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ON THE COMPLETE $GL(n, \mathbf{C})$-DECOMPOSITION OF THE STABLE COHOMOLOGY OF $gl_n(A)$

PHIL HANLON

ABSTRACT. Let A be a graded, associative $\mathbf{C}$-algebra. For each n let $gl_n(A)$ denote the Lie algebra of $n \times n$ matrices with entries from A.

In this paper we extend the Loday-Quillen theorem to nontrivial isotypic components of $GL(n, \mathbf{C})$ acting on the Lie algebra cohomology of $gl_n(A)$. For α and β partitions of some nonnegative integer m let $[\alpha, \beta]_n \in \mathbf{Z}^n$ denote the maximal $GL(n, \mathbf{C})$-weight given by

$$[\alpha, \beta]_n = \sum_i \alpha_i e_i - \sum_j \beta_j e_{n+1-j}.$$

We show that the $[\alpha, \beta]_n$-isotypic component of the Lie algebra cohomology of $gl_n(A)$ stabilizes when $n \to \infty$ and is equal to

$$HRC^*(A) \otimes (\tilde{H}^*(A; \mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)_{S_m}$$

where $\tilde{H}^*(A; \mathbf{C})$ is the reduced Hochschild cohomology of A with trivial coefficients, where $HRC^*(A)$ is the graded exterior algebra generated by the cyclic cohomology of A, where S^α and S^β are the irreducible S_m-modules indexed by α and β and where the action of S_m on $\tilde{H}(A; \mathbf{C})^{\otimes m}$ is the exterior action.

1. Introduction. The problem considered in this paper is to describe the $GL(n, \mathbf{C})$ module structure of the Lie algebra cohomology of $gl_n(A)$ with trivial coefficients for A an associative $\mathbf{C}$-algebra. As it stands, this is too difficult a problem to solve. What we actually look at is the "stable" structure, i.e., the limit of the $GL(n, \mathbf{C})$ structure as $n \to \infty$. Our main theorem will describe each $GL(n, \mathbf{C})$-isotypic component of the Lie algebra cohomology of $gl_n(A)$ stably in terms of the cyclic cohomology of A and the Hochschild cohomology of A with trivial coefficients.

In this kind of stability theory one considers a sequence $(V_n)_{n=1}^\infty$ where V_n is a $GL(n, \mathbf{C})$-module. The question one tries to answer is whether the multiplicities of certain well-chosen irreducible $GL(n, \mathbf{C})$-modules approach a limit as $n \to \infty$. If so, this limit is called the *stable multiplicity* of that sequence of irreducibles in the sequence (V_n). This idea goes back to classical invariant theory where the sequence of irreducibles was taken to be the trivial representations. Examples of this are the work of Schur, Brauer and Weyl on the centralizer algebras of the classical groups acting on the tensor powers of their defining representations (see [**17, 1**

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and **21**]) and Littlewood's work on what he called special series (see [**12**, Chapter 11]). Much more recently there have been two developments which inspired the work in this paper. One development was the proof of Loday and Quillen that the $GL(n, \mathbf{C})$-invariants in the Lie algebra cohomology of $gl_n(A)$ stabilize as $n \to \infty$ to the graded exterior algebra of the cyclic cohomology of A (see [**14**]). The other development was a series of results by Lie theorists and combinatorialists concerning the stability of the mixed-tensor representations in the symmetric, exterior and tensor algebras of the adjoint representation. This work was initiated by Ranee Gupta who first investigated the stability of the mixed-tensor representations in the symmetric algebra of the adjoint representation.

Let α and β be partitions of the same nonnegative integer m. The *mixed-tensor representation* $V_{[\alpha,\beta]_n}$ of $GL(n, \mathbf{C})$ is the irreducible representation with maximal weight

$$[\alpha, \beta]_n = (\alpha_1, \alpha_2, \ldots, \alpha_e, 0, \ldots, 0, -\beta_k, -\beta_{k-1}, \ldots, -\beta_1) \in \mathbf{Z}^n.$$

Let $HL^d(gl_n(A))_{[\alpha,\beta]_n}$ denote the space of maximal weight vectors of weight $[\alpha, \beta]_n$ in the dth-graded piece of the Lie algebra cohomology of $gl_n(A)$. We will show that $HL^d(gl_n(A))_{[\alpha,\beta]_n}$ is isomorphic to $HL^d(gl_{n+1}(A))_{[\alpha,\beta]_{n+1}}$ for n sufficiently large and we will compute the limit,

$$HL^*(gl(A))_{[\alpha,\beta]} = \lim_{n\to\infty} HL^*(gl_n(A))_{[\alpha,\beta]_n}.$$

Our main result will express $HL^*(gl(A))_{[\alpha,\beta]}$ explicitly in terms of the cyclic cohomology of A and the Hochschild cohomology of A with trivial coefficients.

THEOREM 1.1. *Let A be an associative $\mathbf{C}$-algebra which is either finite dimensional or graded and finite dimensional in each graded piece. Then the sequence $(HL^*(gl_n(A)))_{n=1}^{\infty}$ of $GL(n, \mathbf{C})$-modules stabilizes. For α and β partitions of the same number m, the stable $[\alpha, \beta]$-isotypic component is*

$$HL^*(gl(A))_{[\alpha,\beta]} = RHC^*(A) \otimes (\tilde{H}^*(A, \mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)_{S_m}$$

where $RHC^*(A)$ is the graded exterior algebra of the cyclic cohomology of A, where $\tilde{H}^*(A, \mathbf{C})$ denotes the reduced Hochschild cohomology of A with trivial coefficients, where S^α and S^β are the Specht modules indexed by α and β and where V_{S_m} denotes the S_m-invariants in the S_m-module V. Also the action of S_m on $\tilde{H}^*(A, \mathbf{C})^{\otimes m}$ is the exterior action rather than ordinary permutation of tensor positions.

In the case that $\alpha = \beta = \varnothing$ we have that $[\alpha, \beta]_n$ is the zero vector so

$$HL^*(gl_n(A))_{[\varnothing,\varnothing]_n} = HL^*(gl_n(A))^{GL(n,\mathbf{C})}.$$

Theorem 1.1 asserts that

$$\lim_{n\to\infty} HL^*(gl_n(A))^{GL(n,\mathbf{C})} = RHC^*(A)$$

which is a result that can be found in Loday and Quillen [**14**].

If A has an identity then the only nonzero $GL(n, \mathbf{C})$-isotypic component of $HL^*(gl_n(A))$ is the space of $GL(n, \mathbf{C})$-invariants. So by the above remarks, the left-hand side of Theorem 1.1 is $RHC^*(A)$. This agrees with the right-hand side because $\tilde{H}^*(A, \mathbf{C})$ is zero if A has an identity.

Theorem 1.1 provides new information in the case that A is nonunital. In this case, $HL^*(gl_n(A))$ has nontrivial $GL(n, \mathbf{C})$-isotypic components. The stable form of these nontrivial components is predicted by Theorem 1.1.

It is not clear whether Theorem 1.1 will hold for an arbitrary associative $\mathbf{C}$-algebra. The proof we give uses the finite-dimensionality constraints on A in a crucial way. An interesting open problem is to determine how much these constraints can be relaxed.

The paper is organized as follows. In §2 we state definitions and basic facts from homological algebra and the representation theory of $GL(n, \mathbf{C})$. In §3 we prove Theorem 1.1 and in §4 we derive consequences of Theorem 1.1 in terms of Poincaré series and Euler characteristics.

The results in this paper have been achieved independently by B. L. Feigin and B. L. Tsygan. Their proofs can be found in their article *Additive k-theory*, Lecture Notes in Math., vol. 1289, Springer-Verlag, 1987.

2. Background.

PART 1. REPRESENTATIONS OF $GL(n, \mathbf{C})$ AND S_m. We begin by recalling some facts from the representation theory of $GL(n, \mathbf{C})$ (we abbreviate $GL(n, \mathbf{C})$ by G_n). In this paper, all representations of G_n will be *finite dimensional polynomial* representations. We let U_n denote the subgroup of all upper triangular matrices with 1's down the main diagonal and we let H_n be the subgroup of all diagonal matrices.

Let V be a G_n-module. A *maximal weight vector* $v \in V$, with weight $\lambda \in \mathbf{Z}^n$, is a nonzero vector satisfying

(C1) $u \cdot v = v$ for all $u \in U_n$, and

(C2) $\operatorname{diag}(x_1, \ldots, x_n) \cdot v = (x_1^{\lambda_1} \cdots x_n^{\lambda_n})v$ for all $\operatorname{diag}(x_1, \ldots, x_n) \in H_n$.

In condition (C2), $\lambda_1, \ldots, \lambda_n$ denote the coordinates of λ. Given $\lambda \in \mathbf{Z}^n$ we let $M_\lambda(V)$ denote the subspace of V spanned by the maximal weight vectors of weight λ. The G_n-submodule of V generated by $M_\lambda(V)$ is called the λ- isotypic component of V. The following facts are well known.

(F1) The G_n-submodule of V generated by v is irreducible.

(F2) If W is an irreducible G_n-submodule of V then W contains a unique maximal weight vector (up to nonzero scalar multiple).

(F3) Let W and W' be irreducible G_n-submodules of V with maximal weight vectors v and v' of weights λ and λ' respectively. Then W and W' are isomorphic as G_n-modules if and only if $\lambda = \lambda'$.

(F4) Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the maximal weight associated to a maximal weight vector v. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let D_n be the subset of $\mathbf{Z}^n$ consisting of all vectors $(\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We call the vectors in D_n *dominant weights for* G_n. It follows from (F1)–(F4) that the irreducible representations of G_n are indexed by the dominant weights in D_n. We let W_λ denote the G_n-irreducible associated to λ and we let $m_\lambda(V)$ denote the multiplicity of W_λ in V. One can show that

(F5) $m_\lambda(V) = \dim(M_\lambda(V))$.

The formula in (F5) gives a method for computing the multiplicity $m_\lambda(V)$. Usually this is a poor method for computing multiplicities, but in a situation we encounter in this paper it will be useful. Suppose $V = \bigoplus_{r \geq 0} C_r$ is a chain complex with boundary maps $\partial_r \colon C_r \to C_{r-1}$ (so $\partial_{r-1} \circ \partial_r = 0$ for all r). As usual we define the graded homology $H_r = \ker(\partial_r)/\operatorname{im}(\partial_{r+1})$. Assume that each C_r is a G_n-submodule of V and that the maps ∂_r are G_n-equivariant. Then G_n acts on each graded piece H_r of the homology.

For each r and each λ we have $\partial_r(M_\lambda(C_r)) \subseteq M_\lambda(C_{r-1})$. So $\bigoplus_r M_\lambda(C_r)$ is a subcomplex of V. Define $H_{\lambda,r}$ to be the graded homology of this subcomplex. The next result follows easily from (F5).

THEOREM 2.1. *With notation as above we have* $m_\lambda(H_r) = \dim(H_{\lambda,r})$.

In this paper we will encounter a special subclass of the representations of $GL(n, \mathbf{C})$ called *mixed-tensor representations.*

DEFINITION 2.2. Let $\alpha = (\alpha_1, \ldots, \alpha_a)$ and $\beta = (\beta_1, \ldots, \beta_b)$ be partitions of a nonnegative integer m with $l(\alpha) + l(\beta) = a + b \le n$. Define the *mixed-tensor representation* $W_{[\alpha,\beta]_n}$ to be the representation of $GL(n, \mathbf{C})$ with maximal weight

$$[\alpha, \beta]_n = \left(\sum_{i=1}^{a} \alpha_i e_i\right) - \left(\sum_{j=1}^{b} \beta_j e_{n+1-j}\right)$$

where $e_1, e_2, \ldots, e_n$ are the unit coordinate vectors in $\mathbf{Z}^n$.

These mixed-tensor representations were originally studied by physicists. In mathematical terms, they are the representations of G_n whose dominant weights lie in the root lattice or equivalently the representations which occur in the tensor powers of the adjoint representation.

Let $\mathbf{V} = (V_n)_{n=1}^{\infty}$ be a sequence of G_n-modules (so V_n is a G_n-module). We say the sequence is *stable* if for each value of n and each pair α, β there is an injection from the space of maximal weight vectors of weight $[\alpha, \beta]_n$ in V_n to the space of maximal weight vectors of weight $[\alpha, \beta]_{n+1}$ in V_{n+1} which is a surjection for large enough n. Here the value of n where stability occurs depends on α and β. If (V_n) is a stable sequence then for every pair α, β the limit of $m_{[\alpha,\beta]_n}(V_n)$ exists. We define $m_{[\alpha,\beta]}(V)$ to be this limit. During the last few years there has been a great deal of activity investigating stability properties of certain sequences of $GL(n, \mathbf{C})$-modules (see Gupta [**5**], Hanlon [**6**], Stanley [**18**] and Stembridge [**19**]).

There are close connections between the representation theory of $GL(n, \mathbf{C})$ and S_m. The irreducible representations of S_m are indexed by partitions α of m. There is an explicit construction of the irreducible representations called the *Specht modules* (see James [**10**]). We let S^α denote the Specht module indexed by α.

Let $V = \bigoplus_r V_r$ be a graded vector space. There are two actions of S_m on $V^{\otimes m}$ that will interest us in this paper. The first is the usual action of S_m on $V^{\otimes m}$ by permutation of tensor positions. The second, which we call the *exterior action*, is a variant of this defined in the following way.

DEFINITION 2.3. Let $v_i \in V_{r_i}$, $i = 1, 2, \ldots, m$, and let $\sigma \in S_m$. Define $\sigma \circ (v_1 \otimes \cdots \otimes v_m)$ called the *exterior action of* σ on $v_1 \otimes \cdots \otimes v_m$ by

$$\sigma \circ (v_1 \otimes \cdots \otimes v_m) = \left(\prod_{(i,j) \in I(\sigma)} (-1)^{r_i r_j}\right) v_{\sigma^{-1}1} \otimes v_{\sigma^{-1}2} \otimes \cdots \otimes v_{\sigma^{-1}m}$$

where $I(\sigma) = \{(i, j) \colon i < j \text{ and } \sigma i > \sigma j\}$ is the inversion set of σ.

Note that if we start with the tensor algebra of V and project each $V^{\otimes m}$ by the exterior action of S_m we obtain $\mathrm{Sym}(V_{\text{even}}) \otimes \mathrm{Ext}(V_{\text{odd}})$ where V_{even} and V_{odd} are the direct sum of V_r for r even and odd.

PART II. SOME DEFINITIONS AND RESULTS FROM HOMOLOGICAL ALGEBRA. Let N be a complex Lie algebra. The *Lie algebra homology of N* (with trivial coefficients) is the graded vector space $HL_*(N)$ defined by

$$HL_r(N) = \mathrm{Tor}_r^N(\mathbf{C}, \mathbf{C}).$$

There is a complex $(CL_*(N), \partial_*)$ whose homology is $HL_*(N)$ called the *Koszul complex.* It is defined by

$$CL_r(N) = \bigwedge^r N$$

and

$$\partial_r(n_1 \wedge n_2 \wedge \cdots \wedge n_r) = \sum_{i<j} (-1)^{i+j+1} [n_i, n_j] \wedge n_1 \wedge \cdots \wedge \hat{n}_i \wedge \cdots \wedge \hat{n}_j \wedge \cdots \wedge n_r.$$

Let Γ be a reductive group of automorphisms of N, i.e., $[\gamma n_1, \gamma n_2] = \gamma([n_1, n_2])$ for all n_1, $n_2 \in N$ and all $\gamma \in \Gamma$. The group Γ acts on $CL_r(N)$ by

$$\gamma(n_1 \wedge n_2 \wedge \cdots \wedge n_r) = (\gamma n_1) \wedge (\gamma n_2) \wedge \cdots \wedge (\gamma n_r).$$

This action commutes with the differential ∂_r, so this gives a graded action of Γ on $HL_*(N)$.

Let A be an associative $\mathbf{C}$-algebra. Define the Lie algebra $gl_n(A)$ to be the vector space

$$gl_n(A) = gl_n(\mathbf{C}) \otimes_{\mathbf{C}} A$$

with Lie bracket

$$[x, y] = xy - yx \quad \text{for } x, y \in gl_n(A).$$

The group $GL(n, \mathbf{C})$ acts as a group of automorphisms of $gl_n(A)$ by the action $\gamma \cdot (x \otimes a) = (\gamma^{-1} x \gamma) \otimes a$. We will refer to this as the adjoint action of $GL(n, \mathbf{C})$ on $gl_n(A)$. As explained above, this gives a graded action of $GL(n, \mathbf{C})$ on the Lie algebra homology of $gl_n(A)$. Our goal in this paper is to describe the stable $GL(n, \mathbf{C})$-module structure of $HL^*(gl_n(A))$.

A word about notation is in order. Usually, the Lie algebra homology of N is denoted $H_*(N)$. However in this paper we will deal with various kinds of homologies and so we adapt the notation $HL_*(N)$ to emphasize that this is Lie algebra homology.

There are numerous good references to Lie algebra homology. Among them are the original paper on the subject by Koszul [**11**] and the recent book by Guichardet [**4**].

Let A be an associative $\mathbf{C}$-algebra and let M be an A-bimodule. Define a complex $C_r(A, M)$ with boundary maps ∂_r by

$$\begin{aligned} C_r(A, M) &= M \otimes A^{\otimes r}, \\ \partial_r(m \otimes a_1 \otimes \cdots \otimes a_r) &= (m \cdot a_1) \otimes a_2 \cdots \otimes a_r \\ &\quad + \sum_{i=1}^{r-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_r \\ &\quad + (-1)^r (a_r \cdot m) \otimes a_1 \otimes \cdots \otimes a_{r-1}. \end{aligned}$$

It is easy to check that $\partial_{r-1} \circ \partial_r = 0$. Define $H_r(A, M)$ to be the homology of this chain complex. We call $H_r(A, M)$ the *Hochschild homology of A with coefficients in M*. In the case that $M = \mathbf{C}$ is the trivial module, the complex $(C_r(A, \mathbf{C}), \partial_r)$ is called the *Bar resolution*. In the case of the Bar resolution we denote the differential ∂ by b'. We will need the following well-known fact about $H_*(A, \mathbf{C})$ (see for example [**9**]). In this result, $\tilde{H}(A, \mathbf{C})$ denotes the reduced Hochschild homology.

THEOREM 2.4. *If A has an identity element then $\tilde{H}_*(A, \mathbf{C}) = 0$.*

A second case of Hochschild homology that we will use is the case where $M = A$ with bimodule structure being the left and right regular representation. In this case we denote the differential ∂ by b. The rth-graded piece of the complex is $A^{\otimes(r+1)}$. Define $N\colon C_r(A, A) \to C_r(A, A)$ to be the signed cyclic shift,

$$N(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = (-1)^r a_r \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}.$$

Let $I_r(A)$ be the invariants of $\langle N \rangle$ in $C_r(A, A)$ where $\langle N \rangle$ denotes the group generated by N. It is straightforward to check that $N \circ b_r = b_r \circ N$ so b_r can be restricted to a differential on $I_r(A)$. Define the *cyclic homology of A*, $HC_*(A)$, to be the homology of the complex $(I_*(A), b_*)$.

Connes [**2**] and Tsygan [**20**] independently discovered cyclic homology. In the last few years cyclic homology has received a tremendous amount of attention due to its wide variety of applications. The work most pertinent to this paper is that of Loday and Quillen [**14**]. For a survey of cyclic homology and its applications see Loday [**13**].

We will need the notion of graded exterior algebra for both $I_*(A)$ and $HC_*(A)$. In general, if $V = \bigoplus V_r$ is a graded vector space, the *graded exterior algebra* of V is defined to be $\mathrm{Sym}(V_{\mathrm{even}}) \otimes \mathrm{Ext}(V_{\mathrm{odd}})$ where V_{even} denotes the sum of the V_r for r even and V_{odd} denotes the sum of the V_r for r odd. We denote this graded exterior algebra by RV.

In the cases of interest to us, namely $V = \bigoplus I_*(A)$ and $V = \bigoplus HC_*(A)$, the grading that is important is the actual tensor power of A. Recall that this is one greater than the degree assigned to $I_*(A)$ and $HC_*(A)$ above.

DEFINITION 2.5. Define the graded rings $RC_*(A)$ and $RHC_*(A)$ by:

$$RC_*(A) = \mathrm{Sym}(I_{\mathrm{odd}}(A)) \otimes \mathrm{Ext}(I_{\mathrm{even}}(A)),$$
$$RHC_*(A) = \mathrm{Sym}(HC_{\mathrm{odd}}(A)) \otimes \mathrm{Ext}(HC_{\mathrm{even}}(A)).$$

Here the grading on $RC_*(A)$ and $RHC_*(A)$ is given in the following way. If $v_i \in I_{a_i}(A)$ and $w_j \in I_{b_j}(A)$ (where a_i is odd for all i and b_j is even for all j) then $(v_1 \cdots v_r) \otimes (w_1 \wedge \cdots \wedge w_s)$ lies in the $(\sum_{i=1}^r (a_i + 1)) + (\sum_{j=1}^s (b_j + 1))$ graded piece of $RC_*(A)$. The grading on $RHC_*(A)$ is analogous.

The differential b_* on $I_*(A)$ can be extended to a differential $\hat{b}_*$ on $RC_*(A)$ by making it an antiderivation on $RC_*(A)$. So we can form the homology $HRC_*(A)$ of $RC_*(A)$ with respect to $\hat{b}_*$. It is easy to check that $HRC_*(A) = RHC_*(A)$.

Given this grading for $HRC(A)$ we can now state the following important result of Loday and Quillen (Theorem 6.2, p. 584 of [**14**]).

THEOREM 2.6. *Let A be an associative ring. Then the $GL(n, \mathbf{C})$ invariants in $HL_*(gl_n(A))$ are stably isomorphic to $HRC_*(A)$.*

In subsequent sections we will deal with the cohomology rather than homology of the three types of complexes defined above. It will be necessary to know the coboundary operators explicitly and so we end this section with a quick computation of the coboundaries in each case. We use the following result which is easy to prove.

LEMMA 2.7. *Let $C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \cdots$ be a graded complex where each C_r is finite dimensional. Let β_r be a basis for C_r and let D_r be the matrix for ∂_r with respect to the bases β_r and β_{r-1}. Then the cohomology of this complex is $H^r = \ker \delta^r / \operatorname{im} \delta^{r-1}$ where δ^r is the linear transformation which has matrix D_r^t with respect to the bases β_r and β_{r+1}.*

We will use Lemma 2.7 to compute the coboundary for $H^*(A;\mathbf{C})$, $HC^*(A)$ and $HL^*(gl_n(A))$. Throughout we will assume that A is a graded $\mathbf{C}$-algebra which is finite dimensional in each graded piece. So as a vector space A has a direct sum decomposition, $A = \bigoplus_r A_r$, such that $A_r A_s \subseteq A_{r+s}$ and such that each A_r is finite dimensional. Let $B = \{c_1, c_2, \dots\}$ be a basis for A and let the numbers $\mu^c_{d,e}$ be the constants of multiplication in A with respect to B. In other words, for $d, e \in B$ we have

$$de = \sum_{c\in B} \mu^c_{d,e} c.$$

1. *The coboundary for $H^*(A;\mathbf{C})$.* Recall that the Bar resolution for computing $H^*(A;\mathbf{C})$ has rth-graded piece $A^{\otimes r}$ and boundary $b'_r\colon A^{\otimes r} \to A^{\otimes(r-1)}$ given by

$$b'_r(a_1 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r-1} (-1)^{i-1} a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_r.$$

In view of Lemma 2.7, we can compute the Hochschild cohomology of A with trivial coefficients using the complex

$$C \xrightarrow{0} A \xrightarrow{\beta'_1} A^{\otimes 2} \xrightarrow{\beta'_2} A^{\otimes 3} \xrightarrow{\beta'_3} A^{\otimes 4} \to \cdots$$

where β'_r is defined by

$$\beta'_r(a_1 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} (-1)^{i-1} \sum_{d,e\in B} \mu^{a_i}_{d,e} \{a_1 \otimes \cdots \otimes a_{i-1} \otimes (d \otimes e) \otimes \cdots \otimes a_r\}.$$

Here we have taken $a_1, \dots, a_r$ from B.

2. *The coboundary for $HC^*(A)$.* The coboundary for $HC^*(A)$ acting on the complex $I_*(A)$ is very similar to the one just given for $H^*(A;\mathbf{C})$. We denote this coboundary by β_r. It is given by

$$\beta_r(a_1 a_2 \cdots a_r) = \sum_{i=1}^{r} (-1)^{i-1} \sum_{d,e\in B} \mu^{a_i}_{d,e} (a_1 a_2 \cdots a_{i-1} d e a_{i+1} \cdots a_r).$$

3. *The coboundary for $HL^*(gl_n(A))$.* The coboundary for $HL^*(gl_n(A))$ acting on the complex $\bigwedge^*(gl_n(A))$ is slightly more complex. We denote this coboundary δ^*. It is most easily described in terms of the map $w\colon gl_n(A) \to gl_n(A) \wedge gl_n(A)$ which is defined by

$$w(z_{ij} \otimes a) = \sum_{l=1}^{n} \sum_{d,e\in B} \mu^a_{d,e} (z_{il} \otimes d) \wedge (z_{lj} \otimes e).$$

Here z_{ij} is the matrix in $gl_n(\mathbf{C})$ having a 1 in the i,j entry and 0's elsewhere.

Now in terms of w, the coboundary δ^* is given by

$$\delta^r(\eta_1 \wedge \cdots \wedge \eta_r) = \sum_{i=1}^{r} (-1)^{i-1} \eta_1 \wedge \cdots \wedge (w\eta_i) \wedge \cdots \wedge \eta_r.$$

3. Proof of the main result. In this section we prove the main result of this paper which was stated in the Introduction (Theorem 1.1). Before proceeding we must recall the construction of the Specht modules S^α.

Let $\alpha = (\alpha_1, \dots, \alpha_l)$ be a partition of m and let R_α and C_α be the row and column stabilizers of α (we think of R_α and C_α as permuting the squares of the Ferrer's diagram of α). A *Young tableau t of shape α* is a filling of the Ferrer's diagram of α with the numbers $1, 2, \dots, m$. Two tableaux t_1 and t_2 are *row equivalent* if $t_1 = \sigma t_2$ for some $\sigma \in R_\alpha$ (equivalently the fillings are the same up to rearranging the numbers within each row). An *α-tabloid* is a row equivalence class of Young tableaux of shape α. We let V^α denote the complex vector space with basis the set of all α-tabloids. The symmetric group S_m permutes the α-tabloids hence acts on the vector space V^α.

Let t be a Young tableau. Define the vector v_t in V^α by

$$v_t = \sum_{\gamma \in C_\alpha} \operatorname{sgn}(\gamma)\{\gamma t\}.$$

Let S^α denote the subspace of V^α spanned by the vectors v_t. One can show that S^α is an irreducible S_m submodule of V^α which we call the *Specht module* associated to α. A basis for S^α is the set of v_t where t is a standard Young tableau of shape α (a Young tableau is *standard* if its entries increase along rows and columns).

We begin by constructing all maximal weight vectors for $GL(n, \mathbf{C})$ of weight $[\alpha, \beta]_n$ inside the tensor algebra of $gl_n(A)$. This construction will build on the Specht module construction given above. Throughout this discussion $T^*(A)$ will denote the tensor algebra of A and $\tilde{T}^*(A)$ will denote $T^*(A)/T^0(A)$. $\tilde{T}^*(A)$ is the mth tensor power of $\tilde{T}^*(A)$ which we view as a graded vector space according to the grading that it inherits from $T^*(A)$.

DEFINITION 3.1. Let $\mathbf{u} = (a_1 \otimes \cdots \otimes a_r) \in T^r(A)$ (where $r > 0$) and let $i, j \in \{1, 2, \dots, n\}$. Define $Z(i, j; \mathbf{u})$ to be the element of $T^r(gl_n(A))$ given by

$$Z(i,j;\mathbf{u}) = \sum_{(i_1,\dots,i_{r-1})} (z_{i,i_1} \otimes a_1) \otimes (z_{i_1,i_2} \otimes a_2) \otimes \cdots \otimes (z_{i_r,j} \otimes a_r).$$

By a straightforward computation one finds that $gl_n(\mathbf{C})$ acts on these vectors $Z(i, j; \mathbf{u})$ as if they were the matrices z_{ij}. To be precise, we have the following result whose proof is left to the reader.

LEMMA 3.2. *For any $i, j, a, b \in \{1, 2, \dots, n\}$ and any $\mathbf{u} \in T^r(A)$ we have*

$$z_{a,b} \cdot Z(i,j;\mathbf{u}) = \delta_{ib} Z(a,j;\mathbf{u}) - \delta_{aj} Z(i,b;\mathbf{u}).$$

The vectors $Z(i, j; \mathbf{u})$ will be the basic building blocks for the maximal weight vectors in $T(A)$.

DEFINITION 3.3. Let $\mathbf{u}_i \in T^{r_i}(A)$, $i = 1, 2, \dots, m$, and let $\{s\}$ and $\{t\}$ be tabloids of shape α and β respectively where both α and β are partitions of m.

Define the vector $v(\{s\} \otimes \{t\}; \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_m)$ by

$$v(\{s\} \otimes \{t\}; \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_m) = \bigotimes_{i=1}^{m} Z(\rho_i(s), n+1-\rho_i(t); \mathbf{u}_i),$$

where $\rho_i(s)$ denotes the row of s containing the number i. It is easy to see that $v(\{s\} \otimes \{t\}; \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_m)$ is well defined in the sense that it is independent of the particular α-tableau and β-tableau chosen from the tabloids $\{s\}$ and $\{t\}$. It is also easy to see that $v(\{s\} \otimes \{t\}; \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_m)$ is a weight vector for $gl_n(\mathbf{C})$ with weight $[\alpha, \beta]_n$.

We extend this v construction to a degree-preserving linear map $\phi\colon V^\alpha \otimes_{\mathbf{C}} V^\beta \otimes_{\mathbf{C}} \tilde{T}(A)^{\otimes m} \to T^*(gl_n(A))$ by

$$\phi\left(\sum (S \otimes T) \otimes \mathbf{u}\right) = \sum v(S \otimes T; \mathbf{u}).$$

The following theorem is one of the main results in [**6**].

THEOREM 3.4. *Let x be a nonzero vector in $V^\alpha \otimes V^\beta$ and let $\mathbf{u}$ be a nonzero vector in the fth-graded piece of $\tilde{T}(A)^{\otimes m}$. Then $\phi(x \otimes \mathbf{u})$ is a maximal weight vector of weight $[\alpha, \beta]_n$ in $T^f(gl_n(A))$ if and only if $x \in S^\alpha \otimes S^\beta$. Moreover, every maximal weight vector of weight $[\alpha, \beta]_n$ in $T^f(gl_n(A))$ is in the linear span of the vectors*

$$\sigma(J \otimes \phi(x \otimes \mathbf{u})) \tag{3.5}$$

where J is a $GL(n, \mathbf{C})$-invariant in $T^r(gl_n(A))$, $x \in S^\alpha \otimes S^\beta$, $\mathbf{u}$ is in the $(f-r)$th graded piece of $\tilde{T}(A)^{\otimes m}$, $\sigma \in S_f$, and $\sigma(\eta_1 \otimes \cdots \otimes \eta_f) = \eta_{\sigma 1} \otimes \cdots \otimes \eta_{\sigma f}$ for $\eta_1, \ldots, \eta_f \in gl_n(A)$.

Theorem 3.4 gives a method for constructing all maximal weight vectors of weight $[\alpha, \beta]_n$ in $T^*(gl_n(A))$. It is not well understood what linear dependencies hold amongst vectors of the form (3.5). However, if n is greater than $f + l(\alpha) + l(\beta)$ then the exact nature of the linear dependencies is known. The answer in the case $A = \mathbf{C}$ is given by Theorem 4.13 in [**6**]. The answer for arbitrary algebras A can be obtained in exactly the same manner. This question is not directly pertinent to the results in this paper and so we leave it to the interested reader to work out the details.

For each f, define $\Pi_f\colon T^f(gl_n(A)) \to T^f(gl_n(A))$ by

$$\Pi_f(\eta_1 \otimes \cdots \otimes \eta_f) = \frac{1}{f!} \sum_{\sigma \in S_f} \operatorname{sgn}(\sigma) \eta_{\sigma 1} \otimes \cdots \otimes \eta_{\sigma f}.$$

It is clear that the image of Π_f is isomorphic to the fth exterior power of $gl_n(A)$. We will consider Π_f as a map from $T^f(gl_n(A))$ onto $\bigwedge^f(gl_n(A))$. The following result is a major component in the proof of our main theorem.

THEOREM 3.6. *Let $M^f([\alpha, \beta]_n)$ denote the vector space spanned by the maximal weight vectors of weight $[\alpha, \beta]_n$ in $\bigwedge^f(gl_n(A))$. Then there is a naturally defined surjective linear map ψ,*

$$\psi\colon RC^*(A) \otimes (\tilde{T}^*(A)^{\otimes m} \otimes S^\alpha \otimes S^\beta)_{S_m} \twoheadrightarrow M^f([\alpha, \beta]_n).$$

Moreover if n is greater than $f + l(\alpha) + l(\beta)$ then ψ is an isomorphism.

PROOF. For simplicity we define ψ in pieces. First define $\psi_0\colon RC^{r_0}(A) \to \bigwedge^{r_0}(gl_n(A))$ as follows:

(1) If $\gamma = c_1 \otimes c_2 \cdots \otimes c_{r_0} \in C^{r_0}(A)$ define

$$\psi_0(\gamma) = \sum_{i_1,\ldots,i_{r_0}} (z_{i_1 i_2} \otimes c_1) \wedge (z_{i_2,i_3} \otimes c_2) \wedge \cdots \wedge (z_{i_{r_0},i_1} \otimes c_{r_0})$$

where the sum is over all r_0-tuples $(i_1,\ldots,i_{r_0})$ with $1 \le i_j \le n$ for all j. It is easy to see that

$$\psi_0(c_1 \otimes c_2 \otimes \cdots \otimes c_{r_0}) = (-1)^{r_0-1}\psi_0(c_2 \otimes \cdots \otimes c_{r_0} \otimes c_1)$$

so the map ψ_0 is well defined on $C^{r_0}(A)$.

(2) If $\gamma = \gamma_1\gamma_2\cdots\gamma_l \in RC^*(A)$ with $\gamma_i \in C^{p_i}(A)$ define $\psi_0(\gamma)$ by

$$\psi_0(\gamma) = \psi_0(\gamma_1) \wedge \psi_0(\gamma_2) \wedge \cdots \wedge \psi_0(\gamma_l).$$

Since $\psi_0(\gamma_i) \in \bigwedge^{p_i}(A)$ we have

$$\psi_0(\gamma_1\cdots\gamma_i\gamma_{i+1}\cdots\gamma_l) = (-1)^{p_ip_{i+1}}\psi_0(\gamma_1\cdots\gamma_{i+1}\gamma_i\cdots\gamma_l).$$

So ψ_0 is a well-defined map on $RC^*(A)$. Note that $\psi_0(\gamma)$ is a $GL(n;\mathbf{C})$-invariant for all $\gamma \in RC^*(A)$.

Next define $\psi_1\colon (\tilde{T}^*(A)^{\otimes m} \otimes V^\alpha \otimes V^\beta) \to T^f(gl_n(A))$ in the following way. Let $\alpha_i = a_{i1} \otimes \cdots \otimes a_{ir_i} \in T^{r_i}(A)$ and let S and T be α and β tabloids. Define $\psi_1((\alpha_1 \otimes \cdots \otimes \alpha_m) \otimes S \otimes T)$ to be

$$\psi_1((\alpha_1 \otimes \cdots \otimes \alpha_m) \otimes S \otimes T) = \bigwedge_{j=1}^{m} \Omega_j,$$

where

$$\Omega_j = \sum_{i_0,\ldots,i_{r_j}} \left\{ \bigwedge_{l=1}^{r_j} z_{i_{l-1},i_l} \otimes a_{j,l} \right\} \in \bigwedge^{r_j} gl_n(A).$$

Here the sum in Ω_j is over all sequences i_0, $i_1,\ldots,i_{r_j}$ with $i_0 = p_j(S)$, $i_{r_j} = n+1-\rho_j(T)$ and $1 \le i_l \le n$ for $l = 1,2,\ldots,r_j-1$. We extend ψ_1 linearly to all of $(\tilde{T}^*(A)^{\otimes m} \otimes V^\alpha \otimes V^\beta)$.

Let $\tau = (j,j+1)$ be an adjacent transposition in S_m. Then

$$\begin{aligned}
&\psi_1(\tau \cdot ((\alpha_1 \otimes \cdots \otimes \alpha_m) \otimes S \otimes T)) \\
&\quad = (-1)^{r_jr_{j+1}}\psi_1((\alpha_1 \otimes \cdots \otimes \alpha_{j+1} \otimes \alpha_j \otimes \cdots \otimes \alpha_m) \otimes \tau \cdot S \otimes \tau \cdot T) \\
&\quad = (-1)^{r_jr_{j+1}}\Omega_1 \wedge \cdots \wedge \Omega_{j+1} \wedge \Omega_j \wedge \cdots \wedge \Omega_m \\
&\quad = \Omega_1 \wedge \cdots \wedge \Omega_j \wedge \Omega_{j+1} \wedge \cdots \wedge \Omega_m \\
&\quad = \psi_1((\alpha_1 \otimes \cdots \otimes \alpha_m) \otimes S \otimes T)
\end{aligned} \tag{3.7}$$

where the action of τ on $\tilde{T}^*(A)^{\otimes m} \otimes V^\alpha \otimes V^\beta$ is tensor product of the exterior action on $\tilde{T}^*(A)^{\otimes m}$ with the ordinary action of S_m on $V^\alpha \otimes V^\beta$. It follows immediately from equation (3.7) that ψ_1 annihilates every S_m-isotypic component of $\tilde{T}^*(A)^{\otimes m} \otimes V^\alpha \otimes V^\beta$ except the invariants. Consequently ψ_1 can be viewed as a map from $(\tilde{T}^*(A)^{\otimes m} \otimes V^\alpha \otimes V^\beta)_{S^m}$.

Now that we have ψ_0 and ψ_1 we put them together to define ψ,

$$\psi\colon RC^*(A)\otimes(\tilde{T}^*(A)^{\otimes m}\otimes V^\alpha\otimes V^\beta)_{S_m}\to\bigwedge^*(gl_n(A))$$

by $\psi(\gamma\otimes w)=\psi_0(\gamma)\wedge\psi_1(w)$.

We next show that ψ is one-to-one if n is greater than $f+l(\alpha)+l(\beta)-m$. The argument we use is very similar to the argument used in **[6]** to give an exact description of the linear dependencies amongst the maximal weight vectors given in (3.5). The idea of the proof is very simple but to write down a complete proof is very difficult from a notational point of view. We will give a sketch of the proof which demonstrates the main ideas and we leave the details to the interested reader.

We consider $\psi((\gamma_1\cdots\gamma_l)\otimes((\alpha_1\otimes\cdots\otimes\alpha_m)\otimes S\otimes T))=X$ where $\gamma_i=c_{i1}\otimes\cdots\otimes c_{ip_i}\in C^{p_i}(A)$, $\alpha_j=a_{j_1}\otimes\cdots\otimes a_{jr_j}\in\tilde{T}^{r_j}(A)$ where S is an α-tabloid and T is a β-tabloid. We will show that there exist completely decomposable wedge products $w\in T^f(gl_n(A))$ which have nonzero coefficient in X and from which we can read off the γ_i (up to the cyclic equivalence in $C^{P_i}(A)$), the tensors α_j and the vector $(\alpha_1\otimes\cdots\otimes\alpha_m)\otimes S\otimes T$ (up to the tensor product action of S_m). It follows that if w occurs with nonzero coefficient in $\psi((\gamma_1'\cdots\gamma_{l'}')\otimes((\alpha_1'\otimes\cdots\otimes\alpha_{m'}')\otimes S'\otimes T'))$ then $(\gamma_1\cdots\gamma_l)\otimes((\alpha_1\cdots\alpha_m)\otimes S\otimes T)=(\gamma_1'\cdots\gamma_{l'}')\otimes((\alpha_1'\otimes\cdots\otimes\alpha_{m'}')\otimes S'\otimes T')$ where the equality is in $RC^*(A)\otimes(\tilde{T}^*(A)^{\otimes m}\otimes V^\alpha\otimes V^\beta)_{S_m}$. Once we know this is follows easily that ψ is one-to-one. Choose a basis $\{B_1,B_2,\dots\}$ for the vector space $RC^*\otimes(\tilde{T}^*(A)\otimes V^\alpha\otimes V^\beta)_{S_m}$. We have just shown that for each B_i there is a completely decomposable wedge product $w(B_i)$ which occurs with nonzero coefficient in $\psi(B_i)$ and with coefficient 0 in $\psi(B_j)$ for $j\neq i$. Since the completely decomposable wedge products form a basis for $T^*(gl_n(A))$ it follows that no nontrivial linear combination of the B_i's is in the kernel of ψ.

Return now to X and the construction of the wedge product w. Let $H(\alpha,\beta)=\{l(\alpha)+1,l(\alpha)+2,\dots,n-l(\beta)\}$. Note that $H(\alpha,\beta)$ has size $n-l(\alpha)-l(\beta)$. Choose disjoint subsets $R_0,\hat{R}_1,\dots,\hat{R}_m\subseteq H(\alpha,\beta)$ with $|R_0|=r_0$ and $|\hat{R}_i|=r_i-1$ for $i=1,2,\dots,m$. This can be done as $n>f+l(\alpha)+l(\beta)-m$. We need notation for the elements of the sets R_0 and $\hat{R}_i$.

Write R_0 as the disjoint union $R_0=S_1\cup\cdots\cup S_l$ where $S_i=\{s_{i1},\dots,s_{ip_i}\}$ (so $|S_i|=p_i$). In the definition of w below we will refer to s_{i,p_i+1} by which we mean s_{i1}.

Next write $\hat{R}_j=\{t_{j1},\dots,t_{jr_{j-1}}\}$ and let $t_{j0}=\rho_j(S)$ and $t_{jr_j}=n+1-\rho_j(T)$. Let R_j be $\hat{R}_j$ together with t_{j0} and t_{jr_j}. We are now prepared to write down w.

$$w=\bigwedge_{i=1}^{l}\left(\bigwedge_{k=1}^{p_i}(z_{s_{i,k}s_{i,k+1}}\otimes c_{i,k})\right)\wedge\left\{\bigwedge_{j=1}^{m}\left(\bigwedge_{k=1}^{r_j}z_{t_{jk-1}t_{jk}}\otimes a_{jk}\right)\right\}. \tag{3.8}$$

We claim that w occurs with nonzero coefficient in X and that w occurs with coefficient 0 in $\psi((\gamma_1'\cdots\gamma_{l'}')\otimes((\alpha_1'\otimes\cdots\otimes\alpha_{m'}')\otimes S'\otimes T'))$ unless $\gamma_1'\cdots\gamma_{l'}'=\gamma_1\cdots\gamma_l$ in $RC^*(A)$ and $((\alpha_1'\otimes\cdots\otimes\alpha_{m'}')\otimes S'\otimes T')=((\alpha_1\otimes\cdots\otimes\alpha_m)\otimes S\otimes T)$ in $(\tilde{T}^*(A))^{\otimes m}\otimes V^\alpha\otimes V^\beta)_{S_m}$.

The easiest way to see this is with a combinatorial argument. Given a completely decomposable tensor $W=(z_{u_1v_1}\otimes b_1)\wedge\cdots\wedge(z_{u_fv_f}\otimes b_f)$ in $T^f(gl_n(A))$ we associate a directed graph $\Delta(W)$ with points $\{1,2,\dots,n\}$ and f labelled directed edges. There will be an edge directed from u_i to v_i labelled with b_i for $i=1,2,\dots,f$.

For the tensor w written down in (2.8) the graph $\Delta(w)$ consists of l cycles $\Gamma_1,\ldots,\Gamma_l$ and m paths $\Pi_1,\ldots,\Pi_m$. The cycle Γ_i has length p_i and edges labelled $c_{i1},\ldots,c_{ip_i}$ (read cyclically). The jth path Π_j has length r_j and has edges labelled $a_{j1},\ldots,a_{jr_j}$ (read linearly). Most importantly though, the cycles are disjoint from each other and from the paths and the paths intersect only at their initial points and terminal points. So this path-cycle decomposition of $\Delta(w)$ is the only possible one.

Now consider $Y = \psi((\gamma'_1\cdots\gamma'_{l'})\otimes((\alpha'_1\otimes\cdots\otimes\alpha'_{m'})\otimes S'\otimes T'))$ where $\gamma'_i = c'_{i1}\otimes\cdots\otimes c'_{p'_i}\in C^{p'_i}(A)$, $\alpha'_j = a'_{j1}\otimes\cdots\otimes a'_{jr'_j}\in T^{r'_j}(A)$, where S' is an α-tableau and T' is a β-tableau. It follows from the definitions of ψ_0 and ψ_1 that Y is a sum of completely decomposable wedge products. There is one W for each digraph Δ with cycles $\Gamma_1,\ldots,\Gamma_{l'}$, where Γ'_i has length p'_i and edge weights $c'_{i1},\ldots,c'_{ip'_i}$, and paths $\Pi'_1,\ldots,\Pi'_{m'}$, where Π'_i has length r'_i, initial point $p_j(S')$, terminal point $n+1-p_j(T')$ and edge weights $a'_{j'},\ldots,a'_{j_{r'_j}}$. So w has nonzero coefficient in Y if and only if $l=l'$, $m=m'$ and

1. After a suitable reordering of the γ'_i we have that γ_i is obtained from γ'_i by a cyclic shift.

2. There exists τ in S_m such that $\alpha'_{\tau j}=\alpha_j$, $\tau S' = S$ and $\tau T' = T$. It follows that

$$(\gamma'_1\cdots\gamma'_{l'})\otimes((\alpha'_1\otimes\cdots\otimes\alpha'_{m'})\otimes S'\otimes T')_{S_m} = (\gamma_1\cdots\gamma_l)\otimes((\alpha_1\otimes\cdots\otimes\alpha_m)\otimes S\otimes T)$$

and so ψ is one-to-one.

Now we restrict ψ to $RC^*(A)\otimes(\tilde{T}^*(A)^{\otimes m}\otimes S^\alpha\otimes S^\beta)_{S_m}$. Let $x\in S^\alpha\otimes S^\beta$, $\alpha_i\in T^{r_i}(A)$, $c_1\otimes\cdots\otimes c_{r_0}\in T^{r_0}(A)$ and $\pi\in S_{r_0}$. Define $\gamma_1\cdots\gamma_l\in RC^*(A)$, where l is the number of cycles of π, in the following way. There will be one γ_i for each cycle of π. The γ_i corresponding to $(u_1,\ldots,u_d)$ is $\gamma_i = c_{u_1}c_{u_2}\cdots c_{u_d}$ (this γ_i is defined up to a sign). It is easy to see that

$$(3.9)\quad \psi(\gamma_1\cdots\gamma_l\otimes((\alpha_1\otimes\cdots\otimes\alpha_m)\otimes x\otimes y)) = \pm f!\{P_f(\sigma(\pi[c_1\cdots c_{r_0}]\otimes\varphi(\mathbf{u}\otimes x)))\}$$

where $\mathbf{u}=\alpha_1\otimes\cdots\otimes\alpha_m$.

It follows from (3.9) and Theorem 3.4 that the image of ψ is $M^f([\alpha,\beta]_n)$. We have also shown that ψ is one-to-one and this completes the proof of Theorem 2.6.

Recall that the Bar complex for computing $\tilde{H}^*(A,\mathbf{C})$ consists of the graded vector space $\tilde{T}^*(A)$ with differential β'. We can make $\tilde{T}^*(A)^{\otimes m}$ into the tensor product complex with coboundary called β' defined by

$$\beta'(\eta_1\otimes\cdots\otimes\eta_m) = \sum_{i=1}^{m}(-1)^{\sum d_j}\eta_1\otimes\cdots\otimes\beta'(\eta_i)\otimes\cdots\otimes\eta_m$$

where η_j has degree d_j and $\sum d_j$ is over $j\le i$. It is easy to check that β' commutes with exterior action of S_m on $\tilde{T}^*(A)^{\otimes m}$ and the induced action on $\tilde{H}^*(A,\mathbf{C})^{\otimes m}$ is just the exterior action of S_m on the mth tensor power of the graded vector space $\tilde{H}^*(A,\mathbf{C})$.

Recall that $RC^*(A)$ comes equipped with a coboundary ∂^* and that the cohomology is just the cyclic cohomology $HRC^*(A)$. We can now view $RC^*(A)\otimes(\tilde{T}^*(A)^{\otimes m}\otimes S^\alpha\otimes S^\beta)_{S_m}$ as a complex by giving it coboundary T,

$$T(\gamma\otimes(\mathbf{u}\otimes x)) = (\partial\gamma)\otimes(\mathbf{u}\otimes x)+(-1)^r\gamma\otimes((\beta'\mathbf{u})\otimes x)$$

where $\gamma \in RC^r(A)$, $\mathbf{u} \in \tilde{T}^*(A)^{\otimes m}$ and $x \in S^\alpha \otimes S^\beta$. Clearly the cohomology with respect to T is

$$HRC^*(A) \otimes (\tilde{H}^*(A, \mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)_{S_m}. \tag{3.10}$$

Note that this is the right-hand side of the equation given by Theorem 1.1.

Recall that the vector space $M^*([\alpha,\beta]_n)$ is closed under the Lie algebra coboundary δ^* and that $(HL^*(gl_n(A)))_{[\alpha,\beta]_n}$ is defined to be the cohomology of this subcomplex $(M^*([\alpha,\beta]_n), \delta^*)$. In view of this and (3.10), we see that the main theorem (Theorem 1.1) is a direct consequence of the following result.

THEOREM 3.11. *Let n be greater than $f + l(\alpha) + l(\beta) - m$ so that*

$$\psi\colon RC^*(A) \otimes (\tilde{T}^*(A)^{\otimes m} \otimes S^\alpha \otimes S^\beta)_{S_m} \to M^*([\alpha,\beta]_n)$$

is an isomorphism. Then $\psi \circ T = \delta \circ \psi$.

PROOF. First suppose $\gamma = c_1 \otimes c_2 \otimes \cdots \otimes c_{r_0} \in C^{r_0}(A)$. Then

$$T\gamma = \sum_{j=1}^{r_0} (-1)^{j-1} \sum_{d,e} \mu_{d,e}^{c_j} c_1 \otimes \cdots \otimes c_{j-1} \otimes d \otimes e \otimes \cdots \otimes c_{r_0}.$$

So

$$\begin{aligned}
\psi_0 T\gamma = &\sum_{\substack{i_1,\dots,i_{r_0}\\ l}} \sum_{j=1}^{r_0} (-1)^{j-1} \sum_{d,e} \mu_{d,e}^{c_j} (z_{i_1 i_2} \otimes c_1) \wedge \cdots \wedge (z_{i_{j-1} i_j} \otimes c_{j-1}) \\
&\qquad \wedge (z_{i_j l} \otimes d) \wedge (z_{l i_{j+1}} \otimes e) \wedge \cdots \wedge (z_{i_r i_1} \otimes c_r) \\
&= \delta\left(\sum_{i_1,\dots,i_r} (z_{i_1 i_2} \otimes c_1) \wedge \cdots \wedge (z_{i_r i_1} \otimes c_r) \right) \\
&= \delta\psi_0\gamma.
\end{aligned} \tag{3.12}$$

It follows easily that $\psi_0 T(\gamma_1 \cdots \gamma_l) = \delta\psi_0(\gamma_1 \cdots \gamma_l)$ for any $\gamma_1 \cdots \gamma_l \in RC^*(A)$.

Next suppose that $a_1 \otimes a_2 \otimes \cdots \otimes a_r \in T^r(A)$. Then

$$\begin{aligned}
\delta Z(i,j; a_1 \otimes \cdots \otimes a_r) &= \delta\left(\sum_{i_1,\dots,i_{r-1}} (z_{i i_1} \otimes a_1) \wedge \cdots \wedge (z_{i_{r-1} j} \otimes a_r) \right) \\
&= \sum_{s=1}^{r} \sum_{l=1}^{n} \sum_{i_1,\dots,i_{r-1}} \sum_{d,e} (-1)^{s-1} \mu_{d,e}^{a_s} (z_{i i_1} \otimes a_1) \wedge \cdots \\
&\qquad \wedge \{(z_{i_{s-1} l} \otimes d) \wedge (z_{l i_s} \otimes e)\} \wedge \cdots \wedge (z_{i_{r-1} j} \otimes a_r) \\
&= Z\left(i,j; \sum_{s=1}^{r} \sum_{d,e} (-1)^{s-1} \mu_{d,e}^{a_s} a_1 \otimes \cdots \otimes a_{s-1} \otimes d \otimes e \otimes \cdots \otimes a_r \right) \\
&= Z(i,j; \beta'(a_1 \otimes \cdots \otimes a_s)).
\end{aligned}$$

Hence for $\alpha_1 \otimes \cdots \otimes \alpha_m \in \tilde{T}^*(A)^{\otimes m}$ and S, T α- and β-tabloids we have

$$\delta(\psi_1((\alpha_1 \otimes \cdots \otimes \alpha_m) \otimes S \otimes T)) = \psi_1((\beta'(\alpha_1 \otimes \cdots \otimes \alpha_m)) \otimes S \otimes T) \tag{3.13}$$

Now Theorem 3.11 follows immediately from (3.12) and (3.13).

This proves our main theorem, Theorem 1.1.

4. Poincaré series and Euler characteristics. In this section we deduce certain information about the Poincaré series and Euler characteristic of $HL^*(gl(A))$ from Theorem 1.1. For the rest of this section let A be a graded associative $\mathbf{C}$-algebra. So A can be written as a direct sum of finite-dimensional subspaces A_r $(r \geq 0)$ and $A_r A_s \subseteq A_{r+s}$. We will assume that $A_0 = 0$ so in particular A has no identity element. We let $d_r = \dim A_r$ and let $P(A;q)$ denote the Poincaré series of A,

$$P(A;q) = \sum_{r \geq 1} d_r q^r.$$

The grading of A induces a grading on the Koszul complex $\bigwedge gl_n(A)$. More precisely, if $a_i \in A_{r_i}$ for $i = 1, 2, \ldots, f$ we say that $(z_{i_1 j_1} \otimes a_1) \wedge \cdots \wedge (z_{i_f j_f} \otimes a_f)$ is in the $(r_1 + \cdots + r_f)$-graded piece of $\bigwedge^f gl_n(A)$. It is easy to check that the Lie algebra coboundary δ preserves this grading and so we get a bigrading on the Lie algebra cohomology $HL(gl_n(A))$. This bigrading

$$HL(gl_n(A)) = \bigoplus_{f,r} HL^{f,r}(gl_n(A))$$

has first component f being homological degree and second component r being the grading inherited from A. It is easy to check that each $gl_n(\mathbf{C})$-isotypic component $HL_{[\alpha,\beta]_n}(gl_n(A))$ is a bigraded subspace of $HL(gl_n(A))$. Define the *Poincaré series* $P(HL_{[\alpha,\beta]_n}(gl_n(A)); x, q)$ and the *Euler characteristic* $E(HL_{[\alpha,\beta]_n}(gl_n(A)); q)$ by

$$P(HL_{[\alpha,\beta]_n}(gl_n(A)); x, q) = \sum_{f,r} m_{[\alpha,\beta]_n}(HL^{f,r}(gl_n(A)))x^f q^r$$

and

$$E(HL_{[\alpha,\beta]_n}(gl_n(A)); q) = P(HL_{[\alpha,\beta]_n}(gl_n(A)); -1, q).$$

The grading on A yields a similar bigrading on $HRC(A)$ and $H(A;\mathbf{C})$. Define their Poincaré series and Euler characteristics by

$$P(HRC(A); x, q) = \sum_{f,r} \dim(HRC^{f,r}(A))x^f q^r,$$
$$P(H(A;\mathbf{C}); x, q) = \sum_{f,r} \dim(H^{f,r}(A;\mathbf{C}))x^f q^r$$

and

$$E(HRC(A); q) = P(HRC(A); -1, q),$$
$$E(H(A;\mathbf{C}); q) = P(H(A;\mathbf{C}); -1, q).$$

To state the next theorem we need some standard notation from the representation theory of the symmetric group S_m. For $\sigma \in S_m$ we let $j_i(\sigma)$ denote the number of i-cycles of σ and we define the *cyclic indicator of* σ, $Z(\sigma)$, by

$$Z(\sigma) = x_1^{j_1(\sigma)} x_2^{j_2(\sigma)} \cdots x_m^{j_m(\sigma)}.$$

If φ is any class function on S_m, the *cyclic index of* φ, denoted $Z(\varphi)$, is the polynomial

$$Z(\varphi) = \frac{1}{m!} \sum_{\sigma \in S_m} \varphi(\sigma) Z(\sigma).$$

Lastly if $F(y_1, y_2, \dots)$ is a power series in any set of variables define the *composition of F over $Z(\varphi)$*, denoted $Z(\varphi)[F(y)]$, to be the power series obtained by replacing every occurrence of x_i in $Z(\varphi)$ by $F(y_1^i, y_2^i, \dots)$. In short,

$$Z(\varphi)[F(y)] = Z(\varphi)[x_i \to F(y_1^i, y_2^i, \dots)].$$

Recall that χ^α and χ^β denote the irreducible characters of S_m given by the trace of S_m acting on S^α and S^β. Primarily we will be interested in the class function $\varphi = \chi^\alpha\chi^\beta$ by which we mean the class function whose value on a permutation σ is the product $\chi^\alpha(\sigma)\chi^\beta(\sigma)$. Clearly $\chi^\alpha\chi^\beta$ is the trace of S_m acting on $S^\alpha \otimes S^\beta$.

THEOREM 4.1. *For any α and β the Poincaré series $P(HL_{[\alpha,\beta]_n}(gl_n(A)); x, q)$ approaches a limit as n goes to infinity. This limit is given by*

$$P(HL_{[\alpha,\beta]}(gl(A)); x, q) = P(HRC(A); x, q)F(x, q)$$

where

$$F(x,q) = Z(\chi^\alpha\chi^\beta)\begin{bmatrix} x_{2i} \to P(\tilde{H}(A;\mathbf{C}); -x^{2i}, q^{2i}) \\ x_{2i+1} \to P(\tilde{H}(A;\mathbf{C}); x^{2i+1}, q^{2i+1}) \end{bmatrix}.$$

PROOF. In view of Theorem 1.1 it is enough to show that

$$\sum_{f,r} \dim(\tilde{H}(A,\mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)^{f,r}_{S_m} x^f q^r$$
$$= Z(\chi^\alpha\chi^\beta)\begin{bmatrix} x_{2i} \to P(\tilde{H}(A;\mathbf{C}); -x^{2i}, q^{2i}) \\ x_{2i+1} \to P(\tilde{H}(A;\mathbf{C}); x^{2i+1}, q^{2i+1}) \end{bmatrix}.$$

By standard character-theoretic arguments we have

$$\sum_{f,r} \dim(\tilde{H}(A;\mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)^{f,r}_{S_m} x^f q^r$$
$$= \frac{1}{m!}\sum_{\sigma\in S_m} \operatorname{tr}(\sigma|(\tilde{H}(A;\mathbf{C})^{\otimes m} \otimes S^\alpha \otimes S^\beta)^{f,r})x^f q^r \tag{4.2}$$

where $\operatorname{tr}(\sigma|V)$ denotes the trace of σ acting on the vector space V.

Let $\{B_1, B_2, \dots\}$ be a basis for $\tilde{H}(A;\mathbf{C})$. So a basis for $\tilde{H}(A;\mathbf{C})^{\otimes m}$ is $\{B_{i_1} \otimes \cdots \otimes B_{i_m}\}$. We compute the trace of σ with respect to this basis (assuming here that B_i is in the (d_i, r_i)-graded piece of $\tilde{H}(A;\mathbf{C})$. We have a contribution to the trace of σ from $B_{i_1} \otimes \cdots \otimes B_{i_m}$ if and only if $B_{i_l} = B_{i_{\sigma l}}$ for all l. In other words the same B_{i_l} must appear in tensor positions l which lie in each cycle of σ. If this condition is met then the contribution to the trace of σ from a cycle $C = (c_1, \dots, c_s)$ is $(-1)^{d_{c_1}^2(s-1)}$ (recall that the action of S_m on $\tilde{H}(A;\mathbf{C})^{\otimes m}$ is the exterior action).

So the right-hand side of (4.2) is equal to

$$\frac{1}{m!}\sum_{\sigma\in S_m} \chi^\alpha(\sigma)\chi^\beta(\sigma)\prod_{s=1}^{m}\left\{\sum_{d,r}\dim(\tilde{H}^{d,r}(A;\mathbf{C}))((-1)^{s-1}x)^d q^{rs}\right\}$$
$$= Z(\chi^\alpha\chi^\beta)\begin{bmatrix} x_{2i} \to P(\tilde{H}(A;\mathbf{C}); -x^{2i}, q^{2i}) \\ x_{2i+1} \to P(\tilde{H}(A;\mathbf{C}); x^{2i+1}, q^{2i+1}) \end{bmatrix}$$

which completes the proof.

The Euler characteristic has an even simpler form just in terms of the Poincaré series for A. This is described in the next theorem which was proved independently by R. P. Stanley [**18**] using symmetric functions methods.

THEOREM 4.3. *For any α and β we have*

$$E(HL_{[\alpha,\beta]}(gl(A));q)=\prod_{i=1}^{\infty}\frac{F(q)}{1+P(A;q^i)}$$

where

$$F(q)=Z(\chi^\alpha\chi^\beta)\,[-P(A;q)/(1+P(A;q))]\,.$$

PROOF. Let C_n denote the cyclic group of order n. It is easy to see that

$$\sum_f \dim(C^{n-1,f}(A))(-1)^n q^f = Z(C_n)[x_i\to(-1)^{i-1}P(A;q^i)].$$

It follows from R. W. Robinson's Composition Theorem (see Harary and Palmer [**8**] or Robinson [**16**]) that

$$\begin{aligned}&\sum_{d,f}\dim(RC^{d,f}(A))(-1)^d q^f\\ &\quad=\left\{\sum_{n=0}^{\infty}Z(S_n)\left[x_i\to\sum_m(-1)^{(i-1)m}Z(C_m)[x_j\to(-1)^{j-1}x_{ij}]\right]\right\}[P(A;q)]\\ &\quad=\left\{\sum_{n=0}^{\infty}Z(S_n)\left[x_i\to\sum_m Z(C_m)[x_j\to -x_{ij}]\right]\right\}[P(A;q)].\end{aligned}$$

A theorem due to R. C. Read (see [**15**]) implies that

$$\sum_{n=0}^{\infty}Z(S_n)\left[x_i\to\sum_m Z(C_m)[x_j\to x_{ij}]\right]=\prod_l(1-x_l)^{-1}.$$

So

$$\sum_{d,f}\dim(RC^{d,f}(A))(-1)^d q^f=\prod_l(1+P(A;q^l))^{-1}. \tag{4.4}$$

Next note that $E(H(\tilde{A};\mathbf{C});q)=-P(A;q)/(1+P(A;q))$. Combining (4.4) with this observation and Theorem 4.1 gives the result.

One example which is of particular interest to combinatorialists is the case $A=t\mathbf{C}[t]/t^{k+1}$, the truncated polynomial ring without constants. Earlier results of the author showed that $HRC(A)$ is an exterior algebra with k generators of each odd degree $2m+1$. The generators of degree $2m+1$ have bidegree $(2m+1,(k+1)m+i)$, $i=1,2,\dots,k$. It is easy to check that the Hochschild cohomology of A with trivial coefficients is the tensor product of an exterior algebra with generator of bidegree (1,1) and a symmetric algebra with generator of bidegree $(2,k+1)$. The generator of the exterior algebra is t and the generator of the symmetric algebra is $\sum_{i=1}^k t^i\otimes t^{k+1-i}$. So by Theorem 4.1 we have

$$P(HL_{[\alpha,\beta]}(gl(t\mathbf{C}[t]/t^{k+1}));x,q)=\left\{\prod_{m=1}^{\infty}\prod_{i=1}^{k}(1+x^{2m+1}q^{(k+1)m+i})\right\}F(x,q)$$

where

$$F(x,q)=Z(\chi^\alpha\chi^\beta)\left[\begin{array}{c} x_{2i}\to\dfrac{1-(xq)^{2i}}{1-(x^2q^{k+1})^{2i}}-1\\ x_{2i+1}\to\dfrac{1+(xq)^{2i+1}}{1-(x^2q^{k+1})^{2i+1}}-1\end{array}\right]. \tag{4.5}$$

Generating functions of the form (4.5) have been studied by combinatorialists since the time of Littlewood (see [**12**, Chapter 11]). In particular, R. P. Stanley proved that such generating functions have an extremely elegant expression as a rational function (see [**19**, Theorem 6.2]). It would be interesting to know if that factorization has a cohomological interpretation.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109-1003