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SINGULARITIES OF THE SCATTERING KERNEL AND SCATTERING INVARIANTS FOR SEVERAL STRICTLY CONVEX OBSTACLES

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ABSTRACT. Let $\Omega \subset \mathbf{R}^n$ be a domain such that $\mathbf{R}^n \backslash \Omega$ is a disjoint union of a finite number of compact strictly convex obstacles with C^∞ smooth boundaries. In this paper the singularities of the scattering kernel $s(t, \theta, \omega)$, related to the wave equation in $\mathbf{R} \times \Omega$ with Dirichlet boundary condition, are studied. It is proved that for every $\omega \in S^{n-1}$ there exists a residual subset $\mathscr{R}(\omega)$ of S^{n-1} such that for each $\theta \in \mathscr{R}(\omega)$, $\theta \neq \omega$,

$$\operatorname{singsupp} s(t, \theta, \omega) = \{-T_\gamma\}_\gamma,$$

where γ runs over the scattering rays in Ω with incoming direction ω and with outgoing direction θ having no segments tangent to $\partial\Omega$, and T_γ is the sojourn time of γ. Under some condition on Ω, introduced by M. Ikawa, the asymptotic behavior of the sojourn times of the scattering rays related to a given configuration, as well as the precise rate of the decay of the coefficients of the main singularity of $s(t, \theta, \omega)$, is examined.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, n odd, be an open domain with C^∞ smooth boundary $\partial\Omega$ and bounded complement $K = \mathbf{R}^n \backslash \Omega \subset \{x : |x| \leq \rho_0\}$. The scattering operator S for the wave equation in $\mathbf{R} \times \Omega$ with Dirichlet boundary condition is a unitary operator from $L^2(\mathbf{R} \times S^{n-1})$ into $L^2(\mathbf{R} \times S^{n-1})$ and the kernel $s(t - t', \theta, \omega)$ of S-Id is called the scattering kernel (see [10, 11]). For fixed $(\theta, \omega) \in S^{n-1} \times S^{n-1}$ we have $s(t, \theta, \omega) \in \mathscr{S}'(\mathbf{R})$ and

$$s(t, \theta, \omega) = C_n \int_{\partial K} \partial_\tau^{n-2} \partial_\nu w(\langle x, \theta \rangle - t, x; \omega)\, dS_x. \tag{1.1}$$

Here $w(\tau, x; \omega)$ is the solution to the problem

$$\begin{cases} (\partial_\tau^2 - \Delta_x) w = 0 & \text{in } \mathbf{R} \times \Omega, \\ w = 0 & \text{on } \mathbf{R} \times \partial\Omega, \\ w|_{\tau < -\rho_0} = \delta(\tau - \langle x, \omega \rangle), \end{cases} \tag{1.2}$$

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ν is the interior unit normal to $\partial\Omega$ (pointing into Ω),

$$C_n = (-1)^{(n+1)/2} 2^{-n} \pi^{(1-n)},$$

and dS_x is the induced measure on ∂K. The integral (1.1) is interpreted in the sense of distributions (see [11, 26]).

Denoting by $\hat{s}(\lambda, \theta, \omega)$ the Fourier transform of $s(t, \theta, \omega)$ with respect to t,

$$a(\lambda, \theta, \omega) = (2\pi/i\lambda)^{(n-1)/2} \hat{s}(\lambda, \theta, \omega)$$

is called the scattering amplitude. The analysis of the asymptotic behavior of $a(\lambda, \theta, \omega)$ as $\lambda \to \infty$ is closely related to the examination of the singularities of $s(t, \theta, \omega)$.

For $\omega \neq \theta$ and for strictly convex obstacles $s(t, \theta, \omega)$ has only one singularity related to the reflecting ray at $\hat{x} \in \partial K$, $\hat{x}$ being the point where $\nu(\hat{x}) = (\theta - \omega)/\|\theta - \omega\|$. On the other hand, for nonconvex obstacles there exists $\omega \in S^{n-1}$ for which $s(t, -\omega, \omega)$ has at least two different singularities [26, 29].)

For obstacles with arbitrary geometry $\max_t \operatorname{singsupp} s(t, \theta, \omega)$ has been studied in [11, 17, 27]. One of the authors investigated in [16] the set of all singularities of $s(t, \theta, \omega)$ for nonconvex obstacles, making some restrictions on the geometry of the rays incoming with direction ω. These restrictions are too difficult for verification. Nevertheless, some of them are fulfilled for generic obstacles, provided ω, θ are fixed (see [20, 21]).

As suggested in [5, 16], $\operatorname{singsupp} s(-t, \theta, \omega)$ must be related to the sojourn times of the so-called (ω, θ)-rays (see §2 for a precise definition). It is natural to expect that for generic directions (ω, θ) the sojourn times of all ordinary (ω, θ)-rays belong to $\operatorname{singsupp} s(-t, \theta, \omega)$. A (ω, θ)-ray γ is called ordinary if γ has no segments tangent to ∂K. In general a nonconvex obstacle K could admit (generalized) rays with incoming direction ω and outgoing direction θ which have some gliding segments on ∂K. This leads to considerable difficulties when we try to prove the above assertion for nonconvex obstacles.

Throughout this paper we assume

$$(1.3) \qquad \begin{cases} K = \bigcup_{i=1}^{s} K_i, \ K_i \cap K_j = \varnothing \quad \text{for } i \neq j, \\ K_i \quad \text{are compact and strictly convex for } i = 1, \dots, s. \end{cases}$$

For fixed $\omega \neq \theta$, let $\mathscr{L}_{\omega,\theta}$ be the set of all ordinary (ω, θ)-rays with reflections on ∂K. Given $\gamma \in \mathscr{L}_{\omega,\theta}$, denote by x_γ (resp. by y_γ) the first (resp. the last) reflection point of γ. Let Z_ω be a fixed hyperplane so that K is contained in the open half-space H_ω determined by Z_ω and having ω as an inward normal. If γ hits Z_ω at A_γ, then in some neighborhood U_γ of A_γ we can define the map $J_\gamma : U_\gamma \ni u \to \theta(u)$, $\theta(u)$ being the outgoing direction of the $(\omega, \theta(u))$-ray issued from u (see [5, 16] and §3). Denote by T_γ the sojourn time of γ (see [5, 16] and §4 for the definition). Finally, recall that a subset

$\mathscr{R} \subset S^{n-1}$ is called residual if $\mathscr{R}$ is a countable intersection of open and dense subsets of S^{n-1}. Our main result is the following.

Theorem 1.1. *Let $\omega \in S^{n-1}$ be fixed. Then there exists a residual subset $\mathscr{R}(\omega) \subset S^{n-1}$ such that for each $\theta \in \mathscr{R}(\omega)$, $\theta \neq \omega$, we have*

$$\operatorname{sing\,supp} s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathscr{L}_{\omega,\theta}\}. \tag{1.4}$$

Moreover, near $-T_\gamma$ we have

$$\begin{aligned} s(t, \theta, \omega) = {} & (2\pi)^{(1-n)/2}(-1)^{m_\gamma - 1} i^{\sigma_\gamma} \\ & \times \left| \frac{\det dJ_\gamma(A_\gamma)\langle \nu(x_\gamma), \omega\rangle}{\langle \nu(y_\gamma), \theta\rangle} \right|^{-1/2} \delta^{(n-1)/2}(t + T_\gamma) \end{aligned} \tag{1.5}$$

+ smoother terms. Here m_γ is the number of reflections of γ, $\sigma_\gamma \in \mathbf{N}$ is a Maslov index, and $\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^n$.

Equality (1.4) is similar to the Poisson relation for the distribution $\sigma(t) = \sum_j \cos \lambda_j t$, where $\{\lambda_j^2\}_{j=1}^\infty$ are the eigenvalues of the Laplace-Beltrami operator on a compact manifold without boundary [3] or the eigenvalues of the Dirichlet problem for the Laplacian in a bounded domain $\mathscr{O} \subset \mathbf{R}^2$ (see [19, 22]). In these cases the singularities of $\sigma(t)$ are included in the union of the lengths of all periodic geodesics. For this reason we will call $\{T_\gamma : \gamma \in \mathscr{L}_{\omega,\theta}\}$ the scattering length spectrum of K related to ω, θ. Nakamura and Soga [15] established (1.4) for $\theta = -\omega$ and for two disjoint balls $\mathscr{O}_i$, $i = 1, 2$, making some restrictions on $\operatorname{dist}(\mathscr{O}_1, \mathscr{O}_2)$ and the diameters of $\mathscr{O}_i$, $i = 1, 2$.

Formula (1.5) was obtained in [16]. Similar results concerning the singularities of $\sigma(t)$ when t coincides with the period of some periodic geodesic have been proved in [3, 6]. Theorem 1.1 says that for fixed $\omega \in S^{n-1}$ and $\theta \in \mathscr{R}(\theta)$, $\theta \neq \omega$, from the singularities of $s(t, \theta, \omega)$ we can recover as scattering data all sojourn times T_γ, $\gamma \in \mathscr{L}_{\omega,\theta}$ together with the corresponding coefficients c_γ in front of $\delta^{(n-1)/2}(t + T_\gamma)$. On the other hand, from the scattering kernel we cannot determine when θ belongs to $\mathscr{R}(\omega)$.

The proof of Theorem 1.1 is based on three main points. First, we study the topological properties of the map J_α related to a fixed configuration α. We describe the maximal subset $M_\alpha \subset Z_\omega$, where $J_\alpha : M_\alpha \to J_\alpha(M_\alpha)$ becomes a homeomorphism. This leads to the uniqueness of the (ω, θ)-ray associated to α. Second, for generic θ we establish two properties of (ω, θ)-rays. One of them says that the sojourn times of different (ω, θ)-rays are different too. Third, to examine $\operatorname{sing\,supp} s(t, \theta, \omega) \cap [-T, T]$, we introduce a special localization of the problem (1.2) different from those previously used in [16, 17, 27]. This localization depends on T as well as on the results in §§3 and 4 playing a crucial role when we deal with the rays admitting tangent segments. Moreover, our localization enables us to eliminate completely the investigation of the mixed problems with data localized in the shadow domain with respect to ω (see [16]).

The results of Melrose and Sjöstrand [13, 14] imply that for nontrapping obstacles the sojourn times T_γ of all (ω, θ)-rays γ are uniformly bounded for all $(\omega, \theta) \in S^{n-1}$. For trapping obstacles we expect that for suitably chosen ω, θ we have

$$\sup\{T_\gamma : \gamma \in \mathscr{L}_{\omega,\theta}\} = \infty. \tag{1.6}$$

In our case we are able to establish (1.6) making the additional assumption

$$\text{(H)} \qquad \left[\begin{array}{l}\text{For all triples } (i_1, i_2, i_3) \in \{1, \dots, s\}^3,\ i_j \neq i_k,\ j \neq k, \\ \text{the convex hull of } K_{i_1} \cup K_{i_2} \text{ has no common points with } K_{i_3}.\end{array}\right.$$

This condition was introduced by Ikawa in [9], where he proved that for each configuration

$$\alpha = (i_1, \dots, i_k), \qquad i_1 \neq i_k,$$

there exists a unique periodic ray γ_α with successive reflection points $\tilde{x}_j \in \partial K_{i_j}$, $j = 1, \dots, k$. Under condition (H) we show in §6 that for each such α, choosing ω, θ, suitably for every $q \geq 0$ there exists an ordinary (ω, θ)-ray γ_m^{1l} with $m = qk + l$ reflection points

$$x_{pk+j} \in \partial K_{i_j}, \qquad 0 \leq p \leq q,\ 1 \leq j \leq k\ (1 \leq j \leq l \text{ for } p = q).$$

The choice of ω and θ depends on some condition of visibility concerning K_{i_1} and K_{i_l} which we are able to arrange exploiting (H). On the other hand, to guarantee that γ_m^{1l} are ordinary rays, we apply essentially our results in §4.

In our paper we study the asymptotic behavior of the sojourn times T_{qk+l}^{1l} of γ_{qk+l}^{1l} as $q \to \infty$. The leading term of the corresponding asymptotic is merely qd_α, d_α being the length of γ_α (see Theorem 6.5). Consequently, the scattering length spectrum $\{\mathscr{L}_{\omega,\theta} : \omega \neq \theta\}$ determines the lengths of all periodic rays γ_α. For this reason we could consider d_α as scattering invariants. In our previous paper [22] we proved that the lengths of γ_α can be recovered from the singularities of the distribution $\sum_j e^{it\tilde{\lambda}_j}$, $\tilde{\lambda}_j$ being the poles of the scattering matrix, provided that K is a generic obstacle. The above result says that we may obtain the same information from the scattering length spectrum.

Marvizi and Melrose [12] and Colin de Verdiére [2] studied the asymptotic behavior of the lengths of periodic geodesics for strictly convex planar regions $\mathscr{O} \subset \mathbf{R}^2$. In [12] these geodesics approximate the boundary $\partial\mathscr{O}$, while in [2] they approximate an elliptic periodic ray. In our situation γ_α is a hyperbolic ray and in the asymptotic of T_{qk+l}^{1l} we obtain two terms modulo a remainder $O(\delta^q)$ with $0 < \delta < 1$. For two disjoint balls some partial results have been obtained in [15].

In §7 we examine the precise rate of decay of the coefficients c_m^{1l} in front of $\delta^{(n-1)/2}(t + T_m^{1l})$ as $m \to \infty$. We obtain the asymptotic behavior of $\ln|c_{qk+l}^{1l}|$

as $q \to \infty$ with leading term

$$qc_\alpha = -\frac{1}{2} q \sum_{j=1}^{n-1} \ln|\mu_j| .$$

Here μ_j, $|\mu_j| > 1$, are the eigenvalues of the (linear) Poincaré map related to γ_α. Our analysis is based on the representation of the Poincaré map [18, 22] which makes it possible to simplify the calculation of $\det dJ_\gamma(A_\gamma)$ and the examination of the asymptotic of $\ln|c^{1l}_{qk+l}|$.

For two obstacles K_i, $i = 1, 2$, condition (H) is trivial. Then from the scattering length spectrum we can recover the invariants $d = \operatorname{dist}(K_1, K_2)$ and $c_d = -\sum_{j=1}^{n-1} \ln|\mu_j|$ related to the unique periodic ray. Moreover, we may determine the first sequence

$$\lambda_j = -(ic_d)/d + j(d/\pi), \qquad j \in Z,$$

of pseudopoles of the scattering matrix (see [1, 4, 7]). In this case λ_j are connected with the meromorphic continuation of the leading term

$$a_0(\lambda) = \sum_{\gamma \in \mathscr{L}_{\omega,\theta}} c_\gamma e^{-i\lambda T_\gamma}$$

of the scattering amplitude $a(-\lambda, \theta, \omega)$. It is interesting to study the analytic properties of $a_0(\lambda)$ for more than two obstacles.

The paper is organized as follows. In §2 some notation and preliminary facts are given. In §3 we study the map J_α, while in §4 we treat the (ω, θ)-rays for generic θ. §5 is devoted to the proof of Theorem 1.1. In §6 we discuss the existence of (ω, θ)-rays and property (1.6) together with the asymptotic of T^{1l}_{qk+l}. Finally, the asymptotic of $\ln|c^{1l}_{qk+l}|$ is examined in §7. Part of our results have been announced in [23].

2. Preliminaries

2.1. Let l_1 and l_2 be linear segments with a common end point $x \in \partial K$. We say that l_1 and l_2 satisfy the law of reflection at x with respect to ∂K if either l_1 and l_2 lie on a common line tangent to ∂K at x and $l_1 \cap l_2 = \{x\}$ (the case of tangency) or l_1 and l_2 make equal acute angles with $\nu(x)$ lying in a common two-dimensional plane with $\nu(x)$ (the case of a proper reflection).

Let $\omega, \theta \in S^{n-1}$ and let $\gamma = \bigcup_{i=0}^{k} l_i$ be a curve in $\mathbf{R}^n$, where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \dots, k-1$ $(k \geq 1)$, $x_i \in \partial K$ for $0 \leq i \leq k$, l_0 (resp. l_k) is an infinite segment starting at x_1 (resp. at x_k) and having direction $-\omega$ (resp. θ). Then γ will be called an (ω, θ)-ray if the following conditions hold:

(i) the open segments $\overset{\circ}{l}_i$, $i = 0, \dots, k$, have no common points with K, and

(ii) for every $i = 0, \dots, k-1$ the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to ∂K.

Points $x_1, \dots, x_k$ will be called reflection points of γ. Some of them are proper reflection points, and if x_i is among the others, then we say that γ is tangent to ∂K at x_i. This definition of an (ω, θ)-ray is slightly different from that in [20, 21]. If all reflection points of γ are proper ones, then γ will be called an ordinary (ω, θ)-ray.

2.2. Given $\omega \in S^{n-1}$, let Z_ω be a fixed hyperplane so that K is contained in the open half-space H_ω determined by Z_ω and having ω as an inward normal.

For $u \in Z_\omega$ consider the motion of a point with unit velocity starting at u with direction ω and reflecting from ∂K following the usual reflection law or tangent to ∂K. By $S_t(u)$ we denote the shift of u at time $t \geq 0$, and by $N_t(u)$ we denote the velocity vector of the point at the moment t (for $y = S_t(u) \in \partial K$ we identify $N_t(u)$ with $\sigma_y(N_t(u))$, σ_y being the symmetry with respect to the tangent plane to ∂K at y). By $x_1(u), x_2(u), \dots$ we denote the successive reflection points (proper or not) of the ray $\gamma(u) = \{S_t(u) : t \geq 0\}$. By $t_1(u), t_2(u), \dots$ we denote the times of the corresponding reflections. Clearly, these two sequences may be empty, finite, or infinite. By $r(u)$ we denote the number of all reflections of $\gamma(u)$. Thus we have $r(u) = 0, 1, \dots, \infty$.

The following property of $S_t(u)$ is well known in the theory of dispersing billiards (see [24, 25, 28]).

Proposition 2.1. *Let U be an open subset of Z_ω and let $t > 0$ be such that for every $u \in U$ we have $r(u) \geq m$, $t_m(u) < t < t_{m+1}(u)$, and $x_1(u), \dots, x_m(u)$ are proper reflection points of $\gamma(u)$. Assume $x_j(u) \in \partial K_{i_j}$ for every $u \in U$ and $j = 1, \dots, m$. Then $S_t(u)$ is a smooth surface in $\mathbf{R}^n$ and the second fundamental form of $S_t(u)$ with respect to the normal field $\{N_t(u) : u \in U\}$ is positive definite.*

2.3. In this paper smooth means C^∞. For a subset Y of a topological space X by $\overline{Y}$ $(\overset{\circ}{Y})$ we denote the closure (interior) of Y in X. We will use the notation

$$\partial K_i^{\pm} = \{y \in \partial K_i : \langle \nu(y), \omega \rangle \lessgtr 0\}.$$

2.4. Let A be a real $(p \times p)$ matrix and let I be the identity $(p \times p)$ matrix. Recall that

$$|\det A| \leq (1 + \|A - I\|)^p,$$

$\|A\|$ being the norm of the operator $A : \mathbf{C}^p \to \mathbf{C}^p$. If M is a symmetric and positive definite matrix, then for $\lambda > 0$ we have

$$\|(I + \lambda M)^{-1}\| \leq (1 + \lambda\sigma)^{-1}, \qquad \|M(I + \lambda M)^{-1}\| \leq 1/\lambda,$$

where $\sigma = \min \operatorname{spec} M$, $\operatorname{spec} M$ being the spectrum of M.

3. The maps J_α

By a configuration α with length $|\alpha| = m$, $m \geq 1$, we mean a symbol $\alpha = (i_1, i_2, \dots, i_m)$ such that $i_j \in \{1, 2, \dots, s\}$ for all j and $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, m-1$.

Let $\omega \in S^{n-1}$ and let Z_ω be as in subsection 2.2. Consider the sets

$$F_\alpha = \{u \in Z_\omega : r(u) \geq m \text{ and } x_j(u) \in \partial K_{i_j} \text{ for } j = 1, \dots, m\},$$
$$U_\alpha = \{u \in F_\alpha : x_j(u) \text{ is a proper reflection point for all } j \leq m\}.$$

Define the map

$$J_\alpha : F_\alpha \to S^{n-1}$$

by $J_\alpha(u) = N_t(u)$ for arbitrary t with $t_m(u) < t < t_{m+1}(u)$ (resp. $t < \infty$ if $r(u) = m$). This map was introduced by Guillemin [5] in a slightly different context.

In this section we establish some properties of the maps J_α which will be used in the next sections.

Lemma 3.1.

(a) *For every $u \in \overline{F_\alpha}$ there exists a configuration $\beta = (\bar{i}_1, \dots, \bar{i}_p)$ with $p \geq m$, $u \in F_\beta$ such that there is a sequence $p_1 < p_2 < \dots < p_m = p$ with $\overline{i_{p_j}} = i_j$ for all $j = 1, \dots, m$. Moreover, if $x_r(u)$ is a proper reflection point of $\gamma(u)$ for some $r = 1, \dots, p$, then $r = p_j$ for some $j = 1, \dots, m$;*

(b) *J_α can be extended to a continuous map $J_\alpha : \overline{F_\alpha} \to S^{n-1}$;*

(c) $\mathring{F}_\alpha = U_\alpha$.

Proof. The proofs of (a) and (b) are quite elementary and we omit them. To establish (c), first note that U_α is obviously open and $U_\alpha \subset F_\alpha$. Take $u \in \mathring{F}_\alpha$ and suppose $u \notin U_\alpha$. Then $\gamma(u)$ is tangent to ∂K at $x_j(u)$ for some $j = 1, \dots, m$. Let j be the minimal number with this property and for convenience set $x_0(u) = u$ and $t_0(u) = 0$. Choose an arbitrary t with $t_{j-1}(u) < t < t_j(u)$ and an open neighborhood U of u in $\mathring{F}_\alpha$ so small that $t_{j-1}(u') < t < t_j(u')$ for all $u' \in U$. We may arrange $S_t(U)$ to be strictly convex with respect to the normal field $\{N_t(u') : u' \in U\}$ (cf. subsection 2.2 and Proposition 2.1). A simple geometric argument shows that there exists $u' \in U$ so that the straight ray starting at $x_{j-1}(u')$ with direction $N_t(u')$ has no common points with ∂K_{i_j}. Thus $u' \notin F_\alpha$, which gives a contradiction. Therefore $\mathring{F}_\alpha \subset U_\alpha$ and we obtain (c). □

Consider the set

$$L_\alpha = \{u \in \overline{F_\alpha} : N_t(u) = \omega \text{ for every } t \geq 0\}. \tag{3.1}$$

For $u \in L_\alpha$ the ray $\gamma(u)$ goes straightforward with constant direction ω and has common (tangent) points with $\partial K_{i_1}, \dots, \partial K_{i_m}$ and, possibly, with some other ∂K_r. Clearly, L_α is compact, $\overset{\circ}{L}_\alpha = \varnothing$, and $L_\alpha \cap U_\alpha = \varnothing$. In fact, L_α is contained in the boundary (in Z_ω) of the orthogonal projection of K_{i_1} onto Z_ω. Consequently, the set

$$M_\alpha = \overline{F_\alpha} \backslash L_\alpha \tag{3.2}$$

contains $\overset{\circ}{\overline{F_\alpha}}$.

Since $J_\alpha : \overline{F_\alpha} \to S^{n-1}$ is continuous and $\overline{F_\alpha}$ is compact,

$$E_\alpha = J_\alpha(\overline{F_\alpha}) \tag{3.3}$$

is a compact subset of S^{n-1}. Note that $J_\alpha(L_\alpha) = \{\omega\}$, so $J_\alpha(M_\alpha)$ is either E_α or $E_\alpha \backslash \{\omega\}$.

The main result in this section is the following.

Theorem 3.2. *For every configuration α the map*

$$J_\alpha : M_\alpha \to J_\alpha(M_\alpha)$$

is a homeomorphism.

Proof. It is sufficient to prove that if $u \in M_\alpha$, $v \in \overline{F_\alpha}$ with $u \neq v$, then $J_\alpha(u) \neq J_\alpha(v)$. Indeed, assume this true. Then $J_\alpha : M_\alpha \to J_\alpha(M_\alpha)$ is a continuous bijection and it remains to show that J_α^{-1} is continuous. Take a sequence $\{u_k\} \subset M_\alpha$ and $u \in M_\alpha$ so that $J_\alpha(u_k) \to J_\alpha(u)$. Let v be a cluster point of $\{u_k\}$; then $v \in \overline{F_\alpha}$ and clearly $J_\alpha(v) = J_\alpha(u)$. Therefore $v = u$, and this shows that u is the only cluster point of $\{u_k\}$ in $\overline{F_\alpha}$. Thus $u_k \to u$ and $J_\alpha : M_\alpha \to J_\alpha(M_\alpha)$ is a homeomorphism.

Let $\alpha = (i_1, \dots, i_m)$ and let $u \neq v$ be elements of M_α and $\overline{F_\alpha}$, respectively. By Lemma 3.1(a), there are configurations $\beta = (\overline{i_1}, \dots, \overline{i_p})$ and $\gamma = (\overline{\overline{i_1}}, \dots, \overline{\overline{i_q}})$ with $p \geq m$, $q \geq m$, $u \in F_\beta$, $v \in F_\gamma$. Moreover, there exist $p_1 < p_2 < \cdots < p_m = p$ and $q_1 < q_2 < \cdots < q_m = q$ such that $\overline{i_{p_j}} = \overline{\overline{i_{q_j}}} = i_j$ for all $j = 1, \dots, m$, and if $x_r(u)$, $r \leq p$ (resp. $x_r(v)$, $r \leq q$) is a proper reflection point of $\gamma(u)$ (resp. $\gamma(v)$), then $r = p_j$ (resp. $r = q_j$) for some $j = 1, \dots, m$.

Define $\overline{S_t}(u)$ (resp. $\overline{S_t}(v)$) for $t \geq 0$ in the same way as $S_t(u)$ (resp. $S_t(v)$) assuming that after the pth (resp. qth) reflection from ∂K the point u (resp. v) is moving straightforward with constant velocity $J_\alpha(u) = J_\beta(u)$ (resp. $J_\alpha(v) = J_\gamma(v)$) no matter whether it intersects K or not.

Set $x_j = x_{p_j}(u)$ and $y_j = y_{q_j}(v)$ for $j = 1, \dots, m$ and take

$$t > \max(t_p(u), t_q(v))$$

sufficiently large (this will be made clear afterwards). Denote by A_j the tangent plane to ∂K_{i_j} at x_j, by H_j the closed half-space determined by A_j and containing K_{i_j}, by σ_j the symmetry map with respect to A_j, and by δ_j the symmetry map with respect to the tangent plane to ∂K_{i_j} at y_j.

Further, denote by $x_1^{(t)}$ (resp. $y_1^{(t)}$) the point for which we have $x_1 \in [u, x_1^{(t)}]$ and $\|u - x_1^{(t)}\| = t$ (resp. $y_1 \in [v, y_1^{(t)}]$ and $\|v - y_1^{(t)}\| = t$). Let $A_1^{(t)}$ be the plane parallel to Z_ω and passing through $x_1^{(t)}$ and $y_1^{(t)}$. Set $x_2^{(t)} = \sigma_1(x_1^{(t)})$, $y_2^{(t)} = \delta_1(y_1^{(t)})$, $A_2^{(t)} = \sigma_1(A_1^{(t)})$, and $H_2^{(t)} = \sigma_1(H_1^{(t)})$ (see Figure 1).

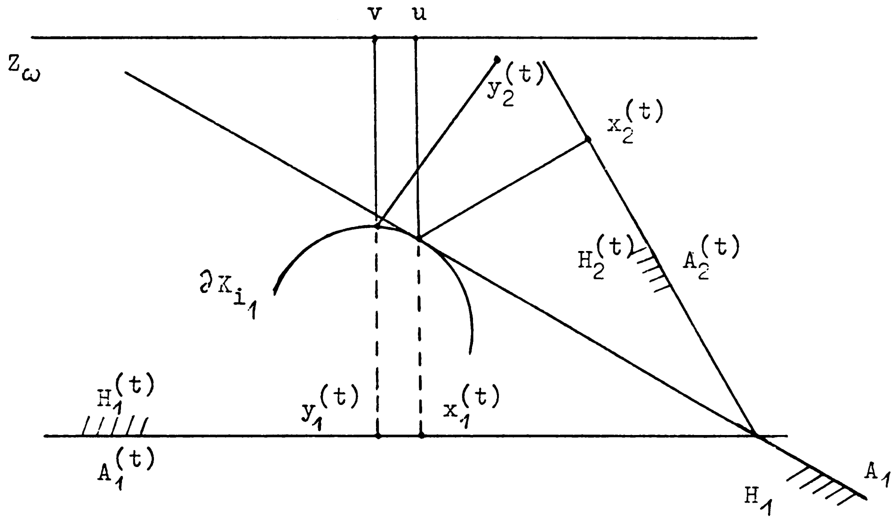

FIGURE 1

Choose t so large that $K_{i_1} \subset \mathring{H}_1^{(t)}$. It is then clear that $K_{i_1} \subset \mathring{H}_2^{(t)}$. Note that if $\gamma(u)$ is tangent to ∂K_{i_1} at x_1, then $x_2^{(t)} = x_1^{(t)}$, $A_2^{(t)} = A_1^{(t)}$, and $H_2^{(t)} = H_1^{(t)}$. In particular, we have

$$y_1 \in \mathring{H}_2^{(t)}. \tag{3.4}$$

On the other hand, $y_1 \in K_{i_1}\setminus\{x\}$ implies $y_1 \in \mathring{H}_1$. By definition $A_2^{(t)} = \sigma_1(A_1^{(t)})$, therefore $\rho(y_1, A_1^{(t)}) \le \rho(y_1, A_2^{(t)})$ and the equality holds if and only if $\gamma(u)$ is tangent to ∂K_{i_1} at x_1. Here $\rho(z, A)$ is the Euclidean distance between z and A. Since $\|y_1 - y_2^{(t)}\| = \|y_1 - y_1^{(t)}\| = \rho(y_1, A_1^{(t)})$, we get

$$\|y_1 - y_2^{(t)}\| \le \rho(y_1, A_2^{(t)}), \tag{3.5}$$

and (3.4) yields

$$y_2^{(t)} \in H_2^{(t)}. \tag{3.6}$$

Moreover, if x_1 or y_1 is a proper reflection point of $\gamma(u)$ or $\gamma(v)$, respectively, then $y_2^{(t)} \in \mathring{H}_2^{(t)}$.

Further, we proceed by induction. In fact we will describe in detail only the next step.

Set $A_3^{(t)} = \sigma_2(A_2^{(t)})$, $H_3^{(t)} = \sigma_2(H_2^{(t)})$, $x_3^{(t)} = \sigma_2(x_2^{(t)})$, and $y_3^{(t)} = \delta_2(y_2^{(t)})$ (see Figure 2). Since $y_2 \in [y_1, y_2^{(t)}]$, by (3.4) and (3.6), we get $y_2 \in \mathring{H}_2^{(t)}$. On the other hand, $y_2 \in \mathring{H}_2$, hence $\rho(y_2, A_2^{(t)}) \le \rho(y_2, A_3^{(t)})$. By (3.5) we find $\|y_2 - y_2^{(t)}\| \le \rho(y_2, A_2^{(t)})$, so

$$\|y_2 - y_3^{(t)}\| \le \rho(y_2, A_3^{(t)}). \tag{3.7}$$

Since $x_2, x_2^{(t)} \in H_2 \cap H_2^{(t)}$, exploiting the fact that $[x_2, x_2^{(t)}]$ is orthogonal to $A_2^{(t)}$, we see that $y_2 \in H_3^{(t)}$. Now (3.7) yields $y_3^{(t)} \in H_3^{(t)}$. Moreover, if at least one of the points x_1, x_2, y_1, y_2 is a proper reflection point, then $y_3^{(t)} \in \mathring{H}_3^{(t)}$.

Since $u \in M_\alpha$ means that some x_j is a proper reflection point of $\gamma(u)$, proceeding in this way we conclude finally that

$$y_{m+1}^{(t)} \in \mathring{H}_{m+1}^{(t)}, \tag{3.8}$$

where $H_{m+1}^{(t)}$ is the half-space determined by $A_{m+1}^{(t)}$ and containing x_m, while $A_{m+1}^{(t)}$ is the plane passing through $x_{m+1}^{(t)}$ and orthogonal to $[x_m, x_{m+1}^{(t)}]$. Let us mention that by our construction we have $x_{m+1}^{(t)} = \overline{S_t}(u)$ and $y_{m+1}^{(t)} = \overline{S_t}(v)$. Thus, by (3.8), $S_t(v) \in H_{m+1}^{(t)}$.

Replacing the roles of u and v and using the same arguments, we see that $\overline{S_t}(u) \in \mathring{G}_{m+1}^{(t)}$. Here $G_{m+1}^{(t)}$ is the half-space determined by $B_{m+1}^{(t)}$ and containing y_m, while $B_{m+1}^{(t)}$ is the plane orthogonal to $[y_m, y_{m+1}^{(t)}]$ and passing through $\overline{S_t}(v) = y_{m+1}^{(t)}$.

Suppose $J_\alpha(u) = J_\alpha(v)$. Choose t so large that $K_{i_m} \subset \mathring{H}_{m+1}^{(t)} \cap \mathring{G}_{m+1}^{(t)}$. Then $A_{m+1}^{(t)}$ and $B_{m+1}^{(t)}$ are parallel, so either $G_{m+1}^{(t)} \subset H_{m+1}^{(t)}$ or $H_{m+1}^{(t)} \subset G_{m+1}^{(t)}$.

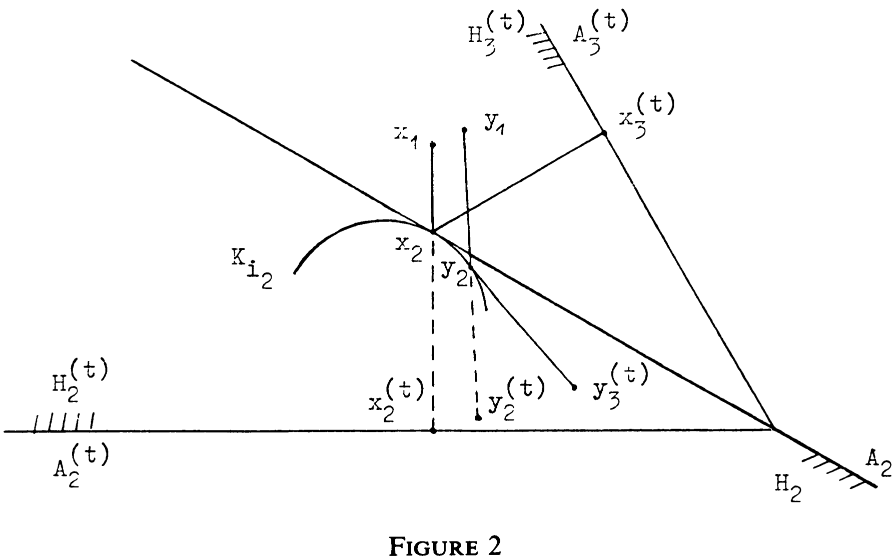

FIGURE 2

Assume, for example, $G_{m+1}^{(t)} \subset H_{m+1}^{(t)}$. Then $\overline{S_t}(u) \in G_{m+1}^{(t)} \subset H_{m+1}^{(t)}$, hence $\overline{S_t}(u) \notin A_{m+1}^{(t)}$, which gives a contradiction. Therefore $J_\alpha(u) \neq J_\alpha(v)$ and the proof of the theorem is complete.

Let $\omega, \theta \in S^{n-1}$ and let γ be an (ω, θ)-ray with successive reflection points $x_1, \dots, x_m$. We say that γ is of type $\alpha = (i_1, \dots, i_m)$ if $x_j \in \partial K_{i_j}$ for every $j = 1, \dots, m$.

Corollary 3.3. *If $\omega \neq \theta$, then for every configuration α there exists at most one (ω, θ)-ray of type α.*

It is clear that in general there could exist some configurations α for which there are no (ω, θ)-rays of type α. The existence of (ω, θ)-rays will be treated in §6.

Fix a configuration $\alpha = (i_1, \dots, i_m)$ and consider the map $J_\alpha : U_\alpha \to S^{n-1}$. For our aims in §5 we will show that $dJ_\alpha(u)$ is invertible for each $u \in U_\alpha$. The proof of this fact is based on the representation of the Poincaré map related to a periodic billiard trajectory found by Petkov and Vogel [18] (see also §§4, 5 in [21]).

Let $u_0 \in U_\alpha$ and let $Q_j = x_j(u_0)$, $j = 1, \dots, m$. For $j = 1, \dots, m-1$ denote by D_j the oriented line $Q_j Q_{j+1}$ with direction $\overrightarrow{Q_j Q_{j+1}}$. By D_m we denote the oriented line through Q_m with direction $\theta = J_\alpha(u_0)$. Choose a point $X_j \in D_j$ so that $\|Q_j X_j\| = 1$ and $\overrightarrow{Q_j X_j}$ is colinear with $\overrightarrow{Q_j Q_{j+1}}$ for $j < m$ and $\overrightarrow{Q_m X_m} = \theta$. By $\sqcap_j$ we denote the hyperplane passing through X_j and orthogonal to D_j. Let σ_j be the symmetry map with respect to the tangent plane A_j to ∂K at Q_j and let $\lambda_j = \|Q_j Q_{j-1}\|$ for $j = 2, \dots, m$, $\lambda_1 = \|u_0 Q_1\| + 1$. Then

$$Z_\omega \times Z_\omega \ni (u, v) \to (u + \lambda_1 v, v) \in Z_1 \times Z_1,$$

where we identify Z_ω and $Z_1 = \sigma_1(\sqcap_1)$ by the orthogonal projection of Z_ω onto Z_1. Let $\pi_j : \sqcap_{j-1} \to T_{Q_j}(\partial K)$ be the projection along the direction $\overrightarrow{Q_{j-1} Q_j}$, $j = 2, \dots, m$, and let G_j be the differential of the Gauss map corresponding to ∂K at Q_j. Consider the symmetric linear map $\widetilde{\psi}_j : \sqcap_j \to \sqcap_j$, given by

$$\langle \widetilde{\psi}_j \sigma_j w, \sigma_j w \rangle = -2\langle \overrightarrow{Q_{j-1} Q_j}, \nu(Q_j) \rangle \langle G_j \pi_j(w), \pi_j(w) \rangle / \|Q_{j-1} Q_j\|$$

for $w \in \sqcap_{j-1}$ (we identify here and below the planes $\sigma_j(\sqcap_{j-1})$ and $\sqcap_j$ according to the orthogonal projection of one of them onto the other). Following the argument and the basis given in [18, 21], we have

(3.9)
$$(dJ_\alpha(u_0))(u) = pr_2 \begin{pmatrix} \sigma_m & \lambda_m \sigma_m \\ \widetilde{\psi}_m \sigma_m & \sigma_m + \lambda_m \widetilde{\psi}_m \sigma_m \end{pmatrix} \cdots \begin{pmatrix} \sigma_1 & \lambda_1 \sigma_1 \\ \widetilde{\psi}_1 \sigma_1 & \sigma_1 + \lambda_1 \widetilde{\psi}_1 \sigma_1 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix}$$

for $u \in Z_\omega$, where $pr_2(u, v) = v$.

Proposition 3.4. *For every configuration* α *we have*

$$(3.10)\quad \begin{aligned}(dJ_\alpha(u_0))(u) = M_m\sigma_m(I_{m-1} + \lambda_m M_{m-1})\sigma_{m-1}(I_{m-2} + \lambda_{m-1}M_{m-2})\\ \cdots \sigma_2(I_1 + \lambda_2 M_1)\sigma_1 u,\end{aligned}$$

where I_i *is the identity map on* $\sqcap_i$ *and for* $i = 1, \dots, m$, $M_i : \sqcap_i \to \sqcap_i$ *is a symmetric positive definite linear map. Moreover,*

$$(3.11)\quad M_1 = \tilde{\psi}_1, \qquad M_i = \sigma_i M_{i-1}(I_{i-1} + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i, \qquad i = 2, \dots, m.$$

The proof is straightforward and we omit it. Consequently, we get

Corollary 3.5. *For every configuration* α *and every* $u_0 \in U_\alpha$ *the map* $dJ_\alpha(u_0)$ *is invertible and* $J_\alpha : U_\alpha \to J_\alpha(U_\alpha)$ *is a diffeomorphism.*

4. (ω, θ)-RAYS FOR GENERIC θ

In this section we prove that for given $\omega \in S^{n-1}$ there exists a residual subset $\mathscr{R} = \mathscr{R}(\omega)$ of S^{n-1} such that for $\theta \in \mathscr{R}$ all (ω, θ)-rays have some special properties. Recall that a subset $\mathscr{R}$ of a topological space X is called residual in X if $\mathscr{R} = \bigcap_{i=1}^\infty U_i$, where U_i are open and dense subsets of X.

Throughout this section $\omega \in S^{n-1}$ will be fixed.

Theorem 4.1. *There exists a residual subset* $\mathscr{R}_1 = \mathscr{R}_1(\omega)$ *of* S^{n-1} *so that for every* $\theta \in \mathscr{R}_1$ *and every configuration* α *if* $u \in F_\alpha$ *and* $J_\alpha(u) = \theta$, *then* $u \in U_\alpha$.

The latter means that if $J_\alpha(u) \in \mathscr{R}_1$ for some $u \in F_\alpha$, then the first $|\alpha|$ reflection points of $\gamma(u)$ are proper ones.

Proof. We are going to construct a sequence $\mathscr{T}_1 \supset \mathscr{T}_2 \supset \cdots \supset \mathscr{T}_m \supset \cdots$ of open and dense subsets of S^{n-1} so that for every m if α is a configuration with $|\alpha| = m$, $u \in F_\alpha$ and $J_\alpha(u) \in \mathscr{T}_m$, then the first m reflection points of $\gamma(u)$ are proper ones.

Set $\mathscr{T}_1 = S^{n-1}\backslash\{\omega\}$. Suppose we have constructed open and dense subsets $\mathscr{T}_1 \supset \cdots \supset \mathscr{T}_m$ of S^{n-1} having the desired properties.

Fix a configuration $\alpha = (i_1, \dots, i_m, i_{m+1})$ and set $\alpha_k = (i_1, \dots, i_k)$ for $k = 1, \dots, m$. Since $\mathscr{T}_m$ is open and dense, $F = S^{n-1}\backslash\mathscr{T}_m$ is compact and $\overset{\circ}{F} = \varnothing$. Let $\beta = \alpha_k$ for some $k = 1, \dots, m$. Then $F \cap E_\beta$ is compact with empty interior (cf. (3.3)). Consider the map $J_\beta : \overline{F_\beta} \to E_\beta \subset S^{n-1}$. Since

$$J_\beta^{-1}(F) = J_\beta^{-1}(F \cap E_\beta) \subset (J_\beta^{-1}(F \cap E_\beta) \cap M_\alpha) \cup L_\alpha,$$

we deduce from Theorem 3.2 that $J_\beta^{-1}(F)$ has empty interior in Z_ω. Hence $F_\alpha \cap J_\beta^{-1}(F)$ is a compact subset of Z_ω with empty interior. Thus $J_\alpha(F_\alpha \cap J_\beta^{-1}(F))$ is compact, and as is easily seen from Theorem 3.2, it has empty interior in S^{n-1}. In this way we have proved that

$$V = \mathscr{T}_m \backslash \bigcup_{k=1}^m J_\alpha(F_\alpha \cap J_{\alpha_k}^{-1}(F))$$

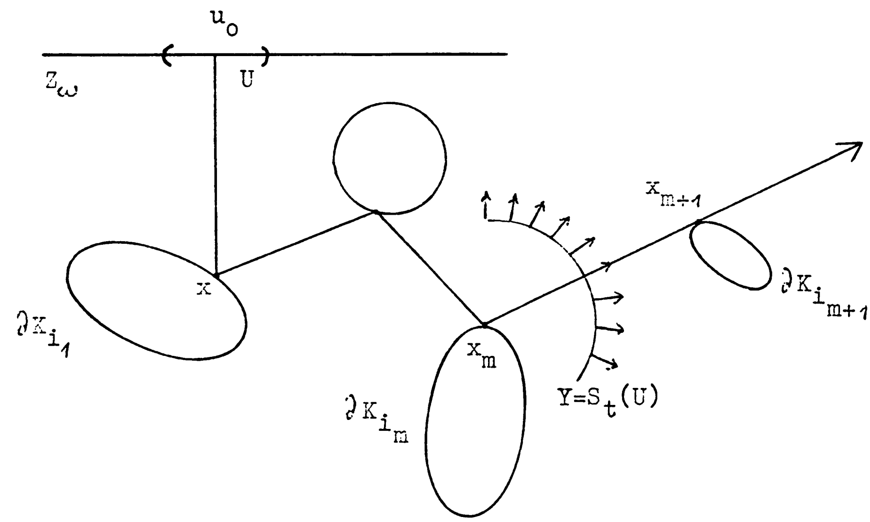

FIGURE 3

is an open and dense subset of S^{n-1}. Note that if $u \in F_\alpha$ and $J_\alpha(u) \in V$, then the first m reflection points of $\gamma(u)$ are proper ones.

Denote by $\mathscr{T}_{m+1}^{(\alpha)}$ the set of $\theta \in V$ such that if $J_\alpha(u) \in \theta$ for some $u \in F_\alpha$, then the first $m+1$ reflection points of $\gamma(u)$ are proper ones. We are going to show that $\mathscr{T}_{m+1}^{(\alpha)}$ is open and dense in V (and therefore in S^{n-1}). Clearly, $\mathscr{T}_{m+1}^{(\alpha)}$ is open. To prove the density, take $\theta \in V \backslash \mathscr{T}_{m+1}^{(\alpha)}$. Then $\theta = J_\alpha(u_0)$ for some $u_0 \in F_\alpha$, the first m reflection points $x_1(u_0), \dots, x_m(u_0)$ of $\gamma(u_0)$ are proper ones, and $\gamma(u_0)$ is tangent to ∂K at the $(m+1)$st reflection point $x_{m+1}(u_0)$. Choose t so that $t_m(u_0) < t < t_{m+1}(u_0)$ and take a neighborhood U of u_0 in Z_ω so small that $U \subset \overset{\circ}{F}_\alpha = U_\alpha$. By Proposition 2.1 it follows that for U sufficiently small $Y = S_t(U)$ is a smooth strictly convex surface in $\mathbf{R}^n$ (see Figure 3). Then for every $u \in U$ the ray $\gamma(u)$ issued from u has exactly m proper reflections from $\partial K_{i_1}, \dots, \partial K_{i_m}$ when the time runs from 0 to t. Note that $\{N_t(u) : u \in U\}$ is a normal field for Y and if $u \in U$, then $S_t(u)$ moves in the direction $N_t(u)$ (cf. subsection 2.2). It is now clear that there exists $u' \in U$ such that the ray starting at $S_t(u') \in Y$ with direction $N_t(u')$ intersects transversely $\partial K_{i_{m+1}}$. This means that $J_\alpha(u') \in \mathscr{T}_{m+1}^{(\alpha)}$. We have shown in this way that $\mathscr{T}_{m+1}^{(\alpha)}$ is open and dense in S^{n-1}.

Setting $\mathscr{T}_{m+1} = \bigcap_{|\alpha|=m+1} \mathscr{T}_{m+1}^{(\alpha)}$, we get an open and dense subset of S^{n-1} with the desired properties. This closes the induction.

Finally, setting $\mathscr{R}_1 = \bigcap_{m=1}^{\infty} \mathscr{T}_m$, we complete the proof of the theorem. □

Let B_a be an open ball with radius $a > 0$ containing K such that Z_ω is tangent to $\overline{B_a}$. For $\theta \in S^{n-1}$ denote by Z_θ the tangent plane to $\overline{B_a}$ such that θ is orthogonal to Z_θ and the half-space H_θ determined by Z_θ and having

θ as an outward normal contains B_a. Let γ be an (ω, θ)-ray with successive reflection points $x_1, \dots, x_m$. The sojourn time T_γ of γ is defined by

$$T_\gamma = \|\pi_\omega(x_1) - x_1\| + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| + \|x_m - \pi_\theta(x_m)\| - 2a, \tag{4.1}$$

where $\pi_\omega : \mathbf{R}^n \to Z_\omega$ and $\pi_\theta : \mathbf{R}^n \to Z_\theta$ are the orthogonal projections. The definition of T_γ was introduced by Guillemin [5]. It is easy to see that T_γ does not depend on a.

Theorem 4.2. *There exists a residual subset $\mathscr{R}_2 = \mathscr{R}_2(\omega)$ of S^{n-1} such that if $\theta \in \mathscr{R}_2$, then for any two different (ω, θ)-rays γ and δ we have $T_\gamma \neq T_\delta$.*

Proof. First, notice that if γ is an (ω, θ)-ray, then there exist $u \in Z_\omega$ and $m \in \mathbf{N}$ so that $x_1(u), \dots, x_m(u)$ are the successive reflection points of γ. Moreover, there is unique $t > t_m(u)$ with $S_t(u) \in Z_\theta$ and then we have $N_t(u) = \theta$ and $T_\gamma = t - 2a$.

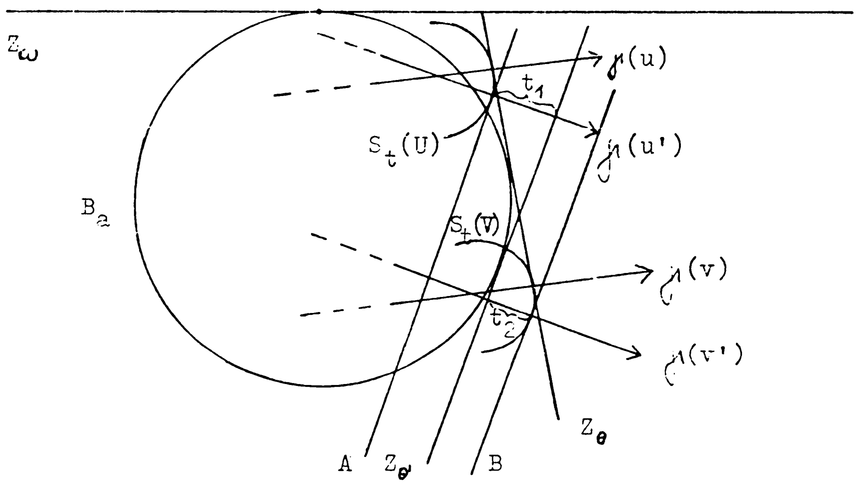

FIGURE 4

Let $u \in Z_\omega$ be such that $1 \leq r(u) < \infty$. Setting $m_1 = r(u)$ and $\theta = N_t(u)$ for $t > t_{m_1}(u)$, choose uniquely t_0 with $S_{t_0}(u) \in Z_\theta$ and set $T_\theta(u) = t_0$.

Further, fix two different configurations α and β and set $m = \max(|\alpha|, |\beta|)$ and

$$\mathscr{L}(\alpha, \beta) = \Big\{\theta \in \mathscr{T}_m : \text{ if } u \in U_\alpha,\ v \in U_\beta,\ r(u) = |\alpha|,\ r(v) = |\beta|, \\ \text{and } J_\alpha(u) = J_\beta(v) = \theta, \text{ then } T_\theta(u) \neq T_\theta(v)\Big\}.$$

Here $\mathscr{T}_m$ is the set defined in the proof of Theorem 4.1. We are going to prove $\mathscr{L}(\alpha, \beta)$ is open and dense in $\mathscr{T}_m$.

To show $\mathscr{T}_m \backslash \mathscr{L}(\alpha, \beta)$ is closed in $\mathscr{T}_m$, consider a sequence $\{\theta_k\} \subset \mathscr{T}_m \backslash \mathscr{L}(\alpha, \beta)$ with $\theta_k \to \theta \in \mathscr{T}_m$. Then for any k there exist $u_k \in U_\alpha$

and $v_k \in U_\beta$ with $r(u_k) = |\alpha|$, $r(v_k) = |\beta|$, $J_\alpha(u_k) = J_\beta(v_k) = \theta_k$, and $T_{\theta_k}(u_k) = T_{\theta_k}(v_k)$. We may assume $u_k \to u \in \overline{F_\alpha}$ and $v_k \to v \in \overline{F_\beta}$. If $u \notin F_\alpha$, then $u \in F_{\alpha'}$ for some α' and $\gamma(u)$ would have some tangent point to ∂K. On the other hand,

$$J_{\alpha'}(u) = J_\alpha(u) = \lim_{k\to\infty} J_\alpha(u_k) = \lim_{k\to\infty} \theta_k = \theta \in \mathcal{T}_m,$$

which is a contradiction with the properties of $\mathcal{T}_m$. So $u \in F_\alpha$ and again by $J_\alpha(u) = \theta \in \mathcal{T}_m$ we get $u \in U_\alpha$. Similarly, $J_\beta(v) = \theta$ and $v \in U_\beta$. It is clear also that $r(u) = |\alpha|$ and $r(v) = |\beta|$. Therefore $\theta \in \mathcal{T}_m \backslash \mathcal{L}(\alpha, \beta)$ and we have shown that $\mathcal{L}(\alpha, \beta)$ is open in $\mathcal{T}_m$.

To establish the density, take $\theta \in \mathcal{T}_m$ and suppose $\theta \notin \mathcal{L}(\alpha, \beta)$. Then there exist $u \in U_\alpha$, $v \in U_\beta$ with $r(u) = |\alpha|$, $r(v) = |\beta|$, $J_\alpha(u) = J_\beta(v) = \theta$, and $T_\theta(u) = T_\theta(v)$. It follows by Proposition 2.1 that for $t = T_\theta(u) = T_\theta(v)$ there exist small open neighborhoods $U \subset U_\alpha$ and $V \subset U_\beta$ of u and v, respectively, in Z_ω such that $S_t(U)$ and $S_t(V)$ are strictly convex surfaces. Note that both $S_t(U)$ and $S_t(V)$ are tangent to Z_θ at $S_t(u)$ and $S_t(v)$, respectively, hence they lie in H_θ (see Figure 4). Further, according to Theorem 3.2, we see that for every $\theta' \in \mathcal{T}_m$ sufficiently close to θ there exist unique $u' \in U$ and $v' \in V$ with $J_\alpha(u') = J_\beta(v') = \theta'$. Denote by F the set of those $\theta' \in \mathcal{T}_m$ with $J_\alpha(u') = J_\beta(v') = \theta'$ for some $u' \in U$ and $v' \in V$ and such that there exists a plane tangent simultaneously to $S_t(U)$ at $S_t(u')$ and to $S_t(V)$ at $S_t(v')$. It is easy to see that $\mathring{F} = \emptyset$. Since $\mathcal{T}_m$ is open in S^{n-1}, we can take $\theta' \in \mathcal{T}_m \backslash F$ arbitrarily close to θ. Choosing such a θ', denote by A (resp. B) the plane tangent to $S_t(U)$ (resp. $S_t(V)$) at $S_t(u')$ (resp. $S_t(v')$). Then the three planes $Z_{\theta'}$, A, and B are mutually parallel. Thus for $t_1 = T_{\theta'}(u') - t$ and $t_2 = T_{\theta'}(v') - t$, we see that $|t_1|$ is the distance between A and $Z_{\theta'}$, while $|t_2|$ is the distance between B and $Z_{\theta'}$. Since $A \neq B$, we have $t_1 \neq t_2$, hence $T_{\theta'}(u') = t + t_1 \neq t + t_2 = T_{\theta'}(v')$. On the other hand, J_α is invertible on U_α, while J_β is invertible on U_β. Consequently, $\theta' \in \mathcal{L}(\alpha, \beta)$ and $\mathcal{L}(\alpha, \beta)$ is dense in $\mathcal{T}_m$.

Setting

$$\mathcal{R}_2 = \bigcap_{\alpha \neq \beta} \mathcal{L}(\alpha, \beta), \tag{4.2}$$

we obtain a residual subset of S^{n-1} having the desired properties. This proves the theorem. □

5. Singularities of the scattering kernel

In this section we prove Theorem 1.1. The scattering kernel $s(t, \theta, \omega)$ is given by (1.1). Let $\rho(t) \in C_0^\infty(\mathbf{R})$, $\rho(t) = 1$ for $|t| \leq 1/2$, $\rho(t) = 0$ for

$|t| \geq 1$. Setting

$$\rho_\varepsilon(t) = \rho(t/\varepsilon), \qquad 0 < \varepsilon \leq 1, \qquad \rho_\varepsilon^{(k)} = \frac{\partial^k \rho_\varepsilon}{\partial t^k}$$

consider

$$\begin{aligned}(5.1) \qquad & (s(t, \theta, \omega), \rho_\varepsilon(t + T_0)e^{-it\lambda}) \\ & = \sum_{k=0}^{n-2} c_k(-i\lambda)^{n-2-k} \int_{\mathbf{R}} \int_{\partial K} e^{i\lambda(t - \langle x, \theta\rangle)} \cdot \rho_\varepsilon^{(k)}(\langle x, \theta\rangle - t + T_0) \\ & \qquad \times \frac{\partial w}{\partial \nu}(t, x, \omega)\, dt\, dS_x,\end{aligned}$$

where $c_k = \text{const}$, $c_0 = C_n$. Obviously, we must study $\partial w/\partial\nu|_{\mathbf{R}\times\partial K}$ only for (t, x) satisfying the relation

$$\begin{aligned}\mathrm{WF}\left(\frac{\partial w}{\partial \nu}\bigg|_{\mathbf{R}\times\partial K}\right) & \cap \{(t, x, 1, -\operatorname{grad}\langle x, \theta\rangle|_{\partial K}) \in T^*(\mathbf{R}\times\partial K)\} \\ & \cap \{(t, x, 1, \xi) \colon (t, x) \in \operatorname{supp} \rho_\varepsilon^{(k)}(\langle x, \theta\rangle - t + T_0)\} \neq \varnothing.\end{aligned}$$

It is well known [14] that the generalized wave front $\mathrm{WF}_b(w)$ is propagating along the outgoing generalized bicharacteristics of the operator $\Box$ issued from $\mathrm{WF}(\delta(-\rho_0 - \langle x, \omega\rangle))$. The projections of these bicharacteristics on $\overline{\Omega}$ will be called generalized rays. They are unions of finite or infinite number segments which are reflecting or tangent to ∂K.

The results about the propagation of singularities [14] show that $(t, x, 1, \eta) \in \mathrm{WF}(\partial w/\partial\nu|_{\mathbf{R}\times\partial K})$ if there exists $\tilde{\eta} \in T_x^*(\overline{\Omega})$ such that $|\tilde{\eta}| = 1$, $\tilde{\eta}|_{T_x(\partial K)} = \eta$, $\langle\tilde{\eta}, \nu(x)\rangle \leq 0$. Therefore $\tilde{\eta}|_{T_x(\partial K)} = -\theta|_{T_x(\partial K)}$ leads to one of the following cases:

(a) $\tilde{\eta} = -\theta$ for $\langle\theta, \nu(x)\rangle > 0$,
(b) $\tilde{\eta} = -\theta + 2\langle\theta, \nu(x)\rangle\nu(x)$ for $\langle\theta, \nu(x)\rangle < 0$,
(c) $\tilde{\eta} = -\theta$ for $\langle\theta, \nu(x)\rangle = 0$.

In cases (a) and (b) the singularities are propagating along a generalized ray γ reflecting at x. Moreover, in case (a) (resp. (b)) the reflecting (resp. the incoming) direction of γ at x is just θ. In case (c) the singularities are propagating along a ray simply tangent at x and having direction θ. Thus we must study the behavior of the generalized rays having at least one (finite or infinite) segment with direction θ.

Fix $T > 0$. We are going to study the singularities T_0, $|T_0| \leq T$, of $s(t, \theta, \omega)$ introducing a partition of unity $\sum_{j=1}^{\infty} \varphi_j(x') = 1$, $\varphi_j(x') \in C_0^\infty(\mathbf{R}^{n-1})$, $x' = (x_1, \dots, x_{n-1})$, depending on T. From

$$x \in \partial K, \qquad (t, x) \in \operatorname{supp} \rho_\varepsilon^{(k)}(\langle x, \theta\rangle - t + T_0), \qquad |T_0| \leq T$$

we conclude that we need to know the singularities of $\partial w/\partial\nu|_{\mathbf{R}\times\partial K}$ for $|t| \leq T_1 = T + \rho_0 + \varepsilon$. Since the singularities of w are propagating with speed 1, we are going to investigate only the generalized rays γ with lengths less than or equal to some fixed number T_2 depending on T and ρ_0. Without loss of generality we may assume $\omega = (0, \dots, 0, 1)$.

Consider the Cauchy problem

$$(5.2)\qquad \begin{cases} \Box v_j = 0, \\ v_j|_{t=\tau} = \varphi_j(x')\delta(\tau - x_n), \\ \left.\dfrac{\partial v_j}{\partial t}\right|_{t=\tau} = \varphi_j(x')\delta'(\tau - x_n) \end{cases}$$

with $\tau < -\rho_0$, τ fixed. It is easy to see that

$$\mathrm{WF}(v_j) \subset \{(t, x, \pm 1, \mp\omega) \in T^*(\mathbf{R}^{n+1})\backslash 0 : \text{ there exist } \hat{x} \text{ and } \sigma \geq 0 \\ \text{so that } \hat{x}_n = \tau,\ t = \tau \pm \sigma,\ x = \hat{x} \pm \sigma\omega\}.$$

Therefore there exists a compact set $F_0' \subset \mathbf{R}^{n-1}$ such that $\operatorname{supp}\varphi_j \cap F_0' = \varnothing$ implies $\operatorname{singsupp} v_j \cap K = \varnothing$. Set

$$F_0 = \{x \in \mathbf{R}^n : x' \in F_0',\ x_n = \tau\}$$

and consider the generalized rays $\gamma(u)$ starting at $u \in F_0$ with direction ω and having lengths less than or equal to T_2. We identify F_0 with Z_ω and choose $\mathscr{R}(\omega) = \mathscr{R}_1(\omega) \cap \mathscr{R}_2(\omega)$, where $\mathscr{R}_i(\omega)$, $i = 1, 2$, are defined in Theorems 4.1 and 4.2. The rays $\gamma(u)$ have at most m_0 reflections and there exists a finite number of configurations α with $|\alpha| \leq m_0$. Below we assume $\theta \in \mathscr{R}(\omega)$ fixed, $\theta \neq \omega$.

Let $\gamma(u)$, $u \in F_\alpha \cap F_0$ be a ray having at least one segment with direction θ starting at $x_r(u)$. According to Theorem 4.1, the successive reflection points $x_i(u)$, $1 \leq i \leq r$, of $\gamma(u)$ are proper ones and $u \in U_\beta$ for some $|\beta| = r$. Exploiting the continuity of the broken Hamiltonian flow related to $\Box$ (see [14]) for rays with uniformly bounded lengths and Corollary 3.3, we conclude that the points $u \in F_0$ with the above property of $\gamma(u)$ form a finite set $\{u_1, \dots, u_N\}$. Thus, if $u_0 \in F_0\backslash\{u_1, \dots, u_N\}$, then the ray $\gamma(u_0)$ has no segments with direction θ among the first m_0 ones. Choosing a sufficienly small neighborhood $\mathscr{O}(u_0)$ of u_0 in $\mathbf{R}^{n-1}$, we arrange the same property for $\gamma(u)$ for all $u \in \mathscr{O}(u_0)$.

Given u_i, $i = 1, \dots, N$, there are two cases. First, assume $x_p(u_i)$ is a proper reflection point for every $p \leq m_0$. Taking a small neighborhood $\mathscr{O}(u_i)$ of u_i, we obtain that for every $u \in \mathscr{O}(u_i)$ the first m_0 reflection points of $\gamma(u)$ are proper ones. Second, let $\gamma(u_i)$ have some tangent segments among the first m_0 ones. Since $\theta \in \mathscr{R}_1(\omega)$, these segments have directions different from θ. The continuity of the broken Hamiltonian flow implies that there exists a neighborhood $\mathscr{O}(u_i)$ of u_i so that for $u \in \mathscr{O}(u_i)$ the ray $\gamma(u)$ has reflection points

$$(5.3)\qquad x_1(u), \dots, x_{r_0}(u),\ x_{r_0+1}(u), \dots, x_q(u), \qquad r_0 \geq 1,$$

where $x_i(u)$, $i = 1, \dots, r_0$ are proper ones, while the segments starting at $x_i(u)$, $i = r_0, \dots, q$, have directions different from θ. Notice that $r_0 = r_0(i)$ does not depend on $u \in \mathscr{O}(u_i)$, while q could depend on it. Finally, assume that $u_i \notin \mathscr{O}(u_j)$ for $i \neq j$.

This procedure leads to a covering $F_0 \subset \bigcup_{u_0 \in F_0} \mathscr{O}(u_0)$ and we may assume

$$F_0 \subset \bigcup_{j=1}^{M} \mathscr{O}(u_j), \qquad N \leq M.$$

Let $\tilde{\mathscr{O}}(u_j) \subset \mathscr{O}(u_j)$ be a neighborhood of u_j. Choose the partition of unity $\{\varphi_j(x')\}_{j=1}^{\infty}$ so that

$$\text{(5.4)} \qquad \operatorname{supp} \varphi_j \subset \mathscr{O}(u_j), \ \varphi_j(x') = 1 \quad \text{for } x' \in \tilde{\mathscr{O}}(u_j), \ 1 \leq j \leq M.$$

Assume that for $M_0 \leq N$ the rays $\gamma(u_i)$, $i = 1, \dots, M_0$, have only proper reflection points among the first m_0 ones; that is, the first case described above holds.

Setting $V_j = v_j - \varphi_j(x')\delta(t - x_n)$, $j = 1, \dots, N$, we have

$$\begin{cases} \Box V_j = (\Delta_{x'}\varphi_j)\delta(t - x_n), \\ V_j|_{t=\tau} = \left.\dfrac{\partial V_j}{\partial t}\right|_{t=\tau} = 0, \end{cases}$$

and the singularities of V_j are propagating along the straight lines $l(u)$ issued from $\mathscr{O}(u_j)\backslash\tilde{\mathscr{O}}(u_j)$ with direction ω.

Fix $1 \leq j \leq N$ and consider the lines $l(u)$ with $u \in \mathscr{O}(u_j)$. Let $l(u)$ meet ∂K transversely for the first time $t(u) > \tau$ at $x_1(u) = (u, t(u)) \in \partial K_{i_j}$. Suitably modifying V_j and φ_j in the interior of K_{i_j}, we introduce two distributions $\tilde{V}_j$ and $\phi_j = \tilde{\varphi}_j(t, x)\delta(t - x_n)$ so that

$$\begin{cases} \Box(\tilde{V}_j + \phi_j) \in C^\infty \quad \text{in } \mathbf{R} \times \Omega, \\ (\tilde{V}_j + \phi_j - v_j)|_{t=\tau} = \left.\dfrac{\partial}{\partial t}(\tilde{V}_j + \phi_j - v_j)\right|_{t=\tau} = 0. \end{cases}$$

We take $\tilde{\varphi}_j(t, x) \in C^\infty(\mathbf{R}^{n+1})$ such that

$$\tilde{\varphi}_j(t, x) = \begin{cases} \varphi_j(x') & \text{for } t \leq t(x'), \ x_n \leq t(x'), \ x' \in \mathscr{O}(u_j), \\ 0 & \text{for } x' \notin \mathscr{O}(u_j) \text{ or } x_n > t(x') + \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is chosen sufficiently small. Similarly extending V_j, we arrange

(5.5)

$$\mathrm{WF}(\tilde{V}_j|_{\mathbf{R}\times\partial K}) \cup \mathrm{WF}\left(\left.\frac{\partial \tilde{V}_j}{\partial \nu}\right|_{\mathbf{R}\times\partial K}\right) \subset \{(t(u), x_1(u), \sigma, -\sigma\omega|_{T_{x_1(u)}(\partial K)}) \in T^*(\mathbf{R} \times \partial K^*_{i_j})\backslash 0 : u \in \mathscr{O}(u_j)\backslash\tilde{\mathscr{O}}(u_j)\}.$$

Introduce W_j and w_j as solutions of the problems:

$$\begin{cases} \Box W_j = 0 & \text{in } \mathbf{R}\times\Omega\,, \\ W_j + \tilde{V}_j = 0 & \text{on } \mathbf{R}\times\partial K\,, \\ W_j|_{t<-\rho_0} = 0\,, & j = 1\,,\ldots\,,N\,, \end{cases}$$

$$\begin{cases} \Box w_j = 0 & \text{in } \mathbf{R}\times\Omega\,, \\ w_j + \phi_j = 0 & \text{on } \mathbf{R}\times\partial K\,, \\ w_j|_{t<-\rho_0} = 0\,, & j = 1\,,\ldots\,,N\,, \end{cases}$$

$$\begin{cases} \Box w_j = 0 \quad \text{in } \mathbf{R}\times\Omega\,, & w_j = 0 \quad \text{on } \mathbf{R}\times\partial K\,, \\ w_j|_{t=\tau} = \varphi_j(x')\delta(t-x_n)\,, & \left.\dfrac{\partial w_j}{\partial t}\right|_{t=\tau} = \varphi_j(x')\delta'(\tau-x_n)\,, \end{cases}$$

for $j = N+1\,,\ldots\,,M$.

After this preparation set

$$\tilde{w} = \sum_{j=1}^{N}(w_j + \phi_j + W_j + \tilde{V}_j) + \sum_{j=N+1}^{M} w_j + \sum_{j>M} v_j\,.$$

A simple argument yields $w - \tilde{w} \in C^\infty(\mathbf{R}\times\overline{\Omega})$. The singularities of W_j are related to the (generalized) rays $\gamma(u)$, with u belonging to $\mathscr{O}(u_j)\backslash\widetilde{\mathscr{O}}(u_j)$, which have no segments with direction θ. Consequently,

$$\left(\bigcup_{j=1}^{N} \mathrm{WF}\left(\left.\frac{\partial(W_j+\tilde{V}_j)}{\partial\nu}\right|_{\mathbf{R}\times\partial K}\right) \cup \bigcup_{j=N+1}^{M} \mathrm{WF}\left(\left.\frac{\partial w_j}{\partial\nu}\right|_{\mathbf{R}\times\partial K}\right)\right) \tag{5.6}$$
$$\cap\, \{(t\,,x\,,1\,,\,-\theta|_{T_x(\partial K)}) : x\in\partial K\,,\ |t|\le T_1\} = \varnothing.$$

Replacing w in (5.1) by $\tilde{w}$ and using (5.6), we are going to study the asymptotics of the integrals

$$I_{k\,,j}(\lambda) = \int_{\mathbf{R}}\int_{\partial K} e^{i\lambda(t-\langle x\,,\theta\rangle)}\rho_\varepsilon^{(k)}(\langle x\,,\theta\rangle - t + T_0)\frac{\partial}{\partial\nu}(w_j+\phi_j)\,dt\,dS_x\,,$$

$j = 1\,,\ldots\,,N$. We have

$$\frac{\partial\phi_j}{\partial\nu} = \frac{\partial\tilde{\varphi}_j}{\partial\nu}\delta(t-x_n) - \langle\nu\,,\omega\rangle\tilde{\varphi}_j\delta'(t-x_n) \quad \text{on } \mathbf{R}\times\partial K.$$

If z_j is a stationary point of $\langle x\,,\theta-\omega\rangle|_{\partial K_{i_j}}$ lying on $\partial K_{i_j}^{+}$, then $\nu(z_j) = (\theta-\omega)/\|\theta-\omega\|$. Moreover, $\partial\tilde{\varphi}_j/\partial\nu$ and $\partial\tilde{\varphi}_j/\partial t$ vanish on some neighborhood of z_j in ∂K. Therefore a stationary phase argument implies

$$I_{k\,,j}(\lambda) = \int_{\mathbf{R}}\int_{\partial K} e^{i\lambda(t-\langle x\,,\theta\rangle)}\rho_\varepsilon^{(k)}(\langle x\,,\theta\rangle - t + T_0)\mathscr{B}w_j\,dt\,dS$$

mod $\mathscr{O}(|\lambda|^{-m})$ for every m with $\mathscr{B}w_j = (\partial/\partial\nu - \langle\nu, \theta\rangle\partial/\partial t)w_j|_{\mathbf{R}\times\partial K}$. Here we have used the equality

$$\int_{\partial K} e^{i\lambda\langle x, \omega-\theta\rangle}\langle\nu, \theta+\omega\rangle\rho_\varepsilon^{(k)}(\langle x, \theta-\omega\rangle + T_0)\, dS_x$$
$$= \int_K \frac{\partial}{\partial(\theta+\omega)}[e^{i\lambda\langle x, \omega-\theta\rangle}\rho_\varepsilon^{(k)}(\langle x, \theta-\omega\rangle + T_0)]\, dx = 0.$$

Fix $M_0+1 \le j \le N$. The wave front $\mathrm{WF}(\mathscr{B}w_j)$ is related to the rays $\gamma(u)$, $u \in \mathscr{O}(u_j)$. As we mentioned above, the first r_0 reflection points of these rays are proper ones. Then by the construction in [16], we take w_j in the form

$$w_j = \sum_{p=1}^{r_0-1} w_{p,j} + w_j'.$$

By using (5.3), the singularities of $\mathscr{B}w_{p,j}$ are related to the finite segments $[x_p(u), x_{p+1}(u)]$ meeting ∂K transversally. Extending $w_{p,j}$ in the interior of ∂K and repeating the argument in [16] with the outgoing Green function, we conclude that $w_{p,j}$ does not contribute to the asymptotic of $I_{k,j}(\lambda)$. On the other hand, the singularities of $\mathscr{B}w_j'$ are connected with the segments issued from $x_p(u)$, $p = r_0, \dots, q$ which have no directions θ. Consequently, $I_{k,j}(\lambda) = \mathscr{O}(|\lambda|^{-m})$, $\forall m$, for $M_0 + 1 \le j \le N$.

To study $I_{k,j}(\lambda)$, $j = 1, \dots, M_0$, notice that $\mathrm{WF}(\mathscr{B}w_j)$ is related to ordinary reflecting rays issued from $\mathrm{WF}(\phi_j|_{\mathbf{R}\times\partial K^+_{i_j}})$. Thus we are in a position to apply the results of Petkov [16] for $I_{k,j}(\lambda)$. The arguments in [16] with trivial modifications work for dimensions $n \ge 3$, n odd. In particular, instead of Lemmas 1 and 2 on p. 321 of [16] we obtain directly the following.

Lemma 5.1. *Let G, P, and Q be $(n-1)\times(n-1)$ matrices. Then*

$$\det\begin{pmatrix} -G & 0 & -I \\ 0 & P & I \\ -I & I & Q \end{pmatrix} = \det(GQP + P - G). \tag{5.6}$$

Proof. First assume G invertible. Then

$$\det\begin{pmatrix} -G & 0 & -I \\ 0 & P & I \\ -I & I & Q \end{pmatrix}\cdot\det G^{-1} = \det\begin{pmatrix} -I & 0 & -I \\ 0 & P & I \\ -G^{-1} & I & Q \end{pmatrix}$$
$$= \det\begin{pmatrix} -I & P \\ -G^{-1} - Q & I \end{pmatrix} = \det(G^{-1}P + QP - I),$$

which implies (5.6). In the case $\det G = 0$, we choose $\varepsilon_0 > 0$ so that $G_\varepsilon = G + \varepsilon I$ is invertible for $0 < \varepsilon < \varepsilon_0$. Applying (5.6) for G_ε and letting $\varepsilon \to 0$, we complete the proof.

As we proved in §3, the map dJ_γ (see §1) is invertible at the point A_γ, where γ hits Z_ω. Moreover, the sojourn times of the different (ω, θ)-rays are different for $\theta \in \mathscr{R}(\omega)$. Therefore, from the asymptotic of $I_{k,j}(\lambda)$, given in [16], we get

$$\operatorname{singsupp} s(t, \theta, \omega) \cap \{|t| \leq T\} = \{-T_\gamma : \gamma \in \mathscr{L}_{\omega,\theta}, \ |T_\gamma| \leq T\}.$$

Since T is arbitrary, we obtain (1.4). The form of the leading singularity at $-T_\gamma$ follows from [16]. This completes the proof of Theorem 1.1.

6. Existence of (ω, θ)-rays and asymptotics of T_m^{1l}

Throughout this section we assume that $K = \bigcup_{i=1}^{s} K_i$ satisfies condition (H) introduced in §1. Let $\mathscr{R}(\omega) = \mathscr{R}_1(\omega) \cap \mathscr{R}_2(\omega)$, where $\mathscr{R}_1(\omega)$ and $\mathscr{R}_2(\omega)$ are the residual subsets of S^{n-1} defined in §4.

Definition 6.1. Let $\alpha = (i_1, \dots, i_k)$ be a configuration. We say that a pair (ω, θ) of unit vectors satisfies the conditions of visibility with respect to α if $\theta \in \mathscr{R}(\omega)$ and the following conditions hold:

(a) for every $x \in \partial K_{i_1}$ the ray starting at x with direction $-\omega$ (resp. ω) has no common point with $K \backslash K_{i_1}$ (resp. K_{i_2});

(b) for every $x \in \partial K_{i_k}$ the ray starting at x with direction θ (resp. $-\theta$) has no common point with $K \backslash K_{i_k}$ (resp. $K_{i_{k-1}}$).

Lemma 6.2. *For every configuration $\alpha = (i_1, \dots, i_k)$ there exist $\omega, \theta \in S^{n-1}$ such that (ω, θ) satisfies the conditions of visibility with respect to α.*

Proof. Take a hyperplane A which separates K_{i_1} and K_{i_2} and such that A is tangent to K_{i_1} at some point x, while A is tangent to K_{i_2} at some point y. Set $\omega' = (y-x)/\|y-x\|$ and denote by Z an arbitrary hyperplane orthogonal to ω' such that the open half-space, determined by Z and having ω' as an inward normal, contains K. Consider the convex cone $C = \{y + t(u-y) : u \in K_{i_1}, t \geq 0\}$. It is easy to see that the orthogonal projection of K_{i_1} on Z is contained in C. Therefore, condition (H) implies

$$\{u - t\omega' : t \geq 0\} \cap (K \backslash K_{i_1}) = \varnothing, \qquad u \in K_{i_1}. \tag{6.1}$$

Indeed, suppose there are $u \in K_{i_1}$, $t > 0$, and $j \neq i_1$ so that $v = u - t\omega' \in K_j$. Let u' be the orthogonal projection of u (and therefore of v) on Z. Since $u, u' \in C$, we have $v \in C$. On the other hand, the definition of C shows that the segment $[y, v]$ contains a point of K_{i_1} which is a contradiction with condition (H). Hence (6.1) holds and the compactness of $K \backslash K_{i_1}$ implies the existence of a number $\varepsilon > 0$ so that

$$\{u - t\omega : t \geq 0\} \cap (K \backslash K_{i_1}) = \varnothing, \qquad u \in K_{i_1},$$

whenever $\|\omega - \omega'\| < \varepsilon$. Choose ω with $\|\omega - \omega'\| < \varepsilon$ and $\langle -\omega, \nu(x)\rangle > 0$. Then it is clear that condition (a) of Definition 6.1 holds.

Since $\mathscr{R}(\omega)$ is dense in S^{n-1}, in a similar way we can find $\theta \in \mathscr{R}(\omega)$ satisfying (b). This completes the proof of the lemma.

Fix a configuration $\alpha = (i_1, \dots, i_k)$ with $i_1 \neq i_k$ and an integer $l = 1, \dots, k$. For $q = 0, 1, \dots$ set

$$\alpha_{q,1} = (\underbrace{i_1, \dots, i_k; \cdots; i_1, \dots, i_k}_{q}; i_1, \dots, i_l).$$

Proposition 6.3. *Let (ω, θ) satisfy the conditions of visibility with respect to $\alpha_{1,l}$. Then for every integer $q \geq 0$ there exists an (ω, θ)-ray of type $\alpha_{q,1}$.*

Proof. Fix $q \geq 0$ and two hyperplanes Z_ω, Z_θ as in the text preceding Theorem 4.2. Set $m = qk+l$, $D = Z_\omega \times \partial K_{i_1} \times \cdots \times \partial K_{i_m} \times Z_\theta$ and define $F : D \to \mathbf{R}$ by

$$F(\xi) = \|z_1 - x_1\| + \sum_{j=1}^{m-1} \|x_j - x_{j+1}\| + \|x_m - z_2\|,$$

where $\xi = (z_1; x_1, \dots, x_m; z_2) \in D$. Since F is continuous, it is clear that there exists $\overline{\xi} = (\overline{z}_1; \overline{x}_1, \dots, \overline{x}_m; \overline{z}_2) \in D$ with $F(\overline{\xi}) = \min F$. Moreover, $\overline{z}_1 = \pi_\omega(\overline{x}_1)$ and $\overline{z}_2 = \pi_\theta(\overline{x}_m)$. Then from condition (a) we deduce that the segment $[\overline{z}_1, \overline{x}_2]$ has no common point with K_{i_1}. For $c > 0$ consider the rotative ellipsoid

$$E_c = \{x \in \mathbf{R}^n : \|\overline{z}_1 - x\| + \|x - x_2\| \leq c\}.$$

Take the minimal $c > 0$ with $E_c \cap K_{i_1} \neq \varnothing$. Therefore, E_c will be tangent to ∂K_{i_1} at some $\overline{x}'_1$. In view of $F(\overline{\xi}) = \min F(\xi)$ we get $\overline{x}_1 = \overline{x}'_1$. Thus the segments $[\overline{z}, \overline{x}_1]$ and $[\overline{x}_1, \overline{x}_2]$ satisfy the law of reflection at x.

Repeating this argument and exploiting conditions (H) and (b), we deduce that $\overline{x}_1, \dots, \overline{x}_m$ are the successive reflection points of an (ω, θ)-ray of type $\alpha_{q,l}$. This proves the proposition.

Remark. In [23] we announced the existence of (ω, θ)-rays under weaker conditions than those given above. However, there are counterexamples showing that this statement in [23] is not true.

Up to the end of this section we will assume that α and l are fixed and (ω, θ) are fixed too so that the conditions of visibility with respect to $\alpha_{1,l}$ are satisfied. This assumption combined with (H) implies the existence of a constant $\kappa_1 = \kappa_1(\alpha, \omega) > 0$ such that if $\gamma(z)$ is a ray issued from $z \in Z_\omega$ with direction ω and having successive reflection points $x_1 \in \partial K_{i_1}$, $x_2 \in \partial K_{i_2}$, $x_3, \dots, x_p$, then

$$\cos\varphi_j \geq \kappa_1, \qquad j = 1, \dots, p-1,$$

where φ_j is the angle between $\nu(x_j)$ and $\overrightarrow{x_j x_{j+1}}$ $(1 \leq j \leq p-1)$.

Let $V_0 \subset Z_\omega$ be the set of points such that every ray $\gamma(z)$ issued from $z \in V_0$ with direction ω has successive reflection points

$$u^{(z)}_{qk+j} \in \partial K_{i_j}, \qquad 0 \leq q \leq m,\ 1 \leq j \leq k\ (1 \leq j \leq l \text{ for } q = m).$$

Of course V_0 depends on m, l but we will not mention this explicitly. Let $\varphi(x)$ be a real smooth function defined in $\mathscr{U}$ so that $|\nabla\varphi| = 1$. We say that φ satisfies condition (P) in $\mathscr{U}$ if the principal curvatures of $\mathscr{L}_\varphi(x) = \{y \in \mathscr{U} : \varphi(y) = \varphi(x)\}$ with respect to $-\nabla\varphi$ are nonnegative for all $x \in \mathscr{U}$.

Choose $\varphi_0(x) = \langle x, \omega\rangle$ and $\mathscr{U}_0 = \{y + \tau\omega,\ y \in V_0 : \tau \geq 0\}$. Obviously, φ_0 satisfies (P) in $\mathscr{U}_0$. Let $\varphi_1(x)$ be a function so that $|\nabla\varphi_1| = 1$ and

$$\varphi_1(x) = \varphi_0(x), \qquad \frac{\partial\varphi_1}{\partial\nu} = -\frac{\partial\varphi_0}{\partial\nu} \quad \text{on } V_1 = \mathscr{U}_0 \cap \partial K_{i_1}^+.$$

We extend φ_1 on $\mathscr{U}_1 = \bigcup_{y\in V_1}\{y + \tau\nabla\varphi_1(y) : \tau \geq 0\}$ so that φ_1 satisfies (P) in $\mathscr{U}_1$ (see [9]). Following this procedure we define successive functions $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{qk+j}$ satisfying (P) respectively in $\mathscr{U}_0, \mathscr{U}_1, \mathscr{U}_2, \dots, \mathscr{U}_{qk+j}$, where

$$\mathscr{U}_{qk+j} = \bigcup_{y\in V_{qk+j}} \{y + \tau\nabla\varphi_{qk+j}(y) : \tau \geq 0\}, \qquad V_{qk+j} \subset \partial K_{i_j}.$$

Let γ_1 and γ_2 be two ordinary reflecting rays issued respectively from $y_0, z_0 \in V_0$ with direction ω and having successive reflection points

$$y_{qk+j},\ z_{qk+j} \in \partial K_{i_j}, \qquad 0 \leq q \leq m,\ 1 \leq j \leq k\ (1 \leq j \leq l \text{ for } q = m).$$

The properties of $\varphi_0, \dots, \varphi_{qk+j}$ combined with the proof of Proposition 3.8 in [9] yield

Lemma 6.4. *There exist constants $C > 0$, $0 < \delta < 1$, independent of γ_1, γ_2, l so that*

$$\|y_i - z_i\| \leq C\delta^{mk+l-i}, \qquad i = 0, \dots, mk + l. \tag{6.2}$$

Constructing rays with m reflections, letting $m \to \infty$, and exploiting (6.2), it is easy to find a unique ray $\gamma^\infty(\omega)$ starting at $x_0^\infty \in Z_\omega$ with direction ω and having infinite number reflection points $x_{qk+j}^\infty \in \partial K_{i_j}$, $q \geq 0$, $1 \leq j \leq k$. The properties of the dispersive billiards [24, 25] (see also Lemma 5.4 in [9]) yield

$$\|x_{qk+j}^\infty - \tilde{x}_j\| \leq C_1\delta^{qk+j}, \qquad q \geq 0,\ 1 \leq j \leq k, \tag{6.3}$$

with C_1 independent of q and j.

Set $d_j = \sum_{p=1}^{j} \|\tilde{x}_{p+1} - \tilde{x}_p\|$, $1 \leq j \leq k$, $d_\alpha = d_k$, $L_m^\infty = \langle x_1^\infty, \omega\rangle + \sum_{p=1}^{m} \|x_{p+1}^\infty - x_p^\infty\|$. Applying (6.3) for $q \geq 0$, $r \geq 0$, we have

$$|(L_{(q+r)k+j}^\infty - (q+r)d_\alpha - d_j) - (L_{qk+j}^\infty - qd_\alpha - d_j)| \leq 2C_1 \sum_{p=1}^{rk+1} \delta^{qk+p+j} \leq C_2\delta^q.$$

Fixing j and introducing $L_{\alpha,\omega,j}^1 = \lim_{q\to\infty}(L_{qk+j}^\infty - qd_\alpha - d_j)$, we get

$$L_{qk+j}^\infty = qd_\alpha + d_j + L_{\alpha,\omega,j}^1 + O(\delta^q), \qquad q \to \infty. \tag{6.4}$$

As above, there exists a unique ray $\gamma^{\infty}(-\theta)$ starting at some $y_0^{\infty} \in Z_\theta$ with direction $-\theta$ and having an infinite number of reflection points $y_{qk+r}^{\infty} \in \partial K_{j_r}$, $q \geq 0$, $1 \leq r \leq k$, where $(j_1, \dots, j_k) = (i_l, i_{l-1}, \dots, i_1, i_k, i_{k-1}, \dots, i_{l+1})$. Therefore, we obtain

$$(6.5)\quad G_{qk}^{\infty} = -\langle y_1^{\infty}, \theta\rangle + \sum_{p=1}^{m} \|y_{p+1}^{\infty} - y_p^{\infty}\| = qd_\alpha + L_{\alpha,-\theta}^{1} + O(\delta^q), \qquad q \to \infty$$

with some constant $L_{\alpha,-\theta}^{l}$.

Now let γ_m^{1l} be an ordinary (ω, θ)-ray with sojourn time T_m^{1l} and $m = qk+l$ reflection points $x_{pk+j}^{(m)} \in \partial K_{i_j}$, $0 \leq p \leq q$, $1 \leq j \leq k$ $(1 \leq j \leq l$ for $p = q)$. Set

$$L_p^{(m)} = \langle x_1^{(m)}, \omega\rangle + \sum_{r=1}^{p} \|x_{r+1}^{(m)} - x_r^{(m)}\|, \quad G_p^{(m)} = -\langle x_m^{(m)}, \theta\rangle + \sum_{r=p+1}^{m-1} \|x_{r+1}^{(m)} - x_r^{(m)}\|,$$

and take $p = [q/2]k + l - 1$. Applying (6.2) twice for the reflection points of γ_m^{1l}, $\gamma^{\infty}(\omega)$ as well as for those of γ_m^{1l}, $\gamma^{\infty}(-\theta)$, we obtain

$$\begin{aligned} &|L_p^{(m)} - L_p^{\infty}| + |G_p^{(m)} - G_{q-[q/2]k}| \\ &\qquad \leq 2\sum_{r=1}^{p+1} \|x_r^{(m)} - x_r^{(m)}\| + 2\sum_{q=1}^{m-p} \|x_{m-r+1}^{(m)} - y_r^{\infty}\| \leq C_3\delta^q. \end{aligned}$$

Since $T_m^{1l} = L_p^{(m)} + G_p^{(m)}$, by (6.4) and (6.5) we get

Theorem 6.5. *Assume* (H) *is fulfilled. Let* $\alpha = (i_1, \dots, i_k)$, $i_1 \neq i_k$, *be a configuration and let* (ω, θ) *satisfy the conditions of visibility with respect to* $\alpha_{1,l}$. *Then we have the asymptotic*

$$(6.6)\qquad T_{qk+l}^{1l} = qd_\alpha + L_{\alpha,\omega,\theta}^{1l} + O(\delta^q), \qquad q \to \infty,$$

where $L_{\alpha,\omega,\theta}^{1l} = L_{\alpha,\omega,l-1}^{1} + L_{\alpha,-\theta}^{l} + d_{l-1}$, $d_0 = 0$, $L_{\alpha,\omega,0}^{1} = L_{\alpha,\omega,k}^{1}$.

Corollary 6.6. *Let* $K = K_1 \cup K_2$, $d = \operatorname{dist}(K_1, K_2)$. *Assume* (ω, θ) *satisfies the conditions of visibility with respect to* $\alpha = (1, 2)$. *Let* γ_m^{ij} *be the* (ω, θ)-*ray with sojourn time* T_m^{ij} *and* m *reflection points* $\{x_p^{(m)}\}$, *where* $x_1^{(m)} \in \partial K_i$, $x_m^{(m)} \in \partial K_j$. *Then we have*

$$(6.7)\qquad T_{2q+i+j-1}^{ij} = 2qd + L_{\omega,\theta}^{ij} + O(\delta^q), \qquad q \to \infty,\ i, j = 1, 2.$$

Note that a particular case of (6.7) for two disjoint balls has been obtained by Nakamura and Soga [15]. The asymptotic (6.6) is similar to this for the lengths of the periodic reflecting rays in bounded strictly convex domains $\mathscr{O} \subset \mathbf{R}^2$ (see [2, 12]).

7. Asymptotic behavior of $|c_m^{1l}|$

Throughout this section we assume (H) is fulfilled and we use freely the notation of §6. Fix $\omega \neq \theta$ and a configuration $\alpha = (i_1, \dots, i_k)$ with $i_1 \neq i_k$.

Assume l is fixed and (ω, θ) satisfies the conditions of visibility with respect to $\alpha_{1,l}$. Then it is easy to find a constant $\kappa = \kappa(K, \omega, \theta) > 0$ such that if γ is an (ω, θ)-ray with successive reflection points $x_{pk+j} \in K_{i_j}$, $1 \leq p \leq q$, $1 \leq j \leq k$, then

$$\cos\varphi_j \geq \kappa, \qquad j = 1, \dots, qk + l, \tag{7.1}$$

φ_j being the angle between $\nu(x_j)$ and $\overrightarrow{x_j x_{j+1}}$ (resp. θ for $j = qk + l$). Set

$$d = \operatorname{diam} K, \qquad d' = \min_{i \neq j} \operatorname{dist}(K_i, K_j), \qquad d_0 = 1/d'.$$

Since K_i are strictly convex, there exist constants $\mu_2 > \mu_1 > 0$ so that

$$\mu_1 \langle v, v\rangle \leq \langle G_x v, v\rangle \leq \mu_2 \langle v, v\rangle, \qquad v \in T_x \partial K,$$

where G_x is the differential of the Gauss map of ∂K at x.

Let $x \in \partial K_i$, $y \in \partial K_j$, $i \neq j$. Suppose the segment $s_{x,y} = [x, y] \subset \Omega$ is transverse to both ∂K_i and ∂K_j. Let $\overline{x}$ be a point lying on the line joining x and y such that $\|x - \overline{x}\| = 1$. Let $\sqcap$ be the hyperplane orthogonal to $s_{x,y}$ and passing through $\overline{x}$. Seting $e = (y - x)/\|y - x\|$, denote by π the projection $\sqcap \to T_x \partial K$ along the direction $-e$. Introduce the symmetric operator $\widetilde{\psi} : \sqcap \to \sqcap$ by

$$\langle \widetilde{\psi} u, u\rangle = 2\langle e, \nu(x)\rangle \langle G_x \pi u, \pi u\rangle, \qquad u \in \sqcap. \tag{7.2}$$

We will say that $\widetilde{\psi}$ is related to $s_{x,y}$. It follows easily that

$$\operatorname{spec} \widetilde{\psi} \subset [2\mu_1 \langle \nu(x), e\rangle,\ 2\mu_2 \langle \nu(x), e\rangle^{-1}]. \tag{7.3}$$

Now we will prove a technical lemma which will be applied several times in this section. We need to introduce some assumptions and notation. Let $\beta = (\overline{i_1}, \dots, \overline{i_p})$ be a configuration and let $x_j, x_j' \in \partial K_{\overline{i_j}}$, $j = 1, \dots, p$, be points so that

$$\|x_j - x_j'\| < Da^j, \qquad j = 1, \dots, p, \tag{7.4}$$

with some constants $D > 0$, $a > 0$. Assume $\langle e_j, \nu(x_j)\rangle \geq \kappa$, $\langle e_j', \nu(x_j')\rangle \geq \kappa$, where $e_j = (x_{j+1} - x_j)/\|x_{j+1} - x_j\|$, $e_j' = (x_{j+1}' - x_j')/\|x_{j+1}' - x_j'\|$, $j = 1, \dots, p$. According to Lemma A in the Appendix, for every $j = 1, \dots, p$ there exists an isometry $A_j : \mathbf{R}^n \to \mathbf{R}^n$ such that $A_j(\sqcap_j') = \sqcap_j$ and

$$\|A_j - I\| < C_1 D(1 + a)a^j, \qquad \|\widetilde{\psi}_j - A_j \widetilde{\psi}_j' A_j^{-1}\| < C_1 D(1 + a)a^j. \tag{7.5}$$

Here $\widetilde{\psi}_j : \sqcap_j \to \sqcap_j$ and $\widetilde{\psi}_j' : \sqcap_j' \to \sqcap_j'$ are the operators related to the segments $s_j = [x_j, x_{j+1}]$ and $s_j' = [x_j', x_{j+1}']$. Let $M_1 : \sqcap_1 \to \sqcap_1$ and $M_1' : \sqcap_1' \to \sqcap_1'$ be arbitrary symmetric positive definite linear operators. Set

$$M_i = \sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1} \sigma_i + \widetilde{\psi}_i, \qquad i = 2, \dots, p, \tag{7.6}$$

where $\lambda_i = \|x_{i-1} - x_i\|$ and σ_i is the symmetry with respect to $\sqcap_i$. Let M_i' be the map defined by (7.6) replacing $\widetilde{\psi}_i$, σ_i, λ_i, x_j by $\widetilde{\psi}_i'$, σ_i', λ_i', x_i', respectively. Finally, set

$$(7.7) \qquad b = (1 + 2\mu_1 \kappa d')^{-1}, \qquad a_1 = \begin{cases} a & \text{for } a \geq 1, \\ \max(a, b) & \text{for } a < 1. \end{cases}$$

Lemma 7.1. *Under the assumptions and notation above there exist constants* $E > 0$, $E' > 0$, *depending only on* κ, K, *and* a *such that*

$$(7.8) \qquad \|M_j - A_j M_j' A_j^{-1}\| < DEa_1^j + b^{2(j-v)} \|M_v - A_v M_v' A_v^{-1}\|,$$

$$(7.9) \qquad \begin{aligned} &|\ln \det(I + \lambda_{j+1} M_j)(I + \lambda_{j+1}' M_j')^{-1}| \\ &\quad < DE' a_1^j + (n-1) d b^{2(j-v)+1} \|M_v - A_v M_v' A_v^{-1}\| \end{aligned}$$

for $1 \leq v \leq j \leq p$.

Proof. First notice that $|\lambda_i - \lambda_i'| < D(1+a)a^i$. Moreover, $\min \operatorname{spec} M_{i-1} \geq \min \operatorname{spec} \widetilde{\psi}_{i-1} (i \geq 2)$ implies (see subsection 2.1)

$$\|(I + \lambda_i M_{i-1})^{-1}\| \leq b, \qquad \|M_{i-1}(I + \lambda_i M_{i-1})^{-1}\| \leq 1/\lambda_i \leq d_0.$$

Introduce $L_i = A_i M_i' A_i^{-1} : \sqcap_i \to \sqcap_i$. Then $\sigma_i' = A_i^{-1} \sigma_i A_i$ and we obtain

$$L_i = \sigma_i B_i L_{i-1} (I + \lambda_i' L_{i-1})^{-1} B_i^{-1} \sigma_i + A_i \widetilde{\psi}_i' A_i^{-1},$$

where $B_i = A_i A_{i-1}^{-1}$. Combining (7.5) with the inequality

$$\|X - B_i Y B_i^{-1}\| \leq 2\|X\| \cdot \|I - B_i\| + \|X - Y\|,$$

we deduce

$$\begin{aligned} \|M_i - L_i\| &\leq \|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - B_i L_{i-1}(I + \lambda_i' L_{i-1})^{-1} B_i^{-1}\| \\ &\quad + \|\widetilde{\psi}_i - A_i \widetilde{\psi}_i' A_i^{-1}\| < C_1 D(1+a)a^i + 2C_1 d_0 D(1+a)^2 a^{i-1} \\ &\quad + \|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\|. \end{aligned}$$

On the other hand, using subsection 2.4, we get

$$\begin{aligned} &\|M_{i-1}(I + \lambda_i M_{i-1})^{-1} - L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\| \\ &\quad \leq |\lambda_i - \lambda_i'| \cdot \|(I + \lambda_i M_{i-1})^{-1} M_{i-1}\| \cdot \|L_{i-1}(I + \lambda_i' L_{i-1})^{-1}\| \\ &\qquad + b^2 \|M_{i-1} - L_{i-1}\| \\ &\quad < Dd_0^2 (1+a) a^i + b^2 \|M_{i-1} - L_{i-1}\|. \end{aligned}$$

Therefore,

$$\|M_i - L_i\| < DE'' a^i + b^2 \|M_{i-1} - L_{i-1}\|, \qquad i = 2, \ldots, p,$$

with $E'' = (1+a)(C_1 + 2d_0(1+a)a^{-1} C_1 + d_0^2) > 0$.

By induction on j we get

$$\|M_j - L_j\| < DE'' \sum_{r=0}^{j-v-1} a^{j-r} b^{2r} + b^{2(j-v)} \|M_v - L_v\| \tag{7.10}$$

for $1 \le v \le j \le p$.

Case 1. $a \ge 1$. Then $a > b^2$ and (7.10) yields

$$\begin{aligned} \|M_j - L_j\| &< DE'' a^j \sum_{r=0}^{j-v-1} (b^2/a)^r + b^{2(j-v)} \|M_v - L_v\| \\ &< DE'' a^j (1 - b^2/a)^{-1} + b^{2(j-v)} \|M_v - L_v\|. \end{aligned}$$

In this case we set $E = E'' a(a - b^2)^{-1}$.

Case 2. $a < 1$. Taking $a_1 = \max(a, b)$, by (7.10) we have

$$\begin{aligned} \|M_j - L_j\| &< DE'' \sum_{r=0}^{j-v-1} a_1^{j+r} + b^{2(j-v)} \|M_v - L_v\| \\ &< DE''(1 - a_1)^{-1} a_1^j + b^{2(j-v)} \|M_v - L_v\|. \end{aligned}$$

Then we set $E = E''(1 - a_1)^{-1}$.

Consequently, we have proved (7.8).

To verify (7.9), we use the estimate

$$\begin{aligned} \det[(I + \lambda_{i+1} M_i)(I + \lambda'_{i+1} M'_i)^{-1}] &\le (1 + \|I - (I + \lambda_{i+1} M_i)(I + \lambda'_{i+1} L'_i)^{-1}\|)^{n-1} \\ &= (1 + \|(\lambda'_{i+1} L_i - \lambda_{i+1} M_i)(I + \lambda'_{i+1} L_i)^{-1}\|)^{n-1} \\ &\le \left(1 + \frac{|\lambda'_{i+1} - \lambda_{i+1}|}{\lambda'_{i+1}} + b\lambda_{i+1} \|M_i - L_i\|\right)^{n-1} \\ &< (1 + D(1 + a) d_0 a^{i+1} + bd\|M_i - L_i\|)^{n-1}. \end{aligned}$$

Similar equality holds for $\det[(I + \lambda_{i+1} M_i)^{-1}(I + \lambda'_{i+1} M'_i)]$ and we obtain

$$\begin{aligned} &|\ln\det(I + \lambda_{i+1} M_i) - \ln\det(I + \lambda'_{i+1} M'_i)| \\ &\quad < (n - 1) \ln(1 + Dd_0(1 + a) a^{i+1} + bd\|M_i - L_i\|) \\ &\quad < (n - 1)(Dd_0 + (1 + a) a^{i+1} + bd\|M_i - L_i\|). \end{aligned}$$

Applying (7.8) for $j = i$, we arrange (7.9) with

$$E' = (n - 1).((a_1^2 + a_1) d_0 + bdE).$$

This completes the proof of Lemma 7.2.

Let $\tilde{x}_1, \dots, \tilde{x}_k$ be the reflection points of the periodic ray γ_α related to α, where $\tilde{x}_j \in \partial K_{i_j}$ for $j = 1, \dots, k$. For convenience we set $\tilde{x}_m = \tilde{x}_j$ for $m = pk + j$, $1 \le j \le k$. Let $\tilde{\tilde{\psi}}_j \colon \tilde{\Pi}_j \to \tilde{\Pi}_j$ be the operator related to

$[\tilde{x}_j, \tilde{x}_{j+1}]$ and let $\mathfrak{M}(\tilde{\Pi}_j)$ be the space of all symmetric positive definite linear operators $M : \tilde{\Pi}_j \to \tilde{\Pi}_j$. Consider the map $\mathscr{F}_j : \mathfrak{M}(\tilde{\Pi}_j) \to \mathfrak{M}(\tilde{\Pi}_{j+1})$ given by $\mathscr{F}_j(M) = \tilde{\sigma}_{j+1} M(I + \tilde{\lambda}_{j+1} M)^{-1} \tilde{\sigma}_{j+1} + \tilde{\tilde{\psi}}_{j+1}$, where $\tilde{\lambda}_j = \|\tilde{x}_j - \tilde{x}_{j-1}\|$ and $\tilde{\sigma}_{j+1}$ is the symmetry with respect to $\tilde{\Pi}_{j+1}$. It is easy to show (cf. the proof of Theorem 6.2 in [22]) that the map $\mathscr{F}_k \circ \mathscr{F}_{k-1} \circ \cdots \circ \mathscr{F}_1 : \mathfrak{M}(\tilde{\Pi}_1) \to \mathfrak{M}(\tilde{\Pi}_1)$ has a unique fixed point $\tilde{M}_1$. Then $\tilde{M}_2 = \mathscr{F}_1(\tilde{M}_1)$ is the unique fixed point of $\mathscr{F}_1 \circ \mathscr{F}_k \circ \mathscr{F}_{k-1} \circ \cdots \circ \mathscr{F}_2$, etc.

Let $m = qk + l$. Introduce the configuration $\alpha_{q,l}$ (cf. the beginning of §6). The corresponding map $J_{\alpha_{q,l}}$ will be denoted briefly by J_q. Recall that $J_q : F_{\alpha_{q,l}} \to S^{n-1}$ and $F_{\alpha_{q,l}} \subset Z_\omega$. It follows by Proposition 6.1 and Corollary 3.3 that there exists a unique $u_q \in F_{\alpha_{q,l}}$ such that $\gamma(u_q)$ is an (ω, θ)-ray of type $\alpha_{q,l}$. Moreover, $\theta \in \mathscr{R}(\omega)$ implies $u_q \in U_{\alpha_{q,l}}$. Thus $dJ_q(u_q)$ is well defined and we will examine the asymptotic of $\det dJ_q(u_q)$ as $q \to \infty$.

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_m, y_{m+1}, \ldots, y_{m+k}$ be the successive reflection points of $\gamma(u_q)$ and $\gamma(u_{q+1})$, respectively. Set $\lambda_i = \|x_{i-1} - x_i\|$, $\lambda_i' = \|y_{i-1} - y_i\|$ and denote by $\tilde{\psi}_i$, $\tilde{\psi}_i'$ the operators related to $[x_i, x_{i+1}]$ and $[y_i, y_{i+1}]$, respectively. Introduce M_i and M_i' by (7.6), where σ_i, σ_i' are the symmetries with respect to Π_i and Π_i'. Setting

$$p = [m/2], \qquad t = [p/2], \tag{7.11}$$

we have $2p \leq m < 2p+1$, $2t \leq p < 2t+1$, hence $4t \leq m < 4t+3$. According to Lemma 6.4 and (6.3), we obtain

$$\|x_i - y_i\| < C\delta^{p-i}, \qquad i = 1, \ldots, t, \tag{7.12}$$

$$\|x_i - y_i\| < C\delta^{i}, \qquad i = t+1, \ldots, p, \tag{7.13}$$

$$\|x_{p+i} - y_{p+k+i}\| < C\delta^{p-i}, \qquad i = 1, \ldots, t, \tag{7.14}$$

$$\|x_{p+i} - y_{p+k+i}\| < C\delta^{i}, \qquad i = t+1, \ldots, m-p, \tag{7.15}$$

$$\|y_{vk+j} - \tilde{x}_j\| < C\delta^{vk+j}, \qquad vk + j \leq p + k, \tag{7.16}$$

where $C > 0$ and $0 < \delta < 1$ depend only on K, ω, and θ. Taking $D = C\delta^p$, $a = 1/\delta > 1$, (7.12) becomes

$$\|x_i - y_i\| < Da^i, \qquad i = 1, \ldots, t,$$

and we are in a position to apply Lemma 7.1 to the sequences $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$. Since

$$\|M_1 - A_1 M_1' A_1^{-1}\| = \|\tilde{\psi}_1 - A_1 \tilde{\psi}_1' A_1^{-1}\| < C_1 D(1+a)a < 2C_1 C\delta^{p-2},$$

applying (7.9) for $v = 1$ and $a_1 = 1/\delta$, we get

$$\begin{aligned}(7.17)\qquad |\ln\det(I+\lambda_{i+1}M_i) - \ln\det(I+\lambda'_{i+1}M'_i)| \\ < DE'\delta^{-i} + 2(n-1)db^{2i-1}C_1C\delta^{p-2} \\ < DE'\delta_1^{p-i} + 2(n-1)dCC_1\delta_1^{p+2i-3} \\ < F_1\delta_1^{p-i}, \qquad i = 1, \dots, t,\end{aligned}$$

where $F_1 = CE' + 2(n-1)dCC_1$ and $\delta_1 = \max(\delta, b) < 1$.

Observe that (7.12) implies $\|x_i - y_i\| < C\delta^i$, $i = 1, \dots, t$. Combining this with (7.13), we can apply Lemma 7.1 for $D = C$, $a = \delta = a_1$, $v = 1$. Then we find

$$(7.18)\qquad |\ln\det(I+\lambda_{i+1}M_i) - \ln\det(I+\lambda'_{i+1}M'_i)| < F_1\delta_1^i, \qquad i = 1, \dots, p,$$

where F_1 and δ_1 are the same as above.

Further, we apply Lemma 7.1 for the sequences $\tilde{x}_1, \dots, \tilde{x}_{p+k}$ and $y_1, \dots, y_{p+k}$. By (7.1) and (7.3) we obtain $\|M'_1\| = \|\tilde{\varphi}'_1\| \le 2\mu_2\kappa^{-1}$ and (7.9) yields

(7.19)

$$|\ln\det(I+\lambda_{p+j+1}M'_{p+j}) - \ln\det(I+\tilde{\lambda}_{p+j+1}\tilde{M}_{p+j})| < F_2\delta_1^{p+j}, \qquad j = 1, \dots, k,$$

where $F_2 > 0$ depends only on K, ω, and θ.

Next, consider the sequences $x_1, \dots, x_p$ and $y_{k+1}, \dots, y_{k+p}$ and denote by $A'_j : \sqcap'_{j+k} \to \sqcap_j$ the corresponding isometries. Applying Lemma 7.1, we find

$$(7.20)\qquad \|M_p - A'_p M'_{p+k} A'^{-1}_p\| < F_3\delta_1^p$$

with some constant $F_3 = F_3(K, \omega, \theta) > 0$. Consider the sequences $x_{p+1}, \dots, x_{p+t}$ and $y_{p+k+1}, \dots, y_{p+k+t}$. Taking into account (7.14) and (7.20), by Lemma 7.1 for $D = C\delta^p$, $a = 1/\delta = a_1$, and $v = p$, we obtain

(7.21)

$$|\ln\det(I+\lambda_{p+j+1}M_{p+j}) - \ln\det(I+\lambda'_{p+k+j+1}M'_{p+j+k})| < F_4\delta_1^{p-j}, \quad j = 1, \dots, t,$$

where F_4 depends only on K, ω, and θ.

Finally, we apply Lemma 7.1 two more times and find

$$(7.22)\qquad |\ln\det(I+\lambda_{p+j+1}M_{p+j}) - \ln\det(I+\lambda'_{p+k+j+1}M'_{p+k+j})| < F_5\delta^j, \qquad j = t+1, \dots, m-p,$$

$$(7.23)\qquad |\ln|\det M_m| - \ln|\det M'_{m+k}|| < F_6\delta^p,$$

where $F_5 > 0$ and $F_6 > 0$ depend only on K, ω, and θ.

Set $F = \max\{F_1, \dots, F_6\}$ and

$$(7.24)\qquad \tilde{c} = -\sum_{j=1}^{k} \ln\det(I+\tilde{\lambda}_{j+1}\tilde{M}_j) < 0.$$

By the representations of $dJ_q(u_q)$ and $dJ_{q+1}(u_{q+1})$ (cf. (3.10)) we obtain

$$(7.25)\qquad \ln|\det dJ_{q+1}(u_{q+1})| = \ln|\det dJ_q(u_q)| - \tilde{c} + \varepsilon_{q,l},$$

where

$$\varepsilon_{q,l} = \sum_{i=1}^{p}[\ln\det(I+\lambda'_{i+1}M'_i) - \ln\det(I+\lambda_{i+1}M_i)]$$
$$+\sum_{i=1}^{k}[\ln\det(I+\lambda'_{p+i+1}M'_{p+i}) - \ln\det(I+\tilde{\lambda}_{p+i+1}\tilde{M}_{p+i})]$$
$$+\sum_{j=1}^{m-p-1}[\ln\det(I+\lambda'_{p+k+j+1}M'_{p+k+j}) - \ln\det(I+\lambda_{p+j+1}M_{p+j})]$$
$$+[\ln|\det M'_{m+k}| - \ln|\det M_m|].$$

Now (7.17), (7.18), (7.19), (7.21), (7.22), and (7.23) yield

$$|\varepsilon_{q,l}| < F\left(2\sum_{i=1}^{t}\delta_1^{p-i} + \sum_{i=t+1}^{p}\delta_1^{i} + \sum_{i=1}^{k}\delta_1^{p+i} + \sum_{j=t+1}^{m-p-1}\delta_1^{j} + \delta_1^{p}\right) \tag{7.26}$$
$$< 6F(1-\delta_1)^{-1}\delta_1^t < F'_0\delta_0^{kq}$$

with $\delta_0 = \delta_1^{1/4}$, $F'_0 = 6F(1-\delta_1)^{-1}\delta_1^{-3/4}$.

Let $x_1^\infty, x_2^\infty, \dots$ be the successive reflection points of the ray $\gamma^\infty(\omega)$ related to the configuration α and constructed in the previous section. Set $\lambda_i^\infty = \|x_{i-1}^\infty - x_i^\infty\|$ and let M_i^∞ be the corresponding symmetric positive definite linear operator. Set $\beta = (i_1, \dots, i_l)$ and

$$c_l(\omega,\theta) = \ln|\det M_l^\infty| + \sum_{i=1}^{l-1}\ln\det(I+\lambda_{i+1}^\infty M_i^\infty) + \sum_{j=1}^{\infty}\varepsilon_{j,l}.$$

By (7.25) we obtain

$$\ln|\det dJ_q(u_q)| = -q\tilde{c} + c_l(\omega,\theta) + \delta_{q,l}$$

with $\delta_{q,l} = -\sum_{j=q}^{\infty}\varepsilon_{j,l} + [\ln|\det dJ_\beta(u_q)| - \ln|\det dJ_\beta(x_0^\infty)|]$. The term in the brackets can be estimated by $F''_0\delta_0^{kq}$ with F''_0 depending only on K, ω, and θ. Therefore, by (7.26) we get

$$|\delta_{q,l}| < F'_0\delta_0^{kq}(1-\delta_0^k)^{-1} + F''_0\delta_0^{kq} < F_0\delta_0^{kq}.$$

Now let γ_m^{1l} be the (ω,θ)-ray with m reflections, introduced in §6, and let c_m^{1l} be the coefficient in front of $\delta^{(n-1)/2}(t+T_m^{1l})$ in formula (1.5) for the leading singularity at $-T_m^{1l}$. Set $c_\alpha = \tilde{c}/2$. The above considerations lead to

Theorem 7.3. *There exist constants* $C_{\alpha,\omega,\theta}^{1l}$, $0<\delta_0<1$, *depending only on* α, ω, θ, *and* $l = 1, \dots, k$ *such that*

$$\ln|c_{qk+l}^{1l}| = qc_\alpha + C_{\alpha,\omega,\theta}^{1l} + O(\delta_0^q), \qquad q\to\infty. \tag{7.27}$$

Remark 7.4. It follows easily by our argument in §6 of [22] that $\mu_\alpha = \prod_{j=1}^{n-1}|\mu_j| = \prod_{j=1}^{k}\det(I+\tilde{\lambda}_{j+1}\tilde{M}_j)$, $|\mu_j|>1$, μ_j, $j=1,\dots,n-1$, being the eigenvalues of the (linear) Poincaré map related to γ_α. Consequently, from the asymptotics (7.27) we can recover μ_α.

Appendix

Here we prove the following lemma, which has been applied in §7. We use some notation from §7.

Lemma A. *Let (x, y) and (x', y') be two pairs of points with $x, x' \in \partial K_i$ and $y, y' \in \partial K_j$, $i \neq j$, and let $\varepsilon > 0$ be such that $\|x - x'\| \leq \varepsilon$ and $\|y - y'\| \leq \varepsilon$. Assume $\langle e, \nu(x)\rangle \geq \kappa$ and $\langle e', \nu(x')\rangle \geq \kappa$ for $e = (y - x)/\|y - x\|$ and $e' = (y'-x')/\|y'-x'\|$. Let $\widetilde{\psi} : \sqcap \to \sqcap$ and $\widetilde{\psi}' : \sqcap' \to \sqcap'$ be the operators related to $[x, y]$ and $[x', y']$, respectively. Then there exists an isometry $A : \mathbf{R}^n \to \mathbf{R}^n$ with $A(\sqcap') = \sqcap$ and $\|A - I\| < C_1\varepsilon$ such that $\|\widetilde{\psi} - A\widetilde{\psi}'A^{-1}\| < C_1\varepsilon$, where $C_1 > 0$ is a constant depending only on κ and K.*

Proof. Let $A_1 : \mathbf{R}^n \to \mathbf{R}^n$ be the translation determined by $\overrightarrow{x'x}$. Set $\nu'' = A_1(\nu(x'))$ and denote by A_2 the rotation with rotation angle

$$\varphi = \arccos\langle \nu(x), \nu(x')\rangle$$

such that $A_2(\nu'') = \nu$, $A_2|_{\{\nu(x), \nu''\}^\perp} = \mathrm{Id}$. Set $e'' = A_2 \circ A_1(e')$ and denote by A_3 the rotation with $A_3(e'') = e$ and $A_3|_{\{e, e''\}^\perp} = \mathrm{Id}$. Set $A = A_3 \circ A_2 \circ A_1$. It is easy to see now that $\|A_i - I\| < \mathrm{const}\,\varepsilon$ for $i = 1, 2, 3$. For example,

$$\|A_2 - I\| = \left\| \begin{pmatrix} 1 - \cos\varphi & \sin\varphi \\ -\sin\varphi & 1 - \cos\varphi \end{pmatrix} \right\| = \sqrt{2(1 - \cos\varphi)} = \|\nu(x) - \nu(x')\|,$$

and using the smoothness of the Gauss map on ∂K and the compactness of ∂K, we find $\|\nu(x) - \nu(x')\| < \mathrm{const}\,\|x - x'\| < \mathrm{const}\,\varepsilon$.

Therefore $\|A - I\| < C_1'\varepsilon$ for some constant $C_1' > 0$.

For convenience set $\tilde{\chi} = A\widetilde{\psi}'A^{-1}$, $G = G_x$, $G' = G_{x'}$. Take $u \in \sqcap$ with $\|u\| = 1$ and set $u' = A^{-1}u$. Then $u' \in \sqcap'$ and $\|u'\| = 1$. Let $\pi : \sqcap \to T_x\partial K$ and $\pi' : \sqcap' \to T_{x'}\partial K$ be the projections along e and e', respectively, and let $v = \pi u$, $v' = \pi' u'$. Then we have

$$\begin{aligned} |\langle \widetilde{\psi} u, u\rangle - \langle \tilde{\chi} u, u\rangle| &= |\langle \widetilde{\psi} u, u\rangle - \langle \widetilde{\psi}' u', u'\rangle| = |2\langle e, \nu(x)\rangle\langle G\pi u, \pi u\rangle \\ &\quad - 2\langle e', \nu(x')\rangle\langle G'\pi' u', \pi' u'\rangle| \leq 2|\langle e, \nu(x)\rangle - \langle e', \nu(x')\rangle|\langle Gv, v\rangle \\ &\quad + 2\langle e', \nu(x')\rangle|\langle Gv, v\rangle - \langle G'v', v'\rangle| < \mathrm{const}\,\varepsilon + 2|\langle Gv, v\rangle - \langle G'v', v\rangle|. \end{aligned}$$

By the smoothness of the Riemannian metric on ∂K we get $|\langle Gv, v\rangle - \langle G'v', v'\rangle| < \mathrm{const}\,\|v - v'\|$. Finally, using $\|\pi\| \leq 1/\kappa$ and $\|\pi'\| \leq 1/\kappa$, we find

$$\|v - v'\| = \|\pi u - \pi' A^{-1} u\| \leq \|\pi - \pi' A^{-1}\| + \|\pi'\|\|A - I\| < \mathrm{const}\,\varepsilon.$$

Therefore $|\langle(\widetilde{\psi} - \tilde{\chi})u, u\rangle| < \mathrm{const}\,\varepsilon$ and this yields $\|\widetilde{\psi} - \tilde{\chi}\| < C_1''\varepsilon$ for some constant $C_1'' > 0$.

Setting $C_1 = \max(C_1', C_1'') > 0$, we get $\|A - I\| < C_1\varepsilon$ and $\|\widetilde{\psi} - A\widetilde{\psi}'A^{-1}\| < C_1\varepsilon$.

REFERENCES

1. C. Bardos, J. C. Guillot, and J. Ralston, *La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion*, Comm. Partial Differential Equations **7** (1982), 905–958.
2. Y. Colin de Verdiére, *Sur les longueurs des trajectories périodiques d'un billiard*, Géométrie Symplectique et de Contact: Autour du Théorème de Poincaré-Birkhoff, Hermann, 1984, pp. 122–139.
3. J. J. Duistermaat and V. Guillemin, *The spectrum of positive elliptic operators and periodic geodesics*, Invent. Math. **29** (1975), 39–79.
4. C. Gérard, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes*, Prépublications de l'Université Paris-Sud, 1986.
5. V. Guillemin, *Sojourn time and asymptotic properties of the scattering matrix*. Publ. Res. Inst. Math. Sci. **12** (1977), 69–88.
6. V. Guillemin and R. Melrose, *The Poisson summation formula for manifolds with boundary*, Adv. in Math. **32** (1979), 204–232.
7. M. Ikawa, *On the poles of the scattering matrix for two strictly convex obstacles*, J. Math. Kyoto Univ. **23** (1983), 127–194.
8. ——, *Precise information on the poles of the scattering matrix for two strictly convex obstacles*, J. Math. Kyoto Univ. **27** (1987), 69–102.
9. ——, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier **38** (1988), 113–146.
10. P. Lax and R. Phillips, *Scattering theory*, Academic Press, New York, 1967.
11. A. Majda, *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*, Comm. Pure Appl. Math. **30** (1977), 165–194.
12. S. Marvizi and R. Melrose, *Spectral invariants and convex planar regions*, J. Differential Geometry **17** (1982), 472–502.
13. R. Melrose, *Singularities and energy decay of acoustical scattering*, Duke Math. J. **46** (1979), 43–59.
14. R. Melrose and J. Sjöstrand, *Singularities in boundary value problems*. I, II, Comm. Pure Appl. Math. **31** (1978), 593–617 and **35** (1982), 129–168.
15. S. Nakamura and H. Soga, *Singularities of the scattering kernel for two balls*, J. Math. Soc. Japan **40** (1988), 205–220.
16. V. Petkov, *High frequency asymptotics of the scattering amplitude for non-convex bodies*, Comm. Partial Differential Equations **5** (1980), 293–329.
17. ——, *Singularities of the scattering kernel*, Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, VI, Pitman, 1984, pp. 288–296.
18. V. Petkov and P. Vogel, *La représentation de l'application de Poincaré correspondant aux rayons périodiques réflechissants*, C. R. Acad. Sci. Paris Ser. A **296** (1983), 633–635.
19. V. Petkov and L. Stojanov, *Periodic geodesics of generic nonconvex domains in* $\mathbf{R}^2$ *and the Poisson relation*, Bull. Amer. Math. Soc. **15** (1986), 88–90.
20. ——, *Propriétés génériques de l'application de Poincaré et des géodesiques périodiques généralisées*, Seminaire Equations aux Dérivées Partielles, Ecole Polytechnique, Exposé XI, 1985–1986.
21. ——, *Spectrum of the Poincaré map for periodic reflecting rays in generic domains*, Math. Z. **194** (1987), 505–518.
22. ——, *Periods of multiple reflecting geodesics and inverse spectral results*, Amer. J. Math. **109** (1987), 619–668.
23. ——, *Singularities of the scattering kernel for non-convex obstacles*, Conférence Equations aux Dérivées Partielles, Saint-Jean-de-Monts, 1987.

24. Ya. G. Sinai, *Developments of Krylov ideas*, An addendum to the book: N. S. Krylov: Works on the Foundations of Statistical Physics, Princeton Univ. Press, 1979.

25. ——, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Uspehi Mat. Nauk **25** (1970), 141–192 (Russian).

26. H. Soga, *Singularities of the scattering kernel for convex obstacles*, J. Math. Kyoto Univ. **22** (1983), 729–765.

27. ——, *Conditions against rapid decrease of oscillatory integrals and their applications to inverse scattering problems*, Osaka J. Math. **23** (1986), 441–456.

28. N. Tchernov, *Construction of transversal foliations for multidimensional semi-scattering billiards*, Functional Anal. Appl. **16** (1982), 33–46 (Russian).

29. K. Yamamoto, *Characterization of a convex obstacle by singularities of the scattering kernel*, J. Differential Equations **64** (1986), 283–293.

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