TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 314, Number 1, July 1989

LIMITATION TOPOLOGIES ON FUNCTION SPACES

PHILIP L. BOWERS

ABSTRACT. Four competing definitions for limitation topologies on the set of continuous functions $C(X, Y)$ are compared.

1. INTRODUCTION

Four competing definitions for limitation topologies on the set $C(X, Y)$ of maps (= continuous functions) from a space X to a space Y are compared. Although all four topologies arise naturally, it is shown that only two of the four topologies are appropriate for describing relationships between elements of $C(X, Y)$ and, in particular, closeness of maps that have been useful in the study of Hilbert space manifolds and other infinite-dimensional nonlocally compact manifolds as influenced by the seminal work of H. Toruńczyk.

The limitation topology as originally described by Toruńczyk [To2] is a convenient language in which to express the fact that, under specified conditions on X and Y, arbitrary maps from X to Y are approximable with cover close control in Y by maps having various desirable properties. For example, if X is a separable metric space and Y is the separable Hilbert space, then maps from X to Y are strongly approximable by embeddings, and this can be stated alternately as the set $\mathrm{Emb}(X, Y)$ of embeddings of X in Y is a dense subset of $C(X, Y)$ whenever $C(X, Y)$ is given the limitation topology. More importantly, the limitation topology has been used as a powerful tool in obtaining quick proofs that certain subsets of $C(X, Y)$ are dense in $C(X, Y)$. These proofs rely on the fact that $C(X, Y)$ becomes a Baire space whenever it is given the limitation topology and this reduces the problem of showing that a subset G is dense in $C(X, Y)$ to observing that G can be written as a countable intersection of open dense subsets. In practice, there is usually a natural decomposition of G as a countable intersection of open subsets and it remains to decide whether each such open set is dense. The equivalence of the following two statements is used both for deciding whether the open sets are dense and for proving that $C(X, Y)$ is a Baire space when given the limitation topology:

$$(*) \qquad E \subset C(X, Y) \text{ is dense in } C(X, Y),$$

Received by the editors May 5, 1987 and, in revised form, February 6, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C35; Secondary 54A10.
Key words and phrases. Limitation topology, modified limitation topology.

(**) for every $f \in C(X, Y)$ and open cover $\mathscr{U}$ of Y (or, perhaps, $\operatorname{im} f$), there exists $g \in E$ $\mathscr{U}$-close to f.

Any topology on $C(X, Y)$ that implies the equivalence of (*) and (**) can vie for the title "Limitation Topology." Unfortunately, Toruńczyk's original wording of the definition of the limitation topology coupled with the equivalence of (*) and (**) led many researchers to hold the misconception that the set

$$\mathscr{S} = \{B(f, \mathscr{U}) \mid f \in C(X, Y),\ \mathscr{U} \in \operatorname{cov}(Y)\}$$

(see §2 for definitions) forms a basis for the limitation topology and/or that each set of the form $B(f, \mathscr{U})$ is open. A secondary goal of this paper is to correct this misconception. Simple examples are constructed that show that this assumption that $\mathscr{S}$ forms a basis for the limitation topology on $C(X, Y)$ in general is invalid. In fact, these examples illustrate that in general $\mathscr{S}$ cannot form a basis for any topology on $C(X, Y)$.

§2 presents the definitions of the four competing topologies on $C(X, Y)$ and two are shown to be acceptable definitions for limitation topologies in that they imply the equivalence of (*) and (**). These two are labeled the *limitation topology* and the *modified limitation topology*, respectively. The equivalence of all four topologies as suggested by Toruńczyk [To1, comment before (A), p. 33] in case X is compact is demonstrated and other descriptions of the limitation topology in terms of metrics are given.

§3 presents the aforementioned examples that show that $\mathscr{S}$ in general cannot form a basis for any topology on $C(X, Y)$. Specifically, it is shown that $\mathscr{S}$ cannot form a basis whenever X is not compact and Y is a noncompact absolute retract.

§4 presents a discussion of subspaces of $C(X, Y)$ and a result of Toruńczyk [To2] that implies that $C(X, Y)$ is a Baire space whenever $C(X, Y)$ is given the (modified) limitation topology and Y possesses a complete metric. It is observed that whenever X is homeomorphic to Y, all four topologies introduced in §2 coincide when restricted to the subset of $C(X, Y)$ that consists of homeomorphisms of X onto Y.

Terminology and notation. If f and g are maps of a space X into a space Y and $\mathscr{U}$ is a collection of subsets of Y, then g is called a *$\mathscr{U}$-approximation to f* or g is said to be *$\mathscr{U}$-close to f* provided for every x in X, $\{f(x), g(x)\}$ is contained in some member of $\mathscr{U}$. Given families $\mathscr{A}$ and $\mathscr{B}$ of subsets of a space Y, $\mathscr{A}$ *refines* $\mathscr{B}$ provided each member of $\mathscr{A}$ is contained in some member of $\mathscr{B}$. Observe that it is not assumed that $\bigcup \mathscr{A} = \bigcup \mathscr{B}$ for $\mathscr{A}$ to refine $\mathscr{B}$. $\mathscr{A}$ *star-refines* $\mathscr{B}$ if $\operatorname{st}\mathscr{A} = \{\operatorname{st}(A, \mathscr{A}) | A \in \mathscr{A}\}$ refines $\mathscr{B}$ where $\operatorname{st}(A, \mathscr{A})$ is the union of all elements of $\mathscr{A}$ that hit A, and $\mathscr{A}$ *double star-refines* $\mathscr{B}$ if $\operatorname{st}^2 \mathscr{A} = \operatorname{st}\operatorname{st}\mathscr{A}$ refines $\mathscr{B}$. $AR(ANR)$ denotes absolute (neighborhood) retract for the class of metrizable spaces. Further terminology and notation is introduced as needed.

2. Limitation topologies

For a topological space Y, $\operatorname{cov}(Y)$ denotes the collection of all open covers of Y. For a map $f: X \to Y$ between spaces and a collection $\mathscr{U}$ of subsets of Y, the *$\mathscr{U}$-neighborhood* of f in $C(X, Y)$ is the set $B(f, \mathscr{U})$ that consists of all maps $g: X \to Y$ that are *$\mathscr{U}$-close* to f. For the map f, $\operatorname{cov}(f)$ denotes the collection of all open in Y coverings of $\operatorname{im} f$. We present four topologies on $C(X, Y)$ that arise quite naturally.

τ denotes the topology on $C(X, Y)$ generated by the $\mathscr{U}$-neighborhoods of f as $\mathscr{U}$ ranges over $\operatorname{cov}(Y)$ and f ranges over $C(X, Y)$. The collection $\mathscr{S} = \{B(f, \mathscr{U}) \mid f \in C(X, Y),\ \mathscr{U} \in \operatorname{cov}(Y)\}$ forms a subbasis of open sets for τ and τ is the smallest topology on $C(X, Y)$ in which each $B(f, \mathscr{U})$, $f \in C(X, Y)$ and $\mathscr{U} \in \operatorname{cov}(Y)$, is open. Closely related to τ is the finer topology τ' that is generated by the $\mathscr{U}$-neighborhoods of f as $\mathscr{U}$ ranges over $\operatorname{cov}(f)$ and f ranges over $C(X, Y)$, and for which $\mathscr{S}' = \{B(f, \mathscr{U}) \mid f \in C(X, Y),\ \mathscr{U} \in \operatorname{cov}(f)\}$ forms a subbasis of open sets.

Define a collection $\mathscr{T}$ of subsets of $C(X, Y)$ by the rule: a subset $U \subset C(X, Y)$ is an element of $\mathscr{T}$ if for every $f \in U$, there exists $\mathscr{U} \in \operatorname{cov}(Y)$ such that $B(f, \mathscr{U}) \subset U$. If U and V are elements of $\mathscr{T}$ and $B(f, \mathscr{U}) \subset U$ and $B(f, \mathscr{V}) \subset V$ for $\mathscr{U}$, $\mathscr{V} \in \operatorname{cov}(Y)$, then $B(f, \mathscr{W}) \subset U \cap V$ for any common refinement $\mathscr{W} \in \operatorname{cov}(Y)$ of $\mathscr{U}$ and $\mathscr{V}$. This ensures that $U \cap V$ is an element of $\mathscr{T}$ and obviously implies that $\mathscr{T}$ is a topology on $C(X, Y)$. Closely related to $\mathscr{T}$ is the finer topology $\mathscr{T}'$ defined by the rule: $U \subset C(X, Y)$ is an element of $\mathscr{T}'$ if for every $f \in U$, there exists $\mathscr{U} \in \operatorname{cov}(f)$ such that $B(f, \mathscr{U}) \subset U$.

Though $\mathscr{S}$ and $\mathscr{S}'$ form subbases for the topologies τ and τ', respectively, examples constructed in §3 show that in general they do not form bases. Observe that each $\mathscr{T}$-open subset of $C(X, Y)$ is a union of elements of $\mathscr{S}$ but in general not every union of elements of $\mathscr{S}$ is $\mathscr{T}$-open. In particular, the elements of $\mathscr{S}$ themselves are not necessarily $\mathscr{T}$-open. This is demonstrated, for example, by our first example of §3 where $X = N$ and $Y = \mathbf{R}$. The subset $B(g_+, \mathscr{U})$ is not $\mathscr{T}$-open since $f \in B(g_+, \mathscr{U})$ and there is no $\mathscr{W} \in \operatorname{cov}(Y)$ such that $B(f, \mathscr{W}) \subset B(g_+, \mathscr{U})$. Similarly, each $\mathscr{T}'$-open subset is a union of elements of $\mathscr{S}'$ but in general not every such union is $\mathscr{T}'$-open. The following relationships occur among the four topologies τ, τ', $\mathscr{T}$, and $\mathscr{T}'$ on $C(X, Y)$:

$$\begin{array}{ccc} \mathscr{T} & \subset & \mathscr{T}' \\ \cap & & \cap \\ \tau & \subset & \tau'. \end{array}$$

In general, all the containments are proper; however, Proposition 2.4 and Lemma 2.5 imply that all four topologies are equal to the compact-open topology on $C(X, Y)$ whenever X is compact metrizable and Y is metrizable.

Even though $B(f, \mathscr{U})$ may fail to be $\mathscr{T}$-open, it does have nonempty interior in the topology $\mathscr{T}$ (as proved below) and this is precisely what provides the equivalence of $(*)$ and $(**)$ from the Introduction whenever $C(X, Y)$ is topologized by $\mathcal{T}$. Because of Proposition 2.3 below, we shall call $\mathcal{T}$ the *limitation topology* on $C(X, Y)$ and $\mathcal{T}'$ the *modified limitation topology* on $C(X, Y)$.

2.1. **Lemma.** *Let Y be a paracompact space, $f \in C(X, Y)$, and $\mathcal{U} \in \operatorname{cov}(f)$. Then $B(f, \mathcal{U})$ has nonempty interior in $\mathcal{T}'$ and if $\bigcup \mathcal{U} = Y$, then $B(f, \mathcal{U})$ has nonempty interior in $\mathcal{T}$.*

Proof. Let $Y' = \bigcup \mathcal{U}$ and define the *derived set of* $B(f, \mathcal{U})$, denote $B_*(f, \mathcal{U})$, by

$$B_*(f, \mathcal{U}) = \{g \in B(f, \mathcal{U}) \mid \exists \mathcal{W} \in \operatorname{cov}(Y') \text{ such that } B(g, \mathcal{W}) \subset B(f, \mathcal{U})\}.$$

$B_*(f, \mathcal{U})$ is nonempty since it contains the map f. Let $g \in B_*(f, \mathcal{U})$ and $\mathcal{W} \in \operatorname{cov}(Y')$ such that $B(g, \mathcal{W}) \subset B(f, \mathcal{U})$. Since Y is paracompact, there exists an open cover $\mathcal{V}$ of Y' that star-refines $\mathcal{W}$. If $h \in B(g, \mathcal{V})$, then

$$B(h, \mathcal{V}) \subset B(g, \operatorname{st} \mathcal{V}) \subset B(g, \mathcal{W}) \subset B(f, \mathcal{U}),$$

implying that $h \in B_*(f, \mathcal{U})$. Hence $B(g, \mathcal{V}) \subset B_*(f, \mathcal{U})$ implying that $B_*(f, \mathcal{U})$ is $\mathcal{T}'$-open and, if $\bigcup \mathcal{U} = Y$, that $B_*(f, \mathcal{U})$ is $\mathcal{T}$-open.

2.2. **Corollary.** *Let Y be a paracompact space. The collection $\mathcal{S}_*(f) = \{B_*(f, \mathcal{U}) \mid \mathcal{U} \in \operatorname{cov}(Y)\}$ forms a local $\mathcal{T}$-base of $\mathcal{T}$-open sets at f for each $f \in C(X, Y)$ and the collection $\mathcal{S}_* = \{B_*(f, \mathcal{U}) \mid f \in C(X, Y),\ \mathcal{U} \in \operatorname{cov}(Y)\}$ forms a basis for $\mathcal{T}$. The collection $\mathcal{S}_*'(f) = \{B_*(f, \mathcal{U}) \mid \mathcal{U} \in \operatorname{cov}(f)\}$ forms a local $\mathcal{T}'$-base of $\mathcal{T}'$-open sets at f for each $f \in C(X, Y)$ and the collection $\mathcal{S}_*' = \{B_*(f, \mathcal{U}) \mid f \in C(X, Y),\ \mathcal{U} \in \operatorname{cov}(f)\}$ forms a basis for $\mathcal{T}'$.*

2.3. **Proposition.** *Let Y be a paracompact space. If $C(X, Y)$ is topologized by either $\mathcal{T}$ or $\mathcal{T}'$ and $E \subset C(X, Y)$, then the following are equivalent:*

- $(*)$ *E is dense in $C(X, Y)$,*
- $(**)$ *for every $f \in C(X, Y)$ and $\mathcal{U}$, where $\mathcal{U} \in \operatorname{cov}(Y)$ if $\mathcal{T}$ is used and $\mathcal{U} \in \operatorname{cov}(f)$ if $\mathcal{T}'$ is used, there exists $g \in E$ $\mathcal{U}$-close to f.*

Proof. $(*)$ implies $(**)$ is a consequence of the fact that $B(f, \mathcal{U})$ has nonempty interior. $(**)$ implies $(*)$ is a consequence of the definitions of $\mathcal{T}$-open and $\mathcal{T}'$-open sets.

For an example illustrating the failure of 2.3 whenever $C(X, Y)$ is topologized by τ, see §3.

We now prove that under restrictions on X and Y, the four topologies τ, τ', $\mathcal{T}$, and $\mathcal{T}'$ are all equal to the compact-open topology on $C(X, Y)$.

2.4. **Proposition** (Toruńczyk). *If X is a compact metrizable space and Y is a metrizable space, then $\mathcal{S} = \mathcal{S}'$ forms a basis for the compact-open topology and $\mathcal{S}(f) = \{B(f, \mathcal{U}) \mid \mathcal{U} \in \operatorname{cov}(Y)\}$ forms a local base of open sets at f in the compact-open topology. Furthermore, the four topologies τ, τ', $\mathcal{T}$, and $\mathcal{T}'$ all coincide with the compact-open topology.*

Before proving 2.4, we prove a preliminary lemma.

2.5. Lemma. *If X is a compact metrizable space, Y is a metrizable space, and $f \in B(g, \mathscr{U}) \in \mathscr{S}$, then there exists a cover $\mathscr{V} \in \operatorname{cov}(Y)$ such that $B(f, \mathscr{V}) \subset B(g, \mathscr{U})$.*

Proof. Fix a metric on Y and for each $y \in \operatorname{im} f$ let $\delta(y) \in [0, \infty]$ be the supremum of all $\delta \geq 0$ for which the following holds: for every $x \in f^{-1}(y)$, there exists $U \in \mathscr{U}$ such that $g(x) \in U$ and $N_\delta(y) \subset U$, where $N_\delta(y)$ denotes the δ-neighborhood of y in Y. The compactness of X guarantees that $\delta(y) > 0$ for all $y \in \operatorname{im} f$ and we claim that there exists $\varepsilon > 0$ such that $\varepsilon < \delta(y)$ for every $y \in \operatorname{im} f$. If not, then there exist a sequence $\{y_i\}_{i=1}^{\infty}$ in $\operatorname{im} f$ and $y \in \operatorname{im} f$ such that $y_i \to y$ and $\delta(y_i) \to 0$. We may assume without loss of generality that $y_i \in N_{\bar{\delta}}(y)$ for each i where $\bar{\delta} = \delta(y)/2$ and that $\delta(y_i) < \bar{\delta}$ for each i. Notice that $N_{\bar{\delta}}(y_i) \subset N_{\delta(y)}(y)$ for each i. From the definition of δ, since $\delta(y_i) < \bar{\delta}$, there exists $x_i \in f^{-1}(y_i)$ such that for every $U \in \mathscr{U}$ with $g(x_i) \in U$, $N_{\bar{\delta}}(y_i)$ is not contained in U. Since X is compact, we may assume by passing to a subsequence if necessary that $x_i \to x$ for some $x \in X$. By continuity, $f(x) = y$ and hence there exists $U \in \mathscr{U}$ such that $g(x) \in U$ and $N_{\delta(y)}(y) \subset U$. Since $g(x_i) \to g(x)$ and $y_i \to y$, there exists a positive integer n with $\{g(x_n), y_n\} \subset U$. We have $g(x_n) \in U$ and $N_{\bar{\delta}}(y_n) \subset N_{\delta(y)}(y) \subset U$ which contradicts our definition of x_n. Hence ε exists. Let $\mathscr{V}$ be the open cover of Y that consists of the $\varepsilon/2$-neighborhoods of the points of Y. It is a straightforward argument that shows that $B(f, \mathscr{V}) \subset B(g, \mathscr{U})$.

Proof of Proposition 2.4. Observe that $\mathscr{S} \subset \mathscr{S}'$. For $B(f, \mathscr{U}) \in \mathscr{S}'$, let $\mathscr{U}^* = \mathscr{U} \cup \{Y - \operatorname{im} f\}$ and observe that $B(f, \mathscr{U}) = B(f, \mathscr{U}^*)$. Since X is compact, $\mathscr{U}^* \in \operatorname{cov}(Y)$ and hence $\mathscr{S}' \subset \mathscr{S}$. Then $\mathscr{S} = \mathscr{S}'$.

In order to show that $\mathscr{S}(f)$ forms a local base of open sets at f in τ, it suffices to show that for finitely many elements, say $B(g_1, \mathscr{U}_1), \ldots, B(g_k, \mathscr{U}_k)$, of $\mathscr{S}$ each of which contains f, there exists $\mathscr{V} \in \operatorname{cov}(Y)$ for which $B(f, \mathscr{V})$ is contained in $\bigcap_{i=1}^k B(g_i, \mathscr{U}_i)$. For this, use 2.5 to find $\mathscr{V}_i$ for $i = 1, \ldots, k$ such that $B(f, \mathscr{V}_i) \subset B(g_i, \mathscr{U}_i)$ and let $\mathscr{V}$ be a common refinement of $\mathscr{V}_1, \ldots, \mathscr{V}_k$ covering Y. Then $B(f, \mathscr{V}) \subset B(g_i, \mathscr{V}_i)$ for $i = 1, \ldots, k$.

It is obvious that since $\mathscr{S}(f)$ forms a local basis of open sets at f for each f in $C(X, Y)$, $\mathscr{S}$ forms a basis for τ, and it is easy to show that subsets open in the compact-open topology on $C(X, Y)$ are τ-open (show this for subbasic sets first). For the other direction, if $f \in B(g, \mathscr{U}) \in \mathscr{S}$, then the proof of 2.5 shows that there exists an $\varepsilon > 0$ such that $f \in B(f, \varepsilon) \subset B(g, \mathscr{U})$ where $B(f, \varepsilon) = \{h \in C(X, Y) \mid h \text{ is } \varepsilon\text{-close to } f\}$ (assuming a fixed metric on Y). Since X is compact, the compact-open topology on $C(X, Y)$ is metrizable via the sup metric and each $B(f, \varepsilon)$ is open in the compact-open topology. Hence $B(g, \mathscr{U})$ is open in the compact-open topology and thus τ and the compact-open topology coincide. Since $\mathscr{S} = \mathscr{S}'$, we conclude $\tau = \tau'$ and since further 2.5 holds, we conclude $\mathscr{T} = \tau$, and finally $\mathscr{T} \subset \mathscr{T}' \subset \tau'$ implies $\mathscr{T}' = \tau$.

If Y is metrizable, then there are other descriptions of the limitation topology. Let $\operatorname{metr}(Y)$ denote the collection of all bounded compatible metrics for

Y. For a metric $\rho \in \operatorname{metr}(Y)$ and map $f \in C(X, Y)$, let $B_\rho(f, 1)$ denote the set $\{g \in C(X, Y) \mid \rho(f, g) < 1\}$ where $\rho(f, g)$ denotes the supremum of all distances $\rho(f(x), g(x))$ for $x \in X$. It easily follows from [Du, IX, 9.4, p. 196] that, for each $B(f, \mathscr{U}) \in \mathscr{S}$, there exists a metric $\rho \in \operatorname{metr}(Y)$ such that $B_\rho(f, 1) \subset B(f, \mathscr{U})$; conversely, if $\rho \in \operatorname{metr}(Y)$ is given, then $B(f, \mathscr{U}) \subset B_\rho(f, 1)$ where $\mathscr{U} \in \operatorname{cov}(Y)$ consists of the open ρ-balls in Y of radius $1/3$. Therefore an alternative description of the limitation topology whenever Y is metrizable is that a subset $U \in C(X, Y)$ is $\mathscr{T}$-open if and only if, for every $f \in U$, there exists a metric $\rho \in \operatorname{metr}(Y)$ such that $B_\rho(f, 1) \subset U$. $E \subset C(X, Y)$ is $\mathscr{T}$-dense if and only if for every $f \in C(X, Y)$ and metric $\rho \in \operatorname{metr}(Y)$, there exists $g \in E$ such that $\rho(f, g) < 1$. Furthermore, we have the following result.

2.6 **Proposition.** *Let Y be a metrizable space. The collection $\{B_\rho(f, 1) \mid f \in C(X, Y),\ \rho \in \operatorname{metr}(Y)\}$ forms a basis for $\mathscr{T}$.*

Proof. We need only show that $B_\rho(f, 1)$ is $\mathscr{T}$-open for $f \in C(X, Y)$ and $\rho \in \operatorname{metr}(Y)$. Let $g \in B_\rho(f, 1)$ and choose a positive number $\varepsilon < 1 - \rho(f, g)$ and let $\mathscr{U} \in \operatorname{cov}(Y)$ consist of the open ρ-balls in Y of radius $\varepsilon/3$. Then $B(g, \mathscr{U}) \subset B_\rho(f, 1)$.

There is a similar result for the modified limitation topology $\mathscr{T}'$. For a map $f \in C(X, Y)$, let $\operatorname{metr}(f)$ denote the collection of all bounded compatible metrics for open subspaces of Y that contain $\operatorname{im} f$; that is, $\operatorname{metr}(f) = \bigcup\{\operatorname{metr}(U) \mid \operatorname{im} f \subset U,\ U \text{ is open in } Y\}$.

2.7. **Proposition.** *Let Y be a metrizable space. The collection $\{B_\rho(f, 1) \mid f \in C(X, Y),\ \rho \in \operatorname{metr}(f)\}$ forms a basis for $\mathscr{T}'$.*

The proof of 2.7 is similar to that of 2.6.

Finally, we mention a result of Toruńczyk [To2]. For $\alpha \in C(Y, (0, \infty))$, $\rho \in \operatorname{metr}(Y)$, and $f \in C(X, Y)$, we write $\overline{B}_\rho(f, \alpha) = \{g \in C(X, Y) \mid \rho(f(x), g(x)) \leq \alpha(f(x)) \text{ for all } x \in X\}$. Each $\overline{B}_\rho(f, \alpha)$ is $\mathscr{T}$-closed and, if ρ is fixed, then $\{\overline{B}_\rho(f, \alpha) \mid f \in C(X, Y),\ \alpha \in C(Y, (0, \infty))\}$ forms a basis of $\mathscr{T}$-closed subsets of $C(X, Y)$ [To2].

3. The examples

The following simple example illustrates that in general neither $\mathscr{S}$ nor $\mathscr{S}'$ can form a basis for any topology on $C(X, Y)$. Let X denote the natural numbers $\mathbf{N}$ with the discrete topology and Y the real numbers $\mathbf{R}$ with the usual euclidean topology. Define maps f, g_+, $g_- \in C(X, Y)$ by the equations $f(X) = \{0\}$, $g_+(n) = n$ for $n \in X$, and $g_-(n) = -n$ for $n \in X$. For each natural number n, let U_n denote the open interval $(-1/n, n)$ of $\mathbf{R}$ and U_{-n} the open internal $(-n, 1/n)$ of $\mathbf{R}$, and let $\mathscr{U} = \{U_i\}$, an open cover of Y. Observe that $f \in B(g_+, \mathscr{U}) \cap B(g_-, \mathscr{U})$. If either $\mathscr{S}$ or $\mathscr{S}'$ forms a basis for a topology on $C(X, Y)$, then there is a map $h \in C(X, Y)$ and an open in Y

covering $\mathscr{W}$ of $\operatorname{im} h$ such that

(3.1) $$f \in B(h, \mathscr{W}) \subset B(g_+, \mathscr{U}) \cap B(g_-, \mathscr{U}).$$

Since $f \equiv 0$ is $\mathscr{W}$-close to h, there exists an element W in $\mathscr{W}$ that contains 0. Choose $w \in W$, $w < 0$. Define $h' \in C(X, Y)$ by

$$h'(n) = \begin{cases} h(n) & \text{if } h(n) \notin W, \\ w & \text{if } h(n) \in W. \end{cases}$$

Observe that $h'(n) = w$ for all but finitely many n since the fact that h is $\mathscr{U}$-close to both g_+ and g_- forces $h(n) \to 0$ as $n \to \infty$. Obviously, $h' \in B(h, \mathscr{W})$; however, h' is not $\mathscr{U}$-close to g_+, for any such map cannot take on any fixed negative value infinitely often. This contradicts 3.1. Hence, for this particular example, neither $\mathscr{S}$ not $\mathscr{S}'$ is a basis for any topology on $C(X, Y)$.

The form of this example can be used to generate many similar examples. In particular, we have the following proposition.

3.2. **Proposition.** *If X is a noncompact metrizable space and Y is a noncompact AR, then $\mathscr{S}$ and $\mathscr{S}'$ do not form bases for any topologies on $C(X, Y)$.*

Proof. Fix a metric on Y and choose a sequence $\{x_n\}_{n=1}^{\infty}$ of X and a double sequence $\{y_n\}_{-\infty}^{\infty}$ of Y (indexed by the integers) of pairwise distinct elements that have no convergent subsequences in X and Y, respectively. Choose a double sequence $\{\varepsilon_n\}_{-\infty}^{\infty}$ of positive numbers for which $\varepsilon_n \to 0$ and $\varepsilon_{-n} \to 0$ as $n \to \infty$ and for which $\{N_{\varepsilon_n}(y_n)\}_{-\infty}^{\infty}$ is pairwise disjoint, where $N_\varepsilon(y)$ denotes the ε-neighborhood of y in Y. For each integer $n \neq 0$, let U_n denote $N_{\varepsilon_n}(y_0) \cup N_{\varepsilon_n}(y_n)$ and let $U_0 = Y - \{y_n \mid n \neq 0\}$, and let $\mathscr{U}$ denote the open cover of Y that consists of the sets U_n for all integers n. Use the fact that Y is an AR to obtain maps g_+, $g_- \in C(X, Y)$ that satisfy $g_+(x_n) = y_n$ and $g_-(x_n) = y_{-n}$ for each positive integer n. Let f be the constant map $f(X) = \{y_0\}$ and observe that f is $\mathscr{U}$-close to both g_+ and g_- and, if $h \in C(X, Y)$ is $\mathscr{U}$-close to both g_+ and g_-, then $h(x_n) \to y_0$ as $n \to \infty$. If either $\mathscr{S}$ or $\mathscr{S}'$ is a basis for a topology on $C(X, Y)$, then there is a map $h \in C(X, Y)$ and an open in Y covering $\mathscr{W}$ of $\operatorname{im} h$ for which

$$f \in B(h, \mathscr{W}) \subset B(g_+, \mathscr{U}) \cap B(g_-, \mathscr{U}).$$

Let $Z = \bigcup \mathscr{W}$. Since Z is an ANR, there is a covering $\mathscr{V}$ of Z refining $\mathscr{W}$ such that $\mathscr{V}$-close maps into Z are $\mathscr{W}$-homotopic. Since $f \equiv y_0$ is $\mathscr{W}$-close to h, there exists $W \in \mathscr{W}$ with $y_0 \in W$. Hence y_0 is an element of Z and there exists $V \in \mathscr{V}$ with $y_0 \in V$. Since Y is a nontrivial AR, there is a point v in V different from y_0. Define a map h'' on $\{x_n\}_{n=1}^{\infty}$ by

$$h''(x_n) = \begin{cases} h(x_n) & \text{if } h(x_n) \notin V, \\ v & \text{if } h(x_n) \in V \end{cases}$$

and observe that $h''(x_n) = v$ for all but finitely many values of n. Since h'' is $\mathscr{V}$-close and hence $\mathscr{W}$-homotopic to $h \mid \{x_n\}_{n=1}^{\infty}$, the controlled version of the

Homotopy Extension Theorem implies that h'' extends to a map $h' \in C(X, Y)$ that is $\mathscr{W}$-close to h. Thus $h'(x_n) \to v \neq y_0$ as $n \to \infty$. This contradicts our assumption on h and $\mathscr{W}$, thus showing that neither $\mathscr{S}$ nor $\mathscr{S}'$ forms a basis for a topology on $C(X, Y)$.

The first example of this section provides a simple-minded example illustrating the failure of 2.3 whenever $C(X, Y)$ is topologized by τ. With X, Y, g_+, g_-, and $\mathscr{U}$ as in the first paragraph of this section, let E denote the set of all maps g in $C(X, Y)$ such that $\operatorname{im} g$ misses a neighborhood of 0. It is easy to see that for every $f \in C(X, Y)$ and $\mathscr{V} \in \operatorname{cov}(Y)$, there exists $g \in E$ $\mathscr{V}$-close to f. However, E is not τ-dense in $C(X, Y)$ since E has empty intersection with the nonempty τ-open set $B(g_+, \mathscr{U}) \cap B(g_-, \mathscr{U})$. This example is somewhat artificial and so we offer another example that illustrates the power of the limitation topology $\mathscr{T}$ vis-à-vis the impotence of τ in the study of the topology of the separable Hilbert space l_2. For an arbitrary separable metrizable space X, any map $f\colon X \to l_2$ is approximable, with cover close control in l_2, by Z-maps (i.e., maps f such that the closure of $\operatorname{im} f$ is a Z-set). This means, precisely, that for each map $f\colon X \to l_2$ and open cover $\mathscr{U}$ of l_2, there exists a Z-map $g\colon X \to l_2$ $\mathscr{U}$-close to f. This can be stated conveniently in terms of $(*)$ as the set E of Z-maps of X into l_2 is $\mathscr{T}$-dense in $C(X, l_2)$. The proof of this fact goes as follows. Let $\{e_i\}$ be a countable dense set of maps from the Hilbert cube I^∞ into l_2, which exists since $C(I^\infty, l_2)$ is separable, and for each i let E_i be the set of maps in $C(X, l_2)$ each of whose image misses a neighborhood of the image of e_i. It is the case that E contains $\bigcap_{i=1}^\infty E_i$ and since $C(X, Y)$ is a Baire space when topologized by $\mathscr{T}$ (see 4.1), it suffices to show that each E_i is $\mathscr{T}$-open and $\mathscr{T}$-dense. It is easy to see that each E_i is $\mathscr{T}$-open and since any compact subset of l_2, in particular, $\operatorname{im} e_i$, is a (strong) Z-set, it follows that $(**)$ holds for each i. Hence each E_i is $\mathscr{T}$-open and $\mathscr{T}$-dense and we conclude that $(**)$ holds for E. This actually proves something stronger than that E is $\mathscr{T}$-dense, namely, that E contains a $\mathscr{T}$-dense G_δ-subset. This result is important in that it can be coupled with the fact that the set of closed embeddings $\overline{\operatorname{Emb}}(X, l_2)$ of a completely metrizable space X into l_2 is a $\mathscr{T}$-dense G_δ-subset of $C(X, Y)$ to obtain the result that the set of closed Z-embeddings of such X into l_2 is $\mathscr{T}$-dense in $C(X, Y)$. This result was used in Toruńczyk's original proof of his topological charcterization of l_2 [To2].

Contrast the results of the previous paragraph with the situation for the topology τ. It remains true that each E_i is τ-open; however, even though $(**)$ holds for each E_i, we cannot use this to prove that $(**)$ holds for E. The reader may construct particular examples similar to our simple-minded example of the previous paragraph that show that E_i is not necessarily τ-dense even though $(**)$ holds for E_i. The useful fact that $(**)$ holds for E (and the more useful fact that E contains a dense G_δ-subset in some suitable topology on $C(X, Y)$) is not reflected by the use of the topology τ and cannot be expressed conveniently in the language of τ.

4. Subspaces and the Baire property

Let F be a subset of $C(X,Y)$. It easily is seen that the subset U of F is open in the subspace topology $\mathscr{T}_F$ $(\mathscr{T}_F')$ on F determined by $\mathscr{T}$ $(\mathscr{T}')$ if and only if, for each $f \in U$, there exists $B(f,\mathscr{U}) \in \mathscr{S}$ $(\mathscr{S}')$ with $B(f,\mathscr{U}) \cap F \subset U$. This follows, for example, from the fact that any $U \subset F$ for which each $f \in U$ has $B(f,\mathscr{U})$ with $B(f,\mathscr{U}) \cap F \subset U$ may be written as

$$U = F \cap \left[\bigcup \{B_*(f,\mathscr{U}) \mid B(f,\mathscr{U}) \cap F \subset U\}\right].$$

Consider the case where $X = Y$ is metrizable and F is the group of autohomeomorphisms of X, denoted in the literature by $\operatorname{Auth} X$ ($\operatorname{Homeo}(X)$ is reserved for denoting $\operatorname{Auth} X$ equipped with the compact-open topology). We shall see below that in this case each set of the form $H(f,\mathscr{U}) = B(f,\mathscr{U}) \cap \operatorname{Auth} X$, where $f \in \operatorname{Auth} X$ and $\mathscr{U} \in \operatorname{cov}(X)$, is open in the subspace topology on $\operatorname{Auth} X$ determined by $\mathscr{T}$, which equals that determined by $\mathscr{T}'$. This implies that the subspace topology on $\operatorname{Auth} X$ determined by τ and τ' both equal that determined by $\mathscr{T}$ and also implies the well-known fact (see [BP, IV, §1]) that the collection $\mathscr{B} = \{H(f,\mathscr{U}) \mid f \in \operatorname{Auth} X,\ \mathscr{U} \in \operatorname{cov}(X)\}$ forms a basis for a topology on $\operatorname{Auth} X$, namely the topology $\mathscr{T}_{\operatorname{Auth} X}$, in which the collection $\{H(f,\mathscr{U}) \mid \mathscr{U} \in \operatorname{cov}(X)\}$ forms a basis of open neighborhoods for f. This example probably has reinforced the misconception that $\mathscr{S}$ forms a basis for the limitation topology on $C(X,Y)$. To see that $H(f,\mathscr{U})$ is $\mathscr{T}_{\operatorname{Auth} X}$-open where $f \in \operatorname{Auth} X$ and $\mathscr{U} \in \operatorname{cov}(X)$, fix a metric $\rho \in \operatorname{metr}(X)$ and a homeomorphism $g \in H(f,\mathscr{U})$ and define for each $x \in X$ the number $\beta(x)$:

$$\beta(x) = \sup\{\delta > 0 \mid \exists U \in \mathscr{U} \text{ such that } f(x) \in U \text{ and } N_\delta(g(x)) \subset U\}.$$

$\beta\colon X \to (0,\infty)$ is a lower semicontinuous function and since $0 < \beta$, a well-known result of C. Dowker [Du, VIII, 4.3] provides a continuous $\alpha\colon X \to (0,\infty)$ such that $0 < \alpha(x) < \beta(x)$ for each $x \in X$. Easily, $\overline{B}_\rho(g,\alpha\circ g^{-1})$ is contained in $B(f,\mathscr{U})$ forcing $H(g,\mathscr{V})$ to be contained in $H(f,\mathscr{U})$ where $\mathscr{V} \in \operatorname{cov}(X)$ satisfies $B(g,\mathscr{V}) \subset \overline{B}_\rho(g,\alpha\circ g^{-1})$. It follows that $H(f,\mathscr{U})$ is $\mathscr{T}_{\operatorname{Auth} X}$-open.

We close with an extremely useful and important lemma due to H. Toruńczyk. First, a definition. For a subset $F \subset C(X,Y)$ and metric $\rho \in \operatorname{metr}(Y)$, F_ρ denotes the ρ-closure of F; that is, $f \in F_\rho$ if and only if f is the ρ-uniform limit of maps from F.

4.1. **Toruńczyk's Lemma** (see [To2]). *Let X and Y be spaces with Y completely metrizable and topologize $C(X,Y)$ using either the limitation topology $\mathscr{T}$ or the modified limitation topology $\mathscr{T}'$. Let F be a subspace of $C(X,Y)$ and U_n for each positive integer n an open subset of $C(X,Y)$. If $U_n \cap F$ is dense in F for each n, then F is in the closure of $\bigcap_{n=1}^\infty U_n \cap F_\rho$, where ρ is any metric in* $\operatorname{metr}(Y)$. *In particular, $C(X,Y)$ is a Baire space.*

In [To2], Lemma 4.1 with $C(X, Y)$ topologized by $\mathcal{T}$ appears without proof. For completeness, we give a short proof of 4.1 in case $C(X, Y)$ is topologized by $\mathcal{T}'$.

Proof. If ρ and ρ' are metrics on Y, then $\rho+\rho'$ is an equivalent metric on Y for which $F_{\rho+\rho'} \subset F_\rho$. This allows us to assume, without loss of generality, that the metric $\rho \in \operatorname{metr}(Y)$ is complete for if ρ is not complete, we may replace ρ by $\rho+\rho'$ for any complete metric $\rho' \in \operatorname{metr}(Y)$. We therefore assume that ρ is a complete metric on Y.

It suffices to show that if $f \in F$ and $\mathcal{U} \in \operatorname{cov}(f)$, then there exists a map $f' \in \bigcap_{n=1}^{\infty} U_n \cap F_\rho$ that is $\mathcal{U}$-close to f. Inductively define a sequence of triples $\{(f_i, \mathcal{A}_i, \mathcal{B}_i)\}$ that satisfy the following conditions:

(i) $f_1 \in U_1 \cap B_*(f, \mathcal{B}_1) \cap F$ where $\mathcal{A}_1 = \mathcal{B}_1 = \mathcal{U}$;

(ii) $f_i \in U_i \cap B_*(f_{i-1}, \mathcal{B}_i) \cap F$ where $\mathcal{B}_i \in \operatorname{cov}(f_{i-1})$ has ρ-mesh less than 2^{-i}, $\mathcal{B}_i$ double-star refines the cover $\mathcal{A}_i \in \operatorname{cov}(f_{i-1})$ and $\bigcup \mathcal{B}_i = \bigcup \mathcal{A}_i$ $(i \geq 2)$;

(iii) $\mathcal{A}_{i+1}$ refines $\mathcal{B}_i$ and $B(f_i, \mathcal{A}_{i+1}) \subset U_i \cap B_*(f_{i-1}, \mathcal{B}_i)$ $(i \geq 1$ and where $f_0 = f)$.

The basis step (i) is possible since $U_1 \cap F$ is dense in F and the derived set $B_*(f, \mathcal{B}_1)$ is $\mathcal{T}'$-open in $C(X, Y)$ (see the proof of 2.1). For the inductive step, suppose the triples $(f_j, \mathcal{A}_j, \mathcal{B}_j)$ for $1 \leq j \leq i$ have been chosen satisfying (i), (ii), and (iii). Since f_i is a member of the $\mathcal{T}'$-open set $U_i \cap B_*(f_{i-1}, \mathcal{B}_i)$, there exists a cover $\mathcal{A}_{i+1} \in \operatorname{cov}(f_i)$ refining $\mathcal{B}_i$ such that (iii) holds. Let $\mathcal{B}_{i+1}$ be a double star-refinement of $\mathcal{A}_{i+1}$ by open sets in Y covering $\bigcup \mathcal{A}_{i+1}$ and of ρ-mesh less than $2^{-(i+1)}$. Since $U_{i+1} \cap F$ is dense in F and the derived set $B_*(f_i, \mathcal{B}_{i+1})$ is $\mathcal{T}'$-open, we may choose f_{i+1} so that (ii) holds for f_{i+1}.

By (ii) and our choice of the covers $\mathcal{B}_i$, it is clear that the sequence $\{f_i\}$ is a ρ-Cauchy sequence of maps. Since ρ is a complete metric, the sequence $\{f_i\}$ coverges uniformly with respect to ρ to a map $f' \in C(X, Y)$. Hence $f' \in F_\rho$ and it remains to show that $f' \in U_i$ for each i and that f' is $\mathcal{U}$-close to f. First, use (ii) to show that given $k \geq i$, f_k is $\operatorname{st}\mathcal{B}_{i+1}$-close to f_i. This forces f' to be $\operatorname{st}^2 \mathcal{B}_{i+1}$-close to f_i and hence $\mathcal{A}_{i+1}$-close to f_i. (iii) now implies that $f' \in U_i$. Finally, since f' is $\mathcal{A}_2$-close to f_1, (iii) again implies that f' is $\mathcal{B}_1$-close to $f_0 = f$. Since $\mathcal{B}_1 = \mathcal{U}$, we are done.

Addendum. For another point of view on topologies on $C(X, Y)$, see [Mc1]. The limitation topology on $C(X, Y)$ is the fine uniform topology of [Mc1]. Proposition 2.6 of this paper is the same as [Mc1, Proposition 1.1] and a proof that $C(X, Y)$ with the limitation topology is a Baire space if Y is completely metrizable appears in McCoy's paper [Mc1, Theorem 1.4]. Some fairly general conditions are given in [Mc1] that imply that certain topologies on $C(X, Y)$ are exactly the compact-open topology. See in particular [Mc1, Proposition 1.2]. In another paper by R. A. McCoy [Mc2], a study is made of the convergence of sequences in the topology τ on $C(X, Y)$. In that paper, τ is labeled the *open-cover topology*.

References

[BP] C. Bessaga and A. Pelczynski, *Selected topics in infinite dimensional topology*, PWN, Warsaw, 1975.

[Du] J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

[Mc1] R. A. McCoy, *Fine topology on function spaces*, Internat. J. Math. Math. Sci. **9** (1986), 417–427.

[Mc2] ——, *The open-cover topology on function spaces*, Fund. Math. **104** (1979), 69–73.

[To1] H. Toruńczyk, *On CE-images of the Hilbert cube and characterization of Q-manifolds*, Fund. Math. **106** (1980), 31–40.

[To2] ——, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), 247–262.

Department of Mathematics, The Florida State University, Tallahassee, Florida 32306