TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 304, Number 1, November 1987

HOLOMORPHIC KERNELS AND COMMUTING OPERATORS

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ABSTRACT. Necessary and sufficient conditions in terms of operator polynomials are obtained for an m-tuple $T = (T_1, \ldots, T_m)$ of commuting bounded linear operators on a separable Hilbert space $\mathscr{H}$ to extend to an m-tuple $S = (S_1, \ldots, S_m)$ of operators on some Hilbert space $\mathscr{K}$, where each S_i is realized as a $*$-representation of the adjoint of a multiplication operator on the tensor product of a special type of functional Hilbert spaces. Also, necessary and sufficient conditions in terms of operator polynomials are obtained for T to have a commuting normal extension.

0. Introduction. In this paper, some results in [**1** and **2**] for a single bounded linear operator T on a separable Hilbert space $\mathscr{H}$ are generalized to m commuting operators on $\mathscr{H}$. In [**1**], Agler introduces a special class of functional Hilbert spaces $\mathscr{M}$ and describes conditions under which an operator T on $\mathscr{H}$ extends to $M^{*(\infty)}$, where M denotes the multiplication operator on $\mathscr{M}$, and $M^{*(\infty)}$ denotes the countable direct sum of M^* with itself. Special cases of spaces $\mathscr{M}$, of which the classical Hardy space is the prototype, are considered in [**1, 2**]. The relevant conditions for the kind of extension of T referred to above are expressed in terms of the positivity of certain operator polynomials involving T and T^*. These conditions are closely related to the reproducing kernel associated with the space $\mathscr{M}$. In [**2**], necessary and sufficient conditions are also given for a contraction T to be subnormal. These conditions are really the requirement that a certain sequence of polynomials in T and T^* be positive. It is natural to seek generalizations of these results to m commuting operators $T_1, \ldots, T_m$ on $\mathscr{H}$. An appropriate model for this generalization is obtained by constructing a finite tensor product of spaces $\mathscr{M}$ and exploiting the well-known fact that the reproducing kernel of such a tensor product is the product of the reproducing kernels of the individual spaces. Suitable modifications of the reproducing kernels which render them holomorphic on the unit polydisc can be used to describe analogous extension results for $T = (T_1, \ldots, T_m)$. The question of T having a commuting normal extension $N = (N_1, \ldots, N_m)$ turns out to have a direct link with the multi-dimensional Hausdorff Moment Problem from the theory of probability.

Received by the editors March 21, 1986.
1980 *Mathematics Subject Classification*. Primary 47A20; Secondary 47B20, 47B38.
Key words and phrases. Positive definite, regular analytic model atom, kernel, tensor product, commuting normal extension, $*$-dilation.
This work constitutes part of the author's Ph. D. dissertation written at Indiana University under the direction of John B. Conway.

0002-9947/87 $1.00 + $.25 per page

§1 fixes some notation and introduces the idea of a positive definite matrix of functions defined on the cartesian product of two unit polydiscs. §2 develops the necessary theory of holomorphic kernels in the setting of the tensor product of a finite number of functional Hilbert spaces and derives general results pertaining to extensions of T. §3 presents applications of these results to several concrete examples. §4 deals with the existence of a commuting normal extension for T and concludes with some general discussion.

1. Preliminaries. The set of bounded linear operators on a Hilbert space $\mathscr{H}$ will be denoted by $\mathscr{B}(\mathscr{H})$; $\mathscr{H}^{(n)}$ will denote the direct sum of $\mathscr{H}$ with itself n times. An m-tuple $(b_1, \ldots, b_m)$ of commuting elements in a C^*-algebra will be denoted by b, while b^* will stand for $(b_1^*, \ldots, b_m^*)$. Similarly, any m-tuple $(z_1, \ldots, z_m)$ of complex numbers will be abbreviated to z, with z^* having the obvious meaning. For any m-tuple $(s_1, \ldots, s_m)$ of nonnegative integers, b^s will denote $b_1^{s_1} b_2^{s_2} \cdots b_m^{s_m}$, while z^s will stand for $z_1^{s_1} z_2^{s_2} \cdots z_m^{s_m}$. For any subset A of the complex plane $\mathbf{C}$, A^* and A^m will respectively mean $\{z \in \mathbf{C} : z^* \in A\}$ and the cartesian product of A with itself m times. For any open set G in $\mathbf{C}^m$, $H(G)$ will be the set of holomorphic functions on G and $L^2(G, \mu)$ will be the square integrable functions with respect to measure μ on G. In general, various abbreviations used in the paper will be clear from the context in which they appear. All the Hilbert spaces occurring below are separable. The arguments here parallel those in [**1**, **2**].

We begin by introducing a functional calculus for (b, b^*), where for $1 \leqslant i \leqslant m$, the spectrum $\sigma(b_i)$ of b_i is contained in some open disc G_i in $\mathbf{C}$, with center at $z_i = 0$. If $f \in H((G_1 \times \cdots \times G_m)^2)$, define

$$(1)\quad f(b, b^*) = \left(\frac{1}{2\pi i}\right)^{2m} \int_{\gamma_m^*} \cdots \int_{\gamma_1^*} \int_{\gamma_m} \cdots \int_{\gamma_1} f(z, w)(w_m - b_m^*)^{-1} \cdots (w_1 - b_1^*)^{-1}(z_m - b_m)^{-1} \cdots (z_1 - b_1)^{-1} dz_1 \cdots dz_m \, dw_1 \cdots dw_m,$$

where γ_i is any finite system of Jordan arcs surrounding $\sigma(b_i)$ and lying in G_i. We note in particular the following two consequences of the above definition.

(a) If $p(z, w) = \sum_{s,t} c_{st} z^s w^t$ is a polynomial in $2m$ complex variables (z, w), then

$$(2)\quad p(b, b^*) = \sum_{s,t} c_{st} b^{*t} b^s.$$

(b) If f_1, $f_2 \in H(G_1 \times \cdots \times G_m)$ and $f \in H((G_1 \times \cdots \times G_m)^2)$, then

$$(3)\quad f_2(b^*) f(b, b^*) f_1(b) = g(b, b^*),$$

where $g(z, w) = f_1(z) f_2(w) f(z, w)$.

Relation (3) can be checked easily by verifying it first for powers of z and w and then noting that any $f \in H((G_1 \times \cdots \times G_m)^2)$ can be expressed as

$$(4)\quad f(z, w) = \sum_{s,t} a_{st} z^s w^t,$$

where the series on the right-hand side converges uniformly to f on compact subsets of $(G_1 \times \cdots \times G_m)^2$ [9].

DEFINITION 1.1. If D is the open unit disc in $\mathbf{C}$, then an $n \times n$ matrix $[g_{ij}]$ of functions defined on $D^m \times D^m$ is *positive definite* if for all positive integers p, all vectors C_r in $\mathbf{C}^{(n)}$, and all points $\lambda^{(r)}$ in D^m $(1 \leqslant r \leqslant p)$,

$$\sum_{1 \leqslant r,\, s \leqslant p} \langle G_{rs} C_r, C_s \rangle_{\mathbf{C}^{(n)}} \geqslant 0, \tag{5}$$

where G_{rs} is the $n \times n$ matrix $[g_{ij}(\lambda^{(r)}, \lambda^{(s)})]$ and where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{C}^{(n)}}$ denotes the inner product in $\mathbf{C}^{(n)}$.

LEMMA 1.2. *Let $[g_{ij}]$ be positive definite as in Definition* 1.1. *Also let $g_{ij}(z^*, w)$ be holomorphic on $D^m \times D^m$ for $1 \leqslant i, j \leqslant n$. Then for any r such that $0 < r < 1$, there exist functions f_{il}, $1 \leqslant i \leqslant n$, $1 \leqslant l$, defined and holomorphic on $(rD)^m$ such that*

$$g_{ij}(z, w) = \sum_{l=1}^{\infty} f_{jl}^*(z) f_{il}(w) \tag{6}$$

for $1 \leqslant i, j \leqslant n$, and where the series on the right converges uniformly on compact subsets of $(rD)^m \times (rD)^m$.

PROOF. Define A_{ij} on $L^2((rD)^m, (\mathrm{Area})^m)$ by

$$A_{ij}(f)(w) = \int_{(rD)^m} g_{ij}(z, w) f(z)\, d(\mathrm{Area})^m(z). \tag{7}$$

Further, let $A = [A_{ij}] \in \mathscr{B}((L^2((rD)^m, (\mathrm{Area})^m))^{(n)})$. Since g_{ij} is bounded on $(rD)^m \times (rD)^m$, A is a compact operator. Using the fact that $[g_{ij}]$ is positive definite, it is easy to see that A is a positive operator. If $\{t_l\}$ is the sequence of nonzero eigenvalues of A and $h_l = \oplus_{i=1}^n h_{il}$, the corresponding eigenfunctions, then (6) holds for $f_{il} = \sqrt{t_l}\, h_{il}$, as can be seen by identifying

$$\mathscr{N}_n\big(L^2((rD)^m \times (rD)^m, (\mathrm{Area})^m \times (\mathrm{Area})^m)\big)$$

with $\mathscr{C}_2((L^2((rD)^m, (\mathrm{Area})^m))^{(n)})$, where the former denotes the Hilbert space of $n \times n$ matrices with entries p_{ij} from $L^2((rD)^m \times (rD)^m, (\mathrm{Area})^m \times (\mathrm{Area})^m)$ and with the inner product

$$\langle [p_{ij}], [q_{ij}] \rangle = \sum_{1 \leqslant i,\, j \leqslant n} \int_{(rD)^m \times (rD)^m} p_{ij}(z, w) q_{ij}^*(z, w)\, d(\mathrm{Area})^m \times (\mathrm{Area})^m(z, w);$$

and the latter denotes the Schmidt class of operators on $(L^2((rD)^m, (\mathrm{Area})^m))^{(n)}$. (This is indeed the matricial analog of Theorem 2.4.4 in [**10**].) Note that the series in (6) converges in the norm $\|\cdot\|$ of $L^2((rD)^m \times (rD)^m, (\mathrm{Area})^m \times (\mathrm{Area})^m)$. Since h_l is an eigenfunction corresponding to the nonzero eigenvalue t_l, using (7) the functions f_{il} are seen to be holomorphic on $(rD)^m$. That the series in (6) converges uniformly on compact subsets of $(rD)^m \times (rD)^m$ follows by noting that if $K_1 \times \cdots \times K_{2m}$ is any compact subset of $(rD)^m \times (rD)^m$, then for any $f \in H((rD)^m \times (rD)^m)$, and any point z_0 of $K_1 \times \cdots \times K_{2m}$,

$$\|f\| \geqslant \pi^m (r_1 r_2 \cdots r_{2m}) |f(z_0)|,$$

where r_i is the distance of K_i from the boundary of rD. $\square$

DEFINITION 1.3. Let T be a tuple of m commuting operators on a Hilbert space $\mathscr{H}$. We say *T extends to S* if there exists a Hilbert space $\mathscr{K}$, $S_i \in \mathscr{B}(\mathscr{K})$ $(1 \leqslant i \leqslant m)$, and an isometry V from $\mathscr{H}$ into $\mathscr{K}$ such that Range V is invariant for each S_i and $T_i = V^*S_iV$ for every i. (It follows then that $p(T,T^*) = V^*p(S,S^*)V$ for any $p(T,T^*)$, $p(S,S^*)$ as in (2).)

THEOREM 1.4. *Let B be a C^*-algebra with identity* 1 *and let b be a tuple of m commuting elements in B. If T is a tuple of m commuting elements in $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space, then the following are equivalent.*

(i) *There exist a Hilbert space $\mathscr{K}$ and a $*$-representation π: $B \to \mathscr{B}(\mathscr{K})$ with $\pi(1) = 1$; and T extends to $\pi(b) = (\pi(b_1), \dots, \pi(b_m))$.*

(ii) *For any positive integer n and n^2 polynomials p_{ij} in $2m$ complex variables,*

$$[p_{ij}(b,b^*)] \geqslant 0 \text{ in } \mathscr{N}_n(B) \quad \text{implies} \quad [p_{ij}(T,T^*)] \geqslant 0 \text{ in } \mathscr{B}(\mathscr{H}^{(n)}), \tag{8}$$

where $\mathscr{N}_n(B)$ denotes the class of $n \times n$ matrices with entries from B. (Given B, $\mathscr{N}_n(B)$ is equipped with a unique C^*-norm.)

The proof of Theorem 1.4 follows the same lines as in Theorem 1.5 in [**1**] and uses the Stinespring Representation Theorem and the Arveson Extension Theorem [**4**, Theorem 1.2.3]. The proof is omitted.

2. Kernel functions and kernels. We now introduce a special class of functional Hilbert spaces [**1**].

DEFINITION 2.1. An *analytic model atom $\mathscr{M}$ over D* is a Hilbert space of analytic functions on D satisfying the following properties.

(i) For any $\lambda \in D$, $f, g \in \mathscr{M}$, and $\alpha, \beta \in \mathbf{C}$, $(\alpha f + \beta g)(\lambda) = \alpha f(\lambda) + \beta g(\lambda)$.

(ii) For every $\lambda \in D$, there is a constant c_λ such that $|f(\lambda)| \leqslant c_\lambda \|f\|_{\mathscr{M}}$ for f in $\mathscr{M}$. Here $\|\cdot\|_{\mathscr{M}}$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathscr{M}}$ of $\mathscr{M}$.

(iii) If f is holomorphic on a neighborhood of the closure $\overline{D}$ of D, then $f \in \mathscr{M}$.

(iv) The map M defined on $\mathscr{M}$ by $(Mf)(z) = zf(z)$, $f \in \mathscr{M}$, is a bounded operator on $\mathscr{M}$.

Note that by virtue of (ii) above, there exists for any $\lambda \in D$ a function κ_λ in $\mathscr{M}$ such that $f(\lambda) = \langle f, \kappa_\lambda \rangle_{\mathscr{M}}$ for any $f \in \mathscr{M}$. The *kernel functions* κ_λ give rise to the *kernel* $\kappa(\lambda, \mu)$ of $\mathscr{M}$ defined by $\kappa(\lambda,\mu) = \langle \kappa_{\mu^*}, \kappa_\lambda \rangle_{\mathscr{M}} = \kappa_{\mu^*}(\lambda)$. Note that $\kappa(\lambda, \mu) \in H(D^2)$.

DEFINITION 2.2. An analytic model atom $\mathscr{M}$ over D is called *regular* if it satisfies the following properties.

(i) Polynomials in z are dense in $\mathscr{M}$.

(ii) The kernel $\kappa(\lambda, \mu)$ does not vanish anywhere on D^2, and κ is symmetric, that is $\kappa(\lambda,\mu) = \kappa(\mu,\lambda)$.

(iii) The operator $\lambda - M \in \mathscr{B}(\mathscr{M})$ is Fredholm for every $\lambda \in D$.

Let $\mathscr{M}_1, \dots, \mathscr{M}_m$ be m regular analytic model atoms over D with kernels $\kappa_1, \dots, \kappa_m$ respectively. The tensor product $\mathscr{M} = \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_m$ is also a functional Hilbert space [**3**] with the "kernel" κ given by

$$\kappa(\lambda,\mu) = \kappa_1(\lambda_1,\mu_1)\kappa_2(\lambda_2,\mu_2) \cdots \kappa_m(\lambda_m,\mu_m).$$

The kernel κ is related to the "kernel functions" $\kappa_\lambda = \kappa_{\lambda_1} \otimes \cdots \otimes \kappa_{\lambda_m}$ by $\kappa(\lambda, \mu) = \langle \kappa_{\mu^*}, \kappa_\lambda \rangle_{\mathscr{M}}$. If M_i denotes multiplication by z on $\mathscr{M}_i$, then define a corresponding operator $\underline{M}_i$ on $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_m$ by $\underline{M}_i = 1 \otimes \cdots \otimes M_i \otimes \cdots \otimes 1$. We use $\underline{M}$ to denote $(\underline{M}_1, \ldots, \underline{M}_m)$. Note that $\sigma(\underline{M}_i) = \sigma(M_i) = \overline{D}$.

LEMMA 2.3. *Let $\mathscr{M} = \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_m$, where each M_i is a regular analytic model atom over D. If $p(z, w)$ is any polynomial, then*

$$\left\langle p(\underline{M}^*, \underline{M}) \kappa_\lambda, \kappa_\mu \right\rangle_{\mathscr{M}} = p(\lambda^*, \mu) \kappa(\lambda^*, \mu).$$

PROOF. Use the fact that for any $f \in \mathscr{M}$, it follows that $\langle f, \underline{M}_j^* k_\lambda \rangle_{\mathscr{M}} = \langle \underline{M}_j f, k_\lambda \rangle_{\mathscr{M}} = \lambda_j f(\lambda) = \lambda_j \langle f, k_\lambda \rangle_{\mathscr{M}} = \langle f, \lambda_j^* \kappa_\lambda \rangle_{\mathscr{M}}$. $\square$

LEMMA 2.4. *Let $\mathscr{M}$ be as in Lemma 2.3. Then for any positive integer n and n^2 polynomials p_{ij}, $[p_{ij}(\underline{M}^*, \underline{M})] \geqslant 0$ in $\mathscr{B}(\mathscr{M}^{(n)})$ if and only if $[p_{ij}(z^*, w) \kappa(z^*, w)]$ is positive definite.*

PROOF. Use Lemma 2.3 and argue as in Proposition 2.5 in [**1**]. $\square$

We now arrive at one of the principal results of the paper.

THEOREM 2.5. *Let $\mathscr{M}$ be the Hilbert space with kernel κ obtained by tensoring m regular analytic model atoms $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_m$ over D. Let, for any extended integer n $(0 < n \leqslant \infty)$, $M_i^{*(n)}$ denote the direct sum of M_i^* with itself n times, where M_i denotes multiplication by z on $\mathscr{M}_i$. Let $\mathscr{H}$ be a Hilbert space and let T be a tuple of m commuting operators on $\mathscr{H}$ such that $\sigma(T_i) \subset D$ $(1 \leqslant i \leqslant m)$. If $\frac{1}{\kappa}(T, T^*) \geqslant 0$, then there exist a Hilbert space $\mathscr{K}$ and a $*$-representation π: $\mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{K})$ with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$ and where $\pi(\underline{M}_i^*)$ is unitarily equivalent to $M_i^{*(n_i)} \oplus W_i$ with $\sigma(W_i) \subset \partial D$, the boundary of D, and $0 < n_i \leqslant \infty$ $(1 \leqslant i \leqslant m)$. (One of the summands in $M_i^{*(n_i)} \oplus W_i$ may be absent.)*

PROOF. Assume $\frac{1}{\kappa}(T, T^*) \geqslant 0$. Using Lemmas 1.2 and 2.4 and arguing as in Theorem 2.3 of [**1**], it is not difficult to show that for any positive integer n and n^2 polynomials p_{ij}, $[p_{ij}(\underline{M}^*, \underline{M})] \geqslant 0$ in $\mathscr{B}(\mathscr{M}^{(n)})$ implies $[p_{ij}(T, T^*)] \geqslant 0$ in $\mathscr{B}(\mathscr{H}^{(n)})$. By Theorem 1.4, there exist a Hilbert space $\mathscr{K}$ and a $*$-representation π: $\mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{K})$ with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$. (Note the identification of $\mathscr{B}(\mathscr{M}^{(n)})$ with $\mathscr{N}_n(\mathscr{B}(\mathscr{M}))$.) It is a consequence of Theorem 2.8 in [**1**] that $\pi(\underline{M}_i^*)$ is unitarily equivalent to $M_i^{*(n_i)} \oplus W_i$ with $\sigma(W_i) \subseteq \partial D$ and $0 < n_i \leqslant \infty$. Indeed, the $*$-representation π: $\mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{K})$ gives rise to a $*$-representation π_i: $\mathscr{B}(\mathscr{M}_i) \to \mathscr{B}(\mathscr{K})$ defined by $\pi_i(A) = \pi(1 \otimes \cdots \otimes A \otimes \cdots \otimes 1)$, $A \in \mathscr{B}(\mathscr{M}_i)$; and Theorem 2.8 in [**1**] is directly applicable. The assertion $\sigma(W_i) \subseteq \partial D$ is essentially a consequence of the fact that $\lambda - M_i$ is Fredholm for every λ in D. $\square$

REMARK. With V having the same meaning as in Definition 1.3, Theorem 2.5 enables us to write T_i as $T_i = V^* \pi(\underline{M}_i^*) V = V^* U_i^* (M_i^{*(n_i)} \oplus W_i) U_i V$, where U_i is some unitary operator. The assumption $\sigma(T_i) \subset D$ can further be utilized to conclude that $T_i = V^* U_i^* M_i^{*(n_i)} U_i V$ (see the proof of Theorem 2.3 in [**1**]).

We now state the analog of Theorem 3.1 in [**1**].

THEOREM 2.6. *Let $\mathscr{M}$, κ, T have the same meaning as in Theorem 2.5. Suppose also that $\frac{1}{\kappa}$ extends to a holomorphic function on a neighborhood of $D^m \times D^m$. Then the following are equivalent.*

(i) $\frac{1}{\kappa}(T, T^*) \geqslant 0$.

(ii) *There exist a Hilbert space $\mathscr{K}$ and a $*$-representation π: $\mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{K})$ with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$, and where $\pi(\underline{M}_i^*)$ is unitarily equivalent to $M_i^{*(n_i)} \oplus W_i$ with $\sigma(W_i) \subseteq \partial D$ and $0 < n_i \leqslant \infty$ $(1 \leqslant i \leqslant m)$.*

PROOF. That (i) implies (ii) follows from Theorem 2.5. The proof that (ii) implies (i) is a result of

$$\tfrac{1}{\kappa}(T, T^*) = V^* \tfrac{1}{\kappa}(\pi(\underline{M}^*, \underline{M}))V = V^* \pi\left(\tfrac{1}{\kappa}(\underline{M}^*, \underline{M})\right)V.$$

Indeed, $\frac{1}{\kappa}(\underline{M}^*, \underline{M}) \geqslant 0$ as follows by considering a sequence of polynomials converging to $\frac{1}{\kappa}$ uniformly on compact subsets of a neighborhood of $D^m \times D^m$ (see (4)) and using Lemma 2.4. Since π is a $*$-representation, $\pi(\frac{1}{\kappa}(\underline{M}^*, \underline{M}))$ is positive as well. $\square$

3. Applications. The classical Hardy space $(H^2(\mu_1))$ and Bergman space $(H^2(\mu_2))$ are among a series of regular analytic model atoms $H^2(\mu_n)$ over D discussed in [**2**]. We recall that $H^2(\mu_n)$ is the completion of polynomials in $L^2(D, \mu_n)$ and the measures μ_n are defined in a recursive fashion starting with the normalized Lebesgue measure on ∂D. The kernel κ_{μ_n} of $H^2(\mu_n)$ is given by $\kappa_{\mu_n}(z, w) = 1/(1 - zw)^n$. The following is the analog of Theorem 1.10 in [**2**].

THEOREM 3.1. *Let T be a tuple of m commuting operators on a Hilbert space $\mathscr{H}$. Also, let M_i denote multiplication by z on $H^2(\mu_{\kappa_i})$. Then the following are equivalent.*

(i) *There exist a Hilbert space $\mathscr{K}$ and a $*$-representation*

$$\pi \colon \mathscr{B}\left(H^2(\mu_{k_1}) \otimes \cdots \otimes H^2(\mu_{k_m})\right) \to \mathscr{B}(\mathscr{K})$$

with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$, and where $\pi(\underline{M}_i^*)$ is unitarily equivalent to $M_i^{*(n_i)} \oplus W_i$ with W_i unitary and $0 < n_i \leqslant \infty$ $(1 \leqslant i \leqslant m)$.

(ii) $(\prod_{i=1}^m (1 - z_i w_i)^{p_i})(T, T^*) \geqslant 0$ *for all p_i such that $0 \leqslant p_i \leqslant k_i$.*

PROOF. The proof that (i) implies (ii) is similar to the proof that (ii) impliles (i) in Theorem 2.6. We now prove (ii) implies (i). Suppose $(\prod_{i=1}^m (1 - z_i w_i)^{p_i})(T, T^*) \geqslant 0$ for all p_i such that $0 \leqslant p_i \leqslant k_i$. Note that for $0 < s < 1$,

$$\begin{aligned}
\left(\prod_{i=1}^m (1 - z_i w_i)^{k_i}\right)(sT, sT^*) &= \left(\prod_{i=1}^m \left(1 - s^2 z_i w_i\right)^{k_i}\right)(T^*, T^*) \\
&= \left(\prod_{i=1}^m \left(1 - z_i w_i + (1 - s^2) z_i w_i\right)^{k_i}\right)(T, T^*) \\
&= \left(\prod_{i=1}^m \left(\sum_{j=0}^{k_i} \binom{k_i}{j} (1 - s^2)^{k_i - j} w_i^{k_i - j} (1 - z_i w_i)^j z_i^{k_i - j}\right)\right)(T, T^*).
\end{aligned}$$

Expanding the bracketed function and appealing to (3) repeatedly, it is clear that the last expression is a positive operator. Since each T_i is a contraction, $\sigma(sT_i) \subset D$ for each i. It follows by Theorem 2.5 that there exist a Hilbert space $\mathscr{K}_0$ and a

$*$-representation π_0: $\mathscr{B}(H^2(\mu_{k_1}) \otimes \cdots \otimes H^2(\mu_{k_m})) \to \mathscr{B}(\mathscr{K}_0)$ with $\pi_0(1) = 1$ such that sT extends to $\pi_0(\underline{\tilde{M}}^*)$. But then by Theorem 1.4 we have that $[p_{ij}(\underline{M}^*, \underline{M})] \geqslant 0$ in $\mathscr{N}_n(\mathscr{B}(H^2(\mu_{k_1}) \otimes \cdots \otimes H^2(\mu_{k_m})))$ implies $[p_{ij}(sT, sT^*)] \geqslant 0$ in $\mathscr{B}(\mathscr{H}^{(n)})$. Letting $s \to 1$, we see that $[p_{ij}(\underline{M}^*, \underline{M})] \geqslant 0$ implies $[p_{ij}(T, T^*)] \geqslant 0$. So by Theorem 1.4 again, there exist a Hilbert space $\mathscr{K}$ and a $*$-representation

$$\pi\colon \mathscr{B}\left(H^2(\mu_{k_1}) \otimes \cdots \otimes H^2(\mu_{k_m})\right) \to \mathscr{B}(\mathscr{K})$$

with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$. As before, $\pi(\underline{M}_i^*)$ has the form $M_i^{*(n_i)} \oplus W_i$ with $\sigma(W_i) \subseteq \partial D$ and $0 < n_i \leqslant \infty$ $(1 \leqslant i \leqslant m)$. Combining the proof of Theorem 2.8 in **[1]** with the fact that each M_i is essentially unitary (see Proposition 1.3 in **[2]**), it is not difficult to show that each W_i is unitary. □

COROLLARY. *Let T be a tuple of m commuting operators on a Hilbert space $\mathscr{H}$. Let, for each i, M_i denote the multiplication operator M on the Hardy space $H^2(\mu_1)$. Then the following are equivalent.*

(i) *There exist a Hilbert space $\mathscr{K}$ and a $*$-representation*

$$\pi\colon \mathscr{B}\left(H^2(\mu_1) \otimes \cdots \otimes H^2(\mu_1)\right) \to \mathscr{B}(\mathscr{K})$$

with $\pi(1) = 1$ such that T extends to $\pi(\underline{M}^*)$, where $\pi(\underline{M}_i^*)$ is unitarily equivalent to $M_i^{*(n_i)} \oplus W_i$ with W_i unitary and $0 < n_i \leqslant \infty$ $(1 \leqslant i \leqslant m)$.

(ii) *$T_1, \ldots, T_m$ are m commuting contractions such that $T = (T_1, \ldots, T_m)$ has a regular unitary dilation* **[13]**.

PROOF. In both (i) and (ii), the necessary and sufficient conditions are

$$\left(\prod_{i=1}^{m} (1 - z_i w_i)^{p_i}\right)(T, T^*) \geqslant 0$$

for $0 \leqslant p_i \leqslant 1$ $(1 \leqslant i \leqslant m)$. □

REMARK 1. In case $m = 1$, the results here reduce to those in **[1, 2]**.

REMARK 2. Theorems 2.5, 2.6, and 3.1 allow an infinite number of interesting possibilities. Thus if M_1 denotes the unilateral shift and M_2 denotes the Bergman shift, the necessary and sufficient conditions for (T_1, T_2) to extend to $(\pi(\underline{M}_1^*), \pi(\underline{M}_2^*))$ with $\pi(1) = 1$ are

$$\left((1 - z_1 w_1)^{p_1}(1 - z_2 w_2)^{p_2}\right)\left(T_1, T_2, T_1^*, T_2^*\right) \geqslant 0$$

for $0 \leqslant p_1 \leqslant 1$, $0 \leqslant p_2 \leqslant 2$. If $\sigma(T_i) \subset D$ for $i = 1, 2$, then the relevant conditions are

$$\left((1 - z_1 w_1)(1 - z_2 w_2)^2\right)\left(T_1, T_2, T_1^*, T_2^*\right) \geqslant 0.$$

The modification of Theorem 3.1 and in particular of conditions (ii) there when some of the operators T_i have their spectra inside D is obvious and will not be stated.

REMARK 3. The unitary part in the representation of $\pi(\underline{M}_i^*)$ in Theorem 3.1 can be removed to get $T_i = V^* U_i^* M_i^{*(n_i)} U_i V$ by requiring that T_i^m tend to zero strongly as $m \to \infty$.

4. Commuting normal extension. In [**2**], Agler showed that for $T \in \mathscr{B}(\mathscr{H})$, the following are equivalent.

(i) $\|T\| \leqslant 1$ and T is subnormal.

(ii) $(1 - zw)^n(T, T^*) \geqslant 0$ for all $n \geqslant 1$.

(In this connection, the statement on p. 212 in [**2**] of a result of Sz.-Nagy needs to be corrected.)

In view of our work in the previous sections and the above result, it is natural to look for conditions for $T = (T_1, \ldots, T_m)$ to have a commuting normal extension, that is, for m commuting normal operators $N_1, \ldots, N_m$ to exist so that T extends to $N = (N_1, \ldots, N_m)$. In what follows, we may assume without any loss of generality that for each i, $\|T_i\| = \|N_i\|$. The following theorem is a consequence of the solution of the Hausdorff Moment Problem in several dimensions.

THEOREM 4.1. *Let T be a tuple of m commuting operators on a Hilbert space $\mathscr{H}$. Then the following are equivalent.*

(i) $\|T_i\| \leqslant 1$ *for* $1 \leqslant i \leqslant m$ *and T has a commuting normal extension N.*

(ii) *For any choice of nonnegative integers k_i $(1 \leqslant i \leqslant m)$,*

$$\left(\prod_{i=1}^{m} (1 - z_i w_i)^{k_i}\right)(T, T^*) \geqslant 0.$$

PROOF. That (i) implies (ii) can be checked easily by utilizing $N_i N_j = N_j N_i$, $N_i N_j^* = N_j^* N_i$, and $\|N_i\| = \|T_i\|$ for all i and j. Conversely suppose (ii) holds. Let $N = \{0, 1, 2, \ldots\}$, and for p, $n \in N^m$ such that $n_i \geqslant p_i$ $(1 \leqslant i \leqslant m)$, write

$$|p| = p_1 + \cdots + p_m, \qquad \binom{n}{p} = \binom{n_1}{p_1}\binom{n_2}{p_2}\cdots\binom{n_m}{p_m}.$$

Then conditions (ii) can be written as $\sum_p (-1)^{|p|}\binom{k}{p} T^{*p} T^p \geqslant 0$ for all $k \in N^m$. Let $s \in N^m$ and let u be any vector in $\mathscr{H}$. Then we obviously have

$$\sum_p (-1)^{|p|}\binom{k}{p}\langle T^{p+s}u, T^{p+s}u\rangle_{\mathscr{H}} \geqslant 0 \quad \text{for all } k \in N^m.$$

Here $p + s = (p_1 + s_1, \ldots, p_m + s_m)$. Now define a function φ_u from N^m to the real line $\mathbf{R}$ by $\varphi_u(n) = \langle T^n u, T^n u\rangle_{\mathscr{H}} = \|T^n u\|^2_{\mathscr{H}}$. Then the conditions just derived can be expressed as $\Delta_1^{k_1}\Delta_2^{k_2}\cdots\Delta_m^{k_m}\varphi_u(s) \geqslant 0$ for any s, $k \in N^m$, where the difference operator Δ_i acts on φ_u by $\Delta_i\varphi_u(s) = \varphi_u(s) - \varphi_u(s_1, \ldots, s_i + 1, \ldots, s_m)$. (In other words, the function φ_u is "completely monotone" on N^m.) It follows then by a result of T. N. Hildebrandt and I. J. Schoenberg [**6**] that there exists a positive measure μ_u on $[0, 1]^m$ such that

$$\langle T^{*s}T^s u, u\rangle_{\mathscr{H}} = \|T^s u\|^2_{\mathscr{H}} = \varphi_u(s) = \int_{[0,1]^m} x^s \, d\mu_u(x).$$

If for any measurable set $A \subset [0, 1]^m$ and $u \in \mathscr{H}$, we define $\langle \rho(A)u, u\rangle_{\mathscr{H}} = \mu_u(A)$, then the above implies that $T^{*s}T^s = \int_{[0,1]^m} x^s \, d\rho_u(x)$. A simple change of variables and an application of a result of A. Lubin [**8**] now yield that T has a commuting normal extension. That each T_i is a contraction is obvious, and the proof is complete. □

We conclude this section by trying to unify the ideas implicit in the preceding analysis. First a few definitions.

DEFINITION 4.2. Let S be a semigroup with identity and with involution $*$. A function ψ: $S \to \mathcal{B}(\mathcal{H})$ is called *positive definite* if $\sum_{s,t}\langle\psi(s^*t)f(t), f(s)\rangle_{\mathcal{H}} \geqslant 0$ for all functions f: $S \to \mathcal{H}$ such that f has finite support.

DEFINITION 4.3. Let S be as in Definition 4.2, and let $\mathcal{H}$ be a Hilbert space. A function ψ: $S \to \mathcal{B}(\mathcal{H})$ is said to be $*$*-dilatable* if there exist a Hilbert space $\mathcal{K}$, a $*$-preserving semigroup homomorphism φ: $S \to \mathcal{B}(\mathcal{K})$, and a bounded linear operator V: $\mathcal{H} \to \mathcal{K}$ such that $\psi(s) = V^*\varphi(s)V$, $s \in S$.

THEOREM 4.4. *Let T be a tuple of m commuting operators on $\mathcal{H}$. Let $N = \{0, 1, 2, \dots\}$ and $\tilde{N}$ be the semigroup $N^m \times N^m$ with multiplication as coordinatewise addition and with the involution $*$ defined by $(p, q)^* = (q, p)$, p, $q \in N^m$. Let ψ: $\tilde{N} \to \mathcal{B}(\mathcal{H})$ be defined by $\psi(p, q) = T^{*q}T^p$. Then the following are equivalent.*

(i) *T has a commuting normal extension.*

(ii) *ψ is positive definite on $\tilde{N}$.*

(iii) *ψ is $*$-dilatable on $\tilde{N}$.*

(iv) *Multi-dimensional Halmos-Bram conditions* [**7**] *hold.*

(v) *Multi-dimensional Embry conditions* [**8**] *hold.*

(vi) *For any $u \in \mathcal{H}$, the function φ_u: $N^m \to \mathbf{R}$ defined by $\varphi_u(p) = \|S^p u\|^2_{\mathcal{H}}$, where $S_i = \alpha T_i$ ($\alpha \neq 0$) are so chosen that $\|\alpha T_i\| \leqslant 1$ ($1 \leqslant i \leqslant m$), is completely monotone.*

PROOF. The equivalence of (i) and (vi) can be deduced from the proof of Theorem 4.1. That (ii) implies (iii) can be proved by specializing a result of F. H. Szafraniec [**11**] to the semigroup $\tilde{N}$ and then applying a result of Sz.-Nagy [**12**]. The rest of the implications are either easy to prove or well known. □

We remark that in the one-dimensional case, a parallel version of the above results with the semigroup $\tilde{N}$ replaced by the additive group of complex numbers and the map ψ replaced by the map $\eta(s) = e^{-s^*T^*}e^{sT}$ ($s \in \mathbf{C}$), can be found in [**5**]. We see from Theorem 4.4 that for m contractions $T_1, \dots, T_m$ on $\mathcal{H}$, conditions (ii) in Theorem 4.1 are equivalent to requiring the function ψ as defined in Theorem 4.4 to be positive definite. It is interesting to note that the partial fulfillment of these conditions leads to the kind of results stated in Theorem 3.1.

ACKNOWLEDGMENTS. I wish to thank John B. Conway for his constant guidance throughout the preparation of this work. Thanks are also due to W. Szymanski who pointed out to me the one-dimensional version of a portion of Theorem 4.4.

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