabla\varphi|^2]\, dx + \int_D \varphi^2 u_n^2 d\mu_n \right\} \\ &\le \limsup_{n\to\infty} F_{\mu_n}(u_n) + \int_D [2\varphi u \nabla\varphi\nabla u + u^2|\nabla\varphi|^2]\, dx = F_\mu(u). \end{aligned}$$

Therefore (2.24) is proved when u has compact support on D. In the general case $u \in H_0^1(D)$, there exists a sequence (v_n) of functions in $H_0^1(D)$ with compact support in D which converges to u strongly in $H_0^1(D)$ and such that the sequence (v_n^2) is increasing and converges to u^2 pointwise quasi everywhere on D. By the monotone convergence theorem and by the lower semicontinuity of F_+^D we obtain

$$F_+^D(u) \le \liminf_{n\to\infty} F_+^D(v_n) \le \lim_{n\to\infty} F_\mu^D(v_n) = F_\mu^D(u),$$

which proves (2.24).

Condition (b) follows now from (2.23) and (2.24).

The implication (b)$\Rightarrow$(c) is trivial.

Let us prove that (c) implies (a). Assume (c). We prove that

$$F_\mu \leq F_- \quad \text{on } H^1(R^d). \tag{2.25}$$

Let $u \in H^1(R^d)$ with $F_-(u) < \infty$. By (2.22) there exists a sequence (u_n) converging to u in $L^2(R^d)$ such that

$$\liminf_{n\to\infty} \int_{R^d} |\nabla u_n|^2 dx \leq \liminf_{n\to\infty} F_{\mu_n}(u_n) = F_-(u) < +\infty.$$

We may assume that $u_n \in H^1(R^d)$ and that (u_n) converges to u weakly in $H^1(R^d)$.

Let ψ be a nonincreasing function on R_+ of class C^∞ such that $0 \leq \psi \leq 1$, $\psi(t) = 1$ for $0 \leq t \leq 1$, and $\psi(t) = 0$ for $t \geq 2$. For every $k \in N$, let $\varphi_k \in C_0^\infty(R^d)$ be defined by

$$\varphi_k(x) = \psi(|x|/k). \tag{2.26}$$

Let us fix $k \in N$ and a bounded open set D containing the support of φ_k, so that $(F_{\mu_n}^D)$ Γ-converges to F_μ^D in $L^2(D)$. Then $(\varphi_k u_n)$ converges to $\varphi_k u$ in $L^2(D)$ as $n \to \infty$, hence

$$\begin{aligned}
F_\mu(\varphi_k u) &= F_\mu^D(\varphi_k u) \leq \liminf_{n\to\infty} F_{\mu_n}^D(\varphi_k u_n) \\
&= \liminf_{n\to\infty} \left\{ \int_D [\varphi_k^2 |\nabla u_n|^2 + 2\varphi_k u_n \nabla\varphi_k \nabla u_n + u_n^2 |\nabla\varphi_k|^2]\, dx + \int_D \varphi_k^2 u_n^2 d\mu_n \right\} \\
&\leq \liminf_{n\to\infty} F_{\mu_n}(u_n) + \int_{R^d} [2\varphi_k u \nabla\varphi_k \nabla u + u^2 |\nabla\varphi_k|^2]\, dx.
\end{aligned}$$

Since $(\nabla\varphi_k)$ converges to 0 uniformly on R^d and $(\varphi_k u)$ converges to u in $H^1(R^d)$, by the lower semicontinuity of F_μ we obtain

$$F_\mu(u) \leq \liminf_{k\to\infty} F_\mu(\varphi_k u) \leq \liminf_{n\to\infty} F_{\mu_n}(u_n) = F_-(u),$$

which proves (2.25).

Let us prove that

$$F_+ \leq F_\mu \quad \text{on } H^1(R^d). \tag{2.27}$$

Let $u \in H^1(R^d)$ with $F_\mu(u) < \infty$ and let (φ_k) be the sequence in $C_0^\infty(R^d)$ defined in the previous step of the proof.

Let us fix $k \in N$ and a bounded open set D containing the support of φ_k so that $(F_{\mu_n}^\infty)$ Γ-converges to F_μ^D in $L^2(D)$. Since $\varphi_k u \in H_0^1(R^d)$, there exists a sequence (u_n) converging to $\varphi_k u$ in $L^2(D)$ such that

$$F_\mu(\varphi_k u) = F_\mu^D(\varphi_k u) = \lim_{n\to\infty} F_{\mu_n}^D(u_n) < \infty.$$

Then $u_n \in H_0^1(D)$ for n large enough. If we extend u_n to R^d by putting $u_n = 0$ outside D we obtain that $u_n \in H^1(R^d)$ and (u_n) converges to $\varphi_k u$ in $L^2(R^d)$, hence

$$F_+(\varphi_k u) \leq \limsup_{n\to\infty} F_{\mu_n}(u_n) = \lim_{n\to\infty} F_{\mu_n}^D(u_n) = F_\mu(\varphi_k u).$$

Since $(\varphi_k u)$ converges to u strongly in $H^1(R^d)$ and $(\varphi_k^2 u^2)$ is increasing and converges to u^2 quasi everywhere in R^d, by the monotone convergence theorem and by the lower semicontinuity of F_+ we obtain

$$F_+(u) \le \liminf_{k\to\infty} F_+(\varphi_k u) = \lim_{k\to\infty} F_\mu(\varphi_k u) = F_\mu(u),$$

which proves (2.27).

Condition (a) now follows from (2.25) and (2.27).

Let us mention some general properties of γ-convergence as established in [**8**].

γ-convergence in $\mathcal{M}_0$ is *metrizable* [**8**, Proposition 4.9], and $\mathcal{M}_0$ is *compact* under γ [**8**, Theorem 4.14].

The γ-convergence of a sequence μ_n to μ in $\mathcal{M}_0$ can be characterized in variational terms [**8**, Proposition 4.10], as convergence for every bounded open set $D \subset R^d$ and every $f \in L^2(D)$ of the minimum values m_n to m, where

$$m_n \equiv \min_{v\in H_0^1(D)} \left[F_{\mu_n}^D(v) + \int_D fv\,dx\right],$$

$$m \equiv \min_{v\in H_0^1(D)} \left[F_{\mu}^D(v) + \int_D fv\,dx\right].$$

However, more conveniently for our present purposes, Γ-convergence (hence γ-convergence) can be characterized in terms of strong convergence of the resolvents in $L^2(D)$, as we shall see in the next proposition.

With each $\mu \in \mathcal{M}_0$, each open set D in R^d, and each $\lambda > 0$ we associate the Moreau-Yosida approximation $(F_\mu^D)_\lambda$ of F_μ^D, defined for every $f \in L^2(D)$ by

$$(F_\mu^D)_\lambda(f) \equiv \min_{v\in H_0^1(D)} \left[F_\mu^D(v) + \lambda\int_D (v-f)^2 dx\right]. \tag{2.28}$$

PROPOSITION 2.9. *Let (μ_n) be a sequence in $\mathcal{M}_0$, let $\mu \in \mathcal{M}_0$, let D be open in R^d, and let $\lambda > 0$. The following conditions are equivalent:*

(a) *the functionals $F_{\mu_n}^D$ Γ-converge to F_μ^D in $L^2(D)$ as $n \to \infty$;*

(b) *the resolvent operators $R_\lambda^D(\mu_n)$ converge strongly to $R_\lambda^D(\mu)$ in $L^2(D)$ as $n \to \infty$;*

(c) *the Moreau-Yosida approximations $(F_{\mu_n}^D)_\lambda$ converge pointwise to $(F_\mu^D)_\lambda$ in $L^2(D)$ as $n \to \infty$.*

To prove Proposition 2.9 we need

LEMMA 2.2. *Suppose that*

$$F_\mu^D(u) \le \liminf_{n\to\infty} F_{\mu_n}^D(u_n) \tag{2.29}$$

for every $u \in L^2(D)$ and every sequence (u_n) converging to u strongly in $L^2(D)$. Then (2.29) holds for every $u \in L^2(D)$ and every sequence (u_n) converging to u weakly in $L^2(D)$.

PROOF. Let $u \in L^2(D)$ and let (u_n) be a sequence converging to u weakly in $L^2(D)$ such that

$$\liminf_{n\to\infty} F_{\mu_n}^D(u_n) < \infty.$$

Then $u_n \in H_0^1(D)$ for infinitely many n and

$$\liminf_{n\to\infty} \int_D |\nabla u_n|^2\, dx < \infty,$$

so we may assume that (u_n) converges to u weakly in $H_0^1(D)$. Let (φ_k) be the sequence in $C_0^\infty(R^d)$ defined by (2.26). Let us fix $k \in N$. Then $\varphi_k u_n \in H_0^1(D)$ and $(\varphi_k u_n)$ converges to $\varphi_k u$ weakly in $H_0^1(D)$ and, by Rellich's theorem, strongly in $L^2(D)$. Therefore, by the hypothesis,

$$\begin{aligned} F_\mu^D(\varphi_k u) &\le \liminf_{n\to\infty} F_{\mu_n}^D(\varphi_k u_n) \\ &= \liminf_{n\to\infty} \Big\{ \int_D [\varphi_k^2 |\nabla u_n|^2 + 2\varphi_k u_n \nabla\varphi_k \nabla u_n + u_n^2 |\nabla \varphi_k|^2]\, dx \\ &\qquad\qquad + \int_D \varphi_k^2 u_n^2 d\mu_n \Big\} \\ &\le \liminf_{n\to\infty} F_{\mu_n}^D(u_n) + \int_D [2\varphi_k u \nabla\varphi_k \nabla u + u^2 |\nabla\varphi_k|^2]\, dx. \end{aligned}$$

In the last inequality we have used the fact that, by Rellich's theorem, (u_n) converges to u in $L^2(D_k)$ where $D_k = \{x : x \in D, \nabla\varphi_k(x) \ne 0\}$. Since $(\nabla\varphi_k)$ converges to 0 uniformly on R^d and $(\varphi_k u)$ converges to u in $H_0^1(D)$, we then obtain

$$F_\mu^D(u) \le \liminf_{k\to\infty} F_\mu^D(\varphi_k u) \le \liminf_{n\to\infty} F_{\mu_n}^D(u_n),$$

and the lemma follows.

We will now proceed with the

PROOF OF PROPOSITION 2.9. By Lemma 2.2 the equivalence of (a), (b), and (c) could be obtained as a consequence of an abstract result ([**2**, Theorem 3.26] with $X = L^2(D)$). However we prefer to give a more direct proof here.

Let $F_n = F_{\mu_n}^D, F = F_\mu^D$ and, correspondingly, $(F_n)_\lambda = (F_{\mu_n}^D)_\lambda, (F)_\lambda = (F_\mu^D)_\lambda$.

(a)$\Longrightarrow$(b) Assume (a). Let $f \in L^2(D)$, $u_n = \lambda R_\lambda^D(\mu_n) f$, and $u = \lambda R_\lambda^D(\mu) f$. We have to prove that (u_n) converges to u strongly in $L^2(D)$.

By the definition of Γ-convergence there exists a sequence (v_n) converging to u in $L^2(D)$ such that $F(u) = \lim_{n\to\infty} F_n(v_n)$.

By the minimum property of u_n (Proposition 2.1) we have

$$F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \le F_n(v_n) + \lambda \int_D (v_n - f)^2 dx,$$

hence

$$\begin{aligned} &\limsup_{n\to\infty} \left[F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \right] \\ &\quad \le \lim_{n\to\infty} \left[F_n(v_n) + \lambda \int_D (v_n - f)^2 dx \right] = F(u) + \lambda \int_D (u - f)^2 dx. \end{aligned} \tag{2.30}$$

Since

$$F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \ge \int_D |\nabla u_n|^2 dx + \frac{\lambda}{2} \int_D u_n^2 dx - \lambda \int_D f^2 dx,$$

the sequence (u_n) is bounded in $H_0^1(D)$, hence it contains a subsequence which converges weakly in $H_0^1(D)$ to a function $w \in H_0^1(D)$. We will relabel this subsequence as (u_n) again. By Lemma 2.1, we have

$$F(w) \le \liminf_{n\to\infty} F_n(u_n). \tag{2.31}$$

Moreover,

$$\int_D (w-f)^2 dx \le \liminf_{n\to\infty} \int_D (u_n - f)^2 dx. \tag{2.32}$$

Hence, by using (2.30), we obtain

$$\begin{aligned} F(w) + \lambda \int_D (w-f)^2 dx &\le \liminf_{n\to\infty} \left[F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \right] \\ &\le \limsup_{n\to\infty} \left[F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \right] \\ &\le F(u) + \lambda \int_D (u-f)^2 dx. \end{aligned}$$

Since u is the unique minimum point of problem (2.28) (see Proposition 2.1), we obtain $w = u$ and

$$F(u) + \lambda \int_D (u-f)^2 dx = \lim_{n\to\infty} \left[F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \right]. \tag{2.33}$$

From (2.31), (2.32), (2.33), and equality $u = w$ it follows that

$$\int_D (u-f)^2 dx = \lim_{n\to\infty} \int_D (u_n - f)^2 dx,$$

and hence that (u_n) converges to $u = w$ strongly in $L^2(D)$. The limit being independent of the subsequence, the whole original sequence (u_n) converges to u in $L^2(D)$ and (b) is proved.

(b)$\Rightarrow$(c) Assume (b). Let $f \in L^2(D)$, $u_n = \lambda R_\lambda^D(\mu_n) f$, and $u = \lambda R_\lambda^D(\mu) f$. By taking the test function $u = u_n$ in the equation satisfied by u_n, we obtain

$$F_n(u_n) + \lambda \int_D u_n^2\, dx = \lambda \int_D f u_n dx.$$

Since u_n is the minimum point of problem (2.28) we have

$$(F_n)_\lambda f = F_n(u_n) + \lambda \int_D (u_n - f)^2 dx = \lambda \int_D f(f - u_n)\, dx.$$

In the same sense we obtain

$$(F)_\lambda f = \lambda \int_D f(f-u)\, dx.$$

By (b), the sequence (u_n) converges to u in $L^2(D)$, hence $(F_n)_\lambda f$ converges to $(F)_\lambda f$ as $n \to \infty$, and (c) is proved.

(c)$\Rightarrow$(a) Assume (c). We set

$$G_n(v) = F_n(v) + \lambda \int_D v^2 dx,$$

$$G(v) = F(v) + \lambda \int_D v^2 dx.$$

By condition (c) we have

$$\min_{v\in H_0^1(D)}\left[G(v)+\int_D fv\,dx\right]=\lim_{n\to\infty}\min_{v\in H_0^1(D)}\left[G_n(v)+\int_D fv\,dx\right] \tag{2.34}$$

for every $f\in L^2(D)$.

Let us consider the following functionals on $L^2(D)$:

$$F_+(u)=\inf\left\{\limsup_{n\to\infty}F_n(u_n)\colon\ u_n\to u \text{ in } L^2(D)\right\},$$

$$F_-(u)=\inf\left\{\liminf_{n\to\infty}F_n(u_n)\colon\ u_n\to u \text{ in } L^2(D)\right\}.$$

In order to prove that (F_n) Γ-converges to F in $L^2(D)$, it is enough to show that

$$F\le F_- \quad \text{on } L^2(D), \tag{2.35}$$

$$F_+\le F \quad \text{on } H_0^1(D)\cap L^2(D,\mu). \tag{2.36}$$

Let us prove (2.35). Let $u\in L^2(D)$. Since G is convex and lower semicontinuous on $L^2(D)$ we have (see for example [**3**, Proposition 4.1])

$$G(u)=\sup_{f\in L^2(D)}\inf_{v\in L^2(D)}\left[G(v)+\int_D f(v-u)\,dx\right]. \tag{2.37}$$

If (u_n) converges to u in $L^2(D)$, by (2.34) we have for every $f\in L^2(D)$

$$\min_{v\in H_0^1(D)}\left[G(v)+\int_D f(v-u)\,dx\right]$$

$$=\lim_{n\to\infty}\min_{v\in H_0^1(D)}\left[G_n(v)+\int_D f(v-u_n)\,dx\right]\le\liminf_{n\to\infty}G_n(u_n).$$

By taking the supremum with respect to $f\in L^2(D)$ in the previous inequality and by using (2.37) we obtain

$$G(u)\le\liminf_{n\to\infty}G_n(u_n).$$

Hence

$$F(u)\le\liminf_{n\to\infty}F_n(u_n).$$

Since (u_n) is an arbitrary sequence converging to u in $L^2(D)$ we obtain $F(u)\le F_-(u)$ and (2.35) is proved.

Let us prove (2.36). Let $f\in L^2(D)$,

$$u_n=-\tfrac{1}{2}R_\lambda^D(\mu_n)f, \quad \text{and} \quad u=-\tfrac{1}{2}R_\lambda^D(\mu)f.$$

By Proposition 2.1 we have

$$G(u)+\int_D fu\,dx=\min_{v\in H_0^1(D)}\left[G(v)+\int_D fv\,dx\right],$$

$$G_n(u_n)+\int_D fu_n\,dx=\min_{v\in H_0^1(D)}\left[G_n(v)+\int_D fv\,dx\right].$$

Therefore by (2.34) we have

$$\lim_{n\to\infty}\left[G_n(u_n)+\int_D fu_n\,dx\right]=G(u)+\int_D fu\,dx<\infty. \tag{2.38}$$

Since

$$G_n(u_n) + \int_D f u_n \, dx \geq \int_D |\nabla u_n|^2 dx + \frac{\lambda}{2} \int_D u_n^2 \, dx - \frac{1}{2\lambda} \int_D f^2 dx,$$

the sequence (u_n) is bounded in $H_0^1(D)$, hence it contains a subsequence, which by relabelling we may still denote by (u_n), which converges weakly in $H_0^1(D)$ to a function $w \in H_0^1(D)$. By the inequality $F \leq F_-$ and Lemma 2.2 we have

$$F(w) \leq \liminf_{n\to\infty} F_n(u_n). \tag{2.39}$$

Since

$$\int_D w^2 dx \leq \liminf_{n\to\infty} \int_D u_n^2 dx, \tag{2.40}$$

we have

$$\begin{aligned} G(w) + \int_D f w \, dx &\leq \lim_{n\to\infty} \left[G_n(u_n) + \int_D f u_n \, dx \right] = G(u) + \int_D f u \, dx \\ &= \min_{v \in H_0^1(D)} \left[G_n(v) + \int_D f v \, dx \right]. \end{aligned}$$

By the uniqueness of the minimum point we have $w = u$, hence (u_n) converges to u weakly in $H_0^1(D)$. By (2.38), (2.39), and (2.40) we have

$$\int_D u^2 dx = \lim_{n\to\infty} \int_D u_n^2 dx,$$

and hence (u_n) converges to u strongly in $L^2(D)$. It follows from this argument that the entire original sequence (u_n) must converge strongly to u in $L^2(D)$ and we obtain easily from (2.38) that $F_+(u) \leq F(u)$ for every $u \in H_0^1(D) \cap L^2(D, \mu)$ of the form $u = R_\lambda^D(\mu) f$ with $f \in L^2(D)$.

Since the range of $R_\lambda^D(\mu)$ is dense in $H_0^1(D) \cap L^2(D, \mu)$, for every $u \in H_0^1(D) \cap L^2(D, \mu)$ there exists a sequence (f_n) in $L^2(D)$ such that the functions $v_n = R_\lambda^D(\mu) f_n$ converge to u in $H_0^1(D)$ and in $L^2(D)$. By the lower semicontinuity of F_+, we obtain

$$F_+(u) \leq \liminf_{n\to\infty} F_+(v_n) \leq \lim_{n\to\infty} F(v_n) = F(u)$$

and inequality (2.36) is proved.

Condition (a) follows now from (2.35) and (2.36), and Proposition 2.9 is proved.

We conclude the present section by stating the following theorem which follows immediately from Propositions 2.8 and 2.9.

THEOREM 2.1. *Let (μ_n) be a sequence in $\mathcal{M}_0$, Let $\mu \in \mathcal{M}_0$, and let $\lambda > 0$. The following conditions are equivalent*:

(a) *(μ_n) γ-converges to μ;*

(b) *the resolvent operators $R_\lambda(\mu_n)$ converge to $R_\lambda(\mu)$ strongly in $L^2(R^d)$ as $n \to \infty$;*

(c) *the resolvent operators $R_\lambda^D(\mu_n)$ converge to $R_\lambda^D(\mu)$ strongly in $L^2(D)$ as $n \to \infty$ for every open set D in R^d;*

(d) *the resolvent operators $R_\lambda^D(\mu_n)$ converge to $R_\lambda^D(\mu)$ strongly in $L^2(D)$ as $n \to \infty$ for every bounded open set D in R^d.*

This result will be convenient for connecting γ-convergence with the probabilistic notion of *stable* convergence to be defined in the next section.

3. We shall use the concept of a randomized stopping time for Brownian motion. Some notations and results will be taken from [**4**, §2].

Let $C = \mathbf{C}([0,\infty], R^d)$, the space of continuous R^d-valued functions of nonnegative time, endowed with the topology of uniform convergence on compact time sets. C is the sample space for standard Brownian motion (B_t), where $B_t\colon C \to R^d$ is the projection map defined by $B_t(\omega) = \omega(t), \omega$ denoting a typical point or "sample path" in C. The relevant σ-algebras on C are $\mathcal{F}_t = \sigma(B_t\colon 0 \le s \le t)$, and $\mathcal{G}_t = \mathcal{F}_{t+}$. We let $\mathcal{G} = \mathcal{G}_\infty = \mathcal{F}_\infty = \sigma(B_s\colon 0 \le s < \infty)$.

A *stopping time* τ with respect to the fields $(\mathcal{G}_t)$ is defined as usual to be a map $\tau\colon C \to [0,\infty]$, such that $\{\tau \le t\}$ is in $\mathcal{G}_t$ for all t, $0 \le t < \infty$. A *randomized stopping time* T is defined to be a map $T\colon C \times [0,1] \to [0,\infty]$, such that T is a stopping time with respect to the σ-algebras $(\mathcal{G}_t \times \mathcal{B}_1)$, where $\mathcal{B}_1$ denotes the Borel sets on $[0,1]$. We shall require $T(\omega,\cdot)$ to be *nondecreasing and left continuous* on $[0,1]$, with $T(\omega,0) = 0$, for every ω in C. When convenient we shall regard an ordinary stopping time τ also as a randomized one, by setting $\tau(\omega,a) = \tau(\omega)$ for all a in $(0,1]$. If T is a randomized stopping time, then $T(\cdot,a)$ is an ordinary stopping time, for each a in $[0,1]$.

A randomized stopping time T can be expressed by an equivalent object, the *stopping time measure* F induced by T. F is the map $F\colon C \times \mathcal{B} \to [0,1]$, where $\mathcal{B} =$ the Borel sets of $[0,\infty]$, defined by

$$F(\omega,[0,t]) = \sup\{a\colon T(\omega,a) \le t\} \tag{3.1}$$

and the condition that $F(\omega,\cdot)$ be a measure on $\mathcal{B}$.

We shall often write $F(\cdot,(t,\infty])$ as $F((t,\infty])$ or as F_t. If P denotes a probability measure on $(C,\mathcal{A})$, and m_1 denotes Lebesgue measure on $[0,1]$, then F_t is a version of the conditional probability of $\{T > t\}$ using the probability $P \times m_1$, with respect to the σ-algebra $\mathcal{G} \times \{\emptyset,[0,1]\}$. $F(\omega,\cdot)$ is thus a version of the conditional distribution of T given the entire path ω. Thus for any bounded measurable Z on $C \times [0,\infty]$, we have

$$\int Z(\omega,T(\omega,s))P(d\omega)m_1(ds) = \int Z(\omega,t)F(\omega,dt)P(d\omega). \tag{3.2}$$

We can recover T from F by

$$T(\omega,a) = \inf\{t\colon F(\omega,[0,t]) \ge a\}. \tag{3.3}$$

Furthermore, given *any* map $F\colon C \times \mathcal{B} \to [0,1]$ such that

$$F(\omega,\cdot) \text{ is a probability for each } \omega \text{ in } C, \tag{3.4}$$

we can *define* T by (3.3). T will be a randomized stopping time, provided that

$$F(\cdot,[0,t]) \text{ is } \mathcal{G}_t\text{-measurable for each } t. \tag{3.5}$$

Any $F\colon C \times \mathcal{B} \to [0,1]$ such that (3.4) and (3.5) hold will be called a stopping time measure. We see that there is a complete correspondence between the notions of stopping time measure and randomized stopping time.

DEFINITION 3.1. A sequence T of randomized stopping times will be said to converge *stably* to a limit T, *with respect to a probability measure* P on $(C,\mathcal{G})$, if for each $A \in \mathcal{G}$, $T_n|_{A\times[0,1]}$ converges in distribution to $T|_{A\times[0,1]}$ with respect to

$P \times m_1$. If F_n, F are the stopping time measures for T_n, T, we will also say that F_n converges stably to F with respect to P.

It is shown in [**3**] that this convergence is defined by a *compact* topology.

Until now we have discussed arbitrary probabilities P on $(C, \mathcal{G})$. Since our interest is in Brownian motion, we now consider P^ν, the usual Wiener measure on $(C, \mathcal{G})$ with initial probability distribution ν on R^d, that is

$$P^\nu(B_0 \in A) = \nu(A) \tag{3.6}$$

for any Borel set A in R^d. We will refer to such a probability measure $P = P^\nu$ as a *Brownian* probability on C. We will denote the usual heat semigroup by P_t, where P_t acts both on measures and functions as a Markov operator, so that the distribution of B_t under P^ν is νP_t, and $P_t h(x) = E^x[h \circ B_t]$ for any bounded Borel h on R^d.

DEFINITION 3.2. If T_n converges stably to T, for one, and hence for all, probability measures P^ν such that $\nu \ll m, m \ll \nu$, where m denotes Lebesgue measure on R^d, then we will simply say that T_n converges *stably* to T. If F_n, F are the stopping time measures for T_n, T, we will also say F_n converges stably to F.

The fact that $T_n \to T$ stably for one ν with $\nu \ll m,\ m \ll \nu$ implies that $T_n \to T$ stably for every λ with $\lambda \ll m$, follows readily from the fact that $P^\lambda \ll P^\nu$ (see also Lemma 3.1 below).

A statement will be said to hold *almost surely* (a.s.) on C if it holds P^x-a.e. for every x in R^d. If $T_n \to T$ stably, as in Definition 3.2, we see easily that although T is not uniquely determined a.s. by $T_n, T \circ \theta_t$ is uniquely determined a.s. for every $t > 0$, where θ_t denotes the usual shift operator, so that $\theta_t(\omega)(s) = \omega(t+s)$.

In what follows we will often follow a *convention* of employing the same letters E and P for expectations and probabilites on the randomized space $C \times [0,1]$, that is, with respect to $P \times m_1$, as we do on C with respect to P.

LEMMA 3.1. *Let T_n, T be randomized stopping times, and let P be a probability measure on $(C, \mathcal{G})$ such that $T_n \to T$ stably with respect to P. Let F be the stopping time measure associated with T. Let $Z\colon C \times [0,\infty] \to R$ be given. We will write $Z(\cdot, t) = Z_t$ where convenient. Suppose Z is bounded and $\mathcal{G} \times \mathcal{B}$-measurable on $C \times [0,\infty]$.*

(i) *Suppose $Z(\omega, \cdot)$ is usc (lsc) for P-a.e. ω. Then*

$$\limsup_{n\to\infty} \left(\liminf_{n\to\infty}\right) \int Z_{T_n}\, dP \le \ (\ge) \int Z_T\, dP.$$

(ii) *Suppose that for P-a.e. ω, $F(\omega, \{t\colon Z(\omega,\cdot)$ is discontinuous at $t\}) = 0$. Then*

$$\lim_{n\to\infty} \int Z_{T_n}\, dP = \int Z_T\, dP.$$

Lemma 3.1 follows from Theorem 7 in [**21**].

COROLLARY TO LEMMA 3.1. *Let T_n, T be randomized stopping times, and let P be a probability measure on $(C, \mathcal{G})$.*

(i) *Let σ be a $\mathcal{G}_t$-stopping time. If $T_n \to T$ stably with respect to P then $T_n \wedge \sigma \to T \wedge \sigma$ stably with respect to P.*

(ii) *If there exists τ_j a sequence of $\mathcal{G}_t$-stopping times such that $\tau_j \uparrow \infty$ and $T_n \wedge \tau_j \to T \wedge \tau_j$ stably with respect to P for each j, then $T_n \to T$ stably with respect to P.*

PROOF. (i) Given $f \in \mathbf{C}([0,\infty])$, Y bounded measurable, let $Z_t(\omega) = f(t \wedge \sigma(\omega))Y(\omega)$. Since $Z_{T_n} = f(T_n \wedge \sigma(\omega))Y(\omega)$, and $Z_T = f(T \wedge \sigma(\omega))Y(\omega)$, the result follows at once from Lemma 3.1.

(ii) Given $f \in \mathbf{C}([0,\infty])$, Y bounded measurable, for every $\varepsilon > 0$ there exists j such that

$$E\left[\sup_{t \geq \tau_j} |f(t) - f(\tau_j)|\right] \leq \varepsilon.$$

Then $|E[f(T_n)Y] - E[f(T_n \wedge \tau_j)Y]| \leq \varepsilon \|Y\|_\infty$ and $|E[f(T)Y] - E[f(T \wedge \tau_j)Y]| \leq \varepsilon \|Y\|_\infty$. Since $E[f(T_n \wedge \tau_j)Y] \to E[f(T \wedge \tau_j)Y]$, the corollary follows.

In order to consider the probabilistic solution of the Dirichlet problem in a region with many small absorbing holes, one needs to show that a stable limit for the diffusion also implies a corresponding convergence for the Dirichlet problem solutions u_n, which are expressed in terms of expected values of boundary values by (6.11) below, with M replaced by M_n. The next theorem applies to arbitrary randomized stopping times T_n, not just those associated with multiplicative functionals M_n (see below), but in particular it shows that the solutions of (6.11) converge weakly in $L^2(D)$, for arbitrary multiplicative functionals M_n, that is, not just for those of §4, and for bounded measurable boundary values as well as boundary values in $H^1(D)$. By Remark 3.3 we see that the convergence is actually strong in $L^2(D)$. Finally, by Remark 3.2, the convergence preserves integrals with respect to finite measures in H^{-1}. These convergence results follow from Proposition 5.12 of [**8**] when the M_n are induced by measures as in §4 and the boundary values are in $H^1(D)$.

THEOREM 3.1. *Let T_n and T be randomized stopping times, with corresponding stopping time measures F_n and F, respectively. Let ν be a probability measure on R^d, and $P = P^\nu$, the Wiener measure with initial distribution ν. Suppose $T_n \to T$ stably with respect to P. Let K be a closed set in R^d, and let τ denote the first entrance time of K. Suppose that $P \times m_1(T = \tau) = 0$, that is, that $\int F(\omega, \{\tau(\omega)\})P(d\omega) = 0$. Let $Y \in L^1(C, \mathcal{G}, P)$ with $Y = 0$ on $\{\tau = \infty\}$. Define signed measures ψ_n and ψ by the equations*

$$(3.7) \quad \int h\, d\psi_n = \int h \circ B_\tau F_n((\tau,\infty])Y\, dP, \qquad \int h\, d\psi = \int h \circ B_\tau F((\tau,\infty])Y\, dP,$$

for every bounded Borel function h on R^d.

Then $\psi_n \to \psi$ in total variation norm as $n \to \infty$.

PROOF. As usual we use $\|\psi\|$ to denote the total variation norm of a signed measure ψ. Clearly $\|\psi_n\| \leq \|Y\|_1, \|\psi\| \leq \|Y\|_1$. Thus the collection of Y's for which the theorem is true is *closed* in $L^1(C, \mathcal{G}, P)$. Hence it is enough to prove the theorem when $Y = 0$ on $\{\tau > \varsigma\}$, where $\varsigma =$ the first entrance time of the complement of some large ball. Thus we may assume that K contains the complement of some ball, and hence that $\tau < \infty$ almost surely.

Since $\int_{\{\tau=0\}} F(\{0\})\,dP = 0$, by Lemma 3.1 we have $\limsup_{n\to\infty} \int F_n(\{0\})\,dP = 0$. Clearly

$$\|\psi_n - \psi\| \le \int |F_n((\tau,\infty]) - F((\tau,\infty])||Y|\,dP,$$

so if $Y = 0$ on $\{\tau > 0\}$ we see easily that Theorem 3.1 holds. Thus we may, by decomposing Y into two parts, assume that $Y = 0$ on $\{\tau = 0\}$. Then, discarding part of ν, we may assume $P^\nu(\tau = 0) = 0$ also. Thus we assume $\tau > 0$, P-a.e.

Let W_k be open sets in R^d, $W_k \downarrow K$. Let τ_k denote the first entrance time of W_k. Then $\tau_k < \tau$ on $\{\tau > 0\}$, and $\tau_k \uparrow \tau$ everywhere. Clearly $\mathcal{G}$ is generated by the two σ-algebras $\mathcal{G}_\tau$ and $\theta_\tau^{-1}(\mathcal{G})$. Using again the fact that the set of Y's for which the theorem is true is closed in L^1-norm and the fact that Brownian motion has predictable σ-algebras, we see that it is enough to prove Theorem 3.1 when Y is of the form $U(V \circ \theta_\tau)$, where U is bounded and $\mathcal{G}_{\tau_k}$-measurable for some k, and V is bounded and $\mathcal{G}$-measurable. We assume $|U| \le 1$, $|V| \le 1$ without loss of generality, and consider fixed U, V, k. Let M denote the collection of bounded Borel functions h on R^d, having sup norm ≤ 1, and having support in a fixed compact subset S of R^d. In order to prove Theorem 3.1 it is clearly sufficient to prove that

$$\int h \circ B_\tau F_n((\tau,\infty])Y\,dP \to \int h \circ B_\tau F((\tau,\infty])Y\,dP \tag{3.8}$$

as $n \to \infty$, *uniformly* over all h in M.

Suppose not. We will show a contradiction. There exists some $\delta > 0$, and a sequence h_{n_i} in M, such that

$$\left|\int h_{n_i} \circ B_\tau F_{n_i}((\tau,\infty])Y\,dP - \int h_{n_i} \circ B_\tau F((\tau,\infty])Y\,dP\right| > \delta \quad \text{for all } i. \tag{3.9}$$

Passing to a subsequence and relabelling, we may assume that h_n converges to a limit h in the weak*-topology on $L^\infty(R^d, \mathcal{B}'', \lambda)$, where $\mathcal{B}''$ denotes the Borel sets in R^d, and λ denotes the distribution of B_τ with respect to P^μ, where μ is chosen so that $m \ll \mu$. We note that if λ^x denotes the distribution of B_τ with respect to P^x, then $\lambda^x \ll \lambda$ for every x in K^c.

Let $\varepsilon > 0$ be given. Choose $j \ge k$ such that $P \times m_1(\tau_j \le T \le \tau) < \varepsilon/2$, that is, $\int F([\tau_j,\tau])\,dP < \varepsilon/2$. By Lemma 3.1, there exists n_0 such that for every $n \ge n_0$,

$$\int F_n([\tau_j,\tau])\,dP < \frac{\varepsilon}{2}. \tag{3.10}$$

Then, for $n \ge n_0$,

$$\left|\int h_n \circ B_\tau F_n((\tau,\infty])Y\,dP - \int h_n \circ B_\tau F_n((\tau_j,\infty])Y\,dP\right| < \frac{\varepsilon}{2}. \tag{3.11}$$

$$\begin{aligned}
&\left|\int h_n \circ B_\tau F_n((\tau_j,\infty])Y\,dP - \int h \circ B_\tau F_n((\tau_j,\infty])Y\,dP\right| \\
&\quad = \left|\int (h_n - h) \circ B_\tau F_n((\tau_j,\infty])U(V \circ \theta_\tau)\,dP\right| \\
&\quad = \left|\int E[(h_n - h) \circ B_\tau F_n((\tau_j,\infty])U(V \circ \theta_\tau) \mid \mathcal{G}_\tau]\,dP\right| \\
&\quad = \left|\int UF_n((\tau_j,\infty])((h_n - h)g) \circ B_\tau\,dP\right|,
\end{aligned}$$

where $g(x) = E^x[V]$, so that g is Borel on R^d and $|g| \leq 1$ everywhere on R^d. Thus

$$\begin{aligned}
&\left|\int h_n \circ B_\tau F_n((\tau_j,\infty])Y\,dP - \int h \circ B_\tau F_n((\tau_j,\infty])Y\,dP\right| \\
&\quad = \left|\int E\left[UF_n((\tau_j,\infty])((h_n-h)g)\circ B_\tau \mid \mathcal{G}_{\tau_j}\right]dP\right| \\
&\quad = \left|\int UF_n((\tau_j,\infty])E^{B_{\tau_j}}[((h_n-h)g)\circ B_\tau]\,dP\right| \\
&\quad \leq \int |E^{B_{\tau_j}}[((h_n-h)g)\circ B_\tau]|\,dP.
\end{aligned}$$

For each x in $K^c, h_n - h \to 0$ in the weak*-topology on $L^1(R^d, \mathcal{B}'', \lambda^x)$. Thus $E^{B_{\tau_j}}[((h_n-h)g)\circ B_\tau] \to 0$, P-a.e. This shows that

$$\left|\int h_n \circ B_\tau F_n((\tau_j,\infty])Y\,dP - \int h \circ B_\tau F_n((\tau_j,\infty])Y\,dP\right| \to 0 \quad \text{as } n\to\infty. \tag{3.12}$$

By (3.10), for every $n \geq n_0$,

$$\left|\int h \circ B_\tau F_n((\tau_j,\infty])Y\,dP - \int h \circ B_\tau F_n((\tau,\infty])Y\,dP\right| < \frac{\varepsilon}{2}. \tag{3.13}$$

By (3.11), (3.12), and (3.13),

$$\left|\int h_n \circ B_\tau F_n((\tau,\infty])Y\,dP - \int h \circ B_\tau F_n((\tau,\infty])Y\,dP\right| \to 0 \quad \text{as } n\to\infty. \tag{3.14}$$

By the same argument (applied to the constant sequence (F)),

$$\left|\int h_n \circ B_\tau F((\tau,\infty])Y\,dP - \int h \circ B_\tau F((\tau,\infty])Y\,dP\right| \to 0 \quad \text{as } n\to\infty. \tag{3.15}$$

By Lemma 3.1, with $Z(\omega,t) = h\circ B_{\tau(\omega)}(\omega)Y(\omega)\mathbf{1}_{(\tau(\omega),\infty]}(t)$, we have

$$\left|\int h \circ B_\tau F_n((\tau,\infty])Y\,dP - \int h \circ B_\tau F((\tau,\infty])Y\,dP\right| \to 0 \quad \text{as } n\to\infty. \tag{3.16}$$

(3.14), (3.15), and (3.16) contradict (3.9), so Theorem 3.1 is proved.

REMARK 3.1. The conclusion of Theorem 3.1 is also true if τ is any stopping time of form $\sigma + u$, where σ is any stopping time and $u > 0$. This result was proved in [**4**] for the case $\sigma = 0$. The proof is a simpler version of the one just given. It seems possible that this result could be proved in a more general form.

Now that we have obtained some properties of weak convergence of general randomized stopping times, we turn to the more restricted class of randomized stopping times that we shall actually work with, namely the *multiplicative functionals.*

DEFINITION 3.3. A stopping time measure M will be called a multiplicative functional if for every $s \geq 0,\ t \geq 0$,

$$M_{t+s} = M_t(M_s \circ \theta_t) \quad \text{almost surely.} \tag{3.17}$$

We shall assume, unless otherwise stated, that any multiplicative functional M is *exact* [**26**, **6**]: for every $t > 0$, and every sequence ε_k of positive real numbers with $\varepsilon_k \downarrow 0$,

$$\lim_{k\to\infty} M_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = M_t \quad \text{almost surely.} \tag{3.18}$$

We note in passing that if T is the randomized stopping time associated with M then (3.17) implies at once that on $\{T > t\}$, $T(\omega, 1-b) = t + T(\theta_t(\omega), 1 - b/M_t(\omega))$ a.s.

If τ is a first hitting time or a first entrance time, and M is the associated multiplicative functional, then $M_t = \mathbf{1}_{(t,\infty]} \circ \tau$ gives the stopping time measure asssociated with τ, and M satisfies (3.17). M also satisfies (3.18) if τ is a first hitting time but not if τ is a first entrance time. For a general treatment of multiplicative functionals see [**6**].

Associated with any multiplicative functional M there is a sub-Markov *semigroup* $Q_t(M)$, defined by

(3.19) $$(Q_t(M)h)(x) = E^x[h \circ B_t M_t], \quad \text{for every bounded Borel function } h.$$

As usual, we also define $\nu Q_t(M)$ for any finite measure ν by the equation

$$\int h \, d(\nu Q_t(M)) = \int Q_t(M) h \, d\nu.$$

The semigroup $Q_t(M)$ characterizes M *uniquely* (cf. Proposition 1.9 in [**6**]).

Although we will refer to $Q_t(M)$ as a semigroup, it should be stressed that $Q_0(M)$ will not in general be the identity operator.

If P_t is the usual heat semigroup associated with Brownian motion, clearly $P_t = Q_t(1)$. Also, one shows easily from (3.18) that

(3.20) $$Q_t(M)h = \lim_{\varepsilon \downarrow 0} P_\varepsilon Q_{t-\varepsilon}(M)h.$$

Furthermore, if $h \geq 0$, the limit in (3.20) is *decreasing*. It follows that for any $h \geq 0$ the function $Q_t(M)h$ is *upper semicontinuous*.

The following fact is sometimes useful:

LEMMA 3.2. *Let M be a stopping time measure, λ a probability measure with $m \ll \lambda$, C an uncountable set of times such that for any $t \in C$ and any $s \geq 0$, $M_{t+s} = M_t(M_s \circ \theta_t)$, P^λ-a.e. Then for any $u > 0$, $M(\{u\}) = 0$, P^λ-a.e.*

PROOF. There exists s, $0 < s < t$, such that $\int M(\{s\}) \, dP^\lambda = 0$ and $t \equiv u - s \in C$. Then $M(\{s\}) = 0$, P^λ-a.e., and hence $M(\{s\}) \circ \theta_t = 0$, a.s. Thus $M(\{u\}) = M_t(M(\{s\}) \circ \theta_t) = 0$, P^λ-a.e., proving Lemma 3.2.

We now wish to consider in what sense a limit of multiplicative functionals is again a multiplicative functional. The definition of convergence we wish to use is that given in Definition 3.2, stable convergence of stopping time measures. As noted after Definition 3.2, this definition of convergence does not specify the limit uniquely. However, we now show:

THEOREM 3.2. *Let F^n, F be stopping time measures such that $F^n \to F$ stably and such that for every n and every $s \geq 0$, $t \geq 0$,*

(3.21) $$F^n_{t+s} = F^n_t(F^n_s \circ \theta_t), \quad P^\lambda\text{-a.e.},$$

where λ is a fixed probability measure with $m \ll \lambda$. There exists a multiplicative functional M (satisfying (3.17) and (3.18)) such that $F^n \to M$ stably. M is unique a.s.

PROOF. We wish to prove first that for every $s \geq 0$, $t > 0$,

(3.22) $$F_{t+s} = F_t(F_s \circ \theta_t), \quad P^\lambda\text{-a.e.}$$

Let $C = \{t\colon \int F(\{t\})\,dP^\lambda = 0\}$. We will prove (3.22) for $t \in C$ first. By right continuity it is enough to prove (3.22) when s is such that $\int F(\{t+s\})\,dP^\lambda = 0$ and $\int F(\{s\})\,dP^\lambda = 0$. (3.22) is true if for every Y in $L^1(C, \mathcal{G}, P^\lambda)$,

$$\int Y F_{t+s}\,dP^\lambda = \int Y F_t(F_s \circ \theta_t)\,dP^\lambda,$$

or by the usual density argument, if this equality holds for every $Y = W(V \circ \theta_t)$, where W is bounded and $\mathcal{G}_t$-measurable, and V is bounded and $\mathcal{G}$-measurable.

We have, using Lemma 3.1,

$$\begin{aligned}\int Y F_{t+s} dP^\lambda &= \lim_{n\to\infty} \int Y F^n_{t+s} dP^\lambda = \lim_{n\to\infty} \int W(V\circ\theta_t) F^n_t (F^n_s \circ \theta_t)\,dP^\lambda \\ &= \lim_{n\to\infty} \int W F^n_t E^{B_t}[V F^n_s]\,dP^\lambda = \lim_{n\to\infty} \int E^x[V F^n_s]\psi^n(dx),\end{aligned}$$

where ψ^n is the signed measure on R^d defined by

$$\int h\,d\psi^n = E^\lambda[h \circ B_t W F^n_t] \quad \text{for any } h \text{ bounded Borel on } R^d. \tag{3.23}$$

Let ψ be the signed measure on R^d defined by $\int h\,d\psi = E^\lambda[h \circ B_t W F_t]$. By Remark 3.1 (the case proved in [**4**]),

$$\|\psi^n - \psi\| \to 0 \quad \text{as } n \to \infty. \tag{3.24}$$

Clearly we may approximate ψ in total variation norm by $g\,d\lambda$, where g is a bounded Borel function on R^d. Applying Lemma 3.1 to $E^\lambda[g \circ B_0 V F^n_s]$,

$$\int E^x[V F^n_s]\psi(dx) \to \int E^x[V F_s]\psi(dx) \quad \text{as } n \to \infty. \tag{3.25}$$

$$\int E^x[V F_s]\psi(dx) = E^\lambda[W F_t E^{B_t}[V F_s]] = E^\lambda[Y F_t (F_s \circ \theta_t)].$$

Thus, by (3.24) and (3.25),

$$\int Y F_{t+s}\,dP^\lambda = \int Y F_t(F_s \circ \theta_t)\,dP^\lambda,$$

and (3.22) is proved for $t \in C$. But then by Lemma 3.2, (3.22) is proved for all cases, since for any $t > 0$,

$$\int F(\{t\})\,dP^\lambda = 0. \tag{3.26}$$

Having established (3.22), the proof of Theorem 3.2 is finished by showing that for *any* stopping time measure F, such that (3.22) holds for a fixed probability λ with $m \ll \lambda$, we can find a multiplicative functional M such that (3.17) and (3.18) hold, and such that $M = F$, P^ν-a.e., for any $\nu \ll m$. This is a familiar type of regularization argument (cf. [**26**]), but appears to need checking, since initially F can be arbitrary on a set of P^λ-measure 0, and thus may be very badly behaved with respect to P^x for some x.

As a consequence of (3.22), we see easily that for any $t > 0$, and any sequence of positive real numbers ε_k with $\varepsilon_k \downarrow 0$, we have

$$F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} \quad \text{is almost surely decreasing in } k. \tag{3.27}$$

In particular $F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$ has a limit as $k \to \infty$, almost surely.

Let Y denote the limit of $F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$ as $k \to \infty$. By (3.22), $Y \geq F_t, P^\lambda$-a.e. On the other hand, if $\nu \ll m$, we have

$$\int Y\, dP^\nu = \lim_{k\to\infty} E^\nu[F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}] = \lim_{k\to\infty} E^{\nu_k}[F_{t-\varepsilon_k}],$$

where $\nu_k = \nu P_{\varepsilon_k}$. Since $\|\nu_k - \nu\| \to 0$ as $k \to \infty$, $\int Y\, dP^\nu = E^\nu[F_{t-}] = \int F_t\, dP^\nu$ by (3.26). This proves that if $\nu \ll m,\ t > 0$,

$$\lim_{k\to\infty} F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = F_t, \quad P^\nu\text{-a.e.} \tag{3.28}$$

We now define $N_t(\omega)$, for $t > 0$ and ω in C, by

$$N_t(\omega) = \inf\{F_{t-r} \circ \theta_r : r \text{ rational, } 0 < r < t\}. \tag{3.29}$$

If follows easily that for every $t > 0$, for every $s \geq 0$,

$$N_{t+s} \leq N_t \quad \text{everywhere.} \tag{3.30}$$

For any ε_k real positive, $\varepsilon_k \downarrow 0$, we find from (3.30) that for $t > 0$,

$$N_t = \lim_{k\to\infty} F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} \quad \text{a.s.,} \tag{3.31}$$

and hence that for any $\nu \ll m$,

$$N_t = F_t, \quad P^\nu\text{-a.e.} \tag{3.32}$$

From (3.31) and (3.22), for $t > 0,\ s \geq 0$,

$$N_{t+s} = N_t F_s \circ \theta_t, \quad \text{a.s.} \tag{3.33}$$

We define M, a stopping time measure, by

$$M_t = N_{t+} \quad \text{for } 0 \leq t < \infty, \qquad M_\infty = 1. \tag{3.34}$$

For $t > 0$, $N_{t+s+1/k} = N_t F_{s+1/k} \circ \theta_t$ almost surely, by (3.33). Thus $M_{t+s} = N_t F_s \circ \theta_t$ almost surely, using the right continuity of M and F. Hence $M_{t+s} = N_{t+s}$ almost surely, by (3.33) again. This shows that for $0 < t < \infty$,

$$M_t = N_t \quad \text{a.s.} \tag{3.35}$$

Fix $u > 0$, consider t, $0 < t < u$, and let $s = u - t$. Applying (3.33), $N_u = N_t F_{u-t} \circ \theta_t$, almost surely. Letting $t \downarrow 0$ through a sequence gives $N_u = M_0 N_u$ almost surely, by (3.34) and (3.31). Hence, by (3.34), $M_r = M_0 M_r$ almost surely for all $r \geq 0$. This proves that (3.17) holds for our M, when $t = 0$.

When $t > 0$, $M_{t+s} = N_{t+s} = N_t F_s \circ \theta_t$ almost surely by (3.35) and (3.33), and $F_s \circ \theta_t = N_s \circ \theta_t = M_s \circ \theta_t$ by (3.32) and (3.35), so (3.17) holds in all cases.

$M_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = N_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$ almost surely by (3.35) and (3.32), so (3.18) holds by (3.31) and (3.35). This proves Theorem 3.2.

REMARK 3.2. Let $M(n), M$ be multiplicative functionals such that $M(n) \to M$ stably. Let λ be a probability measure in $\mathcal{M}_0$ (defined in §1). Then $M(n) \to M$ stably with respect to P^λ.

PROOF. By Proposition 2.5 we may assume $\lambda \in \mathcal{M}_2$ (defined in §2) and it clearly does no harm to assume $m \ll \lambda$ also. Let F be a limit point of $(M(n))$ with respect to P^λ. F is a stopping time measure but not necessarily a multiplicative functional, although (3.22) must hold. We must show that $F = M$, P^λ-a.e. By choosing a subsequence and relabelling, we may assume that $M(n) \to F$ stably with respect to P^λ. Fix $u > 0$. It follows from the results in [**25**] that for any $\varepsilon > 0$, for $s > 0$ sufficiently small there exists a stopping time τ with $0 \le \tau \le u$ such that if λ_1 denotes the distribution of B_s with respect to P^λ and λ_2 denotes the distribution of B_τ with respect to P^{λ_1}, then $\|\lambda - \lambda_2\| < \varepsilon$. For each $t > 0$ and each n,

$$E^\lambda[M_{t+u}(n) \circ \theta_s] = E^{\lambda_1}[M_{t+u}(n)] = E^{\lambda_1}[M_\tau(n) M_{t+u-\tau}(n) \circ \theta_\tau],$$

by [**6**, 4.14]. Thus $E^\lambda[M_{t+u}(n) \circ \theta_s] \le E^{\lambda_1}[M_t(n) \circ \theta_\tau] = E^{\lambda_2}[M_t(n)]$, so

$$E^\lambda[M_{t+u}(n) \circ \theta_s] \le E^\lambda[M_t(n)] + \varepsilon.$$

By Lemma 3.1, $E^\lambda[M_{t+u} \circ \theta_s] \le E^\lambda[F_{t-}] + \varepsilon$. It follows that for any $t > 0$, $\lim_{s\downarrow 0} E^\lambda[M_{t-s} \circ \theta_s] \le E^\lambda[F_{t-}] = E^\lambda[F_t]$ by (3.26). The same argument used to prove (3.28) now shows $F_t = \lim_{k\to\infty} M_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}, P^\lambda$-a.e. and hence $F = M$, P^λ-a.e., proving the remark.

DEFINITION 3.4. Let M be a multiplicative functional, $Q_t(M)$ the corresponding sub-Markov semigroup. The resolvent $R_\alpha(M), \alpha > 0$, associated with M is defined by

(3.36)

$$R_\alpha(M) = \int_0^\infty e^{-\alpha t} Q_t(M)\, dt, \quad \text{i.e.} \quad R_\alpha(M)h(x) = E^x\left[\int_0^\infty e^{-\alpha t} M_t h \circ B_t\, dt\right].$$

We note that the usual resolvent equation argument shows that if M and N are multiplicative functionals and $R_\alpha(M) = R_\alpha(N)$ for one $\alpha > 0$ then $R_\beta(M) = R_\beta(N)$ for every $\beta > 0$.

We may consider $Q_t(M)$ and $R_t(M)$ as defined initially for h bounded Borel, and then extend to h in $L^p(R^d, m)$, since $Q_t(M)$ is a contraction for $1 \le p \le \infty$.

THEOREM 3.3. *Let M_n and M be multiplicative functionals. The following statements are equivalent:*

(i) *$M_n \to M$ stably as $n \to \infty$;*

(ii) *$R_\alpha(M_n) \to R_\alpha(M)$ strongly on $L^2(R^d, m)$ for each $\alpha > 0$ as $n \to \infty$.*

PROOF. Assume (i). Let λ be a probability measure with $m \ll \lambda$, $\lambda \ll m$. By Lemma 3.2 and Remark 3.1, for any $t > 0$,

$$\|\lambda Q_t(M_n) - \lambda Q_t(M)\| \to 0 \quad \text{as } n \to \infty, \tag{3.37}$$

where $Q_t(M)$ is defined by (3.19).

$Q_t(M_n), Q_t(M)$ are dominated by P_t, and hence have kernels $q_t(n), q_t$ which are dominated by the kernel p_t for P_t. By (3.20) and (3.37) we see that for every $x, \|(q_t(n)(x,\cdot) - q_t(x,\cdot))^+\|_1 \to 0$, and hence by (3.37) $q_t(n) \to q_t$ in $L^2(D \times R^d)$ for any compact D. It follows that $Q_t(M_n) \to Q_t(M)$ strongly on $L^2(R^d)$, and (ii) follows.

Conversely, assume (ii). Let N be a stable limit point of M_n. Then $R_\alpha(N) = R_\alpha(M)$ on $L^2(R^d, m)$. $Q_t(N)$ and $Q_t(M)$ are clearly strongly right continuous on $L^2(R^d, m)$ as functions of t. $Q_t(N) = Q_t(M)$ on $L^2(R^d, m)$ for a.e. t and hence for all t since as functions of t they have the same Laplace transforms. Hence, by (3.20), $Q_t(N) = Q_t(M)$ for all t, as sub-Markov operators. Thus $N = M$ by uniqueness, so (i) holds, and Theorem 3.3 is proved.

REMARK 3.3. Our argument show that if $M_n \to M$ stably and $f_n \to f$ weakly in $L^2(R^d)$ then for $t > 0$, $Q_t(M_n)f_n \to Q_t(M)f$ strongly in $L^2(D)$ for any compact D.

4. The spaces $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ were defined in §2. We will now define the multiplicative functional associated with any measure in $\mathcal{M}_0$. This is a standard construction for measures in $\mathcal{M}_1$. First we must recall the notion of the additive functional A_t associated with a general measure μ. A_t is additive in the sense that $A_{t+s} = A_t + A_s \circ \theta_t$ a.s. for all s, t. To begin with, let μ be a measure with a bounded density f with respect to Lebesgue measure on R^d. In this case the additive functional $A_t(\mu)$ associated with μ is simply

$$A_t(\mu) = \int_{[0,t]} f \circ B_s \, ds, \tag{4.1}$$

and the multiplicative functional associated with μ is then $M_t(\mu) = \exp(-A_t(\mu))$.

In order to state the general construction, it is convenient to introduce further potential operators: For each $\lambda > 0$, and each open set $D \subset R^d$, let G_λ^D denote the resolvent operator for the semigroup of Brownian motion killed on D^c, where we interpret G_λ^D as an operator from measures to functions. That is,

$$G_\lambda^D \mu(x) = \int_{\mathbf{R}^d} \int_{[0,\infty]} e^{-\lambda t} p_t(x,y) \, dt \mu(dy), \tag{4.2}$$

where $p_t(x,y)$ denotes the transition *density* for Brownian motion killed on D^c.

We will write $G_\lambda^{R^d}$ simply as G_λ. For $d \geq 3$, we will also allow $\lambda = 0$, and write $G_0 = G$ in this case, so that G is again just the classical potential operator, up to a constant factor. More generally, when D is a Green region, we will allow $\lambda = 0$, and of course G^D is just the classical Green operator for D in this case, up to a constant factor.

Consider μ in $\mathcal{M}_2$. It is easy to show that a finite measure μ is in $\mathcal{M}_2$ if and only if $G_\lambda \mu$ is bounded and continuous for one, and hence every $\lambda > 0$. For such measures, standard techniques (for a systematic development see [**6**, Chapters IV and VI, also **14**]) show that there is a continuous additive functional $A_t(\mu)$ associated with μ, characterized by the condition that for every $\lambda > 0$ and every $x \in R^d$,

$$G_\lambda \mu(x) = E^x \left[\int_{[0,\infty]} e^{-\lambda t} dA_t(\mu) \right]. \tag{4.3}$$

Clearly (4.3) agrees with (4.1) in the special case. We note that when $G\mu$ exists, $A_t(\mu)$ is the increasing process whose existence is guaranteed by the Doob-Meyer decomposition theorem applied to the supermartingale $G\mu \circ B_t$, such that $G\mu \circ B_t + A_t(\mu)$ is a martingale.

Since $A_t(\mu)$ is finite and continuous, $A_0(\mu) = 0$, and hence $A_t(\mu)$ is exact, in the sense that for every $t > 0$ and every sequence $\varepsilon_k \downarrow 0$, $A_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} \to A_t$ a.s. We also have the following useful fact [**6**, 6.3.1, 4.2.13]: if $\nu \in \mathcal{M}_2, q$ is bounded Borel, and $\mu = q\nu$, then

$$A_t(\mu) = \int_{[0,t]} q \circ B_s dA_s(\nu). \tag{4.4}$$

Next we consider $\mu \in \mathcal{M}_1$. We define $A_t(\mu)$ to be the limit of $A_t(\mu_n)$, where μ_n is any sequence of measures in $\mathcal{M}_2$ with $\mu_n \uparrow \mu$. $A_t(\mu)$ clearly is an additive functional, and is independent of the choice of the sequence μ_n. (4.4) continues to hold, so it is a routine exercise using the monotone convergence theorem to show that A_{t+} is an exact additive functional, and is continuous in t on the interval of times for which it is finite. We define the multiplicative functional $M(\mu)$ associated with μ, satisfying (3.17) and (3.18), by $M_t(\mu) = \exp(-A_{t+})$.

Before defining $M(\mu)$ for $\mu \in \mathcal{M}_0$, we must prove some facts.

LEMMA 4.1. *Let μ_1 and μ_2 be in $\mathcal{M}_0$. Then $\mu_1 \sim \mu_2$ if and only if $\mu_1(V) = \mu_2(V)$ for all finely open sets V.*

PROOF. Assume $\mu_1 \sim \mu_2$. Let D be a bounded open set. By the proof of 1.XI.10 in [**12**] we can choose a bounded continuous Green potential $q \equiv G^D\nu$ (even with $\nu \ll m$) such that for any finely open subset V of D, if h is the reduction (defined in [**12**, 1.III.4]) of q on V^c, then $V = \{q > h\}$, up to a polar set. If $v_k \equiv (k(q-h)) \wedge 1$ then $v_k \in H^1(R^d)$ and $v_k^2 \uparrow \mathbf{1}_{\{q>h\}}$, so (1.6) shows that $\mu_1(V) = \mu_2(V)$. The same relation for arbitrary V then follows.

Now suppose that μ_1, μ_2 are such that $\mu_1(V) = \mu_2(V)$ for every finely open set V. As mentioned earlier, any function u in $H^1(R^d)$ is quasi continuous, so that for any $\varepsilon > 0$ and any Green region D there is a set B_ε of capacity (relative to D, say) less than ε, such that the restriction of u to B_ε^c is continuous. We may enlarge B_ε to make it fine closed without changing its capacity. u is finely continuous on D at each point of the complement of B_ε, a finely open set. It follows that u on R^d is fine continuous at each point of a finely open set in R^d whose complement is polar. Let φ be any nonnegative continuous function on R and let $f = \varphi \circ u$. The inverse image under f of any open set differs from a finely open set by a polar set, so $\int_{[0,\infty)} \mu_1(\{f > t\})\,dt = \int_{[0,\infty)} \mu_2(\{f > t\})\,dt$, or

$$\int f\,d\mu_1 = \int f\,d\mu_2. \tag{4.5}$$

In particular, taking $\varphi(x) = x^2$ proves Lemma 4.1.

A direct proof of (4.5) from (1.6) is also easy.

REMARK 4.1. Let $\mu_1, \mu_2 \in \mathcal{M}_0$ and say f is good if (4.5) holds. Let D be a Green region, W a finely open subset of D, such that for any probability $\nu \ll m$ on R^d with $q \equiv G^D\nu$ bounded and continuous, and any finely open subset V of W, we have $q - h$ good, where h denotes the reduction of q on V^c. Then $\mu_1 = \mu_2$ on all finely open subsets of W. Indeed, $q - q \wedge (h + \alpha)$ is good by [**12**, 1.XI.16], so $(q - h) \wedge \alpha$ is the difference of good functions, and is easily seen to be good. The argument of Lemma 4.1 now applies, so $\mu_1(V) = \mu_2(V)$ as claimed.

Let $\mu \in \mathcal{M}_2$, ν any probability measure on R^d. Let D be a Green region, σ the first exit time of D. Let τ_0 and τ_1 be finite stopping times $\leq \sigma$, with $\tau_0 \leq \tau_1$. Let ν_i denote the distribution on R^d of B_{τ_i} with respect to P^ν. $G^D\mu \circ B_t + A_t(\mu)$ is a martingale for $t \leq \sigma$. Hence

$$\int G^D\mu\,d\nu_0 - \int G^D\mu\,d\nu_1 = E^\nu\left[\int_{[\tau_0,\tau_1]} dA_t(\mu)\right],$$

and so

$$\int G^D(\nu_0 - \nu_1)\,d\mu = E^\nu\left[\int_{[\tau_0,\tau_1]} dA_t(\mu)\right]. \tag{4.6}$$

Obviously (4.6) remains true when $\mu \in \mathcal{M}_1$ and when τ_0, τ_1 are randomized.

LEMMA 4.2. *Let $\mu_1, \mu_2 \in \mathcal{M}_1$. Then $\mu_1 \sim \mu_2$ if and only if $M(\mu_1) = M(\mu_2)$ a.s.*

PROOF. Assume $\mu_1 \sim \mu_2$. Let ν be a probability such that $G^D\nu$ is bounded for each Green region D. Let τ be any finite stopping time. Let D be bounded open, and σ the first exit time of D. Let $\tau_0 = 0$, $\tau_1 = \tau \wedge \sigma$. Then

$$\int G^D(\nu - \nu_1)\,d\mu_i = E^\nu\left[\int_{[0,\tau\wedge\sigma]} dA_t(\mu_i)\right] \quad \text{for } i = 1, 2,$$

by (4.6). Letting D expand to R^d, we see by (4.5) that

$$E^\nu\left[\int_{[0,\tau]} dA_t(\mu_1)\right] = E^\nu\left[\int_{[0,\tau]} dA_t(\mu_2)\right].$$

This in turn implies $A_t(\mu_1) = A_t(\mu_2)$ P^ν-a.e., by Theorem 2.6 of [**15**], so $M(\mu_1) = M(\mu_2)$ P^ν-a.e. By exactness, $M(\mu_1) = M(\mu_2)$ a.s.

Conversely, assume $M(\mu_1) = M(\mu_2)$ a.s. Let D and σ be as above. For every stopping time $\tau \leq \sigma$, and every probability measure ν on D, if ν_1 denotes the distribution of B_τ with respect to P^ν then by (4.6),

$$\int G^D(\nu - \nu_1)\,d\mu_1 = \int G^D(\nu - \nu_1)\,d\mu_2,$$

and hence $\mu_1 \sim \mu_2$ by Remark 4.1, proving the lemma.

DEFINITION 4.1. For any $\mu \in \mathcal{M}_0$, let $M(\mu)$ be defined a.s. by $M(\mu')$, where $\mu' \in \mathcal{M}_1$ with $\mu \sim \mu'$. We denote the randomized stopping time associated with $M(\mu)$ by $T(\mu)$.

We note that $M(\mu)$ has been defined in terms of μ by a probabilistic construction. As in §3, we can then define the resolvent $R_\lambda(M(\mu))$ corresponding to the semigroup associated with $M(\mu)$. At the same time, we can consider the resolvent $R_\lambda(\mu)$ defined by the variational problem discussed in §2. We now prove:

THEOREM 4.1. *Let μ be in $\mathcal{M}_0, \lambda > 0$. Then $R_\lambda(\mu) = R_\lambda(M(\mu))$.*

PROOF. As usual we may take $\mu \in \mathcal{M}_1$. Let μ be expressed as $\mu = q\,d\nu$, where $\nu \in \mathcal{M}_2$ and $q\colon R^d \to [0,\infty]$ is Borel. Let $\mu_n = (q \wedge n)\,d\nu$. Then $\mu_n \uparrow \mu$, so $R_\lambda(\mu_n) \to R_\lambda(\mu)$ strongly, by Theorem 2.1. Also $M_t(\mu_n) = \exp(-A_t(\mu_n))$ decreases pointwise to $\exp(-A_t(\mu)) = M_{t-}(\mu)$, so as a random measure $M(\mu_n)$ converges weakly to $M(\mu)$ on $[0,\infty]$, pointwise for each ω. Thus $M(\mu_n) \to M(\mu)$ stably for any probability measure P^ν, so in particular $M(\mu_n) \to M(\mu)$ stably. Thus $R(M(\mu_n)) \to R(M(\mu))$ strongly, by Theorem 3.3. Hence it is enough to show $R_\lambda(\mu) = R_\lambda(M(\mu))$ for $\mu \in \mathcal{M}_2$.

Accordingly, let μ be a measure in $\mathcal{M}_2$. We must show that $R_\lambda(\mu) = R_\lambda(M(\mu))$. Let $\nu_k = \mu P_{1/k}$, where P_t is the usual Brownian motion semigroup. The usual arguments (cf. [**6**, IV.3.8]) show that for any $x \in R^d$,

$$E^x[(A_t(\nu_k) - A_t(\mu))^2] \to 0 \quad \text{as } k \to \infty. \tag{4.7}$$

Let $\tilde{k}_j$ be any subsequence. We can find a subsequence k_i of $\tilde{k}_j$, such that $A_t(\nu_{k_i}) \to A_t(\mu)$ pointwise P^x-a.e., for all rational t. Thus as a measure $M(\nu_{k_i})$ converges weakly to $M(\mu)$ on $[0,\infty]$, for P^x-a.e. ω. Hence $M(\nu_{k_i})$ converges stably to $M(\mu)$. Since $\tilde{k}_j$ was any subsequence of the original sequence, we see that $M(\nu_k) \to M(\mu)$ stably, for all $x \in R^d$.

In particular we have shown that $M(\nu_k) \to M(\mu)$, so $R_\lambda(M(\nu_k)) \to R_\lambda(M(\mu))$ strongly. By [**8**, Proposition 4.12], ν_k γ-converges to μ, so by Theorem 2.1, $R_\lambda(\nu_k) \to R_\lambda(\mu)$ strongly. Thus it is sufficient to consider μ in $\mathcal{M}_2$ such that μ has a C^∞ density with respect to Lebesgue measure. In this case both $R_\lambda(\mu)$ and $R_\lambda(M(\mu))$ are defined by the same classical differential equation, so Theorem 4.1 is proved.

COROLLARY. *For any open set D, let σ be the first exit time of D. Let $M^D(\mu)$ be the multiplicative functional corresponding to $T(\mu) \wedge \sigma$. Then for any $\lambda > 0$, $R^D_\lambda(\mu) = R_\lambda(M^D(\mu))$.*

PROOF. The same argument used in the proof of Theorem 4.1 can be used again. Or by Proposition 2.2 and the method of Example 4.1 below we can just apply Theorem 4.1 to the measure $\mu + \infty_E$, where $E = D^c$.

THEOREM 4.2. *A sequence $\mu_n \in \mathcal{M}_0$ γ-converges if and only if the corresponding sequence $M(\mu_n)$ converges stably.*

PROOF. By Theorem 2.1, μ_n γ-converges if and only if the resolvents $R_\lambda(\mu_n)$ converge strongly. $M(\mu_n)$ converges if and only if $R_\lambda(M(\mu_n))$ converges strongly by Theorem 3.3. Since $R_\lambda(\mu_n) = R_\lambda(M(\mu_n))$ by Theorem 4.1, the result is proved.

REMARK 4.2. Since γ-convergence and stable convergence are now linked, we see that the Corollary to Lemma 3.1 gives a probabilistic proof of Theorem 2.1.

EXAMPLE 4.1. Let K be any Borel set in R^d, $d \geq 3$. Let λ be a probability, $\lambda \ll m$, $m \ll \lambda$. Let τ be the first hitting time of K, and let ν denote the distribution of B_τ on $\{\tau < \infty\}$ with respect to P^λ, i.e. ν is the swept measure of λ on K.

THEN: $\infty\nu \sim \infty_K$, and $M(\infty\nu)$ is the stopping time measure for τ, where ∞_K is given by Definition 2.2, and $\infty\nu(A) = \infty$ if $\nu(A) > 0$, $\infty\nu(A) = 0$ otherwise.

PROOF. Clearly it does not change $\infty\nu$ if we replace λ by any probability which is mutually absolutely continuous with respect to λ. Thus without loss of generality we assume that λ has a bounded density with respect to m. Then $G\lambda$ is bounded and $\geq G\nu$. Let $\sigma = \inf\{t\colon A_t(\nu) > 0\}$. We claim:

(a) $\sigma = \tau, P^x$-a.e., for every $x \in R^d$, and

(b) $\nu(V) > 0$ for any finely open set V such that $V \cap K$ is not polar.

PROOF OF (a). Clearly $G\lambda = G\nu$, quasi everywhere on K. Thus, by (4.6), $E^\lambda[A_\tau(\nu)] = 0$. Hence $A_\tau(\nu) = 0$, P^λ-a.e., so $\tau \leq \sigma$, P^λ-a.e.

Let ψ denote the distribution of B_σ on $\{\sigma < \infty\}$ with respect to P^λ. Then $G\nu \geq G\psi$. By (4.6), $\int G(\lambda - \psi)\,d\nu = 0$. Thus $G\psi = G\lambda \geq G\nu$, ν-a.e. Hence by the domination principle, $G\psi \geq G\nu$. Thus $G\psi = G\nu$. Since $E^\lambda[(\sigma - \tau)\mathbf{1}_{\{\tau<\infty\}}] = \int G(\nu - \psi)\,dm = 0$, this proves $\tau = \sigma$, P^λ-a.e. Since $\tau = \lim_{t\downarrow 0} \tau \circ \theta_t$ and $\sigma = \lim_{t\downarrow 0} \sigma \circ \theta_t$, and $\tau \circ \theta_t = \sigma \circ \theta_t$, P^x-a.e., for every $x \in R^d$, (a) is proved.

PROOF OF (b). Let V be finely open. As noted in the proof of Lemma 4.1, we can find measures ρ_1, ρ_2 such that $G\rho_1$ is bounded and continuous, $G\rho_1 \geq G\rho_2$, and $Y \equiv \{G\rho_1 > G\rho_2\}$ differs from V by a polar set. Let φ_i be the swept measure of ρ_i on K. Then $G\varphi_1 \geq G\varphi_2$. Let $W = \{G\varphi_1 > G\varphi_2\}$. Then $W \cap K = V \cap K$ up to a polar set. $\int G(\varphi_1 - \varphi_2)\, d\nu = \int G(\varphi_1 - \varphi_2)\, d\lambda$. W is finely open, so $\lambda(W) > 0$. Thus $\nu(W \cap K) > 0$. This proves (b). (a) and (b) clearly imply the result.

A similar construction in R^2 shows that $M(\infty_K) =$ the first hitting time of K in this case also.

5. In this section we shall illustrate the earlier results by proving some facts relating to Theorem 5.10 of [**8**].

LEMMA 5.1. *Let u_n and ν_n be two sequences in $\mathcal{M}_0$ such that μ_n γ-converges to μ and ν_n γ-converges to ν. Let Z be a finely open set in R^d. Suppose that $\mu_n = \nu_n$ on all finely open subsets of Z. Then $\mu = \nu$ on all finely open subsets of Z.*

PROOF. We may rephrase the lemma as follows: let μ_n, μ be in $\mathcal{M}_0$, such that μ_n γ-converges to μ. Let Z be a finely open set in R^d. Let $\nu_n = \mathbf{1}_Z \mu_n$, and let ψ be any γ-limit point of ν_n. Then $\psi = \mu$ on finely open subsets of Z. Clearly we may assume that $Z \subset$ a bounded open set D.

Replacing μ_n and μ by equivalent measures, we may assume that $\mu_n, \mu \in \mathcal{M}_1$. Let $M(n) = M(\mu_n)$, $M = M(\mu)$, $N(n) = M(\nu_n)$, $N = M(\psi)$. By relabelling, assume $N(n) \to N$ stably. Let $\tau =$ the first exit time of Z. Fix $t > 0$, and let $Y = \mathbf{1}_{\{\tau > t\}}$. Let ν be any probability measure, $\nu \ll m$. By Lemma 3.2, for $s > 0$, $M(\{s\}) = 0$ and $N(\{s\}) = 0$, P^ν-a.e., and hence, by Lemma 3.1, for any $H \in L^1(C, \mathcal{G}, P^\nu)$,

$$\int Y H \mathbf{1}_{[0,s]}\, dM(n)\, dP^\nu \to \int Y H \mathbf{1}_{[0,s]}\, dM\, dP^\nu$$

and

$$\int Y H \mathbf{1}_{[0,s]}\, dN(n)\, dP^\nu \to \int Y H \mathbf{1}_{[0,s]}\, dN\, dP^\nu.$$

Since $M_\mu(n) = N_u(n)$ for $u < \tau$, we have, for $0 < s \leq t$, $\int Y H \mathbf{1}_{[0,s]}\, dM\, dP^\nu = \int Y H \mathbf{1}_{[0,s]}\, dN\, dP^\nu$. It follows that $M_s = N_s$, P^ν-a.e., on $\{\tau > t\}$, for $0 < s \leq t$. Hence $M_s = N_s$, P^ν-a.e., on $\{\tau > s\}$. Thus $A_{s+}(\mu) = A_{s+}(\psi)$ for $0 < s < \tau$, in the notation of §4. Hence $A_s(\mu) = A_s(\psi)$ for $0 \leq s \leq \tau$. It follows from (4.6) that for every stopping time $\sigma \leq \tau$, if ν_1 denotes the distribution of B_σ with respect to P^ν, then $\int G^D(\nu - \nu_1)\, d\mu = \int G^D(\nu - \nu_1)\, d\psi$. Hence, by Remark 4.1, $\mu = \psi$ on any finely open subset of Z, so Lemma 5.1 is proved.

DEFINITION 5.1. For any measure $\mu \in \mathcal{M}_0$, the *set of finiteness* $W(\mu)$ for μ is the union of all finely open sets V such that $\mu(V) < \infty$.

LEMMA 5.2. *Let μ_n, μ be in $\mathcal{M}_0$, such that μ_n γ-converges to μ. Let $W = W(\mu)$. Let A be a Borel set in R^d such that $\mu(\text{fine-}\partial A) = 0$. Let H be the fine-interior of A. Suppose that* $(\text{fine-}\partial H) \cap (\text{fine-}\partial W)$ *and* $(\text{fine-}\partial A - (\text{fine-}\partial H)) \cap W^c$ *are polar. Let $\nu_n = \mathbf{1}_A \mu_n, \nu = \mathbf{1}_A \mu$. Then ν_n γ-converges to ν.*

PROOF. Let $G =$ fine-interior of A^c. Let $\varphi_n = \mathbf{1}_{A^c} \mu_n$, $\varphi = \mathbf{1}_{A^c} \mu$. Let ψ, λ be any γ-limit points of ν_n, φ_n, respectively. By Lemma 5.1, $\nu \leq \psi$ and $\varphi \leq \lambda$ on all finely open sets. Also, by Lemma 5.1, since $\chi_G \psi =$ the limit of $\chi_G \nu_n$ on finely open subsets of G, $\psi(G) = 0$. Similarly $\lambda(H) = 0$.

Since $\nu_n + \varphi_n = \mu_n$ γ-converges to μ, we must have $\psi + \lambda \equiv \mu$, so $\psi + \lambda = \mu = \nu + \varphi$ on all finely open sets. It follows that $\psi = \nu$ and $\lambda = \varphi$ on all finely open subsets of W.

Now let S be any finely open set. Let

$$Z = S - [\{(\text{fine-}\partial H) \cap (\text{fine-}\partial W)\} \cup \{(\text{fine-}\partial A - \text{fine-}\partial H) \cap W^c\}].$$

Let $x \in Z$. We consider four cases.

Case (i). $x \in H$. Then $\nu(Z \cap H) = \psi(Z \cap H)$ by Lemma 5.1.

Case (ii). $x \in G$. Then $\nu(Z \cap G) = \psi(Z \cap G) = 0$.

Case (iii). $x \in W$. Then $\nu(Z \cap W) = \psi(Z \cap W)$.

The remaining possible case is $x \in \text{fine-}\partial H$, x not in fine-closure W. Let D be a finely open set, $D \subset Z$, $x \in D$, such that $D \cap W = \varnothing$. $D \cap H \neq \varnothing$, so $\nu(D \cap H) = \mu(D \cap H) = \infty = \psi(D \cap H)$. We have shown that in every case, x is contained in a finely open subset D of Z with $\nu(D) = \psi(D)$. Since the fine topology has the quasi-Lindelöf property, and ν and ψ are in $\mathcal{M}_0$, $\nu(Z) = \psi(Z)$, so $\nu(S) = \psi(S)$. Thus $\nu \equiv \psi$, and Lemma 5.2 is proved.

As a corollary, we see that if μ is Radon, so that $W^c = \varnothing$, then ν_n γ-converges to ν whenever $\mu(\text{fine-}\partial A) = 0$, in particular when $\mu(\partial A) = 0$. This is a special case of a more general criterion obtained in [**8**, §5].

For a general $\mu \in \mathcal{M}_0$, we note that the condition that $(\text{fine-}\partial A - \text{fine-}\partial H) \cap W^c$ be polar is trivially satisfied when $A \subset$ the fine closure of its fine interior, for example when A is an open or closed ball.

6. In this section we give some results relating to the probabilistic solution of the μ-Dirichlet problem.

Let $\mu \in \mathcal{M}_2$. Let M denote $M(\mu)$, and let T denote the randomized stopping time corresponding to M. Let τ be a stopping time, $\tau \leq$ the first exit time of some Green region D. Let ν be any probability measure on R^d, and let ν_1 be the distribution of $B_{\tau \wedge T}, \psi$ the distribution of B_τ on $\{T > \tau\}$, both distributions with respect to $P^\nu \times m_1$. Let h be a bounded Borel function on R^d. By Fubini,

$$\begin{aligned} E^\nu\left[\int_{[0,\tau\wedge T]} h \circ B_t \, dA_t(\mu)\right] &= E^\nu\left[\int_{[0,\tau]} h \circ B_t M_t \, dA_t(\mu)\right] \\ &= E^\nu\left[\int_{[0,\tau]} h \circ B_t M(dt)\right]. \end{aligned}$$

Thus by (4.6) and (4.4)

$$\int G^D(\nu - \nu_1) h \, d\mu = \int h \, d\nu_1 - \int h \, d\psi. \tag{6.1}$$

LEMMA 6.1. *Let $\mu \in \mathcal{M}_2$. Let D be open in R^d, $u \in H^1_{\text{loc}}(D), u$ μ-harmonic on D. Let τ be a stopping time, $\tau \leq$ the first exit time of some compact subset K of D. Then for quasi every $x \in D$,*

$$u(x) = E^x[u \circ B_\tau M_\tau]. \tag{6.2}$$

PROOF. Clearly we may assume that D is bounded. By Proposition 2.6, u can be made continuous on D by changing the values of u on a polar set. Thus we assume that u is continuous and bounded.

For any v which is C^∞ with compact support in D, since u is μ-harmonic we have

$$\int u(-\Delta v)\, dm = -\int uv\, d\mu. \tag{6.3}$$

Let ν and λ be probability measures with compact support in D, such that $G^D\nu = G^D\lambda$ outside a compact subset of D. Let φ be C^∞ with compact support on R^d, φ nonnegative and radially symmetric, $\int \varphi\, dm = 1$. Define φ_δ for $\delta > 0$ by $\varphi_\delta(x) = \varphi(x/\delta)/\delta^d$. Then for δ small, $\nu_\delta \equiv \varphi_\delta * \nu$ and $\lambda_\delta \equiv \varphi_\delta * \lambda$ have compact support in D, and $G^D\nu_\delta = G^D\lambda_\delta$ outside a compact subset of D. Letting $v = G^D(\nu_\delta - \lambda_\delta)$ in (6.3),

$$\int u\, d\nu_\delta - \int u\, d\lambda_\delta = -\int uG^D(\nu_\delta - \lambda_\delta)\, d\mu. \tag{6.4}$$

Letting $\delta \to 0$, since $\nu_\delta \to \nu$, $\lambda_\delta \to \lambda$ weakly, we have $\int u\, d\nu - \int u\, d\lambda$ as the limit of the left side of (6.4).

$G^D\nu_\delta \uparrow G^D\nu$, $G^D\lambda_\delta \uparrow G^D\lambda$ pointwise, and $\int G^D\nu\, d\mu < \infty$, $\int G^D\nu\, d\mu < \infty$ since $G^D\mu$ is bounded. Since $|u|$ is bounded, we have $-\int uG^D(\nu - \lambda)\, d\mu$ as the limit of the right side of (6.4), by the dominated convergence theorem. Thus

$$\int u\, d\nu - \int u\, d\lambda = -\int uG^D(\nu - \lambda)\, d\mu. \tag{6.5}$$

In particular, when $\nu = \delta_x$, and $\lambda = \nu_1$ as in (6.1), (6.5) holds. (6.5) gives $\int u\, d\nu = \int u\, d\psi$, and hence (6.2), proving Lemma 6.1.

LEMMA 6.2. *Let $\mu \in \mathcal{M}_2$. Let $M = M(\mu)$. Let D be open in R^d, $u \in H^1_{\text{loc}}(D) \cap L^2_{\text{loc}}(D,\mu)$. Suppose that for any open ball K with compact closure in D,*

$$u(x) = E^x[u \circ B_\tau M_\tau], \quad \textit{for } m\text{-a.e. } x \in K, \tag{6.6}$$

where $\tau = \tau_K$ denotes the first exit time of K.

Then u is μ-harmonic on D.

PROOF. Let K be fixed. Let w be the solution to the μ-Dirichlet problem on K with data u on ∂K. Let K_n be a sequence of balls with the same center as K such that $K_n \subset K$ and $K_n \uparrow K$. Let τ_n be the first exit time of K_n. Then $\tau_n \uparrow \tau$. Define $\psi_x, \psi_x(n)$ by $\int h\, d\psi_x = E^x[h \circ B_\tau M_\tau]$, $\int h\, d\psi_x(n) = E^x[h \circ B_{\tau_n} M_{\tau_n}]$. $\int w\, d\psi_x(n) \to \int u\, d\psi_x$ as $n \to \infty$. $\psi_x(n)$ converges to ψ_x in energy norm, so $\psi_x(n) \to \psi_x$ in $H^{-1}(R^d)$, and $\int w\, d\psi_x(n) \to \int u\, d\psi_x$ as $n \to \infty$. But $\int w\, d\psi_x(n) = w(x)$ for q.e. x, by Lemma 6.1, while $\int u\, d\psi_x = u(x)$ for m-a.e. x by (6.6). Thus $w = u$, m-a.e. Thus $w = u$ q.e., and Lemma 6.2 is proved.

THEOREM 6.1. *Let $\mu \in \mathcal{M}_0$, $M = M(\mu)$. Let D be open in R^d. Let u be in $H^1_{\text{loc}}(D) \cap L^2_{\text{loc}}(D,\mu)$. The following statements are equivalent:*

(i) *u is locally μ-harmonic on D;*

(ii) *if τ is a stopping time, $\tau \leq$ the first exit time of an open set U with compact closure in D, then for quasi every $x \in D$, if $\lambda \in H^{-1}(R^d)$, where λ is the distribution of B_τ with respect to P^x (in particular if $\tau \geq$ the first exit time of some open set around x), then*

$$u(x) = E^x[u \circ B_\tau M_{\tau-}]; \tag{6.7}$$

(iii) *for every* $\nu \in \mathcal{M}_2$ *with* $\nu(D^c) = 0$, *and every open ball* K *with compact closure in* D,

$$\int u\,d\nu = E^{\nu}[u \circ B_{\tau} M_{\tau-}], \text{ where } \tau = \tau_K, \text{ the first exit time of } K. \tag{6.8}$$

PROOF. (i)$\Rightarrow$(ii) Let U, τ be given. We may assume $\mu \in \mathcal{M}_1$ and U is a finite union of open balls. There exist measures $\mu_n \in \mathcal{M}_2$ with $\mu_n \uparrow \mu$. Let $u_n^{\pm}$, $n = 1, 2, 3$, be the solutions to the μ_n-Dirichlet problem on U with fixed data $u^{\pm}$ on ∂G. By Lemma 6.2, if $M(n) = M(\mu_n)$, for quasi every x in U we have

$$u_n^{\pm}(x) = E^x[u^{\pm} \circ B_{\tau} M_{\tau}(n)]. \tag{6.9}$$

By Proposition 2.7, $u_n^+ - u_n^-$ converges to u q.e. on U. Consider $x \in U$ such that $u_n^+(x) - u_n^-(x) \to u(x)$ and such that (6.9) holds for all n. Suppose λ is in $H^{-1}(R^d)$, where λ is the distribution of B_{τ} with respect to P^x. Then $E^x[u^{\pm} \circ B_{\tau} M_{\tau}(1)] < \infty$. Since $M_{\tau}(n) \downarrow M_{\tau-}$ as $n \to \infty$, the dominated convergence theorem gives

$$\lim_{n \to \infty} E^x[u^{\pm} \circ B_{\tau} M_{\tau}(n)] = E^x[u^{\pm} \circ B_{\tau} M_{t-}].$$

Thus (ii) holds.

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(i) Given K, $\tau = \tau_K$, define $w(x) = E^x[u \circ B_{\tau} M_{\tau-}]$ for every $x \in D$. Let $A = \{w > u\}$. If A is not polar, we can find $\nu \in \mathcal{M}_2$ with compact support in K and $\nu(A^c) = 0$. Then $\int w\,d\nu > \int u\,d\nu$. But $\int w\,d\nu = E^{\nu}[u \circ B_{\tau} M_{\tau-}] = \int u\,d\nu$ by (6.8), contradiction. Thus A is polar. Similarly $\{w < u\}$ is polar. Hence $w = u$ quasi everywhere. Thus, for every choice of K, letting $\tau = \tau_K$, for quasi every $x \in D$,

$$u(x) = E^x[u \circ B_{\tau} M_{\tau-}]. \tag{6.10}$$

Now let K be fixed, $\tau = \tau_K$. Let $\mu_n \in \mathcal{M}_2$, $\mu_n \uparrow \mu$. Let $u_n^{\pm}$ solve the μ_n-Dirichlet problem on K with data $u^{\pm}$ on ∂K. Again $u_n^+ - u_n^- \to f$ quasi everywhere, where f is the solution of the μ-Dirichlet problem on K with data u on ∂K. As in the earlier argument, for every x in K, the dominated convergence theorem shows

$$\lim_{n \to \infty} E^x[u^{\pm} \circ B_{\tau} M_{\tau}(n)] = E^x[u^{\pm} \circ B_{\tau} M_{\tau-}].$$

By (6.10) we then have $u = f$ quasi everywhere on K. Thus u is μ-harmonic locally on D. This proves Theorem 6.1.

Let D be a bounded open set. Consider the μ-Dirichlet problem for u on D with data g on ∂D. As usual we assume $g \in H^1(R^d)$ and $u - g \in H_0^1(D)$. Let D_n be open, $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. Let τ_n, τ be the first exit times of D_n, D respectively. Then $\tau_n \uparrow \tau$. Fix $x \in D$. Let $\psi_x(n), \psi_x$ be defined by $\int h\,d\psi_x = E^x[h \circ B_{\tau} M_{\tau-}]$, $\int h\,d\psi_x(n) = E^x[h \circ B_{\tau_n} M_{\tau_n-}]$, for any h bounded Borel on R^d. Since $M_{\tau_n-} \downarrow M_{\tau-}$, it is easy to see that $\psi_x(n) \to \psi_x$ in $H^{-1}(R^d)$, so that $\int u\,d\psi_x(n) \to \int g\,d\psi_x$ as $n \to \infty$. For q.e. x, $u(x) = \int u\,d\psi_x(n)$ for all n. Thus $u(x) = \int g\,d\psi_x = E^x[g \circ B_{\tau} M_{\tau-}]$, so we have shown:

REMARK 6.1. The solution u of the μ-Dirichlet problem on D with data g is given, for quasi every $x \in D$, by

$$u(x) = E^x[g \circ B_{\tau} M_{\tau-}], \quad \text{where } \tau \text{ is the first exit time of } D. \tag{6.11}$$

We recall that a point $x \in R^d$ is called a *regular* Dirichlet point for μ if every u which is μ-harmonic near x is continuous at x and vanishes there. On the other hand, a point x is called *permanent* for a multiplicative functional M if $P^x(M_0 = 1) = 1$. The Blumenthal 0-1 law shows that if x is not permanent then $P^x(M_0 = 1) = 0$.

THEOREM 6.2. *A point x is regular for μ if and only if $P^x(M_0(\mu) = 0) = 0$.*

PROOF. (i) Suppose x is regular. Let K be a small ball centered at x. Let u be the solution of the μ-Dirichlet problem with data $\equiv 1$ on ∂K. For quasi every y in K,

$$u(y) = E^y[M_{\tau-}], \quad \text{where } \tau \text{ denotes the first exit time of } K. \tag{6.12}$$

For any $t > 0$, and any $z \in K$,

$$\begin{aligned} E^z[\mathbf{1}_{\{\tau>t\}} u \circ B_t] &= E^z[\mathbf{1}_{\{\tau>t\}} E^{B_t}[M_{\tau-}]] \\ &= E^z[\mathbf{1}_{\{\tau>t\}} E[M_{\tau-} \circ \theta_t \mid \mathcal{G}_t]] \geq E^z[E[\mathbf{1}_{\{\tau>t\}} M_t M_{\tau-} \circ \theta_t \mid \mathcal{G}_t]] \\ &= E^z[E[\mathbf{1}_{\{\tau>t\}} M_{(t+\tau\circ\theta_t)-} \mid \mathcal{G}_t]] = E^z[E[\mathbf{1}_{\{\tau>t\}} M_{\tau-} \mid \mathcal{G}_t]] \\ &= E^z[\mathbf{1}_{\{\tau>t\}} E[M_{\tau-} \mid \mathcal{G}_t]] = E^x[E[M_{\tau-} \mid \mathcal{G}_t]] - E^z[\mathbf{1}_{\{\tau\leq t\}} E[M_{\tau-} \mid \mathcal{G}_t]]. \end{aligned}$$

Thus for $z \in K$, $t > 0$,

$$E^z[\mathbf{1}_{\{\tau>t\}} u \circ B_t] \geq E^z[M_{\tau-}] - P^z(\tau \leq t). \tag{6.13}$$

Since u is continuous and $u(y) \to 0$ as $y \to x$, for q.e. y, and $0 \leq u \leq 1$, we see that $E^x[\mathbf{1}_{\{\tau>t\}} u \circ B_t] \to 0$ as $t \to 0$. Since $P^x(\tau \leq t) \to 0$ as $t \to 0$, taking $z = x$ in (6.13) we have $E^x[M_{\tau-}] = 0$, so $M_{\tau-} = 0$, P^x-a.e. Since this is true for all K, we must have $M_0 = 0$, P^x-a.e.

(ii) Suppose that $M_0 = 0$, P^x-a.e. Let K be a fixed open ball centered at x, $\tau = \tau_K$. Let ψ_y be defined by $\int h\, d\psi_y = E^y[h \circ B_\tau M_{\tau-}]$, for h bounded Borel on R^d, $y \in K$. Let $u(y)$ be defined for all $y \in R^d$ by (6.12). For any $z \in K$, $u(z) \leq P^x(\tau \leq t) + E^z[\mathbf{1}_{\{\tau>t\}} M_{\tau-}]$.

$$\begin{aligned} E^z[\mathbf{1}_{\{\tau>t\}} M_{\tau-}] &\leq E^z[\mathbf{1}_{\{\tau>t\}} M_{(t+\tau\circ\theta_t)-}] \\ &\leq E^z[M_{(t+\tau\circ\theta_t)-}] = E^z[M_t M_{(\tau\circ\theta_t)-} \circ \theta_t] \leq E^z[(M_{\tau-}) \circ \theta_t]. \end{aligned}$$

Thus $\limsup_{z\to x} u(z) \leq P^x(\tau \leq t) + E^x[(M_{\tau-}) \circ \theta_t]$, for every $t > 0$. $\tau > 0$, P^x-a.e. Let σ be a positive, measurable function taking rational values such that $\sigma < \tau$, P^x-a.e. Then $M_{\tau-} \circ \theta_t \leq M_\sigma \circ \theta_t$. By (3.18), $M_\sigma \circ \theta_t \to M_\sigma$, P^x-a.e. Thus

$$\limsup_{t\to 0} E^x[(M_{\tau-}) \circ \theta_t] \leq E^x[M_\sigma] \leq E^x[M_0] = 0.$$

Thus $\limsup_{z\to x} u(z) = 0$, so $\psi_z \to 0$ in total variation norm as $z \to x$. The measures ψ_z are dominated by a finite uniform measure on ∂K, so $\psi_z \to 0$ in $H^{-1}(R^d)$, so that for any v in H^1 near ∂K, $\int v\, d\psi_z \to 0$ as $z \to x$. Now let u be an arbitrary μ-harmonic function near x. For K small, and quasi every z near x, $u(z) = \int u\, d\psi_z$. Thus for these z, $u(z) \to 0$ as $z \to x$. This proves Theorem 6.2.

We now give one more criterion for regularity at a point. To avoid trivial details, we assume $d \geq 3$.

LEMMA 6.3. *Let $d \geq 3$, $\mu \in \mathcal{M}_0$, $x \in R^d$. Then x is regular for μ if and only if for every finely open set V containing x, $G\rho(x) = \infty$, where $\rho = \mathbf{1}_V \mu$.*

PROOF. We may take $\mu \in \mathcal{M}_1$. Suppose $G\rho(x) < \infty$ for some V. Let τ be the first hitting time of V^c. By (4.6), $E^x[A_\tau(\mu)] < \infty$. Thus $A_{0+}(\mu) < \infty$, P^x-a.e., so $M_0(\mu) > 0$, P^x-a.e., and so x is not regular.

Conversely, suppose $M_0(\mu) > 0$, P^x-a.e. Let τ be the first time $A_t(\mu) \geq 1$. $\tau > 0$, P^x-a.e. Since Brownian motion has predictable σ-fields, we can find a stopping time σ with $\sigma < \tau$, P^x-a.e., and $P^x(\sigma > 0) = \alpha > 0$. $E^x[A_\sigma(\mu)] < \infty$, so by (4.6), $\int G(\nu - \nu_1)\, d\mu < \infty$, where $\nu = \delta_x$ and ν_1 is the distribution of B_σ with respect to P^ν. By [**12**, 1.XI.4], fine-limit$_{y \to x}$ $G(\nu - \nu_1)(y)/G\nu(y) = 1 - \alpha$, so we can find a fine-open set V containing x such that on V, $G(\nu - \nu_1) \geq \beta G\nu$, where $\beta > 0$. Thus $\int_V G\nu\, d\mu < \infty$, or $G\rho(x) < \infty$, for $\rho = 1_V \mu$. This proves Lemma 6.3.

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