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QUADRATURE AND HARMONIC L^1-APPROXIMATION IN ANNULI

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ABSTRACT. Open sets D in R^N $(N \geq 3)$ with the property that $\overline{D}$ is a closed annulus $\{x: r_1 \leq \|x\| \leq r_2\}$ are characterized by quadrature formulae involving mean values of certain harmonic functions. One such characterization is used to give a criterion for the existence of a best harmonic L^1 approximant to a function which is subharmonic (and satisfies some other conditions) in an annulus.

1. INTRODUCTION

Several authors have shown that, in various ways, balls in R^N $(N \geq 2)$ are characterized by the volume mean value property of harmonic functions. A very elegant characterization of this kind has been given by Kuran [10]. Denoting Lebesgue measure on R^N by λ, we can state his result as follows: if D is a domain in R^N such that $\lambda(D) < +\infty$ and if there is a point $x \in D$ such that $h(x) = (\lambda(D))^{-1} \int_D h\, d\lambda$ for every function that is harmonic and integrable in D, then D is a ball of centre x. Another result of this kind is due to Goldstein, Haussmann and Rogge [9]: if D is a bounded, open set in R^N such that $R^N \backslash \overline{D}$ is connected and if

$$h(x) = (\lambda(\overline{D}))^{-1} \int_{\overline{D}} h\, d\lambda \tag{1.1}$$

for some $x \in R^N$ and every function h harmonic in R^N, then $\overline{D}$ is a closed ball of centre x. (Note that here we cannot conclude that D is a ball.) In the first part of this paper we give similar theorems in which the conclusion is that $\overline{D}$ is a ball or an annulus centered at the origin, that is, $\overline{D} = \{x: r_1 \leq \|x\| \leq r_2\}$, where $0 \leq r_1 < r_2$. One of our results (Theorem 2.3(i)) improves the result from [9] stated above. It will be convenient to work only with $N \geq 3$; for quadrature formulae characterizing plane annuli, we refer to Sakai's work [11, Example 1.2]. Kuran's result [10], quoted above, is central to the proof given by Goldstein, Haussmann and Rogge [9], but the short, simple proof in [10] seems not to be adaptable to our requirements. Our proofs therefore differ substantially from that in [9]. The proof of one of our results (Theorem 2.2)

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was suggested by a paper of Avci [1] in which a somewhat similar theorem is stated. Unfortunately, both the statement and proof in [1] contain serious errors, and substantial modifications are required to prove Theorem 2.2.

In the second part of the paper we use one of our characterizations of annuli by quadrature formulae to establish a criterion for a subharmonic function satisfying certain regularity conditions in an annulus to have a best harmonic L^1 approximant. Our result is analogous to that given for the unit ball in [9].

2. Characterization of annuli by means of quadrature formulae

We start with some notation and terminology. Throughout this section we work in the Euclidean space R^N, where $N \geq 3$. We denote the Euclidean norm of a point $x \in R^N$ by $\|x\|$. If $0 \leq r_1 < r_2$, we put

$$A(r_1, r_2) = \{x \in R^N : r_1 < \|x\| < r_2\}$$

and call $A(r_1, r_2)$ an *annulus*. Further, we call $A(r_1, r_2)$ an *r-annulus* if

$$r = \left(\frac{2}{N} \frac{r_2^N - r_1^N}{r_2^2 - r_1^2}\right)^{1/(N-2)},$$

and we then call $\overline{A}(r_1, r_2)$, the closure of $A(r_1, r_2)$, a *closed r-annulus*. Note that if $A(r_1, r_2)$ is an r-annulus, then $r_1 < r < r_2$. By allowing $r_1 = 0$ we admit the possibility that a closed r-annulus is a closed ball.

If $\rho > 0$, then we denote by $B(x, \rho)$ and $S(x, \rho)$ the open ball and the sphere of centre x and radius r in R^N. The origin of R^N is denoted by O. We use σ to denote surface area measure on $S(O, \rho)$ normalized so that $\sigma(S(O, \rho)) = 1$. If $f \in L^1(S(O, \rho))$, let

$$M(f, \rho) = \int_{S(O,\rho)} f \, d\sigma. \tag{2.2}$$

The volume of $B(O, 1)$ and the surface area of $S(O, 1)$ are denoted by v_N and σ_N. Note that $\sigma_N = N v_N$.

Proposition 2.1. *If s is subharmonic and integrable in an r-annulus $A(r_1, r_2)$, then*

$$(\lambda(A(r_1, r_2)))^{-1} \int_{A(r_1,r_2)} s \, d\lambda \geq M(s, r). \tag{2.3}$$

We use a method of Beardon [2] to prove this result. It is well known (see, e.g. [5, p. 24]) that $M(s, t)$ is a convex function of t^{2-N} for $t \in (r_1, r_2)$. Hence

$$M(s, t) = \phi \circ f(t) \qquad (r_1 < t < r_2), \tag{2.4}$$

where $f(t) = t^{2-N}$ and ϕ is convex on (r_2^{2-N}, r_1^{2-N}). Using the definition of f, the equation

$$(r_2^N - r_1^N)^{-1} \int_{r_1}^{r_2} N t^{N-1} \, dt = 1, \tag{2.5}$$

and Jensen's inequality, we obtain

$$\begin{aligned}
M(s,r) &= \phi\circ f\left(\left(\frac{2}{N}\frac{r_2^N - r_1^N}{r_2^2 - r_1^2}\right)^{1/(N-2)}\right) \\
&= \phi\left(\frac{N}{2}\frac{r_2^2 - r_1^2}{r_2^N - r_1^N}\right) \\
&= \phi\left((r_2^N - r_1^N)^{-1}\int_{r_1}^{r_2} f(t)Nt^{N-1}\,dt\right) \\
&\le (r_2^N - r_1^N)^{-1}\int_{r_1}^{r_2} \phi\circ f(t)Nt^{N-1}\,dt \\
&= (r_2^N - r_1^N)^{-1}\int_{r_1}^{r_2} M(s,t)Nt^{N-1}\,dt \\
&= (\lambda(A(r_1,r_2)))^{-1}\int_{A(r_1,r_2)} s\,d\lambda.
\end{aligned}$$

Corollary 2.1. *If h is harmonic and integrable in an r-annulus $A(r_1,r_2)$, then*

$$(\lambda(A(r_1,r_2)))^{-1}\int_{A(r_1,r_2)} h\,d\lambda = M(h,r). \tag{2.6}$$

This follows immediately by applying Prposition 2.1 to both h and $-h$.

The corollary can also be proved directly by using the fact that $M(h,t) = \alpha t^{2-N} + \beta$ for $t \in (r_1,r_2)$, where α and β are constants and noting that the left-hand side of (2.6) is equal to

$$\begin{aligned}
(r_2^N - r_1^N)^{-1}\int_{r_1}^{r_2} M(h,t)Nt^{N-1}\,dt &= \frac{\alpha N}{2}\left(\frac{r_2^2 - r_1^2}{r_2^N - r_1^N}\right) + \beta \\
&= M(h,r),
\end{aligned}$$

by (2.1).

Our next result is a partial converse to Corollary 2.1. For each $x \in R^N$ define

$$h_x(\xi) = \|x-\xi\|^{2-N} \qquad (\xi \in R^N) \tag{2.7}$$

(so that $h_x(x) = +\infty$). Thus h_x is the fundamental superharmonic function of R^N with pole x, and h_x is harmonic in $R^N\backslash\{x\}$.

Theorem 2.1. *Let D be a nonempty open set in R^N such that $\lambda(\overline{D}) < +\infty$ and let $r > 0$. If for all $x \in R^N\backslash\overline{D}$, we have*

$$(\lambda(\overline{D}))^{-1}\int_{\overline{D}} h_x\,d\lambda = M(h_x,r), \tag{2.8}$$

then $\overline{D}$ is a closed r-annulus or a closed ball of centre O.

We shall deduce Theorem 2.1 from the folowing similar result, which is in a more convenient form for our applications.

Theorem 2.2. *Let D be a nonempty open set in R^N such that $\lambda(\overline{D}) < +\infty$ and let $c > 0$. If for all $x \in R^N \backslash \overline{D}$, we have*

$$(\lambda(\overline{D}))^{-1} \int_{\overline{D}} h_x \, d\lambda \in \{h_x(O), c\}, \tag{2.9}$$

then $\overline{D}$ is a closed r-annulus with $r = c^{-1/(N-2)}$ or $\overline{D}$ is a closed ball of centre O.

To prove Theorem 2.2, we put $\Omega = (\overline{D})^\circ$ (the interior of the closure of D) and note first that $D \subseteq \Omega \subset \overline{D}$, so that $\overline{D} = \overline{\Omega}$ and hence

$$\partial(R^N \backslash \overline{\Omega}) = \partial\overline{\Omega} = \partial\overline{D} = \overline{D} \backslash (\overline{D})^\circ = \overline{\Omega} \backslash \Omega = \partial\Omega. \tag{2.10}$$

Now, following Avci [1], define U in R^N by

$$U(x) = \int_{\overline{D}} h_x \, d\lambda. \tag{2.11}$$

Then U is a Newtonian potential in R^N (for, clearly, $U \not\equiv +\infty$). From a known result (see, e.g., [3, p. 228]) it follows that $U \in C^1(R^N)$. From (2.9) and (2.11), we have, since $\overline{\Omega} = \overline{D}$,

$$U(x) \in \{\|x\|^{2-N}\lambda(\overline{D}), c\lambda(\overline{D})\} \qquad (x \in R^N \backslash \overline{\Omega}), \tag{2.12}$$

and since $U \in C^1(R^N)$, it is easy to see that in each connected component ω of $R^N \backslash \overline{\Omega}$, we have either

$$U(x) = \|x\|^{2-N}\lambda(\overline{D}) \qquad (x \in \omega) \tag{2.13}$$

or

$$U(x) = c\lambda(\overline{D}) \qquad (x \in \omega). \tag{2.14}$$

If $x \in \Omega$, then $\Delta U(x) = (2 - N)\sigma_N$, where Δ is the Laplacian operator in R^N. Hence

$$U(x) = v(x) - (2N)^{-1}(N-2)\sigma_N \|x\|^2 \qquad (x \in \Omega), \tag{2.15}$$

where v is harmonic in Ω. From (2.13), (2.14) and (2.15) it follows that

$$\operatorname{grad} U(x) = \begin{cases} \lambda(\overline{D})(2-N)x\|x\|^{-N} \text{ or } O & (x \in R^N \backslash \overline{\Omega}), \\ \operatorname{grad} v(x) - N^{-1}(N-2)\sigma_N x & (x \in \Omega). \end{cases} \tag{2.16}$$

Now define

$$y \wedge z = \sum_{i=1}^{N} \sum_{j=1}^{i-1} (y_i z_j - y_j z_i)(e_i \wedge e_j),$$

where e_i $(i = 1, \ldots, N)$ is the vector in R^N whose ith component is 1 and whose other components are 0, and where $\wedge$ denotes the exterior product. Since $x \wedge x = O$ for any $x \in R^N$, we obtain from (2.16)

$$\operatorname{grad} U(x) \wedge x = \begin{cases} O & (x \in R^N \backslash \overline{\Omega}), \\ \operatorname{grad} v(x) \wedge x & (x \in \Omega). \end{cases} \tag{2.17}$$

Since $\operatorname{grad} U$ is continuous in R^N, so also is the function $\operatorname{grad} U(x) \wedge x$. Hence it follows from (2.17) that $\operatorname{grad} U(x) \wedge x = O$ on $\partial(R^N \backslash \overline{\Omega}) = \partial\Omega$ (see (2.10)). (At the corresponding point of his proof, Avci [1, p. 125], asserts that $\operatorname{grad} U(x) \wedge x = O$ on ∂D; this is false if $\partial D \neq \partial\Omega$—for example, if D has "slits" or "punctures". Thus the statement of Theorem 2 is in error. A further error arises from the fact that equation (2.1) in [1] cannot hold with, for example, $w \equiv 1$.)

Now let H be one of the components of $\operatorname{grad} v(x) \wedge x$ in Ω, that is

$$H(x) = x_j(\partial v/\partial x_i) - x_i(\partial v/\partial x_j) \tag{2.18}$$

for some i and j with $1 \le j < i \le N$. Calculation of ΔH shows that H is harmonic in Ω. Further, H vanishes continuously on $\partial\Omega$, since $\operatorname{grad} v(x) \wedge x = \operatorname{grad} U(x) \wedge x$ in Ω and $\operatorname{grad} U(x) \wedge x$ is continuous in R^N and equals O on $\partial\Omega$. It follows that the function u, defined in R^N by

$$u(x) = \begin{cases} |H(x)| & (x \in \Omega), \\ 0 & (x \notin \Omega), \end{cases} \tag{2.19}$$

is subharmonic in R^N.

For all $x \in R^N$ and all $i = 1, \dots, N$, we have

$$\begin{aligned} \left|\frac{\partial U}{\partial x_i}(x)\right| &= \left|\int_{\overline{D}} \frac{\partial}{\partial x_i} h_x(\xi)\, d\lambda(\xi)\right| \quad [3, \text{p. } 228] \\ &= (N-2)\left|\int_{\overline{D}} (x_i - \xi_i)\|x-\xi\|^{-N}\, d\lambda(\xi)\right| \\ &\le (N-2)\int_{\overline{D}} \|x-\xi\|^{1-N}\, d\lambda(\xi) \\ &\le (N-2)\int_{B(x,1)} \|x-\xi\|^{1-N}\, d\lambda(\xi) + (N-2)\int_{\overline{D}\backslash B(x,1)} d\lambda \\ &\le (N-2)(\sigma_N + \lambda(\overline{D})). \end{aligned} \tag{2.20}$$

Hence $\operatorname{grad} U$ is bounded in R^N, and so there exists $\kappa > 0$ such that

$$u(x) \le \kappa(1 + \|x\|) \qquad (x \in R^N). \tag{2.21}$$

Using the volume mean value inequality for subharmonic functions, we now obtain for all $x \in R^N$

$$\begin{aligned} 0 \le u(x) &\le (v_N \rho^N)^{-1} \int_{B(x,\rho)} u\, d\lambda \\ &\le (v_n \rho^N)^{-1} \int_{B(O,\|x\|+\rho)} u\, d\lambda \\ &\le (v_N \rho^N)^{-1} \kappa(1 + \|x\| + \rho)\lambda(\overline{D}) \\ &\to 0 \qquad (\rho \to +\infty). \end{aligned} \tag{2.22}$$

Hence $u \equiv 0$ in R^N, and so $\operatorname{grad} v(x) \wedge x \equiv O$ in Ω. Therefore $\operatorname{grad} v(x) = \alpha(x)x$ in Ω for some function $\alpha\colon \Omega \to R$.

Avci's calculation [1, pp. 126–127] shows that $\alpha(x) = k\|x\|^{-N}$ for some constant k. It now follows from (2.16) that

$$\operatorname{grad} U(x) = \begin{cases} \lambda(\overline{D})(2-N)x\|x\|^{-N} \text{ or } O & (x \in R^N\backslash\overline{\Omega}), \\ k\|x\|^{-N}x - N^{-1}(N-2)\sigma_N x & (x \in \Omega). \end{cases} \tag{2.23}$$

Since $\operatorname{grad} U$ is continuous in R^N and $\partial(R^N\backslash\overline{\Omega}) = \partial\Omega$, we find that at each point $x \in \partial\Omega$ either

$$k\|x\|^{-N}x - N^{-1}(N-2)\sigma_N x = \lambda(\overline{D})(2-N)x\|x\|^{-N} \tag{2.24}$$

or

$$k\|x\|^{-N}x - N^{-1}(N-2)\sigma_N x = 0. \tag{2.25}$$

Hence if $x \in \partial\Omega$, then $x = O$ or

$$\|x\|^N = (k + (N-2)\lambda(\overline{D}))N\{\sigma_N(N-2)\}^{-1} = \mu, \quad \text{say} \tag{2.26}$$

or

$$\|x\|^N = kN\{\sigma_N(N-2)\}^{-1} = \nu, \quad \text{say.} \tag{2.27}$$

If $k \le (2-N)\lambda(\overline{D})$, then $\nu < \mu \le 0$, and so $\partial\Omega \subseteq \{O\}$, an impossibility as $0 < \lambda(\Omega) < +\infty$. If $(2-N)\lambda(\overline{D}) < k \le 0$, then $\nu \le 0 < \mu$ and hence $\partial\Omega \subseteq \{O\} \cup S(O, r_2)$, where $r_2 = \mu^{1\backslash N}$. This inclusion and the inequalities $0 < \lambda(\Omega) < +\infty$ imply that $B(O, r_2)\backslash\{O\} \subseteq \Omega \subseteq B(O, r_2)$, so that $\overline{D} = \overline{\Omega} = \overline{B}(O, r_2)$. If $k > 0$, then $0 < \nu < \mu$ and we have

$$\partial\Omega \subseteq \{O\} \cup S(O, r_1) \cup S(O, r_2),$$

where $r_1 = \nu^{1/N}$ and $r_2 = \mu^{1/N}$. Given that $0 < \lambda(\Omega) < +\infty$, we find that

$$B(O, r_1)\backslash\{O\} \subseteq \Omega \subseteq B(O, r_1)$$

or

$$\Omega = A(r_1, r_2)$$

or

$$\Omega = B(O, r_2)\backslash E,$$

where $E \subseteq \{O\} \cup S(O, r_1)$ (and E may be empty). Hence

$$\overline{D} = \overline{\Omega} = \overline{B}(O, r_1), \quad \text{or} \quad \overline{A}(r_1, r_2) \quad \text{or} \quad \overline{B}(O, r_2).$$

It remains to show that if $\overline{D} = \overline{A}(r_1, r_2)$ with $r_1 > 0$, then $A(r_1, r_2)$ is an r-annulus with $r = c^{-1/(N-2)}$. In this case $O \in R^N\backslash\overline{D}$, and so by hypothesis (2.9)

$$(\lambda(\overline{D}))^{-1}\int_{\overline{D}} h_O \, d\lambda = c. \tag{2.28}$$

(The alternative that the left-hand side of (2.28) equals $h_o(O) = +\infty$ cannot hold since h_o is bounded in $\overline{D}$ and $0 < \lambda(\overline{D}) < +\infty$.) Since, also,

$$(\lambda(\overline{D}))^{-1} \int_{\overline{D}} h_o \, d\lambda = (r_2^N - r_1^N)^{-1} \int_{r_1}^{r_2} Nt \, dt = \frac{1}{2} N (r_2^N - r_1^N)^{-1} (r_2^2 - r_1^2), \tag{2.29}$$

the result follows.

To deduce Theorem 2.1 from Theorem 2.2, note that

$$M(h_x, r) = \begin{cases} r^{2-N} & (x \in B(O, r)), \\ \|x\|^{2-N} & (x \notin B(O, r)) \end{cases} \tag{2.30}$$

(see, e.g., [6, p. 100]) so that if the hypotheses of Theorem 2.1 are satisfied, then so also are the hypotheses of Theorem 2.2 with $c = r^{2-N}$.

In our next theorem, we use the following terminology. We say that a domain ω is *locally connected to* ∞ if there exists a piecewise linear continuous function $\phi: [0, +\infty) \to \omega$ such that $\|\phi(t)\| \to +\infty$ as $t \to +\infty$.

Theorem 2.3. *Let D be a nonempty open set in R^N such that $\lambda(\overline{D}) < +\infty$, and let $C = \{\omega_j : j \in J\}$ be the set of connected components of $R^N \backslash \overline{D}$ which are not locally connected to ∞. Let $r > 0$.*

(i) *If C is empty and if*

$$(\lambda(\overline{D}))^{-1} \int_{\overline{D}} h \, d\lambda = M(h, r) \tag{2.31}$$

for every function h harmonic in R^N, then $\overline{D}$ is a closed ball of centre O.

(ii) *If C is nonempty and if, for each $j \in J$, there exists $a_j \in \omega_j$ such that* (2.31) *holds for every function h harmonic in $R^N \backslash \{a_j\}$, then $\overline{D}$ is a closed r-annulus* (*and not a closed ball*).

For the proof, we need a definition and a result from [7]. If $y \in R^N$, we denote by S_y any finite sum

$$\sum_m H_m(z - y) \|z - y\|^{2-N-2m} \quad (z \in R^N \backslash \{y\}), \tag{2.32}$$

where H_m is a homogeneous harmonic polynomial of degree m in R^N, and we shall call any such function S_y a *nonessential singular function* (n.e.s.f.). Note that a n.e.s.f. S_y is harmonic in $R^N \backslash \{y\}$.

Lemma 2.1. *Suppose that $\rho > 0$, $\varepsilon > 0$ and y, z are points of R^N with $\|y - z\| < \rho$. If S_z is a n.e.s.f., then there exists a n.e.s.f. S_y such that*

$$|S_y - S_z| < \varepsilon \quad \textit{in } R^N \backslash B(y, \rho). \tag{2.33}$$

For the proof of this lemma we refer to [7, Lemma 4].

To prove Theorem 2.3, it suffices to show that (2.9) holds for each $x \in R^N \backslash \overline{D}$ with $c = r^{2-N}$, for then it will follow from Theorem 2.2 that $\overline{D}$ is either a closed ball of centre O or a closed r-annulus $\overline{A}(r_1, r_2)$ with $r_1 > 0$, and the latter will be the case if and only if C is nonempty. In fact it is enough to prove (2.9) for each $x \in R^N \backslash \overline{D}$ with $x \neq O$, for if $O \in R^N \backslash \overline{D}$ and (2.9) holds for each $x \in R^N \backslash (\overline{D} \cup \{O\})$, then a simple continuity argument shows that (2.9) also holds with $x = O$.

With a function h which is harmonic in the complement of some compact subset of R^N we associate constants α_h and β_h such that $M(h, t) = \alpha_h t^{2-N} + \beta_h$ for all sufficiently large t. Note that if $h = h_x$ for some $x \in R^N$, then $\alpha_h = 1$ and $\beta_h = 0$, and if h is harmonic in R^N, then $\alpha_h = 0$ and $\beta_h = h(O)$. The hypotheses of Theorem 2.3 imply that

$$(\lambda(\overline{D}))^{-1} \int_{\overline{D}} h \, d\lambda \in \{h(O), \alpha_h r^{2-N} + \beta_h\} \tag{2.34}$$

for every function h harmonic in R^N and (if C is nonempty) for every function h harmonic in $R^N \backslash \{a_j\}$ for some $j \in J$. (If $a_j = O$, then $h(O)$ is undefined for functions harmonic in $R^N \backslash \{a_j\}$ and is therefore suppressed in (2.34).) In the case where h is harmonic in R^N, (2.34) is immediate, for then $M(h, r) = h(O)$, and in the case where h is harmonic in $R^N \backslash \{a_j\}$, (2.34) follows since $M(h, r) = h(O)$ or $\alpha_h r^{2-N} + \beta_h$ according as $\|a_j\| > r$ or $\|a_j\| < r$ (and $\|a_j\| = r$ is impossible as (2.31) would fail by the nonexistence of $M(h, r)$ for some functions h harmonic in $R^N \backslash \{a_j\}$).

Now let $x \in R^N \backslash (\overline{D} \cup \{O\})$ and let ω be the connected component of $R^N \backslash \overline{D}$ to which x belongs.

Consider first the case where ω is not locally connected to ∞. Then $\omega = \omega_j$ for some $j \in J$ and so there exist $a_j \in \omega_j$ such that (2.31) holds for every function harmonic in $R^N \backslash \{a_j\}$. There exist finitely many balls $B(x_k, \rho_k)$ $(k = 0, 1, \ldots, K)$ such that

$$x_o = x, \; x_K = a_j, \; x_{k-1} \in B(x_k, \rho_k) \qquad (k = 1, \ldots, K), \tag{2.35}$$

$$B = \bigcup_{k=0}^{K} B(x_k, \rho_k) \subset \omega_j, \qquad O \notin B \text{ if } a_j \neq O.$$

Let $\varepsilon > 0$ and let $S_{x_o} = h_x$. By Lemma 2.1, there exist n.e.s.f.'s S_{x_k} such that

$$|S_{x_{k-1}} - S_{x_k}| < \varepsilon/K \text{ in } R^N \backslash B(x_k, \rho_k) \qquad (k = 1, \ldots, K). \tag{2.36}$$

Hence

$$|h_x - S_{x_K}| \leq \sum_{k=1}^{K} |S_{x_{k-1}} - S_{x_k}| < \varepsilon \quad \text{in } R^N \backslash B. \tag{2.37}$$

Put $h = S_{x_K}$. Then h is harmonic in $R^N\backslash\{x_K\} = R^N\backslash\{a_j\}$ and so (2.34) holds. By (2.37) we have $|h_x - h| < \varepsilon$ in $\overline{D} \cup (R^N\backslash B(O, R))$ for some $R > 0$ (R independent of ε), and hence

$$\left|(\lambda(\overline{D}))^{-1}\int_{\overline{D}} h_x\, d\lambda - (\lambda(\overline{D}))^{-1}\int_{\overline{D}} h\, d\lambda\right| < \varepsilon \tag{2.38}$$

and

$$|M(h_x, t) - M(h, t)| < \varepsilon \qquad (t \geq R). \tag{2.39}$$

Further, if $a_j \neq O$, then $O \in R^N\backslash B$, and so

$$|h_x(O) - h(O)| < \varepsilon. \tag{2.40}$$

If $a_j \neq O$ and the expression on the left-hand side of (2.34) equals $h(O)$, then (2.38) and (2.40) imply that

$$\left|(\lambda(\overline{D}))^{-1}\int_{\overline{D}} h_x\, d\lambda - h_x(O)\right| < 2\varepsilon, \tag{2.41}$$

and hence (2.9) holds. If, on the other hand, the expression on the left-hand side of (2.34) equals $\alpha_h r^{2-N} + \beta_h$, then we proceed as follows (whether or not $a_j = O$). Since $\alpha_{h_x} = 1$ and $\beta_{h_x} = 0$. It follows from (2.39) that $|\beta_h| \leq \varepsilon$ and $|\alpha_h - 1| < 2\varepsilon R^{N-2}$. Using these inequalities and (2.34) and (2.38), we obtain

$$\left|(\lambda(\overline{D}))^{-1}\int_{\overline{D}} h_x\, d\lambda - r^{2-N}\right| < 2\varepsilon(1 + cR^{N-2}), \tag{2.42}$$

so that (2.9) holds with $c = r^{2-N}$.

Next consider the case where ω is locally connected to ∞. There exists a sequence of balls $(B(x_k, \rho_k))$ such that

$$x_o = x, \quad x_{k-1} \in B(x_k, \rho_k) \qquad (k = 1, 2, \ldots), \tag{2.43}$$
$$O \notin B = \bigcup_{k=1}^{\infty} B(x_k, \rho_k) \subset \omega, \qquad \|x_k\| \to \infty,\ \rho_k < 1\ (k = 1, 2, \ldots).$$

Let $\varepsilon > 0$ and let $S_{x_o} = h_x$. Using Lemma 2.1 and an induction argument, we find that there exist n.e.s.f.'s S_{x_k} such that

$$|S_{x_{k-1}} - S_{x_k}| < \varepsilon 2^{-k} \text{ in } R^N\backslash B(x_k, \rho_k) \qquad (k = 1, 2, \ldots). \tag{2.44}$$

If $r_o > 0$, then there exists k_o such that

$$B(O, r_o) \subset R^N\backslash \bigcup_{k=k_o}^{\infty} B(x_k, \rho_k), \tag{2.45}$$

so that the sequence (S_{x_k}) converges uniformly in $B(O, r_o)$. Hence the limit of this sequence h, say, is harmonic in R^N. From (2.44) it follows that

$$|h_x - h| < \varepsilon \quad \text{in } R^N\backslash B \tag{2.46}$$

and hence in $\overline{D} \cup \{O\}$. Hence (2.38) and (2.40) hold. Also, h satisfies (2.34) and $\alpha_h r^{2-N} + \beta_h = \beta_h = h(O)$. From (2.34), (2.38) and (2.40) it follows that (2.41) holds, and hence so does (2.9).

We have now shown that (2.9) holds for all $x \in R^N \backslash (\overline{D} \cup \{O\})$ and hence for all $x \in R^N \backslash \overline{D}$, as required.

Corollary 2.2. *Let D be a nonempty open set R^N such that $\lambda(\overline{D}) < +\infty$ and every connected component of $R^N \backslash \overline{D}$ is locally connected to ∞. If there exists $x_o \in R^N$ such that every function h harmonic in R^N satisfies*

$$h(x_o) = (\lambda(\overline{D}))^{-1} \int_{\overline{D}} h \, d\lambda, \tag{2.47}$$

then $\overline{D}$ is a closed ball centered at x_o.

This corollary improves part of [9, Theorem 1].

The corollary follows from Theorem 2.3(i), since by hypothesis the set C is empty and taking, without loss of generality, $x_o = O$ and any $r > 0$ we have

$$M(h, r) = h(O) = (\lambda(\overline{D}))^{-1} \int_{\overline{D}} h \, d\lambda \tag{2.48}$$

for every function h harmonic in R^N.

As mentioned in §1, Sakai [11], using properties of Cauchy transforms, a tool unavailable in higher dimensions, characterized plane annuli by quadrature formulae.

3. Best harmonic L^1 approximation to subharmonic functions in annuli

Again, we work in R^N with $N \geq 3$, unless the contrary is stated. We also assume throughout this section that $0 < r_1 < r_2$.

We write $A = A(r_1, r_2)$ and $A' = A(r_1', r_2')$, where

$$(r_2')^2 - (r_1')^2 = \tfrac{1}{2}(r_2^2 - r_1^2), \quad (r_2')^N - (r_1')^N = \tfrac{1}{2}(r_2^N - r_1^N). \tag{3.1}$$

Thus if A is an r-annulus, then A' is the r-annulus such that $\lambda(A') = \frac{1}{2}\lambda(A)$. We call A' the *annulus associated to* A. We denote by $C(\overline{A})$ the set of functions which are real valued and continuous in $\overline{A}$ and by $C^2(A)$ and $H(A)$ the sets of functions which are, respectively, twice continuously differentiable in A and harmonic in A. Given a subharmonic function $s \in C(\overline{A})$, we call a function $h^* \in H(A) \cap C(\overline{A})$ a *best harmonic L^1 approximant to s* in A if

$$\int_A |s - h^*| \, d\lambda \leq \int_A |s - h| \, d\lambda \quad \text{for all } h \in H(A) \cap C(\overline{A}). \tag{3.2}$$

The main result of this section is as follows.

Theorem 3.1. *Let $s \in C^2(A) \cap C(\overline{A})$ with $\Delta s > 0$ a.e. in A. A function $h^* \in H(A) \cap C(\overline{A})$ is a best harmonic L^1 approximant to s if and only if*

$$h^*|_{\partial A'} = s|_{\partial A'} \tag{3.3}$$

and

$$s - h^* > 0 \quad a.e. \text{ in } \overline{A} \backslash \overline{A'}, \tag{3.4}$$

where A' is the annulus associated to A.

Further, if (3.3) *and* (3.4) *are satisfied, then h^* is the unique best harmonic L^1 approximant to s.*

The analogous result for the unit ball in R^N is given in [9, Theorem 2].

The 'if' part of Theorem 3.1 is valid under weaker assumptions about s.

Theorem 3.2. *Let $s \in C(\overline{A})$ and suppose that s is subharmonic in A. If there exists $h^* \in H(A) \cap C(\overline{A})$ satisfying* (3.3) *and* (3.4), *where A' is the annulus associated to A, then h^* is the unique best harmonic L^1 approximant to s in A.*

The proof of Theorem 3.2 requires the following lemmas.

Lemma 3.1. *If h is harmonic and integrable in A, then*

$$\int_{A'} h\, d\lambda = \frac{1}{2} \int_A h\, d\lambda. \tag{3.5}$$

To prove this, note that A and A' are both r-annuli, with r given by (2.1). Hence, by Corollary 2.1,

$$\int_A h\, d\lambda = \lambda(A) M(h, r) = 2\lambda(A') M(h, r) = 2 \int_{A'} h\, d\lambda.$$

Lemma 3.2. *Let V be a proper vector subspace of $C(\overline{A})$, and let $s \in C(\overline{A}) \backslash V$ and $h^* \in V$. Then h^* is a best L^1 approximant (among functions in V) to s in A if and only if there exists a Lebesgue measurable function α on the zero-set $Z(s - h^*)$ of $s - h^*$ such that*

$$|\alpha| \leq 1 \quad a.e. \text{ on } Z(s - h^*) \tag{3.6}$$

and

$$\int_{Z(s-h^*)} \alpha h\, d\lambda + \int_{\overline{A} \backslash Z(s-h^*)} h \operatorname{sgn}(s - h^*)\, d\lambda = 0 \tag{3.7}$$

for all $h \in V$.

For the proof of Lemma 3.2, see Singer's book [12, p. 46].

We start the proof of Theorem 3.2 by showing that if there exists h^* satisfying the hypotheses of the theorem, then h^* is a best harmonic L^1 approximant to s in A. Note that we can apply Lemma 3.2 with $V = H(A) \cap C(\overline{A})$, for

(3.3) and (3.4) imply that $s \notin H(A)$. Taking $\alpha \equiv -1$ on $Z(s-h^*)$, we have for all $h \in H(A) \cap C(\overline{A})$, since $s - h^* \leq 0$ in $\overline{A'}$ by the maximum principle,

$$\begin{aligned}\int_{Z(s-h^*)} \alpha h\, d\lambda + \int_{\overline{A}} h \operatorname{sgn}(s-h^*)\, d\lambda &= -\int_{Z(s-h^*)} h\, d\lambda + \int_{\overline{A}\backslash\overline{A'}} h\, d\lambda - \int_{A'\backslash Z(s-h^*)} h\, d\lambda \\ &= \int_{\overline{A}\backslash\overline{A'}} h\, d\lambda - \int_{A'} h\, d\lambda \\ &= \int_A h\, d\lambda - 2\int_{A'} h\, d\lambda = 0,\end{aligned}$$

by Lemma 3.1. The result now follows from Lemma 3.2.

To complete the proof of Theorem 3.2, it remains to prove the uniqueness of h^*. Let h' be a best harmonic L^1 approximant to s. Following the argument in [9, pp. 259–260], we find that

$$(3.8) \qquad (s-h^*)(s-h') \geq 0 \quad \text{on } \overline{A}.$$

We consider two cases.

(i) Suppose that $s - h^* < 0$ in A'. Then $s - h' \leq 0$ in A' by (3.8). Since $s - h^* > 0$ a.e. in $\overline{A}\backslash\overline{A'}$, (3.8) implies that $s - h' \geq 0$ a.e. in $\overline{A}\backslash\overline{A'}$. Since $s - h'$ is continuous in $\overline{A}$, it now follows that $s - h' = 0$ on $\partial A'$, so that $h' = s = h^*$ on $\partial A'$. Hence, by the maximum principle, $h^* = h'$ in A', and therefore $h^* = h'$ in $\overline{A}$.

(ii) Now suppose that $(s-h^*)(x) = 0$ for some $x \in \overline{A'}$. By the maximum principle, $s - h^* = 0$ in $\overline{A'}$. The argument in (i) again shows that $s - h' \geq 0$ a.e. in $\overline{A}\backslash\overline{A'}$ and hence by continuity $s - h' \geq 0$ on $\partial A'$. We now have $h^* = s \geq h'$ on $\partial A'$. If $h^* = h'$ on $\partial A'$, then, as in (i), $h^* = h'$ in $\overline{A}$. Otherwise, there exists $y \in \partial A'$ such that $h^*(y) > h'(y)$, so that by continuity $s - h' > (s-h^*)(y) = 0$ in some ball $B(y, \delta)$, and by the minimum principle $h^* > h'$ in A', so that $s - h' > s - h^* = 0$ in A'. Hence if α is a Lebesgue measurable function with $|\alpha| \leq 1$ in $Z(s-h')$, we have

$$\begin{aligned}&\int_{Z(s-h')} \alpha\, d\lambda + \int_{\overline{A}\backslash Z(s-h')} \operatorname{sgn}(s-h')\, d\lambda \\ &\qquad \geq \lambda(A') + \lambda(B(y,\delta) \cap (\overline{A}\backslash A')) - \lambda(\overline{A}\backslash(A' \cup B(y,\rho))) \\ &\qquad > \lambda(B(y,\delta) \cap (A\backslash A')) > 0,\end{aligned}$$

the penultimate inequality following from the equation $\lambda(A') = \lambda(\overline{A}\backslash A')$. Hence (3.7) fails with $h \equiv 1$, and so by Lemma 3.2, h' is not a best harmonic L^1 approximant to s in A, a contradiction.

In view of Theorem 3.2, in order to complete the proof of Theorem 3.1, it is enough to prove the necessity of the conditions (3.3) and (3.4). This requires the following result [8, Lemma 6].

Lemma 3.3. *Let D be an open set in R^N. If $u \in C^2(D)$ and $\Delta u > 0$ a.e. in D, then $\lambda(Z(u)) = 0$, where $Z(u)$ denotes the zero set of u.*

Now let s be as in Theorem 3.1 and suppose that h^* is a best harmonic L^1 approximant to s in A. We must show that (3.3) and (3.4) hold. Let E_-, E_+ and E_o be the sets of points of $\overline{A}$ at which $s - h^*$ is, respectively, negative, positive, and zero. We have $\Delta(s - h^*) = \Delta s > 0$ a.e. in A, so by Lemma 3.3, $\lambda(E_o) = 0$. Hence, by Lemma 3.2,

$$\int_{\overline{A}} h \operatorname{sgn}(s - h^*)\, d\lambda = 0$$

for all $h \in H(A) \cap C(\overline{A})$. Hence, for all such h

$$\begin{aligned}(3.9) \qquad \int_{E_- \cap A} h\, d\lambda &= \int_{E_+ \cap A} h\, d\lambda = \frac{1}{2}\int_{\overline{A}} h\, d\lambda \\ &= \frac{1}{2}\lambda(\overline{A})M(h, r) = \lambda(E_- \cap A)M(h, r).\end{aligned}$$

Here r is given by (2.1). The penultimate equation follows from Corollary 2.1, and the last equation follows by taking $h \equiv 1$ in (3.9). Now $\partial(E_- \cap A) \subset \partial A \cup E_o$, and hence $\lambda(\overline{E_- \cap A}) = \lambda(E_- \cap A)$, since $\lambda(\partial A \cup E_o) = 0$. Hence

$$(3.10) \qquad \int_{\overline{E_- \cap A}} h\, d\lambda = \lambda(\overline{E_- \cap A})M(h, r)$$

for every $h \in H(A) \cap C(\overline{A})$.

We shall apply Theorem 2.3 to show that $\overline{E_- \cap A}$ is a closed r-annulus. In order to do so we show first that $R^N \backslash (\overline{E_- \cap A})$ consists of exactly two components. Clearly, $R^N \backslash (\overline{E_- \cap A})$ has exactly one unbounded component, and it also has a component containing $B(O, r_1)$. Suppose, if possible, that there is a component ω which is neither unbounded nor contains $B(O, r_1)$. Then ω is disjoint from $\overline{B}(O, r_1)$ and $R^N \backslash \overline{B}(O, r_2)$. Hence $\omega \subset A$. We shall show that $s - h^* = 0$ on $\partial\omega$. Clearly $s - h^* \geq 0$ in ω and so, by continuity, $s - h^* \geq 0$ on $\partial\omega$. If $x \in \partial\omega \cap A$, then $x \in \partial E_- \cap A$ and so $(s - h^*)(x) \leq 0$. Hence $s - h^* = 0$ on $\partial\omega \cap A$. Now suppose, if possible, that there exists $x \in \partial\omega \cap \partial A$ such that $(s - h^*)(x) > 0$. Then $s - h^* > 0$ in $B(x, \rho) \cap \overline{A}$ for some $\rho > 0$. Hence $B(x, \rho) \cap (\overline{E_- \cap A})$ is empty, that is, $B(x, \rho) \subset R^N \backslash (\overline{E_- \cap A})$. Since $B(x, \rho) \cap \omega$ is nonempty, it now follows that $B(x, \rho) \subseteq \omega$ and hence that $\omega \not\subseteq A$, a contradiction. Hence $s - h^* \leq 0$ on $\partial\omega \cap \partial A$ and so $s - h^* = 0$ on $\partial\omega \cap \partial A$. We now have $s - h^* = 0$ on $\partial\omega$. By the maximum principle, $s - h^* \leq 0$ in ω. Since $s - h^* \geq 0$ in ω, we have $s - h^* = 0$ in ω, so that $\omega \subset E_o$, contradicting $\lambda(E_o) = 0$.

We have now established that $R^N \backslash (\overline{E_- \cap A})$ consists of at most two components, one containing $B(O, r_1)$, and one containing $R^N \backslash (O, r_2)$. If there is only one component, then it is locally connected so ∞ and since (3.10) holds for each h harmonic in R^N, Theorem 2.3(i) is applicable and implies that $\overline{E_- \cap A}$ is a ball with centre O, a contradiction.

Hence $R^N \backslash (\overline{E_- \cap A})$ consists of exactly two components, the first locally connected to ∞ and the second bounded and containing O. Since (3.10) holds for all h harmonic in $R^N \backslash \{O\}$, it follows from Theorem 2.3(ii) that $\overline{E_- \cap A}$ is a closed r-annulus. Since $\lambda(\overline{E_- \cap A}) = \frac{1}{2}\lambda(A)$, we must have $\overline{E_- \cap A} = \overline{A'}$, where A' is the annulus associated to A.

It now follows that $s - h^* \leq 0$ in A' and $s - h^* \geq 0$ in $A \backslash \overline{A'}$, so that, by continuity, $s - h^* = 0$ on $\partial A'$. Since $\lambda(E_o) = 0$ and $s - h^* \geq 0$ in $A \backslash \overline{A'}$, we have $s - h^* > 0$ a.e. in $\overline{A} \backslash \overline{A'}$, and the proof of Theorem 3.1 is complete.

In the case $N = 2$, we note that Lemma 2.1 stills holds. (See [4] and the proof of Lemma 4 in [7].) Of course, when $N = 2$, the Newtonian kernel is replaced by the logarithmic kernel. The proof of Theorem 3.1 is then the same as in the case $N \geq 3$, except that we use the quadrature formulae for plane annuli, given in [11, pp. 6–8].

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