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ON THE NONLINEAR EIGENVALUE PROBLEM $\Delta u + \lambda e^u = 0$

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ABSTRACT. The structure of the set $\mathscr{E}$ of solutions of the nonlinear eigenvalue problem $\Delta u+\lambda e^u = 0$ under Dirichlet condition in a simply connected bounded domain Ω is studied. Through the idea of parametrizing the solutions (u, λ) in terms of $s = \lambda \int_\Omega e^u\, dx$, some profile of $\mathscr{E}$ is illustrated when Ω is star-shaped. Finally, the connectivity of the branch of Weston-Moseley's large solutions to that of minimal ones is discussed.

1. Introduction. Our purpose is to study the nonlinear eigenvalue problem (P):

$$-\Delta u = \lambda e^u \quad (\text{in } \Omega) \tag{1.1}$$

under the Dirichlet boundary condition

$$u = 0 \quad (\text{on } \partial\Omega), \tag{1.2}$$

where $\Omega \subset \mathbf{R}^2$ is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and when $\lambda > 0$. We are seeking the solution $h = {}^T(u, \lambda)$ of (P) which is taken in the classical sense so that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. If we fix λ and regard (P) just as a nonlinear elliptic equation, then its solution u is called a section at λ of the original eigenvalue problem.

Our problem arises in differential geometry and also in mathematical physics and has been studied by several authors [**6**, **12**, **5**, **13**, **9**, **19**, **11**, **1**, **2**, **4**]. From these works we know the following, where "branch" means a portion of a one-dimensional manifold in $\mathbf{R} \times C^0(\bar{\Omega})$:

(i) There is a branch $\mathscr{E}_0$ of solutions $(\lambda, u) = (\lambda_t, u_t)$ $(0 \le t < 1)$ for (P), which originates from $(\lambda, u) = (0, 0)$ at $t = 0$ and goes toward $\lambda > 0$ as $t > 0$.

(ii) That branch $\mathscr{E}_0$, without any bifurcation, continues up to $\lambda = \bar{\lambda}$ for some $\bar{\lambda} = \bar{\lambda}(\Omega)$ in $0 < \bar{\lambda} < \infty$ and then turns to $\lambda < \bar{\lambda}$, that is, the bending occurs. In other words, in the parametrization $\mathscr{E}_0 = \{(\lambda_t, u_t) | 0 \le t < 1\}$, there exists a $\bar{t}$ in $(0, 1)$ such that $\lambda_t \uparrow \bar{\lambda}$ as $(t \uparrow \bar{t})$ and $\lambda_t \downarrow$ for $\bar{t} < t < 1$. Furthermore, the component of the solutions for (P) containing $\mathscr{E}_0$ is unbounded.

We set $\underline{\mathscr{E}} = \{(\lambda_t, u_t) | 0 \le t \le \bar{t}\} \subset \mathscr{E}_0$.

(iii) The branch $\underline{\mathscr{E}}$ is minimal in the sense that for any section $u = u(x)$ at $\lambda = \lambda_t$ $(0 \le t \le \bar{t})$, the relation $u_t(x) \le u(x)$ $(x \in \Omega)$ follows. Furthermore, here the equality holds at some $x \in \Omega$ if and only if $u \equiv u_t$.

(iv) When $\lambda > \bar{\lambda}$, there is no section u of (P). On the other hand, for $0 < \lambda < \bar{\lambda}$ there is a section u such that $(u, \lambda) \notin \underline{\mathscr{E}}$. Therefore, at least two sections exist at each λ in $0 < \lambda < \bar{\lambda}$.

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Recently, under certain assumptions for Ω, V. H. Weston and J. L. Moseley have constructed a branch $\mathscr{E}^*$ differing from $\underline{\mathscr{E}}$ by the method of singular perturbations [**22**, **16**]. In the parametrization $\mathscr{E}^* = \{(\lambda_t, u_t) | 2 < t < 3\}$, we have

$$\lambda_t \downarrow 0 \quad \text{and} \quad u_t(x) \to 4 \log |1 - \bar{\delta} g^{-1}(z)| / |g^{-1}(z) - \delta|$$

as $t \uparrow 3$, where $z = x_1 + ix_2 \in \mathbf{C}$ for $x = (x_1, x_2) \in \mathbf{R}^2$. Here, $g\colon D = \{|\varsigma| < 1\} \to \Omega$ is a Riemann mapping, that is, one-to-one and conformal mapping having a diffeomorphic extension $\bar{g}\colon \bar{D} \to \bar{\Omega}$. Furthermore, $\delta \in D$ solves the equation

$$\bar{\delta} = \tfrac{1}{2}(1 - |\delta|^2) g''(\delta) / g'(\delta). \tag{1.3}$$

Henceforth, $\mathscr{E}^*$ is called the branch of Weston-Moseley's large solutions.

The main object of the present paper is to show that if Ω is close to a disc, then $\underline{\mathscr{E}}$ and $\mathscr{E}^*$ are connected to each other and form one branch of solutions, which may be denoted by $\mathscr{E} = \{(\lambda_t, u_t) | 0 \le t < 3\}$.

We note that the branch of large solutions actually connects with that of minimal solutions, in the case $\Omega = D \equiv \{|\varsigma| < 1\}$. In fact, $f(u) = \lambda e^u > 0$ and hence $u > 0$ in Ω. Therefore, by a theorem due to Gidas, Ni, and Nirenberg [**7**], every section $u = u(x)$ of (P) is radially symmetric: $u = u(|x|)$. Consequently, from the results of Gel'fand [**6**] we have $\bar{\lambda}(D) = 2$ and that (P) for $\Omega = D$ has exactly two sections at λ in $0 < \lambda < 2$. Actually, these are given as

$$\left(\frac{\lambda}{8}\right)^{1/2} e^{u/2} = \frac{\rho^{1/2}}{|x|^2 + \rho} \quad \text{with} \quad \rho^{1/2} = \rho_\pm^{1/2} = \left(\frac{\lambda}{2}\right)^{-1/2} \left\{1 \mp \sqrt{1 - \frac{\lambda}{2}}\right\}. \tag{1.4}$$

2. Preliminaries. 1. We first look at Weston-Moseley's theory briefly and afterwards give some remarks.

They make use of the Liouville integral [**14**] for the equation (1.1) to construct asymptotic solutions $u = u^n$ $(n = 1, 2, \dots)$ for (P) as $\lambda \downarrow 0$ under a certain assumption, which we shall describe later. Namely, $u = u^n$ satisfies (1.1) with

$$u^n = O(\lambda^n) \quad (\text{on } \partial\Omega) \text{ as } \lambda \downarrow 0, \tag{1.2$'$}$$

and is given explicitly in terms of the Riemann mapping $g\colon D \to \Omega$. In fact, it behaves like

$$u^n(x) \sim 4 \log |1 - \bar{\delta} g^{-1}(z)| / |g^{-1}(z) - \delta|$$

as $\lambda \downarrow 0$, where $\delta \in D$ solves the equation (1.3).

It holds that the solution $\delta \in D$ of (1.3) is characterized as $\delta = g^{-1}(d)$, where $d \in \Omega$ is a point of maximal conformal radius for Ω [**16**, p. 721]. Therefore, such a $\delta \in D$ exists for each simply connected domain $\Omega \subset \mathbf{R}^2$. Further, d is unique when Ω is convex. (See [**16**, **8**] and also [**20**, **10**].) Now, construct another Riemann mapping $g_N\colon D \to \Omega$ just by composing $\varphi(\varsigma) = (\varsigma - \delta)/(1 - \bar{\delta}\varsigma)$ to g from the right-hand side. Then, $\delta \in D$ can be reduced to $0 \in D$, and (1.3) is equivalent to

$$g_N''(0) = 0. \tag{2.1}$$

In this notation, a simple sufficient condition for the existence of the asymptotic solutions described above has been given by Moseley [**16**]. That is,

$$\alpha = \alpha(d, \Omega) = |g_N''(0)/g_N'(0)| \neq 2. \tag{2.2}$$

Moseley [**16**] further showed $\alpha < 2$ in the case that Ω is convex.

Genuine solutions for (P) are constructed by a Newton-like iteration. Namely, first we pull back the problem (P) in Ω to that in D by $g_N : D \to \Omega$:

$$-\Delta U = \lambda |g_N'|^2 e^U \quad (\text{in } D) \tag{2.3}$$

with

$$U = 0 \quad (\text{on } \partial D). \tag{2.4}$$

Through the Green's function

$$K(x,y) = \frac{1}{2\pi} \log \left| \frac{w-z}{1-\bar{z}w} \right|,$$

where $z = x_1 + ix_2$ and $w = y_1 + iy_2$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, respectively, the above problem is transformed into the integral equation

$$U = K(U) \equiv \lambda \int_D K(x,y)(|g_N'|^2 e^U)(y)\, dy. \tag{2.5}$$

Here, the modified-Newton iteration

$$U_{k+1} = S(U_k) \qquad (k = 0, 1, 2, \dots) \tag{2.6}$$

is applied where $S(U) = (1 - K'_{U_0})^{-1}(K(U) - K'_{U_0}(U))$. In the case that the iteration (2.6) converges in $C^0(\bar{D})$, a genuine solution U^* of (2.3) with (2.4) is obtained. It can be shown that if the starting point U_0 satisfies

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \le \log((1+\Gamma)/\Gamma) - (1+\Gamma)^{-1},$$

then (2.6) converges, where Γ is a positive constant such that

$$||(1 - K'_{U_0})^{-1} K'_{U_0}|| \le \Gamma.$$

Furthermore, we have

$$||U^* - U_0||_{C^0(\bar{D})} \le \log((1+\Gamma)/\Gamma). \tag{2.7}$$

See Weston [**22**, p. 1040].

When the nth asymptotic solution $U^n = u^n \circ g_N$ is taken as a starting point U_0 in the scheme (2.6), we have

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \le C\lambda^n \quad \text{as } \lambda \downarrow 0$$

with a constant $C > 0$ from (1.1) with (1.2′). On the other hand, by the method of Weston [**22**], we get

$$||(1 - K'_{U_0})^{-1}||_{C^0(\bar{D}) \to C^0(\bar{D})} \le C\lambda^{-1} \quad \text{as } \lambda \downarrow 0$$

except for a "pathological case" of Ω. Therefore, for $n \ge 3$ the iteration (2.6) converges to a genuine solution U^* such that

$$||U^* - U_0||_{C^0(\bar{D})} \le C\lambda^{(n-1)/2} \quad \text{as } \lambda \downarrow 0, \tag{2.8}$$

provided that $\lambda > 0$ is small. In fact, we can take $\Gamma = C_1 \lambda^{-1-l}$ $(l \ge 0)$ by

$$||(1 - K'_{U_0})^{-1} K'_{U_0}|| \le 1 + ||(1 - K'_{U_0})^{-1}|| \le C_2 \lambda^{-1} \qquad (\lambda \downarrow 0).$$

Then,

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \le \log((1+\Gamma)/\Gamma) - (1+\Gamma)^{-1}$$

holds if $n = 2l + 3$ and $\lambda \downarrow 0$. Further, then

$$||U^* - U_0||_{C^0(\bar{D})} \leq \log((1+\Gamma)/\Gamma) \leq C_3\lambda^{1+l} = C_3\lambda^{(n-1)/2}.$$

Here, C_1, C_2 and C_3 are positive constants.

By the method of Wente [**21**], it can be shown that the "pathological case" does not arise when $\alpha \equiv |g_N''(0)/g_N'(0)| < 2$. The function $u^* = u_\lambda^* = U^* \circ g_N^{-1}$ becomes a nonminimal section for (P), and the branch $\mathscr{C}^* = \{(\lambda, u_\lambda^*)\}$ of large solutions has been constructed.

From the inequality (2.8) and the concrete expression of U_0 $(= u_n \circ g_N)$, we can derive an important relation,

$$S \equiv \lambda \int_\Omega e^{u_\lambda^*}\, dx = 8\pi + C\lambda + o(\lambda) \quad \text{as } \lambda \downarrow 0, \tag{2.9}$$

with a constant $C = C(d, \Omega)$ defined by

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{n=3}^{\infty} \frac{n^2}{n-2}|a_n|^2, \tag{2.10}$$

where $g_N(\varsigma) = \sum_{n=0}^{\infty} a_n \varsigma^n$ $(a_2 = 0)$. By virtue of Bieberbach's area theorem [**18**, p. 210], we can show the following fact, where $\kappa = \kappa(\varsigma)$ denotes the curvature of $\partial\Omega$ at the point $g_N(\varsigma) \in \partial\Omega$ for $\varsigma \in \partial D$.

PROPOSITION 1. *If $\kappa|g_N'| < 2$ holds everywhere on ∂D, then $C < 0$ follows.* □

In the case that Ω is a disc: $\Omega = \{|z| < R\}$, we have $\kappa|g_N'| \equiv 1$. Further, we note that $C = C(d, \Omega) < 0$ implies that $\alpha = \alpha(d, \Omega)$ $(= 6|a_3/a_1|) < 2$.

The proof of (2.9) with (2.10) and Proposition 1 will be given in Appendices 1 and 2, respectively.

2. We next look over Bandle's theory [**3**] about a priori estimates for solutions and eigenvalues.

Namely, let $h = {}^T(u, \lambda)$ solve (1.1) with $p = \lambda e^u$ (> 0). We consider a surface $\mathscr{M} \equiv (\Omega, d\sigma)$ with the metric $d\sigma^2 = p\, ds^2$ $(= p(dx_1^2 + dx_2^2))$. Then, the surface element and the Gaussian curvature are $d\tau = p\, dx$ $(= p\, dx_1\, dx_2)$ and $K = -(\Delta \log p)/2p = 1/2$, respectively. Bol's inequality is expressed as

$$l(\omega)^2 \geq \tfrac{1}{2}(8\pi - m(\omega))m(\omega) \tag{2.11}$$

for $\omega \subset \Omega$, where $l(\omega) = \int_{\partial\omega} d\sigma$ and $m(\omega) = \int_\omega d\tau$. In the manner of (2.11), we can give the following estimate [**17**].

PROPOSITION 2. *Let $h = {}^T(u, \lambda)$ solve* (P) *and put $S = \lambda \int_\Omega e^u\, dx$. Then, we have*

$$||u||_{C^0(\bar{\Omega})} \leq -2\log(1 - S/8\pi), \tag{2.12}$$

provided that $S < 8\pi$. □

Note that as for λ we have $|\Omega|^{-1}e^{-||u||_{C^0(\bar{\Omega})}}S \leq \lambda \leq \bar{\lambda}$. Estimate (2.12) is also seen in Bandle [**3**, p. 85, Problem]. (Namely, we have only to take $p = \lambda e^u$, $K_0 = 1/2$ and $M = S$, there.) We shall give the third proof in Appendix 3.

We obtain the operator $A_p = -\Delta - p$ under the Dirichlet condition for $p = \lambda e^u$ by linearizing problem (P) with respect to u at the solution $h = {}^T(u, \lambda)$. Let

$\sigma(A_p) = \{\mu_j(p)\}_{j=1}^{\infty}$ $(-\infty < \mu_1(p) < \mu_2(p) \leq \cdots \to +\infty)$ denote its eigenvalues. Then, the relation $\mu_1(p) \geq 0$ holds when $h = {}^T(u, \lambda)$ is minimal, while conversely, $\mu_1(p) > 0$ implies the minimality of h [**4**].

The following proposition is obtained through Schwarz' symmetrization associated with the surface $\mathscr{M}$ [**3**, p. 108]. See also [**17**].

PROPOSITION 3. *For the solution* $h = {}^T(u, \lambda)$ *of* (P), *the inequality* $S \equiv \lambda \int_\Omega e^u\, dx < 4\pi$ *implies* $\mu_1(p) > 0$. □

An immediate consequence is the following.

COROLLARY 1. *Similarly,* $S < 8\pi$ *implies* $\mu_2(p) > 0$.

PROOF. The eigenfunction φ_2 of A_p corresponding to $\mu_2(p)$ has two nodal domains Ω_1 and Ω_2. From the assumption, either $S_1 \equiv \lambda \int_{\Omega_1} e^u\, dx < 4\pi$ or $S_2 \equiv \lambda \int_{\Omega_2} e^u\, dx < 4\pi$ holds. On the other hand, $\mu_2(p)$ may be regarded as the first eigenvalue of the operator $-\Delta - p$ under the Dirichlet condition on Ω_1 or Ω_2. Hence $\mu_2(p) > 0$ follows. □

3. Now, we shall describe our key idea, that is, parametrizing the solution $h = {}^T(u, \lambda)$ of (P) in terms of $S = \lambda \int_\Omega e^u\, dx$ rather than λ. See Nagasaki and Suzuki [**17**] for the background of this idea.

For α in $0 < \alpha < 1$, we set $X = C_0^{2+\alpha}(\bar{\Omega}) \equiv \{v \in C^{2+\alpha}(\bar{\Omega}) | v = 0 \text{ on } \partial\Omega\}$, $Y = C^\alpha(\bar{\Omega})$, $\hat{X} = \overset{X}{\underset{\mathbf{R}}{\times}}$, $\hat{X}_+ = \overset{X}{\underset{\mathbf{R}_+}{\times}}$, and $\hat{Y} = \overset{Y}{\underset{\mathbf{R}}{\times}}$, and define a mapping $\Phi = \Phi(h, S)\colon \hat{X}_+ \times \mathbf{R} \to \hat{Y}$ as

$$\Phi(h, S) = \begin{pmatrix} \Delta u + \lambda e^u \\ \int_\Omega e^u\, dx - \frac{S}{\lambda} \end{pmatrix}$$

for $h = {}^T(u, \lambda)$. Zeros of Φ characterize the solutions $h = {}^T(u, \lambda)$ of (P) such that $S = \lambda \int_\Omega e^u\, dx$. The Fréchet derivative $d_h\Phi\colon \hat{X} \to \hat{Y}$ of Φ with respect to h at (h, S) is given by the matrix

$$d_h\Phi = \begin{pmatrix} \Delta + \lambda e^u & e^u \\ \int_\Omega e^u \cdot dx & \frac{S}{\lambda^2} \end{pmatrix}.$$

For the moment, let $(h, S) \in \hat{X}_+ \times \mathbf{R}$ $(h = {}^T(u, \lambda))$ be a zero point of Φ and set $p = \lambda e^u$.

LEMMA 1. *The operator* $d_h\Phi\colon \hat{X} \to \hat{Y}$ *is invertible if* $\mu_1(p) \geq 0$. □

PROOF. The operator

$$T = d_h\Phi = \begin{pmatrix} -A_p & e^u \\ \int_\Omega e^u \cdot dx & \frac{S}{\lambda^2} \end{pmatrix} : \hat{X} \to \hat{Y}$$

has a selfadjoint extension $\tilde{T}$ in $\overset{L^2(\Omega)}{\underset{\mathbf{R}}{\times}}$ with the domain

$$D(\tilde{T}) = H_0^1(\Omega) \underset{\mathbf{R}}{\times} \cap H^2(\Omega).$$

Therefore, $\tilde{T}$ is invertible if and only if $\operatorname{Ker} \tilde{T} = \{0\}$, but the same is true for $T = d_h\Phi\colon \hat{X} \to \hat{Y}$ by virtue of the elliptic regularity property of A_p.

Hence, suppose that $f = {}^T(v, \rho) \in \hat{X}$ with $(v, \rho) \neq (0, 0)$ is in the kernel of $T = d_h\Phi$. This means that

$$\Delta v + \lambda e^u v + \rho e^u = 0 \quad (\text{in } \Omega), \qquad v = 0 \quad (\text{on } \partial\Omega) \tag{2.13}$$

and

$$\int_\Omega e^u v\, dx + \frac{\rho S}{\lambda^2} = 0. \tag{2.14}$$

Multiply (2.13) by v and integrate

$$T(v) \equiv \int_\Omega |\nabla v|^2\, dx - \lambda \int_\Omega e^u v^2\, dx = \rho \int_\Omega e^u v\, dx = -\frac{\rho^2 S}{\lambda^2}.$$

If $\rho \neq 0$, then $T(v) < 0$ which implies that $\mu_1(p) < 0$. If $\rho = 0$, then $v =$ constant $\times \varphi_1$ ($\not\equiv 0$), where $\varphi_1 > 0$ is the first eigenfunction of A_p. But this is impossible in equation (2.14). □

In the case that $0 \notin \sigma(A_p)$, the spectrum of A_p, the relation (2.13) with (2.14) reduces to

$$\frac{\rho}{\lambda} \int_\Omega p\{1 + A_p^{-1}(p)\}\, dx = 0 \quad \text{with } v = \frac{\rho}{\lambda} A_p^{-1}(p)$$

because $S = \int_\Omega p\, dx$. Therefore, $T = d_h\Phi$ is invertible if and only if

$$I \equiv -\int_\Omega p\{1 + A_p^{-1}(p)\}\, dx \neq 0.$$

Further,

LEMMA 2. *We have $\partial S/\partial\lambda = -I/\lambda$ if $0 \notin \sigma(A_p)$.* □

PROOF. In that case, the section u of (P) is smooth with respect to λ. Actually, we get

$$v = \frac{\partial}{\partial\lambda} u = \frac{1}{\lambda} A_p^{-1}(p),$$

by differentiating (P) in λ. Therefore,

$$\frac{\partial S}{\partial\lambda} = \int_\Omega \{e^u + \lambda e^u v\}\, dx = \frac{1}{\lambda} \int_\Omega p\{1 + A_p^{-1}(p)\}\, dx. \quad \square$$

Under these preparations, we conclude that

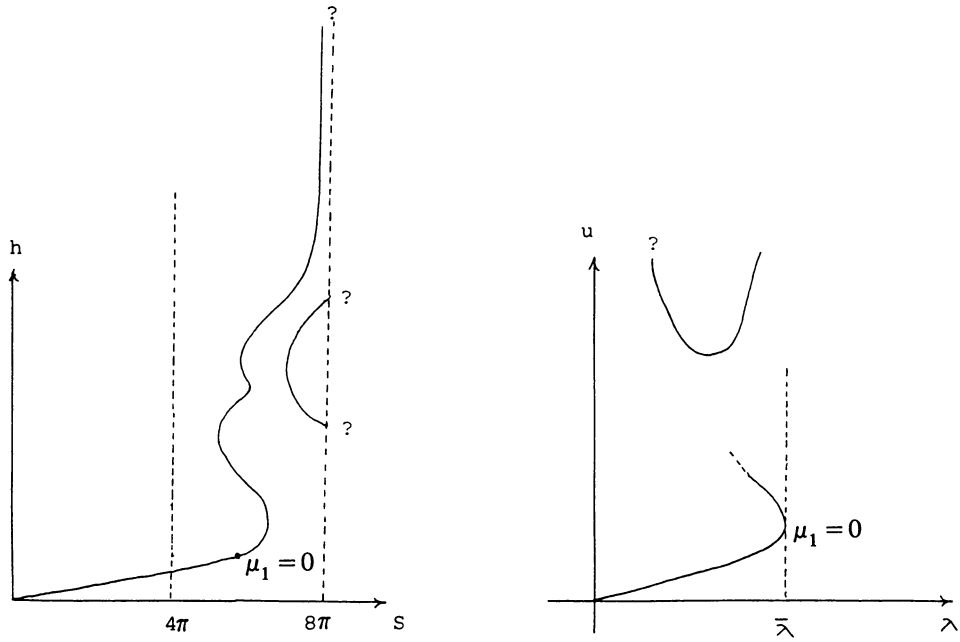
PROPOSITION 4. *In the case of $\Omega = D$, every solution $h = {}^T(u, \lambda)$ of* (P) *is parametrized by $S = \lambda \int_\Omega e^u\, dx \in (0, 8\pi)$. Let it be $h_0(S) = {}^T(u_0(S), \lambda_0(S))$. Then, $d_h\Phi(h_0(S), S)\colon \hat{X} \to \hat{Y}$ is invertible at each $S \in (0, 8\pi)$.* □

PROOF. According to the explicit formula (1.4), every solution $h = {}^T(u, \lambda)$ is reparametrized by $S \in (0, 8\pi)$: $h = h_0(S) = {}^T(u_0(S), \lambda_0(S))$ $(0 < S < 8\pi)$.

The inverse mapping of $S \in (0, 8\pi) \mapsto \lambda_0(S) \in (0, 2)$ is two-valued: $S = S_0^\pm(\lambda)$, where $S_0^\pm(\lambda) \to 4\pi$ as $\lambda \to 2$ and $S_0^+(\lambda) \to 8\pi$, $S_0^-(\lambda) \to 0$ as $\lambda \to 0$. Therefore, $\mu_1(p_0(S)) > 0$ for $0 < S < 4\pi$, $\mu_1(p_0(S)) = 0$ for $S = 4\pi$ and $\mu_1(p_0(S)) < 0$ for $4\pi < S < 8\pi$ from the local theory of Crandall and Rabinowitz [**4**], where $p_0(S) = \lambda_0(S)e^{u_0(S)}$. Hence $d_h\Phi(h_0(S), S)$ is invertible for $0 < S \leq 4\pi$ by Lemma 1. On the other hand, in the case $4\pi < S < 8\pi$ we have $0 \notin \sigma(A_p)$ by Corollary 1. Then, $\partial S_0^+(\lambda)/\partial\lambda \neq 0$ $(0 < \lambda < 2)$ is verified directly by (1.4), so that $d_h\Phi(h_0(S), S)$ $(4\pi < S < 8\pi)$ is invertible by Lemma 2.

3. Theorems and proofs. In what follows, we seek the solutions $(h, S) \in \hat{X}_+ \times \mathbf{R}$ of $\Phi(h, S) = 0$. There is a branch $\underline{\mathscr{S}}$ of zeros of Φ originating from $(h, S) = (0, 0)$, and corresponding to the branch of minimal solutions $\underline{\mathscr{E}}$ for (P) described in §1.

THEOREM 1. *Every zero point (h_0, S_0) of Φ generates a branch $\mathscr{S}_0$ of $\Phi(h, S) = 0$ in the S-h plane, whenever $S_0 < 8\pi$. Each end of $\mathscr{S}_0$ approaches eventually either the hyperplane $S = 8\pi$ or else $(0, 0)$. In the latter case, that is, when $\mathscr{S}_0$ is connected with $\underline{\mathscr{S}}$, the branch formed in this way bends at most once in the λ-u plane.* □



PROOF. Set $p_0 = \lambda_0 e^{u_0}$, where $h_0 = {}^T(u_0, \lambda_0)$. Then, $\mu_2(p_0) > 0$ holds by Corollary 1.

In the case of $\mu_1(p_0) \neq 0$, the implicit function theorem applies to problem (P) with respect to the parameter λ, and (h_0, S_0) generates a branch $\mathscr{S}_0$ of zeros of Φ. In the case $\mu_1(p_0) = 0$, on the other hand, Lemma 1 is available and we get the same conclusion.

Henceforth, we set $p = \lambda e^u$ for $h = {}^T(u, \lambda)$ where $\Phi(h, S) = 0$ holds with some S. We shall show the global behavior of $\mathscr{S}_0$.

Along one direction of that branch $\mathscr{S}_0$, suppose that the relation $S \leq S_1$ always holds with an $S_1 < 8\pi$. Then, we have $\mu_2(p) > 0$ along those zeros of Φ. We shall show that there eventually appears a point (h_1, S_1) in S_0 such that $\mu_1(p_1) > 0$, where $p_1 = \lambda_1 e^{u_1}$. Then, such an (h_1, S_1) lies on the minimal branch $\underline{\mathscr{S}}$, which originates from $(0, 0)$.

To this end, we first show that along that direction with $S \leq S_1$ $(< 8\pi)$, it is impossible for $\mu_1(p) < 0 < \mu_2(p)$ to keep holding. Suppose the contrary. Then, there is a branch $\mathscr{E}_0$ of the solutions of (P) in the λ-u plane corresponding to $\mathscr{S}_0$.

The implicit function theorem holds along the corresponding direction of $\mathscr{C}_0$ with respect to λ from the above assumption. On the other hand, we have an a priori estimate in Proposition 2, so that $\mathscr{C}_0$ continues up to either $\lambda \to +\infty$ or $\lambda \to 0$. However, the estimate $\lambda \leq \bar{\lambda}(\Omega)$ holds and $\lambda \to +\infty$ is impossible. Thus, $\mathscr{C}_0$ continues to $(0,0)$, because $u = 0$ is the unique section at $\lambda = 0$ of (P). However, $\mu_1(p) > 0$ holds near $(0,0)$ on $\mathscr{C}_0$, and hence this case does not occur.

Next we show that when $\mu_1(p_*) = 0$ occurs at some point $(h_*, S_*) \in \mathscr{S}_0$, then $\mu_1(p)$ changes sign near p_* on $\mathscr{S}_0$, where $p_* = \lambda_* e^{u_*}$ for $h_* = {}^T(u_*, \lambda_*)$. This fact, together with the above one, will imply the connectivity of $\mathscr{S}_0$ and $\underline{\mathscr{L}}$ for all cases.

To verify this fact, we recall the local theory of Crandall and Rabinowitz. Namely, in the case $\mu_1(p_*) = 0$, near $h_* = {}^T(u_*, \lambda_*)$ $\mathscr{C}_0$ is parametrized as $\{(\lambda(t), u(t)) |\ |t| < \varepsilon_0\}$ with

$$u(t) = u_* + t\varphi_{1*} + o(t) \quad \text{and} \quad \lambda(t) = \lambda_* + ct^2 + o(t^2),$$

where $\varphi_{1*} > 0$ denotes the first eigenfunction of A_{p_*} [**23**, Theorem 3.2]. Further, the computation of Theorem 4.8 of [**23**] shows that $\ddot{\lambda}(0) < 0$. Here we have

$$-\Delta u(t) = \lambda(t)e^{u(t)} \quad (\text{in } \Omega), \qquad u(t) = 0 \quad (\text{on } \partial\Omega).$$

Hence for $\dot{u}(t) = \partial u(t)/\partial t$ and $p(t) = \lambda(t)e^{u(t)}$ we obtain

$$-\Delta \dot{u}(t) = p(t)\dot{u}(t) + \dot{\lambda}(t)e^{u(t)} \quad (\text{in } \Omega), \qquad \dot{u}(t) = 0 \quad (\text{on } \partial\Omega)$$

so that

$$T(t) \equiv \int_\Omega |\nabla \dot{u}(t)|^2\, dx - \int_\Omega p(t)\dot{u}(t)^2\, dx = \int_\Omega \dot{\lambda}(t)e^{u(t)}\dot{u}(t)\, dx.$$

Because of $\ddot{\lambda}(0) < 0$, we have $\dot{\lambda}(t) \neq 0$ $(0 < |t| < \varepsilon_0)$ for $\varepsilon_0 > 0$ sufficiently small. This means that $\mu_1(p(t)) \neq 0$ $(0 < |t| < \varepsilon_0)$, because $\mu_1(p(t)) = 0$ for $t = t_0$ implies that $\dot{\lambda}(t_0) = 0$ by the local theory. Further, we have

$$T'(0) = \int_\Omega \ddot{\lambda}(0)e^{u_*}\dot{u}(0)\, dx = \ddot{\lambda}(0)\int_\Omega e^{u_*}\varphi_{1*}\, dx < 0$$

with $T(0) = 0$ and hence $\mu_1(p(t)) < 0$ for $0 < t < \varepsilon_0$.

Now, we shall show that $\mu_1(p(t)) > 0$ for $-\varepsilon_0 < t < 0$.

In fact, we have shown that it is impossible for $\mu_1(p) < 0$ to keep holding along the direction of $\mathscr{C}_0$ in consideration. Therefore, in case $\mu_1(p(t)) < 0$ for $-\varepsilon_0 < t < 0$, we have to meet the next point $h_{**} = {}^T(u_{**}, \lambda_{**})$ on $\mathscr{C}_0$ such that $\mu_1(p_{**}) = 0$ for $p_{**} = \lambda_{**}e^{u_{**}}$. But this is impossible, because we must also have $\lambda'' < 0$ at h_{**} from the calculation of [**23**] mentioned above. Thus, we see that along the direction $\mathscr{C}_0$ in consideration, the parameter $t \in (-\varepsilon_0, \varepsilon_0)$ decreases from ε_0 to $-\varepsilon_0$ and that $\mu_1(p(t)) > 0$ holds for $-\varepsilon_0 < t < 0$.

In this way, we have shown that in the case that the relation $S \leq S_1$ $(< 8\pi)$ is preserved along one direction of $\mathscr{S}_0$, (h_0, S_0) connects with $(0,0)$, and furthermore the corresponding branch $\mathscr{C}_0$ in the λ-u plane bends at most once. □

Next, we suppose that Ω is star-shaped with respect to the origin and put $B \equiv \int_{\partial\Omega} ds/(n \cdot x)$, where n denotes the outer unit normal vector on $\partial\Omega$. Then, we have $B \geq 2\pi$, where the equality holds when Ω is a disc.

If $h = {}^T(u, \lambda)$ solves (P), the estimate

$$(S - 2B)^2/B \leq 4B - 4\lambda|\Omega|$$

holds by Rellich's identity, where $S = \lambda \int_\Omega e^u\,dx$ (Bandle [**3**, p. 202]). In particular, $B \le 4\pi$ and $S \ge 8\pi$ imply that

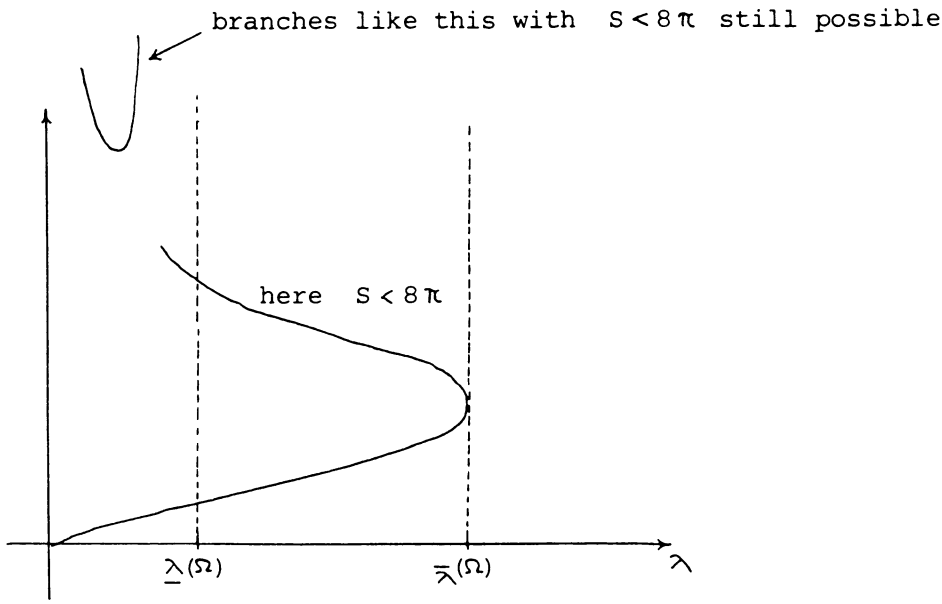
$$(8\pi - 2B)^2/B \le (S - 2B)^2/B \le 4B - 4\lambda|\Omega|,$$

and hence $\lambda \le 8\pi(B - 2\pi)/|\Omega|B$. In other words, $S < 8\pi$ holds when $\lambda > \underline{\lambda}(\Omega) \equiv 8\pi(B - 2\pi)/|\Omega|B$ and $B \le 4\pi$. More precisely,

LEMMA 3. *In the case of $B \le 4\pi$, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda \ge \underline{\lambda} + \varepsilon$ implies $S \le 8\pi - \delta$.* □

Now, the next theorem follows from the previous one.

THEOREM 2. *If Ω is star-shaped with respect to the origin, $B = \int_{\partial\Omega} ds/(n\cdot x) \le 4\pi$ and $\underline{\lambda}(\Omega) < \bar{\lambda}(\Omega)$, then for each λ in $\underline{\lambda} < \lambda < \bar{\lambda}$, the problem* (P) *has exactly two sections, that is, the minimal section and the nonminimal one. In the λ-u plane, these are connected to each other.* □



PROOF. At each $\lambda_0 \in (\underline{\lambda}, \bar{\lambda})$, there exists at least one nonminimal section u_0. Then, $\mu_1(p_0) < 0 < \mu_2(p_0)$ holds for $p_0 = \lambda_0 e^{u_0}$ by Lemma 3 and Corollary 1 to Proposition 3. Hence the implicit function theory applies for (P) at $h_0 = {}^T(u_0, \lambda_0)$. There is a branch $\mathscr{C}_0$ of solutions in the λ-u plane generated by h_0. From Lemma 3, the relation $S \le S_1$ keeps holding in the direction of λ increasing, where $S_1 < 8\pi$. Therefore, from the proof of Theorem 1 h_0 is connected with $(0,0)$ without any bifurcation. The branch $\mathscr{C}$ constructed in this way bends just once. Further, any nonminimal solution $\tilde{h} = {}^T(\tilde{u}, \tilde{\lambda})$ with $\tilde{\lambda} \in (\underline{\lambda}, \bar{\lambda})$ generates a branch $\tilde{\mathscr{C}}$, which is connected with $\mathscr{C}$. Since $\mathscr{C}$ has no bifurcation, we conclude that $\tilde{h} \in \mathscr{C}$. □

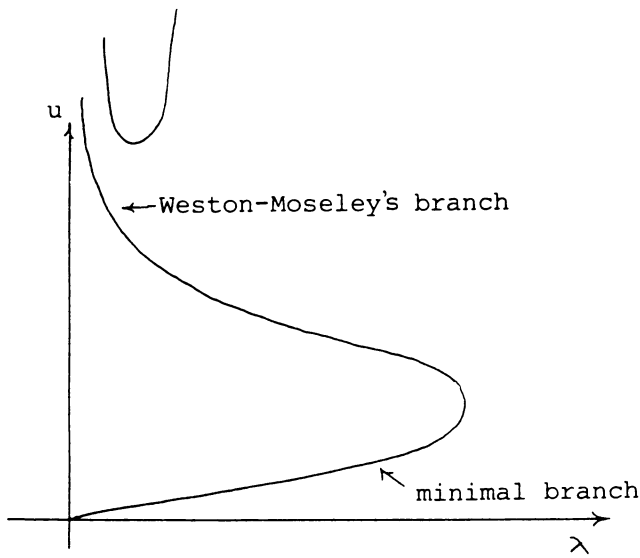
Finally, we shall show our main result, that is, the branch of Weston-Moseley's large solutions connects with that of minimal solutions when Ω is close to a disc.

To this end, let $\omega \subset \mathbf{R}^2$ be a simply connected domain with smooth boundary $\partial\omega$, and let $g_1\colon D \to \omega$ be a Riemann mapping such that $g_1''(0) = 0$. Actually, such

a g_1 exists as we have shown in §2.1. For sufficiently small $|\varepsilon|$, let $g_{N,\varepsilon} = g_{N,\varepsilon}(\varsigma) = \varsigma + \varepsilon g_1(\varsigma)\colon D \to \Omega_\varepsilon$, where $\Omega_\varepsilon = g_{N,\varepsilon}(D)$. Then, $g_{N,\varepsilon}$ becomes a Riemann mapping satisfying $g''_{N,\varepsilon}(0) = 0$. In fact, univalentness follows from Darboux's theorem.

If $|\varepsilon|$ is small, $\alpha_\varepsilon = |g'''_{N,\varepsilon}(0)/g'_{N,\varepsilon}(0)| < 2$ holds, so that the branch of Weston-Moseley's large solutions for (P) can be constructed in Ω_ε, which is denoted by $\mathscr{C}_\varepsilon^* = \{(\lambda, u^*_{\lambda,\varepsilon})\}$. On the other hand, there exists the branch of minimal solutions in Ω_ε denoted by $\underline{\mathscr{C}}_\varepsilon$. Then,

THEOREM 3. *If $|\varepsilon|$ is sufficiently small, $\mathscr{C}_\varepsilon^*$ connects with $\underline{\mathscr{C}}_\varepsilon$. Further, the branch $\mathscr{C}_\varepsilon$ constructed in this way bends just once in the λ-u plane. Namely, we can parametrize $\mathscr{C}_\varepsilon = \{(\lambda_t, u_t) | 0 \le t < 3\}$ as $(u_0, \lambda_0) = (0,0)$ and λ_t increases in $t \in (0, \bar{t})$ and decreases in $t \in (\bar{t}, 3)$ with some $\bar{t} \in (0,3)$. Furthermore, here we have $\lambda_{\bar{t}} = \bar{\lambda}(\Omega_\varepsilon)$.*



PROOF. According to the formulation in §2.3, we can transform problem (P) in Ω_ε to finding zeros of the mapping $\Phi = \Phi_\varepsilon$ defined below. Namely, $X_\varepsilon = C_0^{2+\alpha}(\bar{\Omega}_\varepsilon)$, $Y_\varepsilon = C^\alpha(\bar{\Omega}_\varepsilon)$, $\hat{X}_\varepsilon = \overset{X_\varepsilon}{\underset{\mathbf{R}}{\times}}$, $\hat{X}_{\varepsilon'+} = \overset{X_\varepsilon}{\underset{\mathbf{R}+}{\times}}$, $\hat{Y}_\varepsilon = \overset{Y_\varepsilon}{\underset{\mathbf{R}}{\times}}$, and $\Phi_\varepsilon = \Phi_\varepsilon(h, S)\colon \hat{X}_{\varepsilon'+} \times \mathbf{R} \to \hat{Y}_\varepsilon$, where

$$\Phi_\varepsilon(h, S) = \begin{pmatrix} \Delta u + \lambda e^u \\ \int_{\Omega_\varepsilon} e^u\, dx - \frac{S}{\lambda} \end{pmatrix} \quad \text{for } h = {}^T(u, \lambda).$$

Corresponding to the minimal branch $\underline{\mathscr{C}}_\varepsilon$, there is a branch $\underline{\mathscr{S}}_\varepsilon$ of zeros of Φ_ε in the h-S plane, originating from $(h, S) = (0,0)$. By virtue of Theorem 1, $\underline{\mathscr{S}}_\varepsilon$ approaches eventually the hyperplane $S = 8\pi$. Let $\tilde{\mathscr{S}}_\varepsilon$ be the branch generated by $\underline{\mathscr{S}}_\varepsilon$ in this way.

On the other hand, along the branch $\mathscr{C}_\varepsilon^*$ of large solutions, the quantity $S = \lambda \int_\Omega e^{u^*_{\lambda,\varepsilon}}\, dx$ tends to 8π from below as $\lambda \downarrow 0$ by Proposition 1. Therefore, λ is

parametrized by S and hence $\mathscr{C}_\varepsilon^*$ can be reparametrized as $\mathscr{C}_\varepsilon^* = \{(\lambda(S), u_\varepsilon^*(S)) | S_0 < S < 8\pi\}$ with an $S_0 \in (0, 8\pi)$. Further, S_0 can be taken to be independent of ε in $|\varepsilon| < \varepsilon_1$, where $\varepsilon_1 > 0$ is a small constant, by virtue of (2.9) and (2.10). Actually, (2.9) holds uniformly in ε. Henceforth, we put $h_\varepsilon^*(S) = {}^T(u_\varepsilon^*(S), S)$ $(S_0 < S < 8\pi)$: $\Phi_\varepsilon(h_\varepsilon^*(S), S) = 0$ $(|\varepsilon| < \varepsilon_0,\ S_0 < S < 8\pi)$.

From the Riemann mapping $g_{N,\varepsilon}\colon \Omega_0 \to \Omega_\varepsilon$, the problem (P) on Ω_ε is pulled back to that on $D = \Omega_0$:

$$-\Delta U = \lambda |g'_{N,\varepsilon}|^2 e^U \quad (\text{in } D) \tag{3.1}$$

with

$$U = 0 \quad (\text{on } \partial D). \tag{3.2}$$

Then, Φ_ε is transformed into the operator $F_\varepsilon\colon \hat{X}_{0,+} \times \mathbf{R} \to \hat{Y}_0$ as

$$F_\varepsilon(H, S) = \begin{pmatrix} \Delta U + \lambda |g'_{N,\varepsilon}|^2 e^U \\ \int_{\Omega_0} |g'_{N,\varepsilon}|^2 e^U \, dx - \frac{S}{\lambda} \end{pmatrix},$$

where $H = {}^T(U, \lambda)$.

For the large solution $u^* = u_{\lambda,\varepsilon}^*$, we set $U_{\lambda,\varepsilon}^* = u_{\lambda,\varepsilon}^* \circ g_N$, and $H_\varepsilon^* = {}^T(U_{\lambda,\varepsilon}^*, \lambda)$. Then, H_ε^* is parametrized by $S \in (S_0, 8\pi)$ like h_ε^*, and the relation

$$F_\varepsilon(H_\varepsilon^*(S), S) = 0 \tag{3.3}$$

follows for $|\varepsilon| < \varepsilon_0$ and $S_0 < S < 8\pi$. Furthermore,

$$||U_{\lambda,\varepsilon}^*||_{C^0(\bar{D})} \le -2\log(1 - S/8\pi) \tag{3.4}$$

holds by Proposition 2, so that $\{H_\varepsilon^*(S_1) | \, |\varepsilon| \le \varepsilon_0/2\}$ is compact in $\hat{X}_0$ for each fixed $S_1 \in (S_0, 8\pi)$ by virtue of the elliptic estimate.

Taking a suitable sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, $H_{\varepsilon_j}^*(S_1)$ converges in $\hat{X}_0$. Then, the limit $\tilde{H}_0^*(S_1)$ solves $\Phi_0(\tilde{H}_0^*(S_1), S_1) = F_0(\tilde{H}_0^*(S_1), S_1) = 0$. However, as we have shown in Proposition 4, the zero of $\Phi_0(\cdot, s_1)$ is unique, that is, $h_0(S_1)$. Hence

$$H_\varepsilon^*(S_1) \to h_0(S_1) \quad \text{as } \varepsilon \to 0 \text{ in } \hat{X}_0. \tag{3.5}$$

On the other hand, the branch $\tilde{\mathscr{S}}_\varepsilon$ generated by the minimal one has at least one section at $S = S_1$, which is denoted by $\underline{h}_\varepsilon(S_1) \in \hat{X}_\varepsilon$. Similarly, $\underline{h}_\varepsilon(S_1)$ is transformed into an $\underline{H}_\varepsilon(S_1) \in \hat{X}_0$ through $g_{N,\varepsilon}\colon \Omega_0 \to \Omega_\varepsilon$ with the relation $F_\varepsilon(\underline{H}_\varepsilon(S_1), S_1) = 0$. In the same way, we have

$$\underline{H}_\varepsilon(S_1) \to h_0(S_1) \quad \text{as } \varepsilon \to 0 \text{ in } \hat{X}_0. \tag{3.6}$$

Now, Proposition 4 indicates that the operator $T_0 = d_H F_0(h_0(S_1), S_1)$; $\hat{X}_0 \to \hat{Y}_0$ is invertible. Therefore, the same is true for the operator $T_\varepsilon = d_H F_\varepsilon(H_\varepsilon^*(S_1), S_1)$: $\hat{X}_0 \to \hat{Y}_0$, provided that $|\varepsilon|$ is small. In particular, the equation

$$F_\varepsilon(H, S_1) = 0 \tag{3.7}$$

has the local uniqueness property around the solution $H = H_\varepsilon^*(S_1)$ uniformly in ε. Namely, there exist some $\varepsilon_1 > 0$ and $\kappa > 0$ such that $|\varepsilon| \le \varepsilon_1$, $F_\varepsilon(H, S_1) = 0$ and $||H - H_\varepsilon^*(S_1)||_{X_0} < \kappa$ imply $H = H_\varepsilon^*(S_1)$. Therefore, by virtue of (3.5) and (3.6), we get $H_\varepsilon^*(S_1) = \underline{H}_\varepsilon(S_1)$ to conclude that $\mathscr{S}_\varepsilon^*$ and $\underline{\mathscr{S}}_\varepsilon$, and hence $\mathscr{C}_\varepsilon^*$ and $\underline{\mathscr{C}}_\varepsilon$ connect to each other when $|\varepsilon|$ is sufficiently small.

The latter part of the theorem follows from Theorem 1. □

Appendix I.

PROOF OF (2.9). Let u_0 be the fifth-order asymptotic solution. We first will show that (2.9) is reduced to

$$S_0 \equiv \lambda \int_\Omega e^{u_0}\,dx = 8\pi + C\lambda + o(\lambda) \quad \text{as } \lambda \downarrow 0. \tag{I.1}$$

In fact, then we get

$$\begin{aligned} |S - S_0| &\le \lambda \int_\Omega e^{u_0}\,dx\{e^{\|u-u_0\|_{C^0(\bar D)}} - 1\} \\ &= S_0\{e^{\|u-u_0\|_{C^0(\bar D)}} - 1\} \le C\lambda^2 \end{aligned}$$

by (2.8).

To show (I.1), we put $U = u_0 \circ g_N$. Then U satisfies

$$-\Delta u = \lambda |g_N'|^2 e^U \quad (\text{in } D),$$

so that

$$S_0 = \lambda \int_\Omega e^{u_0}\,dx = \lambda \int_D e^U |g_N'|^2\,dx = -\int_D \Delta U\,dx = -\int_{\partial D} \frac{\partial U}{\partial r}\,ds, \tag{I.2}$$

where $r = |x|$.

The asymptotic solution U is given as

$$e^{-U/2} = \frac{\{|\varsigma|^2 + (\lambda/8)|A(\varsigma)|^2\}}{|G(\varsigma)|^2} \qquad (\varsigma \in D), \tag{I.3}$$

where $G(\varsigma) = G(\varsigma, \lambda) = 1 + \lambda G_1(\varsigma) + \cdots + \lambda^{n-1} G_{n-1}(\varsigma)$ $(n = 5)$ and

$$A(\varsigma) = A(\varsigma, \lambda) = \varsigma \int^{\varsigma} G(\hat\varsigma, \lambda)^2 \frac{g_N^1(\hat\varsigma)}{\hat\varsigma^2}\,d\hat\varsigma,$$

which are described more precisely later [**16**]. Hence

$$\begin{aligned} -\frac{1}{2}\frac{\partial U}{\partial r} e^{-U/2} &= \left\{2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2\right\} \Big/ |G(\varsigma,\lambda)|^2 \\ &\quad - \left\{r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2\right\} \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4, \end{aligned}$$

so that

$$-\frac{1}{2}\frac{\partial U}{\partial r}\bigg|_{r=1,\lambda=0} = 2$$

by $G(\varsigma, 0) = 1$ and $U|_{\partial D} = O(\lambda^n)$. Therefore, we get

$$S_0 = -\int_{\partial D} \frac{\partial U}{\partial r}\,ds = 8\pi + O(\lambda) \quad \text{as } \lambda \downarrow 0. \tag{I.1$'$}$$

Next we have

$$\begin{aligned} &-\frac{1}{2}\frac{\partial}{\partial\lambda}\frac{\partial U}{\partial r} e^{-U/2} + \frac{1}{4}\frac{\partial U}{\partial r}\frac{\partial U}{\partial \lambda} e^{-U/2} \\ &\quad = \frac{\partial}{\partial\lambda}\left\{\left(2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2\right) \Big/ |G(\varsigma,\lambda)|^2\right\} \\ &\qquad - \frac{\partial}{\partial\lambda}\left\{\left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2\right) \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4\right\} \\ &\quad = \mathrm{I} - \mathrm{II} \end{aligned}$$

with

$$\mathrm{I} = \left\{ \frac{1}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 + \frac{\lambda}{8}\frac{\partial}{\partial\lambda}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right\} \Big/ |G(\varsigma,\lambda)|^2 - \left\{ 2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right\} \frac{\partial}{\partial\lambda}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4$$

and

$$\begin{aligned}\mathrm{II} = \Bigg\{ &\left(\frac{1}{8}|A(\varsigma,\lambda)|^2 + \frac{\lambda}{8}\frac{\partial}{\partial\lambda}|A(\varsigma,\lambda)|^2\right)\frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \\ &+ \left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2\right)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Bigg\} \Big/ |G(\varsigma,\lambda)|^4 \\ &- 2\left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2\right)\frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \frac{\partial}{\partial\lambda}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^6.\end{aligned}$$

Therefore, we have

$$\mathrm{I}|_{\lambda=0} = \frac{1}{8}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - 4r\,\mathrm{Re}\,G_1(\varsigma)$$

and

$$\mathrm{II}|_{\lambda=0} = 2r^2\frac{\partial}{\partial r}\,\mathrm{Re}\,G_1(\varsigma),$$

where $A_0(\varsigma) = A(\varsigma, 0)$. Hence

$$\begin{aligned}&-\frac{1}{2}\frac{\partial}{\partial\lambda}\frac{\partial U}{\partial r}\bigg|_{\lambda=0,r=1} \\ &\quad = \left\{\frac{1}{8}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - 4\,\mathrm{Re}\,G_1(\varsigma) - 2\frac{\partial}{\partial r}\,\mathrm{Re}\,G_1(\varsigma)\right\}\bigg|_{|\varsigma|=1}.\end{aligned} \tag{I.4}$$

We recall the relations in [**16**], that is,

$$2\,\mathrm{Re}\,G_1(\varsigma) = \tfrac{1}{8}\{|C_0|^2 + 2\,\mathrm{Re}(-g_N'(0)C_0\varsigma + \bar{C}_0 I_0(\varsigma)) + |-g_N'(0) + \varsigma I_0(\varsigma)|^2\}$$

and

$$A_0(\varsigma) = -g_N'(0) + \varsigma I_0(\varsigma) + C_0\varsigma,$$

where $C_0 \in \mathbf{C}$ is a constant and

$$I_0(\varsigma) = \int_0^\varsigma (g_N'(\hat{\varsigma}) - g_N'(0))\,\frac{d\hat{\varsigma}}{\hat{\varsigma}^2}.$$

Here, g_N is normalized as $g_N'(0) > 0$ so that

$$2\,\mathrm{Re}\,G_1(\varsigma) = \tfrac{1}{8}|A_0(\varsigma)|^2 \quad (\text{on } |\varsigma| = 1). \tag{I.5}$$

Furthermore, $G_1 = G_1(\varsigma)$ is holomorphic in D and hence

$$-\int_{\partial D}\frac{\partial}{\partial r}\,\mathrm{Re}\,G_1\,ds = -\int_D \Delta(\mathrm{Re}\,G_1)\,dx = 0. \tag{I.6}$$

Therefore, the relation (I.1) holds with

$$\begin{aligned}C &= -\int_{\partial D}\frac{\partial}{\partial\lambda}\left(\frac{\partial U}{\partial r}\right)\bigg|_{r=1} ds \\ &= \int_{\partial D}\left\{\frac{1}{4}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - \frac{1}{2}|A_0(\varsigma)|^2\right\} ds.\end{aligned} \tag{I.7}$$

Setting $A_0(\varsigma) = \sum_{n=0}^{\infty} b_n \varsigma^n$, we have

$$\int_{\partial D} |A_0(\varsigma)|^2 \, ds = \sum_{n,m=0}^{\infty} \int_0^{2\pi} b_n \bar{b}_m e^{i(n-m)\theta} \, d\theta = 2\pi \sum_{n=0}^{\infty} |b_n|^2.$$

Similarly,

$$\int_{\partial D} \frac{\partial}{\partial r} |A_0(\varsigma)|^2 \, ds = \sum_{\substack{n,m=0 \\ n+m \geq 1}}^{\infty} (n+m) b_n \bar{b}_m \int_0^{2\pi} e^{i(n-m)} \, d\theta = 4\pi \sum_{n=1}^{\infty} n |b_n|^2,$$

so that

$$C = \left\{ -|b_0|^2 + \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right\} \pi. \tag{I.8}$$

By virtue of $g_N(\varsigma) = \sum_{n=0}^{\infty} a_n \varsigma^n$ with $a_2 = 0$, we have

$$\begin{aligned} A_0(\varsigma) &= -g_N'(0) + C_0 \varsigma + \varsigma I_0(\varsigma) \\ &= -g_N'(0) + C_0 \varsigma + \sum_{n=2}^{\infty} a_{n+1} \frac{n+1}{n-1} \varsigma^n. \end{aligned}$$

Hence $b_0 = -g_N'(0) = -a_1$ and $b_n = a_{n+1}(n+1)/(n-1)$ $(n \geq 2)$. Thus, (2.10) follows. □

Appendix II.

PROOF OF PROPOSITION 1. Taking some constant ξ in $0 < \xi < 1$, we put $g_\xi(\varsigma) = (g_N(\varsigma) - g_N(0))/\xi g_N'(0)$ and $f_\xi(\varsigma) = \varsigma g_\xi'(\varsigma) = \sum_{n=0}^{\infty} d_n \varsigma^n$. Since g_n is univalent in D, so is also g_ξ. Then we have

$$d_n = \frac{1}{n!} f_\xi^{(n)}(0) = n \frac{1}{n!} g_\xi^{(n)}(0) = \frac{n}{\xi} \cdot \frac{a_n}{a_1}.$$

In particular, $d_0 = d_2 = 0$ and $d_1 = 1/\xi$. The relation $C < 0$ follows from

$$\sum_{n=1}^{\infty} \frac{1}{n} |d_{n+2}|^2 \left(= \frac{1}{\xi^2} \sum_{n=3}^{\infty} \frac{n^2}{n-2} \left| \frac{a_n}{a_1} \right|^2 \right) \leq 1. \tag{II.1}$$

We consider the function

$$w_\xi(\varsigma) = \frac{1}{\varsigma} + \sum_{n=1}^{\infty} c_n \varsigma^n,$$

which is holomorphic in $0 < |\varsigma| < 1$, where $c_n = -d_{n+2}/n$. When w_ξ is univalent, the desired inequality (II.1) follows from the area theorem [**18**, p. 210];

$$\sum_{n=1}^{\infty} n |c_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n} |d_{n+2}|^2 \leq 1.$$

The image Γ_r of $c_r = \{|z| = r\}$ $(0 < r < 1)$ by w_ξ is a closed curve. The univalentness of w_ξ follows if Γ_r is a Jordan curve and the winding number of the mapping $\varsigma \in c_r \mapsto w_\xi(\varsigma) \in \Gamma_r$ is -1 for each r close to 1.

In fact, let $\bar{\mathbf{C}}$ be the Riemann sphere $\mathbf{C} \cup \{\infty\}$ and $\mathscr{S}: \mathbf{C} \to \bar{\mathbf{C}}$ be the canonical injection. The pole $\varsigma = 0$ of w_ξ is first order, and hence w_ξ extends conformally as

$\mathscr{S} \circ w_\xi \colon D \to \bar{\mathbf{C}}$. From the above assumption, we can take a mapping $\mathscr{T} \colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, which is nothing but a rotation of the Riemann sphere, so that the image of $\mathscr{T} \circ \mathscr{S} \circ w_\xi \colon D_r = \{|z| < r\} \to \bar{\mathbf{C}}$ does not contain ∞. Therefore, the mapping $\mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi$ is holomorphic in D_r with the image Ω_r surrounded by a Jordan curve γ_r, where $\mathscr{S}_{-1} \colon \bar{\mathbf{C}} \backslash \{\infty\} \to \mathbf{C}$ denote the canonical projection.

This time, the winding number of

$$\varsigma \in c_r \mapsto \mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi(\varsigma) \in \gamma_r$$

is $+1$ and $\mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi$ is univalent in D_r from Darboux's theorem. Therefore, the same is true for w_ξ in $0 < |\varsigma| < 1$, because r can be taken arbitrarily close to 1.

Now, the relation

$$\text{(II.2)} \qquad w'_\xi(\varsigma) = -\frac{1}{\varsigma^3} f_\xi(\varsigma) + \left(\frac{1}{\xi} - 1\right) \frac{1}{\varsigma^2} = -\frac{1}{\varsigma^2} g'_\xi(\varsigma) + \left(\frac{1}{\xi} - 1\right) \frac{1}{\varsigma^2}$$

is derived from $d_0 = d_2 = 1$ and $d_1 = 1/\xi$. In fact, we have

$$(w_\xi(\varsigma) - 1/\varsigma)' = -\varsigma^{-3}(f_\xi(\rho) - d_1\varsigma).$$

Therefore, for $\varsigma = re^{i\theta}$ $(0 \le \theta \le 2\pi)$ we have

$$\frac{\partial}{\partial\theta} w_\xi(re^{i\theta}) = i\varsigma w'_\xi(\varsigma) = -\frac{1}{\varsigma^2}(i\varsigma h'_\xi(\varsigma)) = -\frac{1}{\varsigma^2}\frac{\partial}{\partial\theta} h_\xi(re^{i\theta}),$$

where $h_\xi(\varsigma) = g_\xi(\varsigma) + (1 - 1/\xi)\varsigma$. Hence we get the relation

$$\text{(II.3)} \qquad S_{r,\xi}(\theta) = e^{-2i\theta} T_{r,\xi}(\theta),$$

where

$$S_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} w_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} w_\xi(re^{i\theta})\right| \in S^1$$

and

$$T_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} h_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} h_\xi(re^{i\theta})\right| \in S^1.$$

The holomorphic function $g_\xi = g_\xi(\varsigma)$ is univalent for each $\xi > 0$, so that the winding number of $\varsigma = re^{i\theta} \in c_r \mapsto \tilde{T}_{r,\xi}(\theta) \in S^1$ is equal to $+1$, where

$$\tilde{T}_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} g_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} g_\xi(re^{i\theta})\right|.$$

Therefore, that of $\varsigma = re^{i\theta} \in c_r \mapsto T_{r,\xi}(\theta) \in S^1$ is also $+1$ whenever ξ in $0 < \xi < 1$ is close to 1. Consequently, the winding number of $\varsigma = re^{i\theta} \in c_r \mapsto S_{r,\xi}(\theta) \in S^1$ is equal to -1 by (II.3) when w_ξ is one-to-one on c_r. In this way, we have shown that (II.1) holds if w_ξ is one-to-one on $c_r = \{|z| = r\}$ when ξ and r in $(0, 1)$ are close to 1.

A simple sufficient condition for that is

$$\frac{\partial}{\partial\theta}(\operatorname{Arg} S_{r,\xi}(\theta)) < 0 \qquad (0 \le \theta < 2\pi),$$

namely,

$$\text{(II.4)} \qquad \frac{\partial}{\partial\theta}(\operatorname{Arg} T_{r,\xi}(\theta)) < 2 \qquad (0 \le \theta < 2\pi).$$

When ξ and r in $(0,1)$ are close to 1, (II.4) is implied by

$$\frac{\partial}{\partial\theta}(\operatorname{Arg} T(\theta)) < 2 \qquad (0 \le \theta < 2\pi), \tag{II.5}$$

where $T(\theta) = T_{1,1}(\theta) = g_N'(e^{i\theta})/|g_N'(e^{i\theta})| \in S^1$.

The unit tangent vector e_1 of $\partial\Omega$ at $g_N(e^{i\theta})$ is nothing but $T(\theta)$, and hence

$$e_1(l) = \begin{pmatrix} \cos t(\theta) \\ \sin t(\theta) \end{pmatrix},$$

where $t(\theta) = \operatorname{Arg} T(\theta)$ and $l = \int_0^\theta |g_N'(e^{i\omega})|\, d\omega$ represents the length parameter along $\partial\Omega$. Therefore, the inner unit normal vector on $\partial\Omega$ becomes

$$e_2(l) = \begin{pmatrix} \cos(t(\theta) + \pi/2) \\ \sin(t(\theta) + \pi/2) \end{pmatrix} = \begin{pmatrix} -\sin t(\theta) \\ \cos t(\theta) \end{pmatrix}.$$

Hence

$$e_1'(l) = \begin{pmatrix} -\sin t(\theta) \\ \cos t(\theta) \end{pmatrix} t'(\theta)\, \frac{d\theta}{dl} \qquad (= \kappa e_2(l)),$$

so that $t'(\theta) = \kappa|g_N'|$. In other words, the condition $\kappa|g_N'| < 2$ (on ∂D) implies $C < 0$. □

Appendix III.

PROOF OF PROPOSITION 2. Let $h = {}^T(u, \lambda)$ solve (P) and $S = \lambda \int_\Omega e^u\, dx$. For $t > 0$, set $\Omega_t = \{u > t\}$ and $\Gamma_t = \{u = t\}$. Then, by Green's formula we have

$$D(t) \equiv \int_{\Omega_t} \lambda e^u\, dx = -\int_{\Omega_t} \Delta u\, dx = -\int_{\Gamma_t} \frac{\partial u}{\partial n}\, ds = \int_{\Gamma_t} |\nabla u|\, ds. \tag{III.1}$$

On the other hand, from the co-area formula [**3**, p. 53] follows

$$D(t) = \int_t^\infty dr \int_{\Gamma_r} \lambda e^u \frac{1}{|\nabla u|}\, ds = \lambda \int_t^\infty e^r\, dr \int_{\Gamma_r} \frac{ds}{|\nabla u|},$$

and hence

$$\int_{\Gamma_t} \frac{ds}{|\nabla u|} = -\frac{1}{\lambda} D'(t) e^{-t}. \tag{III.2}$$

From these identities we obtain

$$-\frac{1}{\lambda} D'(t) D(t) e^{-t} \ge \left(\int_{\Gamma_t} ds\right)^2 = |\Gamma_t|^2. \tag{III.3}$$

Next, we have

$$\begin{aligned} |\Omega_t| &= \int_{\Omega_t} 1 \cdot dx = \int_t^\infty dr \int_{\Gamma_r} \frac{ds}{|\nabla u|} = -\frac{1}{\lambda} \int_t^\infty D'(r) e^{-r}\, dr \\ &= \frac{1}{\lambda} D(t) e^{-t} - \frac{1}{\lambda} \int_t^\infty D(r) e^{-r}\, dr. \end{aligned} \tag{III.4}$$

Combining (III.3), (III.4) with the isoperimetric inequality

$$|\Omega_t| \le |\Gamma_t|^2/4\pi,$$

we get

$$D(t) - \int_t^\infty e^{(t-s)} D(s)\, ds \le -\frac{1}{4\pi} D(t) D'(t) \equiv g(t) \qquad (\ge 0).$$

Let $H(t) = \int_t^\infty e^{(t-s)} D(s)\, ds$. Then, $-H'(t) = D(t) - H(t) \le g(t)$ so that $H(t) \le \int_t^\infty g(s)\, ds$. But $D(t) - H(t) \le g(t)$ or

$$D(t) \le g(t) + \int_t^\infty g(s)\, ds = -\frac{1}{4\pi} D(t) D'(t) + \frac{1}{8\pi} D(t)^2.$$

Therefore, $8\pi - D(t) \le -2D'(t)$ or

$$4\pi e^{-t/2} \le -(e^{-t/2} D(t))'. \tag{III.5}$$

Let $t_0 = ||u||_{c^0(\bar{\Omega})}$. Then,

$$\int_0^{t_0} 4\pi e^{-t/2}\, dt = 8\pi(1 - e^{-t_0/2}) \le -[e^{-t/2} D(t)]_{t=0}^{t=t_0} = S,$$

because $D(0) = S$ and $D(t(0)) = 0$. Hence we obtain

$$t_0 = ||u||_{C^0(\bar{\Omega})} \le -2\log(1 - S/8\pi).$$

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