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VARIATIONAL PRINCIPLES FOR HILL'S SPHERICAL VORTEX AND NEARLY SPHERICAL VORTICES

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ABSTRACT. In this paper, vortex rings are regarded as axisymmetric motions without swirl of an incompressible inviscid fluid in space, with vorticity confined to their finite cores. The main results of this paper are (H) Hill's spherical vortex is a "nondegenerate" local maximum of the energy function subject to a fixed impulse, among vortex rings. (N) Norbury's nearly spherical vortex is a "nondegenerate" local maximum of the energy function subject to a fixed impulse, and a fixed circulation. Estimates are established to overcome the discontinuity of vorticity distributions, and the singular behavior of Stoke's stream functions near the axis of symmetry. The spectral analysis involves the use of Legendre's functions.

1. Introduction. In this paper, we consider axisymmetric motions without swirl of an incompressible inviscid fluid in space, with zero velocity at infinity. We shall restrict ourselves to a vortex ring, in which the vorticity is confined to a bounded region (called the core) and its magnitude is equal to the distance to the axis of symmetry.

A vortex ring is said to be *steady* if it moves without change of shape and propagates at a constant speed along its axis of symmetry. Helmholz's vortex ring [**14**, **8**] is steady and it has a solid torus with small cross section as its core. On the other hand, Hill's spherical vortex [**14**, **11**] is steady and it has a solid sphere as its core. Norbury [**18**, **19**] found a family of steady vortex rings which connect Helmholtz's vortex ring to Hill's spherical vortex. The shape of the cores in this family changes at Hill's spherical vortex.

Variational characterizations of steady motions often enable us to draw stability results. This idea in fluid mechanics went back at least to Kelvin [**13**]. The variational characterizations based on stream functions for Helmholtz's rings and Hill's vortex ring have been established by Fraenkel and others [**1**, **9**]. We aim to show that one can also verify variational characterizations based on conserved quantities for Hill's spherical vortex and Norbury's nearly spherical vortex [**19**]. Therefore, one obtains stability results for these vortex rings within the class of axisymmetric motions.

The formulation of our variational principles was suggested by Benjamin [**5**] and Friedman and Turkington [**10**]. Indeed, the main results of this paper are as follows:

(H) *Hill's spherical vortex is a "nondegenerate" local maximum of the energy function subject to a fixed impulse, and a constraint* $Z = 0$.

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(N) *Norbury's nearly spherical vortex is a "nondegenerate" local maximum of the energy function subject to a fixed impulse, a fixed circulation, and a constraint $Z = 0$.*

Notice that one has the impulse [**14**, **6**] as a conserved quantity corresponding to the translation invariance nature of our problem. The constraint $Z = 0$ will be defined in §2, and it is not a translation invariance. Our local maximum ς_A (in (H) or (N)) is "nondegenerate" in the sense that: the negative of increments of energy at ς_A bounds a positive quadratic function of the variations of vorticity within the given constraints near ς_A. One needs such a priori estimates in order to draw stability results from variational principles based on conserved quantities. Our results complement those of Friedman and Turkington [**10**] and Fraenkel and Berger [**9**], in which they provide the existence of steady vortex rings via variational principles.

In [**2**, **3**] Arnold showed how one can establish a priori estimates in a smooth setting, and hence one can obtain a nonlinear stability result. These geometrical considerations have been exploited recently by Marsden and Weinstein [**15**]. They also formulated this method for general Hamiltonian systems and coined it as Casimir-energy method. For more details and applications, see Holm et al. [**12**]. However, one cannot apply this method to motions of vortex patches (in R^2), for the vorticity has discontinuity and the setting is not smooth. In [**21**], Wan and Pulvirenti show how to overcome this difficulty. Indeed, one can reduce this problem to a smooth one in some suitable sense. This idea is used in Tang [**20**] successfully to establish a variational principle and a stability result for Kirchhoff's elliptic vortex.

For axisymmetric flows, we can take a half-plane as a cross section and formulate our problem on this half-plane. This formulation is very similar to that of vortex patches. A steady vortex ring corresponds to a steady translating vortex patch. In principle, a vortex ring near a steady vortex ring allows the possibility of long thin and complicated filaments moving relatively far from the steady one, as that can happen to a vortex patch. For Hill's spherical vortex, a thin spike can grow from the rear stagnation point [**16**]. Thus, we like to use the method in [**21**] to overcome the nonsmoothness nature of our problem. Our problem has another complication due to the singular behavior (of Stokes' stream functions, etc.) near the axis of symmetry. The regularity result for Stokes' stream functions presented as Proposition 2 in the Appendix helps us to resolve this singularity.

Let us now outline the idea of proving our main results (H) and (N). First, by using an implicit function theorem to a suitable mapping, one makes a clever choice of a vortex ς^*, C^1-close to the steady one ς_s and satisfying the same given constraints so that one can prove $E(\varsigma) < E(\varsigma^*)$. Here, $E(\varsigma)$ denotes the energy of the vortex ς. Secondly, by some lengthy computations, one can obtain a second order Taylor expansion for $E(\varsigma^*)$ around ς_s with a nice estimate of the remainder. Thus, $E(\varsigma^*) < E(\varsigma_s)$ follows by proving the negative definiteness of the second order term. For Hill's spherical vortex, this can be carried out by a spectral analysis which involves Legendre's functions of order 1 as eigenfunctions. For a nearly spherical vortex, it suffices to show that its second order term is approximated by that of Hill's spherical vortex in a suitable sense. Combining the above two inequalities, one has $E(\varsigma) < E(\varsigma_s)$.

The precise statements of our main results in this paper are given in §2 as Theorems (H) and (N). The proofs of Theorems (H) and (N) are given in two steps in §§3 and 4 respectively. To make these proofs readable, technical estimates and their consequences are collected and justified in the Appendices.

Finally, I would like to thank the referee for many valuable suggestions, especially the short proof of our regularity results on stream functions (i.e. Proposition 2), based on a transformation discussed in Ni [**17**] and Amick and Fraenkel [**1**].

2. Statement of the main results. Let (r, η, z) be a cylindrical coordinate system on R^3. Throughout this paper, we consider the flow of an incompressible inviscid fluid with unit density in R^3 such that the velocity field is symmetric about the z-axis, tangent to the half-planes $\eta =$ constant, and zero at infinity. It is natural and convenient to represent such an axisymmetric flow on the meridional half-plane $(\eta = 0)$ $M \equiv \{(r,z) | r \geq 0, -\infty < z < \infty\}$ with measure $d\mu = 2\pi r\, dr\, dz$. At any instant, the velocity field on M can be described by a (Stokes') stream function $\psi = \psi(r,z)$, $(v^r, v^z) = (-\psi_z/r, \psi_r/r)$ and its vorticity $\omega = v^r_z - v^z_r$ given by $\omega = -(L\psi)/r$ with

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

We like to use a formulation in terms of a "vortex density" ς (called vortex ς for simplicity) on the half-plane M, defined as $\varsigma(r,z) = \omega(r,z)/r$. We should always assume $\varsigma(r,z)$ is a bounded function with compact support in M. For $\varsigma = -(L\psi)/r^2$, let us take $\psi = \int_M G\varsigma\, d\mu$ as the associated stream function to ς. Here, the Green[1] function G can readily be found as

$$G(r,z,r',z') = \frac{rr'}{8\pi^2}\int_{-\pi}^{\pi} \frac{\cos\eta'\, d\eta'}{[(z-z')^2 + r^2 + r'^2 - 2rr'\cos\eta']^{1/2}}.$$

Now, the kinetic energy E, the impulse P, and the circulation Γ have the following expressions [**14**, **9**, **10**, **5**].

$$E = \frac{1}{2}\int_M \psi\varsigma\, d\mu = \frac{1}{2}\int_M \langle\psi, \varsigma\rangle,$$

$$P = \int_M r^2\varsigma\, d\mu, \qquad \Gamma = \int_M \varsigma\, d\mu.$$

It is well known that $\varsigma, d\mu, E, P$, and Γ are invariant under axisymmetric motions of an incompressible inviscid fluid governed by the vorticity equation: $\varsigma_t + v^r\varsigma_r + v^z\varsigma_z = 0$. Define also a noninvariant quantity $Z = \int_M zr^2\varsigma\, d\mu$, similar to those in [**5**, **10**].

As in [**18**], closeness of vortices can be measured by a L^1-norm, $\|\varsigma\|_1 = \int_M |\varsigma|\, d\mu$. For simplicity, we often take vortices ς from a class $\mathcal{V}$, which consists of vortices ς in the form $\varsigma = \chi_A$, where χ_A denotes the characteristic function of some bounded set $A \subset M$. A vortex $\varsigma = \chi_A \in \mathcal{V}$, with C^1 boundary ∂A in M, is called *steady* if the vortex ς propagates with some constant speed W in the z-direction. This is equivalent to say that the relative stream function, $\tilde{\psi} \equiv \psi - \frac{1}{2}Wr^2 =$ constant on the boundary ∂A of A (flux constant). The *Hill's spherical vortex* ς_H (of radius 1) is a *steady* vortex such that $\varsigma_H = \chi_H$, $H = \{(r,z) \in M | r^2 + z^2 \leq 1\}$ *with* $W_H = 2/15$

[1]Due to different normalizations, the Green function used here and those in [**10**, **18**] are all different by some factors.

[**14**, **11**]. Denote by $\varsigma_N(\varepsilon)$ *Norbury's steady vortex ring* with speed $W(\varepsilon)$ found in [**18**, **19**]. It has the form $\varsigma_N(\varepsilon) = \chi_{H_\varepsilon}$, $H_\varepsilon = \{(r,z) \in M | \tilde{\psi}_N(\varepsilon) \geq \gamma(\varepsilon)\}$ with $\tilde{\psi}_N(\varepsilon) = \psi_N(\varepsilon) - \frac{1}{2}W(\varepsilon)r^2$. $\psi_N(\varepsilon) = \int G\varsigma_N(\varepsilon)\,d\mu$, $\gamma(\varepsilon) > 0$. $\varsigma_N(\varepsilon)$ approaches to Hill's spherical vortex ς_H, $W(\varepsilon) \to W_H$, $\gamma(\varepsilon) \to 0$, and $\psi_N(\varepsilon) \to \psi_H$ pointwise as $\varepsilon \to 0+$. One can assume that $H_\varepsilon \subset H$, by proper scaling if necessary.

Now we are ready to state our main results.

THEOREM (H). *There exist positive constants c_3, δ such that the inequality $E(\varsigma_H) - E(\varsigma) \geq c_3 P(|\varsigma_H - \varsigma|)^2$ holds for $\varsigma \in \mathcal{V}$ when $P(\varsigma) = P(\varsigma_H)$, $Z(\varsigma) = Z(\varsigma_H)(= 0)$, and $P(|\varsigma - \varsigma_H|) < \delta$. Recall that $P(|\xi|) = \int r^2|\xi|\,d\mu$.*

THEOREM (N). *For each small $\varepsilon > 0$, there exist constants $c_3 > 0$, $\delta > 0$ such that $E(\varsigma_N(\varepsilon)) - E(\varsigma) \geq c_3\|\varsigma_N(\varepsilon) - \varsigma\|_1^2$ holds for $\varsigma \in \mathcal{V}$, when $P(\varsigma) = P(\varsigma_N(\varepsilon))$, $\Gamma(\varsigma) = \Gamma(\varsigma_N(\varepsilon))$, $Z(\varsigma) = Z(\varsigma_N(\varepsilon))$ $(= 0)$ and $\|\varsigma_N(\varepsilon) - \varsigma\|_1 < \delta$.*

REMARK 1. The real numbers R acts on $\mathcal{V}$ via translations along z-axis. The functions E, P, Γ, but not Z, are invariant under this action. Here, we take the normalization $Z(\varsigma) = 0$ in Theorems (H), (N) to factor out translations. Otherwise, the estimates are trivially false.

REMARK 2. One expects that, in Theorem (N), $c_3 \to 0$ as $\varepsilon \to 0$, which relates to the fact that one needs to use $P(|\varsigma - \varsigma_H|)$ instead of $\|\varsigma - \varsigma_H\|_1$ in Theorem (H).

REMARK 3. Formally, a steady vortex ring ς_A can be characterized as an extreme point of $E(\varsigma)$ on $\mathcal{V}$, subject to $P(\varsigma) = P(\varsigma_A)$, $\Gamma(\varsigma) = \Gamma(\varsigma_A)$ with the propagation speed $W_A/2$ and the flux constant γ_A as the corresponding Lagrangian multipliers. The flux constant is zero for ς_H and positive for $\varsigma_N(\varepsilon)$. So that the circulation $\Gamma(\varsigma)$ is prescribed in Theorem (N) but not in Theorem (H).

Denote by $\phi_t(\varsigma)$ the vortex at time t, with initial vortex $\varsigma \cdot \Phi_t$ leaves $\mathcal{V}$ invariant. A steady vortex ς_A is said to be *stable* with respect to a metric d if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that to each $\varsigma \in \mathcal{V}$ with $d(\varsigma, \varsigma_A) < \delta$, and to each $t > 0$, $d(\Phi_t(\varsigma), \Phi_{t^*}(\varsigma_A)) < \varepsilon$ for some t^*. Intuitively, it means that $\Phi_t(\varsigma)$ $(t \geq 0)$ remains close to ς_A modulo a translation along the z-axis for ς sufficiently close ς_A. An analogous stability result for soliton solutions of KdV equation has been established by Benjamin in [**4**].

Using methods from Wan and Pulvirenti [**21**] and Tang [**20**], we can obtain the following stability results.

COROLLARY (H). *Hill's spherical vortex ς_H is stable relative to the metric d_H, where*

$$d_H(\varsigma_1, \varsigma_2) = P(|\varsigma_1 - \varsigma_2|) + |Z(\varsigma_1 - \varsigma_2)|.$$

COROLLARY (N). *For sufficiently small $\varepsilon > 0$, the nearly spherical vortex $\varsigma_N(\varepsilon)$ is stable with respect to the metric d_N, where*

$$d_N(\varsigma_1, \varsigma_2) = \|\varsigma_1 - \varsigma_2\|_1 + |P(\varsigma_1 - \varsigma_2)| + |Z(\varsigma_1 - \varsigma_2)|.$$

In what follows, we will establish various inequalities. Thus, for convenience, we will take C as a generic constant, i.e. a constant which may have different values in different places.

3. Proof of Theorem (H). Denote by (ρ, θ) the polar coordinates on $M = \{(r,z) | r \geq 0\}$ with $\rho\cos\theta = z$, $\rho\sin\theta = r$.

The stream function of Hill's spherical vortex ς_H is given by

$$\psi_H = r^2 \cdot \phi_H, \quad \text{with } \phi_H = \begin{cases} \frac{1}{10}\left[\frac{5}{3} - \rho^2\right] & \text{for } \rho < 1, \\ \frac{1}{10}[2/3\rho^3] & \text{for } \rho > 1. \end{cases}$$

(See [**14**], also Remark 4 in Appendix.) Thus, ϕ_H has a constant value $\phi_0 = \frac{1}{15}$ on $\rho = 1$.

Step (1). *Reduction to a radial case* ς^* (i.e. $\varsigma^* = \chi_{D^*}$, $D^* = \{(\rho,\theta)|\rho \le g(\theta)\}$ for some C^1 function g, $0 < g < 1$ on $[0,\pi]$).

From Proposition 2 in Appendix B, to each $\varsigma = \chi_D$, one can write ψ ($= \int_D G\, d\mu$) $= r^2\phi$ with ϕ a C^1 function on M, and ϕ is C^1-close to ϕ_H on $\rho \le 2$ if $P(|\varsigma - \varsigma_H|)$ is small. In what follows, we assume that $P(|\varsigma - \varsigma_H|)$ is sufficiently small.

For any ϕ C^1-close to ϕ_H on $\rho \le 2$, $\mu = (\mu_1, \mu_2)$ close to 0, set $\varsigma^*_{\mu,\phi} = \chi_{D^*}$ with $D^* = \{(r,z) \in M|\rho \le 2,\ \phi - \frac{1}{15} - \mu_1/2 - \mu_2 z \ge 0\}$. Consider the map $F(\mu,\phi) = (P(\varsigma^*_{\mu,\phi}) - P(\varsigma_H),\ Z(\varsigma^*_{\mu,\phi}))$. Computations show that

$$\det D_\mu F(0,\phi_H) = \begin{vmatrix} -20\pi/3 & 0 \\ 0 & -8\pi/3 \end{vmatrix} \ne 0.$$

So for ϕ C^1-close to ϕ_H on $\rho \le 2$, there exists unique $\mu = \mu(\phi)$ close to (0, 0), such that $F(\mu(\phi),\phi) = 0$ by an implicit function theorem. Now define $\varsigma^* = \varsigma^*_{\mu(\phi),\phi} = \chi_{D^*}$ for each ς, with $P(|\varsigma - \varsigma_H|)$ sufficiently small. Observe that $P(\varsigma^*) = P(\varsigma_H)$. $Z(\varsigma^*) = Z(\varsigma_H)$ ($= 0$), and the boundary ∂D^* of ς^* in M is C^1-close to the boundary $\rho = 1$ of ς_H.

Choose numbers ϕ_1, ϕ_2 close to $\phi_0 = \frac{1}{15}$, $\phi_2 < \phi_0 < \phi_1$, such that $P(S) = P(D\backslash D^*) = P(D^*\backslash D)P(T)$, where $P(S)$ means $P(\chi_S)$ etc., and $S = \{(r,s) \in M|\phi_2 < \phi(r,z) < \phi_0\}$, $T = \{\phi_0 < \phi(r,z) < \phi_1\}$. One can readily establish the following formulae (as in the proof of lemma 1 in [**21**]).

(i)

$$\begin{aligned} P(S) = P(T) &= \beta(\phi_2 - \phi_0) + o(|\phi_2 - \phi_0)|) \\ &= \beta(\phi_0 - \phi_1) + o(|\phi_0 - \phi_1|) \qquad \text{with } \beta = -\frac{8\pi}{15}, \end{aligned}$$

(ii)

$$\begin{aligned} \int_S \phi r^2\, d\mu = \phi_0 = P(S) + \frac{1}{2}\beta(\phi_2 - \phi_0)^2 + o(|\phi_2 - \phi_0|), \\ \int_T \phi r^2\, d\mu = \phi_0 = P(T) - \frac{1}{2}\beta(\phi_0 - \phi_1)^2 + o(|\phi_0 - \phi_1|). \end{aligned}$$

(i) and (ii) imply that there exists $c_1 > 0$ ($\approx 15/32\pi$) such that

$$\begin{aligned} \int_T \phi r^2\, d\mu - \int_S \phi r^2\, d\mu &\ge c_1(P(S) + P(T))^2 \\ &= c_1(P(D\backslash D^*) + P(D^*\backslash D))^2 = c_1[P(|\varsigma^* - \varsigma|)]^2. \end{aligned}$$

$\langle \varsigma^* - \varsigma, \psi\rangle \ge \int_T \phi r^2\, d\mu - \int_S \phi r^2\, d\mu$ by using the fact that $\phi > \phi_0$ inside D^*, $\phi < \phi_0$ outside D^* and our chosen properties for S, T. By the above inequalities, and the fact $E(\varsigma^* - \varsigma) \ge 0$, one has

$$E(\varsigma^*) - E(\varsigma) = \langle \varsigma^* - \varsigma, \psi\rangle + E(\varsigma^* - \varsigma) \ge c_1 \left[\int_M r^2 |\varsigma^* - \varsigma|\, d\mu\right]^2.$$

Thus it remains to show that

$$E(\varsigma_H) - E(\varsigma^*) \geq c_2 P(|\varsigma_H - \varsigma^*|)^2 \quad \text{for some constant } c_2 > 0.$$

Indeed,

$$\begin{aligned} E(\varsigma_H) - E(\varsigma) &= (E(\varsigma_H) - E(\varsigma^*)) + (E(\varsigma^*) - E(\varsigma)) \\ &\geq c_2 P(|\varsigma_H - \varsigma^*|)^2 + c_1 P(|\varsigma^* - \varsigma|)^2 \\ &\geq c_3 P(|\varsigma_H - \varsigma|)^2 \quad \text{with } c_3 = \tfrac{1}{2}\min(c_1, c_2). \end{aligned}$$

Step (2). *Second order analysis.* To each C^1 function g on $[0,\pi]$, $0 < g < 1$, define a vortex

$$\chi_g = \begin{cases} 1 & \text{if } \rho \leq g(\theta), \\ 0 & \text{otherwise.} \end{cases}$$

(Thus, $\varsigma_H = \chi_1$.) Set $\psi_g = \int G\chi_g$, $J = d\mu/dx = 2\pi r\rho$ $(dx = dr\,dz)$, and $J_0 = J|_{\rho=1} = 2\pi\sin\theta$. To functions k, k' on $[0,\pi]$, let $\langle\langle k, k'\rangle\rangle = \int_0^\pi \sin\theta k(\theta)k'(\theta)\,d\theta$, and $\|k\|_2^2 = \langle\langle k, k\rangle\rangle$.

The second order Taylor expansion of $E(\chi_{1+h})$ near χ_1 is given by

$$\begin{aligned} E(\chi_{1+h}) - E(\chi_1) = \frac{1}{2}\int_0^\pi \left.\frac{\partial\tilde{\psi}_H}{\partial\rho}\right|_{\rho=1} J_0 h^2\,d\theta \\ +\frac{1}{2}\int_0^\pi\int_0^\pi G_0 J_0 J_0' h h'\,d\theta\,d\theta' + o(\|J_0 h\|_2^2) \end{aligned} \tag{3.1}$$

where, $J_0' = J_0(\theta')$, $h' = h(\theta')$, $G_0 = G(1,\theta;1,\theta')$. We sketch the proof here, for it is a modification of a proof presented in Proposition 2, [**21**].

$$E(\chi_{1+h}) - E(\chi_1) = \langle\chi_{1+h} - \chi_1, \psi_1\rangle + \tfrac{1}{2}\langle\chi_{1+h} - \chi_1, \psi_{1+h} - \psi_1\rangle. \tag{3.2}$$

Using the fact that $\psi_1 = r^2\cdot\phi_H$, one can have

$$(\psi_1)J = (\psi_1\cdot J)(1,\theta) + h(\theta)\left[\psi_1\frac{\partial J}{\partial\rho} + J\frac{\partial\psi_1}{\partial\rho}\right](1,\theta) + o(\sin^2\theta|h(\theta)|).$$

Thus,

$$\begin{aligned} \langle\chi_{1+h} - \chi_1, \psi_1\rangle = \int_0^\pi \psi_1|_{\rho=1} J_0 h\,d\theta + \frac{1}{2}\int_0^\pi \left.\left[\psi_1\frac{\partial J}{\partial\rho} + J\,\frac{\partial\psi_1}{\partial\rho}\right]\right|_{\rho=1} h^2\,d\theta \\ + o(\|J_0 h\|_2^2). \end{aligned} \tag{3.3}$$

From the uniform boundedness of the term $\int_0^\pi (G/\sqrt{\sin\theta}\sqrt{\sin\theta'})^2\,d\theta'$, one can establish

$$(\psi_{1+h} - \psi_1)(\rho', \theta') = \int_0^\pi G_0 J_0(\theta)h(\theta)\,d\theta + o(\sqrt{\sin\theta'}\|J_0 h\|_2) \tag{3.4}$$

and

$$\begin{aligned} \int_0^\pi d\theta' \int_1^{1+h(\theta')} J'\,d\rho' \left\{\int_0^\pi G_0 J_0 h\,d\theta\right\} \\ = \frac{1}{2}\int_0^\pi\int_0^\pi G_0 J_0 J_0' h h'\,d\theta d\theta' + o(\|J_0 h\|_2^2). \end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) imply

$$\langle \chi_{1+h} - \chi_1, \psi_{1+h} - \psi_1 \rangle = \int_0^\pi \int_0^\pi G_0 J_0 J_0' h h' \, d\theta d\theta' + o(\|J_0 h\|_2^2). \tag{3.6}$$

Notice also,

$$\begin{aligned} 0 &= P(\chi_{1+h}) - P(\chi_1) \\ &= \int_0^\pi r^2 \Big|_{\rho=1} J_0 h \, d\theta + \frac{1}{2} \int_0^\pi \left[r^2 \frac{\partial J}{\partial \rho} + J \frac{\partial r^2}{\partial \rho} \right] \Bigg|_{\rho=1} h^2 \, d\theta + o(\|J_0 h_2^2\|). \end{aligned} \tag{3.7}$$

The second order Taylor expansion formula (3.1) follows from equations (3.2), (3.3), (3.6), (3.7) and the relative stream function $\tilde{\psi}_H \equiv \psi_1 - \frac{1}{2} W_H r^2 = 0$ on $\rho = 1$.

Introducing the function $k = J_0 h$, and using $\partial \tilde{\psi}_H / \partial \rho = -\frac{1}{5} \sin^2 \theta$,

$$E(\chi_{1+h}) - E(\chi_1) = \tfrac{1}{2} \langle\langle k, Tk \rangle\rangle + o(\|k\|_2^2) \tag{3.8}$$

where

$$Tk = -\frac{1}{10\pi} k + Kk \quad \text{and} \quad Kk(\theta) = \frac{1}{\sin \theta} \int_0^\pi G_0 k(\theta') \, d\theta',$$

a bounded selfadjoint linear operator.

By Proposition 1 in Appendix A, the selfadjoint operator T has eigenvalues $-1/10\pi + 1/2\pi(2n+1)$, $n = 1, 2, \dots$, with eigenfunctions $P_n^1(\cos\theta)$. Here, $\{P_n^1\}$ denotes the Legendre functions of order 1.

Let $k = k_t + k_n$ be a L^2-orthogonal eigenspace decomposition of T, with $\{k_n\}$ the space spanned by $P_1^1 = \sin\theta$, $P_2^1 = 3 \sin\theta \cos\theta$. T has eigenvalues $\leq -1/60\pi$, when restricting to the eigenspace $\{k_n\}$. Therefore, there exists $\eta > 0$ (small) such that

$$\langle\langle k, Tk \rangle\rangle \leq -(1/70\pi) \|k\|_2^2, \qquad \text{if } \|k_n\|_2 < \eta \|k\|_2. \tag{3.9}$$

$P(\chi_{1+h}) = P(\chi_1)$, $Z(\chi_{1+h}) = Z(\chi_1)$ imply $\|k_n\|_2 = O(\|k\|_2^2)$ (cf. (3.7)). Thus, equations (3.9) and (3.8) give

$$E(\chi_{1+h}) - E(\chi_1) \leq -\frac{1}{80\pi} \|J_0 h\|_2^2. \tag{3.10}$$

Finally,

$$\begin{aligned} P(|\chi_{1+h} - \chi_1|) &= \int_0^\pi \left| \int_1^{1+h} 2\pi\rho \sin\theta \cdot \rho^2 \sin^2\theta \cdot \rho \, d\rho \right| d\theta \\ &\leq C \int_0^\pi \sin^{3/2}\theta |\sqrt{\sin\theta} J_0 h| \, d\theta \qquad (C > 0) \\ &\leq C \|J_0 h\|_2 \quad \text{(by Schwarz inequality)}. \end{aligned} \tag{3.11}$$

From equations (3.10) and (3.11) we obtain the required inequality in Step (2), $E(\varsigma_H) - E(\varsigma^*) \geq c_2 P(|\varsigma_H - \varsigma^*|)^2$ for some constant $c_2 > 0$.

Clearly, combining Step (1) and Step (2), we establish the desired inequality, and thus, we complete the proof of Theorem (H).

4. Proof of Theorem (N). We find it is convenient to use the polar-like coordinates $(-\tilde{\psi}_N(\varepsilon), \alpha)$ on M. Recall that $\tilde{\psi}_N(\varepsilon)$ stands for the relative stream function of Norbury's nearly spherical vortex $\varsigma_N(\varepsilon) = \chi_{H_\varepsilon}$, and α is the angle between the ray $\overrightarrow{X_0X}$ and $\overrightarrow{X_0X_1}$, where $X_0 = (\frac{1}{2}, 0)$, $X_1 = (\frac{1}{2}, 1)$ and $X = (r, z)$.

In what follows, $\varepsilon > 0$ will be fixed and small (to be determined later). Also, for such a $\varepsilon > 0$, ς is sufficiently L^1-close to $\varsigma_N(\varepsilon)$.

Set $I_\varepsilon = d\mu/d(-\tilde{\psi}_N(\varepsilon), \alpha)|_{\partial H_\varepsilon}$, and claim that

$$0 < I_\varepsilon \le 80\pi/r \quad \text{for } (r,z) \in \partial H_\varepsilon. \tag{4.1}$$

Indeed,

$$\begin{aligned} \frac{2\pi}{I_\varepsilon} &= \frac{1}{r}\frac{\partial(\tilde{\psi}_N(\varepsilon), \alpha)}{\partial(r,z)} \\ &= \frac{1}{10}\begin{vmatrix} 2 - 4r^2 - 2z^2 + 2\phi_\varepsilon + r(\phi_\varepsilon)_r & -2rz + r(\phi_\varepsilon)_z \\ \frac{\partial\alpha}{\partial r} & \frac{\partial\alpha}{\partial z} \end{vmatrix}. \end{aligned} \tag{4.2}$$

Thus

$$\frac{2\pi\rho_0}{I_\varepsilon} = \frac{1}{5}\left[\frac{10\gamma_\varepsilon}{r^2}\left(\frac{1}{2} - 2r\right) + r\left(1 - \frac{r}{2}\right) + \frac{1}{2}r(\phi_\varepsilon)_r\left(\frac{1}{2} - r\right) - \frac{1}{2}rz(\phi_\varepsilon)_z\right].$$

Here, we represent ∂H_ε as

$$\tilde{\psi}_N(\varepsilon) = \frac{r^2}{10}[(1 - r^2 - z^2) + \phi_\varepsilon] = \gamma_\varepsilon \qquad (= \gamma(\varepsilon) > 0)$$

with $\gamma_\varepsilon, \phi_\varepsilon, (\phi_\varepsilon)_r, (\phi_\varepsilon)_z \to 0$ as $\varepsilon \to 0$. Let $\rho_0 =$ distance from X_0 to (r,z) $(\in \partial H_\varepsilon)$ (so $\rho_0 \le 2$). Using $(10\gamma_\varepsilon/r^2)(\frac{1}{2} - 2r) \ge -r/10$, one gets

$$\frac{2\pi\rho_0}{I_\varepsilon} \ge \frac{r}{5}\left[1 - \frac{r}{2} - \frac{r}{8}\right] \ge \frac{r}{5}\cdot\frac{1}{4} \quad (\text{for } r \le 6/5).$$

Thus, $0 < I_\varepsilon \le (40\pi)\rho_0/r \le 80\pi/r$.

From equation (4.2),

$$\frac{1}{I_\varepsilon} \to \frac{1}{I_0} \equiv \begin{cases} \dfrac{1}{10\pi}\sin\theta\dfrac{d\alpha}{d\theta}|_{\rho=1} & \text{on } \rho = 1,\ r \ge 0, \\ \dfrac{1}{10\pi}(1 - z^2)\dfrac{d\alpha}{dz}|_{r=0} & \text{on } r = 0,\ |z| \le 1, \end{cases}$$

pointwise, as $\varepsilon \to 0$.

Step (1). *Reduction to a radial case* ς^*. For any ψ C^1-close to $\psi_N(\varepsilon)$ on $\rho \le 2$, $\mu = (\mu_1, \mu_2, \mu_3)$ close to 0, set $\varsigma^*_{\mu,\psi} = \chi_{D^*}$ with

$$D^* = \{(r,z) \in M | \rho \le 2,\ \psi - \tfrac{1}{2}W_N(\varepsilon)r^2 - \tfrac{1}{2}\mu_1 r^2 - \mu_2 z r^2 - \mu_3 \ge 0\}.$$

Consider the map

$$F(\mu, \psi) = (P(\varsigma^*_{\mu,\psi}) - P(\varsigma_N(\varepsilon)),\ Z(\varsigma^*_{\mu,\psi}), \Gamma(\varsigma^*_{\mu,\psi}) - \Gamma(\varsigma_N(\varepsilon))).$$

Computations show

$$D_\mu F(0, \psi_N(\varepsilon)) = \begin{vmatrix} -\int_0^{2\pi} r^2 I_\varepsilon\left(\frac{r^2}{2}\right)d\alpha & -\int_0^{2\pi} r^2 I_\varepsilon(zr^2)\,d\alpha & -\int_0^{2\pi} r^2 I_\varepsilon\,d\alpha \\ -\int_0^{\pi} r^2 z I_\varepsilon\left(\frac{r^2}{2}\right)d\alpha & -\int_0^{2\pi} r^2 z I_\varepsilon(r^2 z)\,d\alpha & -\int_0^{2\pi} r^2 z I_\varepsilon\,d\alpha \\ -\int_0^{2\pi} I_\varepsilon\left(\frac{r^2}{2}\right)d\alpha & -\int_0^{2\pi} I_\varepsilon(r^2 z)\,d\alpha & -\int_0^{2\pi} I_\varepsilon\,d\alpha \end{vmatrix}.$$

Using the estimate $I_\varepsilon \le 80\pi/r$, one can obtain

$$D_\mu F(0, \psi_N(\varepsilon)) \to \begin{vmatrix} -20\pi/3 & 0 & -20\pi \\ 0 & -8\pi/3 & 0 \\ -10\pi & 0 & -\infty \end{vmatrix} \quad \text{as } \varepsilon \to 0.$$

Thus, $\det D_\mu F(0, \psi_N(\varepsilon)) \ne 0$, for ε small. So for ψ C^1-close to $\psi_N(\varepsilon)$ on $\rho \le 2$. There exists a unique $\mu = \mu(\psi)$ close to 0 such that $F(\mu(\psi), \psi) = 0$. Now, set $\varsigma^* = \varsigma^*_{\mu(\psi),\psi} = \chi_{D^*}$. Observe that, for this ς^* $P(\varsigma^*) = P(\varsigma_N(\varepsilon))$, $\Gamma(\varsigma^*) = \Gamma(\varsigma_N(\varepsilon))$, $Z(\varsigma^*) = Z(\varsigma_N(\varepsilon))$ $(= 0)$, and the boundary ∂D^* of ς^* is C^1-close to boundary ∂H_ε of $\varsigma_N(\varepsilon)$ $(= \chi_{H_\varepsilon})$.

Since $\tilde{\psi}(\varepsilon) \le 0$ is a neighborhood of H_ε in the half-plane M. The arguments in **[21]** can be carried through easily. Thus, one gets $\langle \varsigma^* - \varsigma, \psi\rangle \ge c_1 \|\varsigma^* - \varsigma\|_1^2$ for some constant $c_1 > 0$ and it remains to show that

$$E(\varsigma_N(\varepsilon)) - E(\varsigma^*) \ge c_2 \|\varsigma_N(\varepsilon) - \varsigma^*\|_1^2 \quad \text{for some constant } c_2 > 0.$$

Step (2). *Second order analysis.* To each C^1 function $g = g(\alpha)$ on $[0, 2\pi]$ $(-2\gamma(\varepsilon) < g < 0)$, define a vortex

$$\chi_g := \begin{cases} 1 & \text{if } -\tilde{\psi}_N(\varepsilon) \le g(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

(Thus, $\varsigma_N(\varepsilon) = \chi_{-\gamma(\varepsilon)}$.) To functions $k = k(\alpha)$, $k' = k'(\alpha)$ on $[0, 2\pi]$, let $\langle k, k'\rangle = \int_0^{2\pi} k(\alpha)k'(\alpha)\,d\alpha$, and $|k|_2^2 = \langle k, k\rangle$. By arguments as in **[21]**, we obtain the second order Taylor expansion of $E(\chi_{-\gamma(\varepsilon)+h})$ near $\chi_{-\gamma(\varepsilon)}$:

(4.3)
$$E(\chi_{-\gamma(\varepsilon)+h}) - E(\chi_{-\gamma(\varepsilon)}) = \frac{1}{2}\int_0^{2\pi} I_\varepsilon h^2\,d\alpha + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} G_\varepsilon I_\varepsilon I_{\varepsilon'} h h'\,d\alpha\,d\alpha' + o(|I_\varepsilon h|_2^2).$$

where $G_\varepsilon = G|_{\partial H_\varepsilon}$. This formula is easier to get than the analogue formula (3.1) for ς_H, because one needs to worry about the weight $\sin\theta$ in estimating the remainder term in (3.1).

Introducing $\sqrt{I_\varepsilon} h = k$, and noticing $|I_\varepsilon h|_2 \le C(\varepsilon)|k|_2$ (with $C(\varepsilon) \to \infty$, as $\varepsilon \searrow 0$),

$$E(\chi_{-\gamma(\varepsilon)+h}) - E(\chi_{-\gamma(\varepsilon)}) = \tfrac{1}{2}\langle k, T_\varepsilon k\rangle + o(|k|_2^2) \tag{4.4}$$

where $T_\varepsilon k = -k + K_\varepsilon k$, and

$$K_\varepsilon k = \int_0^{2\pi} \mathcal{K}_\varepsilon k'\,d\alpha', \quad \text{with } \mathcal{K}_\varepsilon = G_0\sqrt{I_\varepsilon}\sqrt{I'_\varepsilon}.$$

Define

$$K_0 k = \int_0^{2\pi} \mathcal{K}_0 k'\,d\alpha', \quad \text{with } \mathcal{K}_0 = G_0\sqrt{I_0}\sqrt{I'_0}.$$

One can establish that

$$\int_0^{2\pi} \mathcal{K}_0^4(\alpha,\alpha')\,d\alpha', \qquad \int_0^{2\pi} \mathcal{K}_\varepsilon^4(\alpha,\alpha')\,d\alpha' \le C$$

for some constant $C > 0$ (i.e. independent of α). We illustrate only the proof of the more complicated case: $\int_0^\pi \mathcal{K}_\varepsilon^4\, d\alpha' \leq C$. Indeed by $I_\varepsilon \leq 80\pi/r$ and Lemma 1 (in Appendix B),

$$\int \mathcal{K}_\varepsilon^4\, d\alpha \leq C \int_{s\leq r/2, r\leq r'} \ln^4\left(\frac{r}{s}\right) \frac{d\alpha'}{ds}\, ds + C \int_{s\leq r/2, r'<r} \ln^4\left(\frac{r}{s}\right) \frac{d\alpha'}{d(-s)}\, d(-s)$$
$$+ C \int_{s\geq r/2} \left(\frac{r^{3/2} r'^{3/2}}{s^3}\right)^4 d\alpha'.$$

From a geometric consideration about ∂H_ε as shown in Appendix C, $d\alpha'/ds$, $d\alpha'/d(-s) \leq C$ (i.e. Corollary 2). Observe also

$$\left(\frac{rr'}{s^2}\right)^6 \leq \left(\frac{r}{s}\frac{r+s}{s}\right)^6 \leq [2(2+1)]^6 \qquad (\text{for } r' \leq r+s).$$

Hence, we obtain

$$\int \mathcal{K}_\varepsilon^4\, d\alpha' \leq C.$$

As a consequence of these estimates,

$$\iint |\mathcal{K}_\varepsilon - \mathcal{K}_0|^2\, d\alpha\, d\alpha' \to 0 \quad \text{as } \varepsilon \to 0$$

and K_ε is L^2-close to K_0 as linear transformations.

It is natural to regard $k(\alpha)$ as a function defined on $\rho = 1$, $r \geq 0$ and $r = 0$, $|z| \leq 1$. Identify $k(\alpha)$ as $(k^1(\theta), k^2(z))$ as a L^2-space, where $k^1 = k|_{\rho=1, r\geq 0}\sqrt{d\alpha/d\theta_{\rho=1}}$, $k^2 = k|_{r=0,|z|\leq 1}\sqrt{d\alpha/dz_{r=0}}$. Thus, $K_0(k^1, k^2) = (Kk^1, 0)$ with

$$Kk^1(\theta) = 10\pi \int_0^\pi \frac{G_0}{\sqrt{\sin\theta}\sqrt{\sin\theta'}} k^1(\theta')\, d\theta'.$$

So the operator $T = -I + K_0$ has eigenvalues $-1, -1 + 5/(2n+1)$ with eigenfunctions $(0, k^2(z))$, $(\sqrt{\sin\theta} P_n^1(\cos\theta), 0)$, $n = 1, 2, \ldots$.

Let $k = k_t + k_n$ be a L^2-orthogonal eigenspace decomposition with $\{k_n\}$ the subspace spanned by $(\sqrt{\sin\theta} P_1^1, 0)$ and $(\sqrt{\sin\theta} P_2^1, 0)$. The operator $-I + K_0$ has eigenvalues $\leq -\frac{1}{6}$, when restricting to the eigenspace $\{k_t\}$. Hence, there exists $\eta > 0$ (small) such that

$$\langle k, Tk\rangle \leq -\tfrac{1}{7}|k|_2^2 \quad \text{if } |k_n|_2 < \eta |k|_2.$$

For $T_\varepsilon = -I + K_\varepsilon$ if L^2-close to $T = -I + K_0$, we have

$$\langle k, T_\varepsilon k\rangle \leq -\tfrac{1}{8}|k|_2^2 \quad \text{if } |k_n|_2 < \eta |k|_2. \tag{4.5}$$

From the expansion,

$$0 = P(\chi_{-\gamma(\varepsilon)+h}) - P(\chi_{-\gamma(\varepsilon)})$$
$$= \int_0^{2\pi} r^2|_{\partial H_\varepsilon} I_\varepsilon h\, d\alpha + \frac{1}{2}\int_0^{2\pi} \frac{\partial r^2 I_\varepsilon}{\partial(-\tilde{\psi}_N(\varepsilon))} h^2\, d\alpha + o(|I_\varepsilon h|_2^2)$$

one gets

$$\langle e_1(\varepsilon), k\rangle \leq C(\varepsilon)|k|_2^2, \quad \text{with } e_1(\varepsilon) = r^2|_{\partial H_\varepsilon}\sqrt{I_\varepsilon}. \tag{4.6}$$

Similarly, from the expansion of $Z(\chi_{-\gamma(\varepsilon)+h})$ $(= Z(\chi_{-\gamma(\varepsilon)}))$ one has

$$\langle e_2(\varepsilon), k\rangle \leq C(\varepsilon)|k|_2^2, \quad \text{with } e_2(\varepsilon) = zr^2|_{\partial H_\varepsilon}\sqrt{I_\varepsilon}. \tag{4.7}$$

Now, $e_1(\varepsilon) \to e_1 \equiv \sqrt{10\pi}\ (\sqrt{\sin\theta}P_1^1, 0)$ in L^2-norm as $\varepsilon \to 0$, and $e_2(\varepsilon) \to e_2 \equiv (\sqrt{10\pi}/3)(\sqrt{\sin\theta}P_2^1, 0)$ in L^2-norm as $\varepsilon \to 0$. (Recall that $I_\varepsilon \leq 80\pi/r$ from equation (4.1).)

$$\begin{aligned} |k_n| &= \left|\frac{\langle e_1, k\rangle e_1}{|e_1|_2^2} + \frac{\langle e_2, k\rangle e_2}{|e_2|_2^2}\right| \leq C[|\langle e_1, k\rangle| + |\langle e_2, k\rangle|] \\ &\leq C[\langle |e_1 - e_1(\varepsilon), k\rangle| + |\langle e_1(\varepsilon), k\rangle| + |\langle e_2 - e_2(\varepsilon), k\rangle| + |\langle e_2(\varepsilon), k\rangle|] \\ &= C(\varepsilon)|k|_2^2 + \frac{\eta}{2}|k|_2 \quad \text{for some small but fixed } \varepsilon > 0 \text{ (by (4.6), (4.7))} \\ &\leq \eta|k|_2 \quad \text{for } |k| \text{ small.} \end{aligned} \tag{4.8}$$

Equations (4.4) and (4.5) give

$$E(\chi_{-\gamma(\varepsilon)+h}) - E(\chi_{-\gamma(\varepsilon)}) \leq -\tfrac{1}{17}|k|_2^2. \tag{4.9}$$

Finally,

$$\begin{aligned} |\chi_{-\gamma(\varepsilon)+h} - \chi_{-\gamma(\varepsilon)}|_1 &= \int_0^{2\pi}\left|\int_{-\gamma(\varepsilon)}^{-\gamma(\varepsilon)+h} I_\varepsilon\, d(-\tilde{\psi}_N(\varepsilon))\right| d\alpha \\ &\leq C|I_\varepsilon h|_2 \quad \text{for } |h|_2 \text{ small.} \end{aligned} \tag{4.10}$$

Thus, equations (4.9) and (4.10) imply the required inequality,

$$E(\chi_{-\gamma(\varepsilon)+h}) - E(\chi_{-\gamma(\varepsilon)}) \geq -c_2|\chi_{-\gamma(\varepsilon)+h} - \chi_{-\gamma(\varepsilon)}|_1^2.$$

By combining Steps (1) and (2), we obtain the desired inequality and we complete the proof of Theorem (N).

APPENDICES

(A) *Some integrals involving G.*

Consider the equation $-L\psi/r^2 = \varsigma$ in polar coordinates (ρ, θ),

$$L\psi = \psi_{\rho\rho} - \frac{\cos\theta}{\rho^2\sin\theta}\psi_\theta + \frac{\psi_{\theta\theta}}{\rho^2}.$$

Let ϕ_n be a C^1 function defined by

$$\phi_n = \begin{cases} -\rho^{n+1} + c_1\rho^{n-1} & \text{for } \rho \leq a,\ n = 1, 2, \ldots, \\ c_2\rho^{-n-2} & \text{for } \rho > a, \end{cases}$$

where $c_1 = [(2n+3)/(2n+1)]a^2$, $c_2 = [2/(2n+1)]a^{2n+3}$. Set

$$\xi_n = \begin{cases} (4n+6)\rho^{n-1} & \text{for } \rho \leq a, \\ 0 & \text{for } \rho > a. \end{cases}$$

PROPOSITION 1. (a)

$$\rho^2\sin\theta\phi_n\frac{dP_n}{d\tau} = \int_0^a\int_0^\pi G\xi_n\frac{dP_n}{d\tau}\, d\mu, \qquad \tau = \cos\theta.$$

(b)

$$\frac{1}{2\pi(2n+1)}\sin^2\theta\frac{dP_n}{d\tau} = \int_0^\pi G\frac{dP_n}{d\tau}\sin\theta'\,d\theta'.$$

Here, P_n stands for the Legendre polynomial (indeed, $P_n^1(\tau) = (1-\tau^2)^{1/2}dP_n/d\tau$).

PROOF. Let $\psi_n = \rho^2\sin^2\theta\phi_n dP_n/d\tau$ and $\varsigma_n = \xi_n dP_n/d\tau$. (a) follows by checking $\psi_n \in C^1$, $\psi_n \to 0$ as $\rho \to \infty$, and $-L\psi_n/r^2 = \varsigma_n$. To obtain (b) we differentiate (a) with respect to a at $\rho = a = 1$.

REMARK 4. (1) Putting $n = 1$ in (a), it gives the stream function of Hill's spherical vortex.

(2) (b) implies that G has a series expansion

$$G = \frac{\sin\theta\sin\theta'}{4\pi}\sum\frac{P_n^1(\cos\theta)P_n^1(\cos\theta')}{n(n+1)}.$$

This fact can be found in [**18**, p. 267].

(B) *Regularity results for $-L\psi/r^2 = \varsigma$.*

(1) For $\psi = \int G\varsigma\,d\mu$, estimates about kernel G will provide information about ψ. Let us state Lemma 3.3 in [**10**] as

LEMMA 1. (i) $(0 \le)\ G \le Cr\ln(r/s)$ *for* $s \le r/2$.

(ii) $(0 \le)\ G \le Cr^2r'^2/s^3$ *for* $s \ge r/2$,

where $G = G(r,z,r',z')$, $s^2 = (r-r')^2 + (z-z')^2$.

Arguing as in Lemma 3.4 [**10**], one can establish

COROLLARY 1. *For $\varsigma \in \mathcal{V}$ with $\int\varsigma\,d\mu$, $\int r^2\varsigma\,d\mu \le N$,*

$$(0 <)\ \psi(r,z) \le C(N+1)\min\{r, r^{-1/2}\}.$$

Hence, ψ is bounded and $\psi \to 0$ as $r \to 0$ or $r \to \infty$.

(2) For L is elliptic away from the z-axis, from elliptic theory, we know $\psi = \int G\varsigma\,d\mu$ is C' on $r > 0$ for $\varsigma \in L^\infty$. Notice the function ψ_n/r^2 is C' on $r \ge 0$, where ψ_n is introduced in Appendix A, with $-L\psi_n/r^2 = \varsigma_n \in L^\infty$. Now, define $\phi(r,z) = \psi(r,z)/r^2$ on $r > 0$ for $\psi = \int G\varsigma\,d\mu$.

PROPOSITION 2. (a)

$$|\phi(r,z)| \le C|\varsigma|_\infty^{3/5}P(|\varsigma|)^{2/5}$$

(b)

$$|\phi_r(r,z)|, |\phi_z(r,z)| \le C(|\varsigma|_\infty^{3/5}P(|\varsigma|)^{2/5} + |\varsigma|_\infty^{5/6}P(|\varsigma|)^{1/6})$$

for $r^2 + z^2 \le 3$.

Here, we follow some ideas in [**17, 1**]. Consider ϕ, ς as functions on R^5 via $r^2 = x_1^2 + \cdots + x_4^2$, $z = x_5$. One can readily see $-\varsigma = \Delta_5\phi$, $\phi = \int G_5\varsigma\,dx$, $\int|\varsigma|\,dx = (A_4/2\pi)P(|\varsigma|)$. Here, $\Delta_5 =$ the Laplace operator on R^5, $G_5 = 1/3A_5|x-x'|^3$, with $A_4(A_5) =$ the area of unit sphere in $R^4(R^5)$.

PROOF OF PROPOSITION 2. (a) To each $x \in R^5$,

$$\begin{aligned}|\phi(x)| &\le \int_{|x-x'|<\varepsilon}|G_5\varsigma|\,dx' + \int_{|x-x'|\ge\varepsilon}|G_5\varsigma|\,dx' \\ &\le \frac{C}{2}\left[|\varsigma|_\infty\varepsilon^2 + \frac{1}{\varepsilon^3}\int|\varsigma|\,dx\right] = C|\varsigma|_\infty^{3/5}\left(\int|\varsigma|\,dx\right)^{2/5}\end{aligned}$$

by choosing $\varepsilon^5 = \int|\varsigma|\,dx/|\varsigma|_\infty$.

Thus, (a) follows by observing $\int |\varsigma|\, dx = (A_4/2\pi)P(|\varsigma|)$.

(b) Denote by $\|u\|_{m,p,\Omega} = \sum_{|\alpha|\le m}(\int_\Omega |D^\alpha u|^p\, dx)^{1/p}$ the Sobolev norm of u on the domain Ω. An interior estimate for Laplace operator A_5 gives

$$\|\phi\|_{2,P,\Omega'} \le C(\|\varsigma\|_{0,p,\Omega} + \|\phi\|_{0,p,\Omega}), \tag{B.1}$$

where $\Omega' = \{x \in R^5 | r^2 + z^2 \le 3\}$. $\Omega = \{x \in R^5 | r^2 + z^2 \le 4\}$.

From (a),

$$\|\phi\|_{0,p,\Omega} \le C|\varsigma|_\infty^{3/5}\left(\int |\varsigma|\, dx\right)^{2/5}. \tag{B.2}$$

Notice also,

$$\|\varsigma\|_{0,p,\Omega} \le C|\varsigma|_\infty^{(p-1)/p}\left(\int |\varsigma|\, dx\right)^{1/p}. \tag{B.3}$$

Combining inequalities (B.1), (B.2), (B.3), one gets

$$\|\phi\|_{2,p,\Omega'} \le C\left[|\varsigma|_\infty^{3/5} P(|\varsigma|)^{2/5} + |\varsigma|_\infty^{(p-1)/p} P(|\varsigma|)^{1/p}\right].$$

Thus,

$$|\phi_r|, |\phi_z| \le C\left[|\varsigma|_\infty^{3/5} P(|\varsigma|)^{2/5} + |\varsigma|_\infty^{5/6} P(|\varsigma|)^{1/6}\right] \quad \text{for } r^2 + z^2 \le 3,$$

by using Sobolev inequality with $p = 6$. Hence, we complete the proof of Proposition 2.

C. *A geometric consideration about ∂H_ε.* For small $\varepsilon > 0$, H_ε is a convex set with C^1 boundary ∂H_ε. Here, we give some information about variations of tangent lines along ∂H_ε.

PROPOSITION 3. *There exists $\delta > 0$ such that, for sufficiently small $\varepsilon > 0$, the difference between the angles of tangent lines with the z-axis of two points (r, z), (r', z') on ∂H_ε is smaller than $\pi/2 - \delta$, when $s \le r/2$. (Recall $S^2 = (r - r')^2 + (z - z')^2$.)*

PROOF. Take $m = 1/3$, $r^* = m/2\sqrt{1 + m^2}$, $z^* = 1/2$. It suffices to show that the proposition is valid for $0 < r < r^*$, $z > z^*$. Define a sequence of points $P_i = (r(P_i), z(P_i))$, $i = 1, 2, 3, 4$, counterclockwise on ∂H_ε with $z(P_i) \ge 0$ via the following characterizations:

at P_1, $z(P_1) = 0$, $r(P_1)$ is small,
at P_2, the tangent line of ∂H_ε has slope m,
at P_3, the tangent line of ∂H_ε has slope $1/m$,
at P_4, $r(P_4) = 3m/4\sqrt{1 + m^2}$ (so that its tangent line has a slope that is close to $-\sqrt{16 + 7m^2}/3m < -1/m$).

Thus, the variation of angles along the arc P_2P_4 or the arc P_1P_3 on ∂H_ε is less than $\pi/2 - \delta$ for some $\delta > 0$. Computations show that

$$r(P_2) = (10m\gamma(\varepsilon))^{1/3}(1 + o(1)), \quad r(P_3) = (10\gamma(\varepsilon)/m)^{1/3}(1 + o(1)).$$

As $m^{2/3} < 1/2$, the two points (r, z), (r', z') with $s \le r/2$ lie in either the arc P_2P_4 or P_1P_3. Thus our proposition follows.

COROLLARY 2. *$|ds/d\alpha'| \geq \frac{1}{3}\cos(\pi/2-\delta)$ on ∂H_ε if $s \leq r/2$.*

PROOF. We parametrize $X \in \partial H_\varepsilon$ by its α-coordinate as $X(\alpha') = (\frac{1}{2}, 0) + (\rho_0 \cos\alpha', \rho_0 \sin\alpha')$, where $\rho_0 =$ the Euclidean distant between $(\frac{1}{2}, 0)$ and X. Let $(r, z) = X(\alpha) \in \partial H_\varepsilon$. Thus,

$$\frac{ds}{d\alpha'} = \frac{[X(\alpha') - X(\alpha)]}{s} \cdot \frac{dX}{d\alpha'} = \left|\frac{dX}{d\alpha'}\right| \cos\eta,$$

where η is the angle between $X(\alpha') - X(\alpha)$ and $dX/d\alpha'(\alpha')$. By a mean-value theorem, η equals to the angle between $dX/d\alpha'(\alpha'')$ and $dX/d\alpha'(\alpha')$ for some α'' between α and α'. So, $\eta \leq \pi/2 - \delta$ by the above proposition. Notice also $|dX/d\alpha'| \geq \rho_0 \geq \frac{1}{3}$. Hence, $|ds/d\alpha'| \geq \frac{1}{3}\cos(\pi/2-\delta)$.

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