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A CANONICAL SUBSPACE OF $H^*(BO)$ AND ITS APPLICATION TO BORDISM

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ABSTRACT. A particularly nice canonical subspace of $H^*(BO)$ is defined. The bordism class of a map $f: X \to Y$, where X and Y are compact, closed manifolds, can be determined by the characteristic numbers corresponding to elements of this subspace, and these numbers can be easily calculated. As an application, we study the "fixed-point manifold" of a parameter family of self-maps $F: M \times X \to X$, thus refining to bordism the usual homological analysis of the diagonal which is the basis of the standard Lefschetz fixed point theorem.

1. Introduction. By the work of Thom [**16**], the unoriented bordism class of a manifold is specified by its characteristic numbers. Brown and Peterson [**2**] later introduced a right action of the Steenrod algebra $\mathfrak{A}$ on the Z_2 cohomology $H_*(BO; Z_2)$. (In this paper all cohomology will be with coefficients in Z_2.) In these terms, Thom's results may be formulated as follows:

THEOREM 1.1 (Thom, Brown-Peterson). *$H^*(BO)$ is a free right $\mathfrak{A}$-module. Let $\{x_i\}$ be a basis. Two compact closed manifolds M_1 and M_2 are cobordant if and only if $\langle f^*(x_i), [M_i]\rangle = \langle g^*(x_i), [M_2]\rangle$ for all i, where $f: M_1 \to BO$ (resp. $g: M_2 \to BO$) is the classifying map for the tangent bundle $T_*(M_1)$ over M_1 (resp. $T_*(M_2)$ over M_2).*

More generally, Atiyah [**1**] later introduced the unoriented bordism group $\mathfrak{N}_n(X)$. This consists of equivalence classes of maps $F: M^n \to X$. In this context the corresponding theorem is

THEOREM 1.2. (*See Conner and Floyd* [**6**].) *Maps $F_1: M_1 \to X$, $F_2: M_2 \to X$ are cobordant if and only if*

$$\langle f^*(x_i) \cdot F_1^*(y), [M_1]\rangle = \langle g^*(x_i) \cdot F_2^*(y), [M_2]\rangle$$

for all i and all $y \in H^*(X)$, where $\{x_i\}$, f and g are as above.

In this paper a particularly nice canonical subspace $\mathbf{S}$ of $H^*(BO)$ is defined, with the following properties:

1. $\mathbf{S}$ is a sub-Hopf algebra of $H^*(BO)$;
2. The elements of $\mathbf{S}$ span $H^*(BO)$ under the right action of $\mathfrak{A}$, and so any basis $\{x_i\}$ of $\mathbf{S}$ as a vector space over Z_2 may be used as the basis in Theorems 1.1 and 1.2;
3. $\mathbf{S}$ is closed under the standard (left) action of $\mathfrak{A}$ on $H^*(BO)$;

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4. If t is any characteristic class in $\mathbf{S}$, and E is an n-dimensional bundle whose structure group reduces to the braid group, then $t(E) = 0$;

5. More generally, for E as in (4) and F any bundle, $t(E \oplus F) = t(F)$;

6. $\mathbf{S}$ contains the polynomial algebra generated by the squares $(w_i)^2$;

7. $\mathbf{S}$ itself is a polynomial algebra.

By the work of F. Cohen [**5**] there is a natural isomorphism $\varepsilon\colon \mathfrak{N}_*^{\mathrm{Br}}(X) \to H_*(X; Z_2)$ of braid bordism of X (bordism of maps $M \to X$ with braid structure on the stable normal bundle $\nu(M)$). The natural isomorphism ε sends $[f\colon M \to X]$ to $f_*([M])$. ε induces a natural isomorphism $\mathfrak{N}_*^{\mathrm{Br}}(X) \otimes \mathfrak{N}_*(\mathrm{pt}) \to \mathfrak{N}_*(X)$ sending $[f\colon M \to X] \otimes [W]$ to $f_*([M]) \times [W]$.

In terms of the basis $\{x_i\}$ of $\mathbf{S}$ we derive in §5 an explicit and especially simple calculation of the class in $\mathfrak{N}_*^{\mathrm{Br}}(X) \otimes \mathfrak{N}_*(\mathrm{pt})$ associated with any map $F\colon M \to X$ in $\mathfrak{N}_*(X)$.

$\mathbf{S}$ is defined in terms of braid bundles. The map $B(\mathrm{Br}_\infty) \to BO$ is studied in §2 and the results of F. Cohen [**5**] about $H^*(B(\mathrm{Br}_\infty))$ are recalled. In §3 we define the sub-Hopf algebra $\mathbf{S} \subset H^*(BO)$ and study its relation to $H^*(BO)$ in §4.

In §5 the promised explicit formula for a class in $\mathfrak{N}_*(X)$ (resp. $\mathfrak{N}^*(X)$) as an element of $H_*(X) \otimes \mathfrak{N}_*(\mathrm{pt})$ (resp. $H^*(X) \otimes \mathfrak{N}^*(\mathrm{pt})$) is derived.

In §6 as an application we analyze the fixed point manifold of a parameter family of maps $F\colon M \times X \to X$. This is based on the explicit formula we will derive for the diagonal bordism class $[\Delta\colon M \to M \times M]$, where M is a compact smooth manifold of dimension n. Thus we will refine to bordism the homological analysis of the diagonal Δ in $H_n(M \times M)$ which is the basis of the standard Lefschetz fixed point theorem.

In §7 the low dimensional generators for $\mathbf{S}$ are given.

This paper elaborates on material in my thesis, *A generalization in cobordism of the Lefschetz fixed point theorem*, Polytechnic Institute of New York, 1984.

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2. Braid groups and the map $H^*(BO) \to H^*(B(\mathrm{Br}_\infty))$. The braid group Br_n is the fundamental group of the space of embeddings of the unordered set $\{1, 2, \ldots, n\} \hookrightarrow D^2$, the unit disk in R^2. An element can be represented as a homotopy class of "strands" beginning and ending at a fixed set of n points $p_1, p_2, \ldots, p_n \in D^2$. (For a complete discussion of the braid group, see K. Reidemeister, [**14**].) The "twisting" of the strands defines an element of Σ_n, the permutation group on n objects. If we represent the elements of Σ_n as permutation matrices, Σ_n is a subgroup of $O(n)$, the orthogonal group. The composition is a homomorphism, $i^{(n)}\colon \mathrm{Br}_n \to O(n)$.

By adding strands (mapping $\mathrm{Br}_n \to \mathrm{Br}_{n+k}$) one defines $\mathrm{Br}_\infty = \lim \mathrm{Br}_n$; similarly, by enlarging the defining matrices, we can define $O = \lim O(n)$, and obtain natural homomorphisms:

$$\begin{array}{ccc} \mathrm{Br}_n & \longrightarrow & \mathrm{Br}_\infty \\ \downarrow & & \downarrow \\ O(n) & \longrightarrow & O \end{array}$$

Both Br_n and O_n have algebra structures defined as follows: $\mathrm{Br}_n \times \mathrm{Br}_m \to \mathrm{Br}_{n+m}$ is defined by concatenation; the map $O(n) \times O(m) \to O(n+m)$ is defined

as follows:

$$(A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Br}_n \times \mathrm{Br}_m & \xrightarrow{i^{(n)} \times i^{(m)}} & O(n) \times O(m) \\ \downarrow & & \downarrow \\ \mathrm{Br}_{n+m} & \xrightarrow[i^{(n+m)}]{} & O(n+m) \end{array}$$

Consequently, passing to the limit, the maps $i^{(n)}$ define maps of the classifying spaces $B(\mathrm{Br}_n) \to BO(n)$ (see Milnor and Stasheff, [**12**]). Since the above diagram commutes, we have Hopf algebra structures on $H^*(B(\mathrm{Br}_\infty))$ and $H^*(BO)$ with induced Hopf algebra map $i^*\colon H^*(BO) \to H^*(B(\mathrm{Br}_\infty))$.

Given a bundle $\pi\colon E \to X$, Brown and Peterson [**2**] have defined a right action of the Steenrod algebra $\mathfrak{A}$ via $x \cdot \alpha = \varphi^{-1}(\chi(\alpha) \cdot \varphi(x))$ for $x \in H^*(X)$, where φ is the Thom isomorphism $\varphi\colon H^*(X) \to H^*(T(E))$ and $T(E)$ is the Thom space of E. Passing to the limit, one obtains right actions of $\mathfrak{A}$ on $H^*(BO)$ and $H^*(B(\mathrm{Br}_\infty))$. The map i^* commutes with this right action.

F. Cohen [**5**] has proved this fundamental result:

THEOREM 2.1. *The homomorphism $\theta\colon \mathfrak{A} \to H^*(B(\mathrm{Br}_\infty))$, defined by $\theta(\alpha) = 1 \cdot \alpha$, is an isomorphism.*

Note 1. A similar theorem was proven by S. Bullett [**4**, Theorem 3.1]. An exhaustive description of $H^*(B(\mathrm{Br}_\infty))$ is given by D. Fuks [**7**].

Note 2. In this regard, the work of Mahowald is illuminating [**8**, **9**] (see also Priddy, [**13**]). Mahowald shows that for the canonical map $f\colon \Omega^2 S^3 \to BO$, the Thom spectrum $T(f)$ is the $K(Z_2, 0)$ spectrum, so the cohomology of $T(f)$ is isomorphic to the Steenrod algebra as a module over the Steenrod algebra. Cohen [**5**, Lemma 2.1] has shown that $H^*(\Omega^2 S^3) \simeq H^*(B(\mathrm{Br}_\infty))$.

PROPOSITION 2.2. *θ is a map of coalgebras, i.e. if $\Delta(\alpha) = \sum \alpha' \otimes \alpha''$, then $\theta^*(1 \cdot \alpha) = \sum (1 \cdot \alpha') \otimes (1 \cdot \alpha'')$.*

PROOF. Let

$$\begin{array}{c} \xi^k \\ \downarrow \\ BO(k) \end{array}$$

be the universal $O(k)$ bundle. Then the Whitney sum map $BO(n) \times BO(m) \xrightarrow{\oplus} BO(n+m)$ induces the map

$$T(\xi^n) \wedge T(\xi^m) = T \begin{pmatrix} \xi^n \otimes 1 \oplus 1 \otimes \xi^m \\ \downarrow \\ BO(n) \times BO(m) \end{pmatrix} \xrightarrow{\hat{\oplus}} T(\xi^{n+m}),$$

where $T(\xi)$ is the Thom space of ξ.

Now in cohomology, this induces

$$\hat{\oplus}^*\colon H^*(T(\xi^{n+m})) \to H^*(T(\xi^n)) \wedge H^*(T(\xi^m)),$$

with $\widehat{\oplus}^*(u_{\xi^{n+m}}) = u_{\xi^n} \wedge u_{\xi^m}$. The element $1 \cdot \alpha$ is defined by the relation $\pi^*(1 \cdot \alpha) \cup u_{\xi^{n+m}} = \chi(\alpha) \cdot u_{\xi^{n+m}}$. By the naturality of the Steenrod operations,

$$\begin{aligned}\widehat{\oplus}^*(1 \cdot \alpha) \cup u_{\xi^{n+m}} &= \widehat{\oplus}^*(\chi(\alpha) \cdot u_{\xi^{n+m}}) = \chi(\alpha) \cdot \widehat{\oplus}^*(u_{\xi^{n+m}}) \\ &= \chi(\alpha) \cdot (u_{\xi^n} \wedge u_{\xi^m}) = \sum [\chi(\alpha') \cdot u_{\xi^n}] \wedge [\chi(\alpha'') \cdot u_{\xi^m}],\end{aligned}$$

where $\Delta(\alpha) = \sum \alpha' \otimes \alpha''$. So this last equals

$$\begin{aligned}&\sum \pi^*(1 \cdot \alpha') \cup u_{\xi^n} \wedge \pi^*(1 \cdot \alpha'') \cup u_{\xi^m} \\ &\quad = \sum [\pi^*(1 \cdot \alpha') \wedge \pi^*(1 \cdot \alpha'')] \cup [u_{\xi^n} \wedge u_{\xi^m}],\end{aligned}$$

i.e. $\widehat{\oplus}^*(1 \cdot \alpha) = \sum (1 \cdot \alpha') \wedge (1 \cdot \alpha'')$. □

COROLLARY 2.3. *All primitives of $H^*(B(\mathrm{Br}_\infty))$ are in dimension $2^i - 1$.*

PROOF. The corresponding fact about $\mathfrak{A}$ is proved in Milnor [**10**].

Note. A more direct way of seeing the above result is to note that $H_*(B(\mathrm{Br}_\infty))$ is a primitively generated Hopf algebra with algebra generators concentrated in dimension $2^i - 1$. Thus, the primitives in $H^*(B(\mathrm{Br}_\infty))$ are given by one copy of Z_2 in every dimension of the form $2^i - 1$.

3. Definition and properties of S. Let I denote the ideal in $H_*(BO)$ generated by $i_*(\widetilde{H}_*(B(\mathrm{Br}_\infty)))$ (elements of dimension > 0). Then the projection $\pi\colon H_*(BO) \to H_*(BO)/I$ is a map of Hopf algebras. We define: $\mathbf{S} = \{H_*(BO)/I\}^*$, the dual of $H_*(BO)/I$.

PROPOSITION 3.1. 1. $\mathbf{S}$ *is a Hopf algebra;*

2. *π induces a monomorphism $\pi^*\colon \mathbf{S} \hookrightarrow H^*(BO)$ of Hopf algebras;*

3. *The composite homomorphism $\mathbf{S} \hookrightarrow H^*(BO) \xrightarrow{i^*} H^*(B(\mathrm{Br}_\infty))$ is the zero homomorphism.*

PROOF. 1. This is a standard result about Hopf algebras (see e.g. [**11**]).

2. The functor $\mathrm{Hom}(-, k)$ is right exact for any field k.

3. This follows from the definition of $\mathbf{S}$. □

Note. $\mathbf{S}$ can in fact be realized by the cohomology of a loop space: let $\eta\colon S^3 \to B^3O$ represent a generator of $\pi_3(B^3O) \simeq Z_2$, and let F be the homotopy theoretic fiber $F \to S^3 \xrightarrow{\eta} B^3O$. There is an induced fibration $\Omega^2 S^3 \xrightarrow{\Omega^2\eta} BO \to \Omega F$. Since $(\Omega^{2\eta})_*$ is monic on homology, the Serre spectral sequence collapses. Thus $1 \to H_*(\Omega^2 S^3) \to H_*(BO) \to H_*(\Omega F) \to 1$ is a short exact sequence in the category of Hopf algebras. Dually, $H^*(\Omega F)$ is the Hopf kernel of $(\Omega^{2\eta})^*\colon H^*(BO) \to H^*(\Omega^2 S^3)$. Now, by [**5**], there is a homotopy commutative diagram

$$\begin{tikzcd} B(\mathrm{Br}_\infty) \arrow[r] \arrow[d, "\theta"'] & BO \\ \Omega^2 S^3 \arrow[ur, dashed, "\Omega^2\eta"'] & \end{tikzcd}$$

where θ is a homology isomorphism. Under the isomorphism $\theta^*\colon H^*(\Omega^2 S^3) \to H^*(B(\mathrm{Br}_\infty))$ this same Hopf kernel is identified with $\mathbf{S}$, the Hopf kernel of

$$(\Omega^2\eta \circ \theta)^*\colon H^*(BO) \to H^*(B(\mathrm{Br}_\infty)).$$

The author would like to thank F. Cohen for pointing this out.

From now on we will identify $\mathbf{S}$ with its isomorphic image in $H^*(BO)$. The elements of the sub-Hopf algebra $\mathbf{S} \subset H^*(BO)$ may be characterized in terms of the map

$$\mu\colon B(\mathrm{Br}_\infty) \times BO \to BO \times BO \xrightarrow{\oplus} BO$$

as follows:

PROPOSITION 3.2. *If $t \in H^*(BO)$, then $t \in \mathbf{S}$ iff $\mu^*(t) = 1 \otimes t$ in*

$$H^*(B(\mathrm{Br}_\infty)) \otimes H^*(BO).$$

PROOF. By definition, $t \in \mathbf{S} \subset H^*(BO)$ iff $\langle t, \mu_*(r \otimes s)\rangle = 0$ for all $r \in \widetilde{H}_*(B(\mathrm{Br}_\infty))$ and $s \in H_*(BO)$. That is, iff $\langle \mu^*(t), r \otimes s\rangle = 0$ for all such r, s. Since generally, $\bigoplus^*(t) = t \otimes 1 + \sum t' \otimes t'' + 1 \otimes t$ with $\dim(t')$, $\dim(t'') > 0$, this can only be so if $\mu^*(t) = 1 \otimes t$. □

An immediate consequence of Proposition 3.2 is that $t(E \oplus F) = t(F)$ for any characteristic class $t \in \mathbf{S}$ and E a braid bundle. (Property 5 stated in the introduction.) Another immediate consequence is property 4 ($\mathbf{S}$ is closed under the left action of $\mathfrak{A}$) since $\mu^*(\alpha \cdot t) = \alpha \cdot (1 \otimes t) = \sum(\alpha' \cdot 1) \otimes (\alpha'' \cdot t) = 1 \otimes (\alpha \cdot t)$, where $\Delta(\alpha) = \sum \alpha' \otimes \alpha''$ in $\mathfrak{A}$.

Using Proposition 3.2, we may exhibit some elements of $\mathbf{S}$:

PROPOSITION 3.3. *Every primitive element $x \in H^*(BO)$ with $\dim(x) \neq 2^i - 1$ is in $\mathbf{S}$.*

PROOF. If x is primitive, $\bigoplus^*(x) = x \otimes 1 + 1 \otimes x$. We have to show that $i^*(x)$ vanishes in $H^*(B(\mathrm{Br}_\infty))$. But $i^*(x)$ is primitive in $H^*(B(\mathrm{Br}_\infty))$. By Corollary 2.3, it vanishes. □

Similarly, we prove

PROPOSITION 3.4. *$\mathbf{S}$ contains the polynomial algebra generated by the squares $(w_i)^2$.*

PROOF. $\Delta(w_1) = w_1 \otimes 1 + 1 \otimes w_1$ so $(w_1)^2$ is a primitive in $H^*(BO)$ of dimension $\neq 2^i - 1$. Therefore, by Proposition 3.2, $(w_1)^2 \in \mathbf{S}$.

Suppose inductively that $(w_i)^2 \in \mathbf{S}$ for $i < n$. In other words, $i^*((w_i)^2) = 0$ for $i < n$. Then,

$$\begin{aligned}\Delta(i^*((w_n)^2)) &= i^*\left(\sum (w_i)^2 \otimes (w_{n-i})^2\right)\\ &= i^*((w_n)^2) \otimes 1 + 1 \otimes i^*((w_n)^2),\end{aligned}$$

so $i^*((w_n)^2)$ is primitive in $H^*(B(\mathrm{Br}_\infty))$. Hence $i^*((w_n)^2) = 0$, by Corollary 2.3. But then, since $i^*((w_i)^2) = 0$ for $i \leq n$,

$$\mu^*((w_n)^2) = (i^* \otimes 1)(\Delta(w_n)^2) = (i^* \otimes 1)\left(\sum (w_i)^2 \otimes (w_{n-i})^2\right) = 1 \otimes (w_n)^2.$$

By Proposition 3.2, this shows that $(w_n)^2 \in \mathbf{S}$. □

Note 1. Since $\mathbf{S}$ is a subalgebra of $H^*(BO)$, $\mathbf{S}$ has no elements of finite height. Hence, by the Borel structure theorem [**11**], $\mathbf{S}$ is a polynomial algebra. (Property 7 of the introduction.)

Note 2. It is shown in §7 that there is one generator in each dimension $\neq 2^i - 1$, and the generators through dimension 10 are given there.

In general, if $f\colon X \to BO$ is the classifying map of the orthogonal bundle ξ over X, and $t \in \mathbf{S} \subset H^*(BO)$, we will write $f^*(t)$ as $t(\xi)$. In particular, if $f\colon M \to BO$ is the classifying map of the tangent bundle of M, where M is a smooth manifold, we will write $f^*(t)$ as $t(T_*(M))$.

PROPOSITION 3.5. *Let $g\colon N \to X$ where N is a closed braid manifold and X is a compact closed manifold. If $t \in \mathbf{S}$ and $x \in H^*(X)$ with* $\dim(x) < \dim(N) = n$, *then* $\langle g^*(x) \cdot t(T_*(N)), [N]\rangle = 0$.

PROOF. $\dim(x) + \dim(t) = n$, so $\dim(t) > 0$. Then, by Proposition 3.1, $t(T_*(N)) = 0$. □

The following result shows that certain characteristic numbers of some special maps can be easily calculated. In §4 we will show that these characteristic numbers completely determine the cobordism class of any map, and in §5 we will use the knowledge of the cobordism classes of these special maps to derive a formula for the cobordism class of any map.

THEOREM 3.6. *With $g\colon N \to X$ as above and $t \in \mathbf{S}$, if W is any compact closed smooth manifold, consider the composition*

$$g \circ \mathrm{pr}_2\colon W \times N \to N \to X,$$

where pr_2 *is the projection on the second factor. Then, for $x \in H^*(X)$,*

$$\begin{aligned}\langle (g \circ \mathrm{pr}_2)^*(x) \cdot t(T_*(W \times N)), [W \times N]\rangle \\ = \langle t(T_*(W)), [W]\rangle \cdot \langle g^*(x), [N]\rangle,\end{aligned}$$

if $\dim(x) = n$, 0 *otherwise.*

PROOF. Since $\mathbf{S}$ is a Hopf algebra, $\Delta(t) = t \otimes 1 + \sum t_i \otimes t_j$ with t_i, $t_j \in \mathbf{S}$ and $\dim(t_j) > 0$. Then,

$$\begin{aligned}&(g \circ p_2)^*(x) \cdot t(T_*(W \times N)) \\ &\qquad = t(T_*(W)) \otimes g^*(x) + \sum t_i(T_*(W)) \otimes g^*(x) \cdot t_j(T_*(N)).\end{aligned}$$

Thus,

$$\begin{aligned}&\langle (g \circ p_2)^*(x) \cdot t(T_*(W \times N)), [W \times N]\rangle \\ &\quad = \langle t(T_*(W)), [W]\rangle \cdot \langle g^*(x), [N]\rangle \\ &\qquad + \sum \langle t_i(T_*(W)), [W]\rangle \cdot \langle g^*(x) \cdot t_j(T_*(N)), [N]\rangle.\end{aligned}$$

But by Proposition 3.5, the right-hand terms are all zero if $\dim(x) < n$, and in that case so is the middle term. If $\dim(x) = n$, since $\dim(t_j) > 0$, the only nonzero term is the middle term, and that proves the proposition. □

Similarly, we prove

THEOREM 3.7. *Let $h\colon \widetilde{N} \to X$ be a map with $\widetilde{N}$ closed and $\nu(\widetilde{N}) \to X$ a braid bundle. (This is the stable normal bundle of $\widetilde{N}$ in X.) Then in the composition $h \circ \mathrm{pr}_2\colon W \times \widetilde{N} \to \widetilde{N} \to X$,*

$$\begin{aligned}\langle (h \circ \mathrm{pr}_2)^*(x) \cdot t(\nu(W \times \widetilde{N} \to X)), [W \times \widetilde{N}]\rangle \\ = \langle t(\nu(W)), [W]\rangle \cdot \langle h^*(x), [\widetilde{N}]\rangle,\end{aligned}$$

if $\dim(x) = \dim(\widetilde{N})$, 0 *otherwise.*

PROOF. As in the proof of Proposition 3.5, $\langle h^*(x) \cdot t(\nu(\widetilde{N} \to X)), [\widetilde{N}]\rangle = 0$ if $\dim(x) < \dim(\widetilde{N})$. Now $\nu(W \times \widetilde{N} \to X) = \nu(\widetilde{N} \to X) \oplus \nu(W)$.

The above proof can now be repeated. □

4. The relation between S and $H^*(BO)$. In this section we will show that a basis for $\mathbf{S}$ as a vector space over Z_2 is also a basis for $H^*(BO)$ as a right $\mathfrak{A}$-module. Recall that if $\pi\colon \xi \to BO(n)$ is the universal $O(n)$ bundle, and u_{ξ^n} the Thom class of ξ^n, then, for $x \in H^*(BO)$ and $\alpha \in \mathfrak{A}$, $x \cdot \alpha$ is defined by the relation

$$(x \cdot \alpha) \cup u_{\xi^n} = \chi(\alpha) \cdot (\pi^*(x) \cup u_{\xi^n}).$$

Stong [**15**, p. 94], following Milnor-Moore [**11**], proves the following result: If A is a connected Hopf algebra and M a connected coalgebra which is also a left module over A such that the diagonal map is an A-module map, and if $\nu\colon A \to M$ is a monomorphism, where ν is defined by $a \mapsto a \cdot 1$ with $1 \in M$ the counit of M, then M is isomorphic, as a coalgebra, to $A \otimes N$, where $N = M/\tilde{A}M$, $\tilde{A}$ being the elements of positive degree in A. Further, if $f\colon N \to M$ is any vector space splitting, the isomorphism is given by $a \otimes n \mapsto a \cdot f(n)$. From this result follows

PROPOSITION 4.1. $H_*(BO) = H_*(B(\mathrm{Br}_\infty)) \otimes \mathbf{S}^*$;

2. $H^*(BO) = (H^*(BO)/\widetilde{\mathbf{S}} \cdot H^*(BO)) \otimes \mathbf{S}$, *where* $\widetilde{\mathbf{S}}$ *is the set of elements of positive degree in* $\mathbf{S}$.

PROOF. For 1, take the monomorphism ν to be

$$i_*\colon H_*(B(\mathrm{Br}_\infty)) \to H_*(BO);$$

for 2, take ν to be

$$\mathbf{S} \hookrightarrow H^*(BO). \quad \square$$

Thus, in each dimension,

$$\begin{aligned}\operatorname{rank}(H^*(B(\mathrm{Br}_\infty)) &= \operatorname{rank}(H_*(B(\mathrm{Br}_\infty))) \\ &= \operatorname{rank}(H^*(BO)/\widetilde{\mathbf{S}} \cdot H^*(BO)).\end{aligned}$$

Now, the epimorphism $i^*\colon H^*(BO) \to H^*(B(\mathrm{Br}_\infty))$ can be factored:

$$H^*(BO) \xrightarrow{\pi} H^*(BO)/\widetilde{\mathbf{S}} \cdot H^*(BO) \xrightarrow{\hat{i}^*} H^*(B(\mathrm{Br}_\infty)),$$

where π is the projection, since $i^*|_{\widetilde{\mathbf{S}} \cdot H^*(BO)} = 0$. Therefore, $H^*(B(\mathrm{Br}_\infty)) \simeq H^*(BO)/\widetilde{\mathbf{S}} \cdot H^*(BO)$. But $\mathfrak{A} \simeq H^*(B(\mathrm{Br}_\infty))$ and the isomorphism $\alpha \mapsto 1 \cdot \alpha$ can be used as the above vector space splitting f. Thus we have:

PROPOSITION 4.2. $H^*(BO) \simeq \mathbf{S} \otimes \mathfrak{A}$ *under the map* $s \otimes \alpha \mapsto s \cup (1 \cdot \alpha)$.

The following theorem yields, as a corollary, property 2 of the introduction:

PROPOSITION 4.3. *The elements of* $\mathbf{S}$ *span* $H^*(BO)$ *under the right action of* $\mathfrak{A}$.

PROOF. Every element of $H^*(BO)$ is of the form $\sum s_i \cup (1 \cdot \alpha_i)$ with $s_i \in \mathbf{S}$, $\alpha_i \in \mathfrak{A}$. We want to show that every element $s \cup (1 \cdot \alpha)$ may be written as $\sum s_j \cdot \alpha_j$. We argue by induction on $\dim(\alpha)$. If $\dim(\alpha) = 0$, then $\alpha = 1$, which is of the

required form. Thus we can assume that we can write $s \cup (1 \cdot \alpha)$ as $\sum s_j \cdot \alpha_j$ whenever $\dim(\alpha) < n$. If $\dim(\alpha) = n$,

$$\begin{aligned}\pi^*(s \cdot \alpha) \cup u_\xi &= \chi(\alpha) \cdot (\pi^*(s) \cup u_\xi) \\ &= \sum (\chi(\alpha') \cdot \pi^*(s)) \cup (\chi(\alpha'') \cdot u_\xi),\end{aligned}$$

where $\Delta(\alpha) = \sum \alpha' \otimes \alpha''$. Now $\chi(\alpha') \in \mathbf{S}$ by Proposition 3.4, so:

$$\pi^*(s \cdot \alpha) \cup u_\xi = \sum \pi^*(s') \cup (\chi(\alpha'') \cup u_\xi).$$

But $\chi(\alpha'') \cup u_\xi = (1 \cdot \alpha'') \cup u_\xi$, so

$$\begin{aligned}\pi^*(s \cdot \alpha) \cup u_\xi &= \pi^*(s) \cup (1 \cdot \alpha) \cup u_\xi + \sum \pi^*(s') \cup (1 \cdot \alpha'') \cup u_\xi \\ &= \pi^*(s) \cup (1 \cdot \alpha) \cup u_\xi + \sum \pi^*(s'' \cdot \alpha''') \cup u_\xi,\end{aligned}$$

i.e. $s \cup (1 \cdot \alpha) = s \cdot \alpha + \sum s'' \cdot \alpha'''$. □

COROLLARY 4.4. *A vector space basis for* $\mathbf{S}$ *(over* Z_2*) is also a basis for* $H^*(BO)$ *as a right* $\mathfrak{A}$*-module.*

PROOF. A basis $\{x_\alpha\}$ for $\mathbf{S}$ gives a surjection $\mathbf{S} \otimes \mathfrak{A} \to H^*(BO)$ via $x_\alpha \otimes a \mapsto x_\alpha \cdot a$. This is an isomorphism since, in each dimension, $\operatorname{rank}(\mathbf{S} \otimes \mathfrak{A}) = \operatorname{rank}(H^*(BO))$ by Proposition 4.2. □

5. An explicit formula for the bordism class of a map. Let X be a C.W. complex and $\{\alpha_i\}$ a basis for $H_*(X)$. Choose braid manifolds N_i and maps $f_i \colon N_i \to X$ so that $(f_i)_*([N_i]) = \alpha_i$. The bordism class $[A_i] \otimes [f_i \colon N_i \to X]$ is the class of the map

$$A_i \times N_i \xrightarrow{\mathrm{pr}_2} N_i \xrightarrow{f_i} X,$$

which was analyzed in Theorem 3.6.

In terms of the representation of $\mathfrak{N}_*(X)$ as $\mathfrak{N}_*(\mathrm{pt}) \otimes H_*(X)$ described in the introduction, if $g \colon Y \to X$ is any map of a manifold Y of dimension n, the bordism class of g can be represented as

$$[g \colon Y \to X] = \sum [A_i] \otimes [f_i \colon N_i \to X],$$

with $\dim(\alpha_i) < n$, for some unique classes $[A_i] \in \mathfrak{N}_{n-\dim(\alpha_i)}$. The proposition below gives an explicit formula for $\{[A_i]\}$:

THEOREM 5.1. *If* $\{\beta_i\}$ *is the basis for* $H^*(X)$ *algebraically dual to* $\{\alpha_i\}$*, and* $t \in \mathbf{S}$*, then* $\langle t(T_*(Y)) \cdot g^*(\beta_j), [Y] \rangle = \langle t(T_*(A_j)), [A_j] \rangle$.

PROOF.

$$\begin{aligned}&\langle t(T_*(Y)) \cdot g^*(\beta_j), [Y] \rangle \\ &\quad = \sum \langle t(T_*(A_i \times N_i)) \cdot (f_i \circ \mathrm{pr}_2)^*(\beta_j), [A_i \times N_i] \rangle.\end{aligned}$$

But by Theorem 3.6, this equals

$$\sum \langle t(T_*(A_i)), [A_i] \rangle \cdot \langle f_i^*(\beta_j), [N_i] \rangle.$$

Now $\langle f_i^*(\beta_j), [N_i]\rangle = \langle \beta_j, (f_i)_*([N_i]\rangle = \langle \beta_j, \alpha_i\rangle = \delta_{ij}$. So there is only one nonzero term in the sum, and

$$\langle t(T_*(Y)) \cdot g^*(\beta_j), [Y]\rangle = \langle t(T_*(A_j)), [A_j]\rangle. \quad \square$$

Since a Z_2 basis for $\mathbf{S}$ is a right $\mathfrak{A}$-module basis for $H^*(BO)$, the above formula uniquely determines the $[A_i] \in \mathfrak{N}_*(\mathrm{pt})$.

Similarly, using Theorem 3.7, we can get the *cobordism* representation of g. If X is a manifold, we can take

$$\tilde{f}_i \colon \widetilde{N}_i \to X$$

to be maps of compact closed manifolds such that:

(a) $\alpha_i = (\tilde{f}_i)_*([\widetilde{N}_i])$, and

(b) $\nu(\tilde{f}_i \colon \widetilde{N}_i \to X)$ is a braid bundle (this is stably $(-T_*(\widetilde{N}_i) \oplus T_*(X))$).

The cobordism representation,

$$[g \colon Y \to X] = \sum [B_i] \otimes [\tilde{f}_i \colon \widetilde{N}_i \to X]$$

is similarly determined:

THEOREM 5.2. *With the $\{\beta_i\}$ and t as in the above proposition,*

$$\langle t(\nu(Y \to X))\rangle \cdot \langle g^*(\beta_i), [Y]\rangle = \langle t(\nu(B_i)), [B_i]\rangle.$$

6. Lefschetz-type invariants. Suppose that X is a compact, closed, smooth manifold of dimension n. Let $\{\alpha_i\}$ be a basis for $H_*(X)$. Choose closed manifolds $\widetilde{N}_i$ and maps $\tilde{f}_i \colon \widetilde{N}_i \to X$ such that:

(a) $$(\tilde{f}_i)_*([\widetilde{N}_i]) = \alpha_i, \text{ and}$$

(b) $$\nu(\tilde{f}_i \colon \widetilde{N}_i \to X) \text{ is stably braid.}$$

A number of bases for the homology and cohomology groups of X and M will be used in this section. They are related as follows: $\{\alpha_i\}$ is a basis for $H_*(X)$, $\{\tilde{\alpha}_i\}$ the corresponding Poincaré dual basis for $H^*(X)$, i.e. $\alpha_i = \tilde{\alpha}_i \cap [X]$ and $\{\tilde{\beta}_i\}$ the basis, again for $H^*(X)$, algebraically dual to $\{\alpha_i\}$, i.e. $\langle \tilde{\beta}_i, \alpha_j\rangle = \delta_{ij}$. Finally, $\{\beta_i\}$ is the algebraically dual basis to $\{\tilde{\alpha}_i\}$. It is not hard to see that $\{\beta_i\}$ and $\{\tilde{\alpha}_i\}$ are Poincaré duals also, but we will not use this fact. Similarly, $\{\sigma_i\}$ is a basis for $H_*(M)$, $\{\tilde{\sigma}_i\}$ the corresponding Poincaré dual basis for $H^*(M)$, and $\{\tilde{\tau}_i\}$ the basis for $H^*(M)$ algebraically dual to $\{\sigma_i\}$.

We can get the unique braid cobordism representation of the diagonal $\Delta \colon X \hookrightarrow X \times X$ as

$$\sum [B_{ij}] \otimes \tilde{\alpha}_i \otimes \tilde{\alpha}_j \quad \text{or} \quad \sum [B_{ij}] \otimes [\tilde{f}_i \colon \widetilde{N}_i \to X] \otimes [\tilde{f}_j \colon \widetilde{N}_j \to X],$$

where $[B_{ij}] \in \mathfrak{N}_{n-\dim(\alpha_i)-\dim(\alpha_j)}(\mathrm{pt})$ as follows:

The cobordism class represented by

$$[B_{ij}] \otimes [\tilde{f}_i \colon \widetilde{N}_i \to X] \otimes [\tilde{f}_j \colon \widetilde{N}_j \to X]$$

is the class of the composite map

$$B_{ij} \times \widetilde{N}_i \times \widetilde{N}_j \xrightarrow{\mathrm{pr}_{23}} \widetilde{N}_i \times \widetilde{N}_j \xrightarrow{\tilde{f}_i \times \tilde{f}_j} X \times X.$$

Thus, by Theorem 3.7,

$$\sum_{i',j'} \left\langle [(\tilde{f}_i \times \tilde{f}_j) \circ \mathrm{pr}_{23}]^*(\tilde{\beta}_{i'} \otimes \tilde{\beta}_{j'}) t(\nu(B_{ij} \times \widetilde{N}_i \times \widetilde{N}_j)), [B_{ij} \times \widetilde{N}_i \times \widetilde{N}_j] \right\rangle$$
$$= \sum_{i',j'} \langle t(\nu(B_{ij})), [B_{ij}] \rangle \cdot \langle (\tilde{f}_i)^*(\tilde{\beta}_{i'}), [\widetilde{N}_i] \rangle \cdot \langle (\tilde{f}_j)^*(\tilde{\beta}_{j'}), [\widetilde{N}_j] \rangle.$$

However, $\langle (\tilde{f}_i)^*(\tilde{\beta}_{i'}), [\widetilde{N}_i] \rangle = \delta_{i,i'}$ and $\langle (\tilde{f}_j)^*(\tilde{\beta}_{j'}), [\widetilde{N}_j] \rangle = \delta_{j,j'}$, so this sum reduces to the single term $\langle t(\nu(B_{ij})), [B_{ij}] \rangle$.

But by Theorem 1.2, this must equal

$$\langle \Delta^*(\tilde{\beta}_i \otimes \tilde{\beta}_j) \cdot t(\nu(X \hookrightarrow X \times X)), [X] \rangle.$$

Since $\nu(X \hookrightarrow X \times X) = T_*(X)$ and $\Delta^*(\tilde{\beta}_i \otimes \tilde{\beta}_j) = \tilde{\beta}_i \cup \tilde{\beta}_j$, we have the formula:

$$\langle t(\nu(B_{ij})), [B_{ij}] \rangle = \langle t(T_*(X)) \cdot \tilde{\beta}_i \cdot \tilde{\beta}_j, [X] \rangle.$$

As a cobordism class, $\hat{\Delta}\colon M \times X \to M \times X \times X$ defined by $(m, x) \mapsto (m, x, x)$ is then $1 \otimes \sum [B_{ij}] \otimes \tilde{\alpha}_i \cdot \tilde{\alpha}_j$.

If $F\colon M \times X \to X$ is a parameter family of self maps $f_m\colon X \to X$, we may define $\widehat{F}\colon M \times X \times X \to M \times X \times X$ by $\widehat{F}(m, x, y) = (m, F(m, x), y)$. If $\widehat{F}$ is transverse to $\hat{\Delta}$, $\widetilde{F}^*(1 \otimes \sum [B_{ij}] \otimes \tilde{\alpha}_i \otimes \tilde{\alpha}_j)$ represents the cobordism class of $\{(m, x, y) | F(m, x) = y\}$.

Thus, $\hat{\Delta}^* \circ \widehat{F}^*(1 \otimes \sum [B_{ij}] \otimes \tilde{\alpha}_i \otimes \tilde{\alpha}_j)$ represents the cobordism class of $\{(m, x) | F(m, x) = x\}$. Call this "fixed-point manifold" $\mathrm{Fix}(F)$.

To calculate the cobordism class of $\mathrm{Fix}(F) \subset M \times X$, note that the following diagram commutes:

$$\begin{array}{ccccc} M \times X & \xrightarrow{\hat{\Delta}} & M \times X \times X & \xrightarrow{\widehat{F}} & M \times X \times X \\ \downarrow & & & & \searrow \\ (M \times X) \times (M \times X) & & \underset{1 \otimes \mathrm{flip}_{23} \otimes 1_X}{\longrightarrow} & & M \times (M \times X) \times X \end{array}$$

Let $\{\sigma_i\}$ be a basis for $H_*(M)$, $\{\tilde{\sigma}_j\}$ the corresponding Poincaré dual basis for $H^*(M)$ and $\{\tilde{\tau}_j\}$ the algebraically dual basis, also for $H^*(M)$. Represent $\{\sigma_j\}$ by $\tilde{g}_j\colon \widetilde{M}_j \to M$ so that we have $\sigma_j = (\tilde{g}_j)^*([\widetilde{M}_j])$ and $\nu(\widetilde{M}_j \to M)$ stably braid. In terms of the basis $\{\tilde{\alpha}_i\}$ for $H^*(X)$ and $\{\tilde{\sigma}_j\}$ we have in cohomology:

$$F^*(\tilde{\alpha}_i) = \sum_{j,k} K_i^{j^k} \tilde{\sigma}_j \otimes \tilde{\alpha}_k.$$

Using the above diagram, we can calculate

$$\mathrm{Fix}(F) = \hat{\Delta}^* \circ \widehat{F}^* \left(1 \otimes \sum_{i,j} [B_{ij}] \otimes \tilde{\alpha}_i \otimes \tilde{\alpha}_j \right)$$
$$= \sum_{i,j} K_i^{pk} [B_{ij}] \otimes \tilde{\sigma}_p \otimes (\tilde{\alpha}_k \cup \tilde{\alpha}_j).$$

Now, let $\{\beta_q\}$ be the basis for $H_*(X)$ algebraically dual to $\{\tilde{\alpha}_q\}$.

$$\tilde{\alpha}_k \cup \tilde{\alpha}_j = \sum_q \langle \tilde{\alpha}_k \cup \tilde{\alpha}_j, \beta_q \rangle \cdot \tilde{\alpha}_q.$$

Then we have

$$\mathrm{Fix}(F) = \sum_{i,j,k,p,q} K_i^{pk} \cdot \langle \tilde{\alpha}_k \cup \tilde{\alpha}_j, \alpha_q \rangle \cdot [B_{ij}] \otimes \tilde{\sigma}_p \otimes \tilde{\alpha}_q.$$

We now define the Lefschetz-type invariant

$$\mathfrak{L}(F, t, \tilde{\gamma}, \tilde{\beta}) = \langle t(T_*(\mathrm{Fix}(F) \cdot f^*(\tilde{\gamma} \otimes \tilde{\beta}))), [\mathrm{Fix}(F)] \rangle,$$

where $\tilde{\gamma} \in H^*(M)$, $\tilde{\beta} \in H^*(X)$, $t \in \mathbf{S}$ and $f\colon \mathrm{Fix}(F) \hookrightarrow M \times X$ is the inclusion.

Thus we have

$$\begin{aligned}&\mathfrak{L}(F, t, \tilde{\tau}_r, \tilde{\beta}_s)\\ &\quad = \sum K_i^{pk} \cdot \langle \tilde{\alpha}_j \cdot \tilde{\alpha}_k, \alpha_q \rangle \cdot \langle t(\nu(B_{ij})), [B_{ij}] \rangle \cdot \langle (\tilde{g}_p)^*(\tilde{\tau}_r), [M_p] \rangle \cdot \langle (\tilde{f}_q)^*(\tilde{\beta}_s), [N_q] \rangle,\end{aligned}$$

where the sum is over indices i, j, k, p, q.

But $\langle (\tilde{g}_p)^*(\tilde{\tau}_r), [\widetilde{M}_p] \rangle = \delta_{r,p}$; $\langle (\tilde{f}_q)^*(\tilde{\beta}_s), [\widetilde{N}_q] \rangle = \delta_{q,s}$, so the above sum is simply

$$\mathfrak{L}(F, t, \tilde{\tau}_r, \tilde{\beta}_s) = \sum_{i,j,k} K_i^{rk} \langle \tilde{\alpha}_j, \tilde{\alpha}_k, \beta_s \rangle \cdot \langle t(\nu(B_{ij})), [B_{ij}] \rangle,$$

or

$$\mathfrak{L}(F, t, \tilde{\tau}_r, \tilde{\beta}_s) = \sum_{i,j,k} K_i^{rk} \langle \tilde{\alpha}_j \cdot \tilde{\alpha}_k, \beta_s \rangle \cdot \langle t(T_*(X)) \cdot \tilde{\beta}_i \cdot \tilde{\beta}_j, [X] \rangle.$$

7. The generators of S. To illustrate how generators of $\mathbf{S}$ may be constructed, we exhibit them up through dimension 10.

By Proposition 3.4, $\mathbf{S}$ contains the squares $(w_i)^2$, and by Proposition 3.3, $\mathbf{S}$ also contains any primitive elements of $H^*(BO)$. According to Milnor and Stasheff [**12**], $s_{(k)}$ is primitive in $H^*(BO)$ for any k. In particular, $s_{(5)}$, $s_{(9)}$ are in $\mathbf{S}$. Now, $\Delta(s_{(5,5)}) = s_{(5,5)} \otimes 1 + s_{(5)} \otimes s_{(5)} + 1 \otimes s_{(5,5)}$. Therefore, $(i^* \otimes i^*)(\Delta(s_{(5,5)})) = i^*(s_{(5,5)}) \otimes 1 + i^*(s_{(5)}) \otimes i^*(s_{(5)}) + 1 \otimes i^*(s_{(5,5)})$. Now, since $s_{(5)} \in \mathbf{S}$, $i^*(s_{(5)}) = 0$. Thus, $i^*(s_{(5,5)})$ is primitive in $H^*(B(\mathrm{Br}_\infty))$ and so, by the corollary to Proposition 1, it vanishes. We now compute

$$\begin{aligned}\mu(s_{(5,5)}) &= i^* \otimes 1(\Delta(s_{(5,5)}))\\ &= i^*(s_{(5,5)}) \otimes 1 + i^*(s_{(5)}) \otimes s_{(5)} + 1 \otimes s_{(5,5)} = 1 \otimes s_{(5,5)}.\end{aligned}$$

Thus, by Proposition 3.2, $s_{(5,5)}$ is in $\mathbf{S}$.

Now, $s_{(5)}$ and $s_{(9)}$, being primitive elements in $\mathbf{S}$ are generators of $\mathbf{S}$. There are no polynomial relations between $(w_1)^2$, $(w_2)^2$, $(w_3)^2$ and $(w_4)^2$, all of which are also in $\mathbf{S}$. Finally, using a formula of Brown and Peterson [**3**] for $\partial s_\omega / \partial \sigma_n$, we can show that $s_{(5,5)}$ is indecomposable, and is therefore a generator of $\mathbf{S}$.

We now assert that $\{(w_1)^2, (w_2)^2, s_{(5)}, (w_3)^2, (w_4)^2, s_{(9)}, s_{(5,5)}\}$ is a complete list of polynomial generators through dimension 10.

Since $H^*(BO)$ is a polynomial algebra with generators w_i, the Poincaré polynomial $\mathbf{P}_t(H^*(BO)) = \prod_n 1/(1 - t^n)$. Now since $\mathfrak{A}^*$ is a polynomial algebra in dimensions $= 2^i - 1$ (Milnor, [**10**]),

$$\mathbf{P}_t(\mathfrak{A}) = \prod_{n=2^i-1} \frac{1}{(1 - t^n)}.$$

Since $H^*(BO) \simeq \mathbf{S} \otimes \mathfrak{A}$, $\mathbf{P}_t(H^*(BO)) = \mathbf{P}_t(\mathbf{S}) \cdot \mathbf{P}_t(\mathfrak{A})$. Thus,

$$\mathbf{P}_t(\mathbf{S}) = \prod_{n \neq 2^i - 1} \frac{1}{(1 - t^n)}.$$

Consequently, $\mathbf{S}$ has exactly one polynomial generator in each dimension $\neq 2^i - 1$, so the above list is complete.

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