TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 1, September 1989

THE STRUCTURE OF QUASI-MULTIPLIERS OF C^*-ALGEBRAS

HUAXIN LIN

ABSTRACT. Let A be a C^*-algebra and A^{**} its enveloping W^*-algebra. Let $\mathrm{LM}(A)$ be the left multipliers of A, $\mathrm{RM}(A)$ the right multipliers of A and $\mathrm{QM}(A)$ the quasi-multipliers of A. A question was raised by Akemann and Pedersen [1] whether $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. McKennon [20] gave a nonseparable counterexample. L. Brown [6] shows the answer is negative for stable (separable) C^*-algebras also.

In this paper, we mainly consider σ-unitial C^*-algebras. We give a criterion for $\mathrm{QM}(A) = \mathrm{LM}(A)+\mathrm{RM}(A)$. In the case that A is stable, we give a necessary and sufficient condition for $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. We also give answers for other C^*-algebras.

1. Introduction and preliminaries

Definition 1.1. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. An element x in A^{**} is called a multiplier of A if $xa \in A$ and $ax \in A$ for all $a \in A$. Similarly, x is a left multiplier if $xa \in A$, for all $a \in A$, x is a right multiplier if $ax \in A$, for all $a \in A$, and x is a quasi-multiplier if $axb \in A$, for all $a, b \in A$. We denote the sets of multipliers, left multipliers, right multipliers and quasi-multipliers by $\mathrm{M}(A)$, $\mathrm{LM}(A)$, $\mathrm{RM}(A)$ and $\mathrm{QM}(A)$, respectively.

If $\pi: A \to B(H)$ is a faithful representation, then the extension of π to A^{**} maps $\mathrm{M}(A)$, $\mathrm{LM}(A)$, $\mathrm{RM}(A)$ and $\mathrm{QM}(A)$ isometrically onto the sets of operators in $B(H)$ that satisfy the appropriate multiplication properties relative to $\pi(A)$. Each set $\mathrm{M}(A)$, $\mathrm{LM}(A)$, $\mathrm{RM}(A)$ and $\mathrm{QM}(A)$ is equipped with a natural weak topology.

Definition 1.2. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. The strict topology on A^{**} is generated by the seminorms $x \to \|xa\|$ and $x \to \|ax\|$, $a \in A$. Similarly, we have the left strict topology, generated by the seminorms $\|xa\|$, the right strict topology, generated by $\|ax\|$, and the quasi-strict topology, generated by $\|axb\|$, $a, b \in A$.

Received by the editors June 27, 1986 and, in revised form, March 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.
This paper is the major part of the author's thesis [19] under the supervision of Professor Lawrence G. Brown.

$M(A)$ is the strict closure of A, $LM(A)$ is the left strict closure of A, $RM(A)$ is the right closure of A and $QM(A)$ is the quasi-strict closure of A. For detailed expositions of these results the reader is referred to [1, 2, 8 and 21].

$LM(A)$, $RM(A)$ and $QM(A)$ are norm closed subspaces in A^{**}; $QM(A)$ is *-invariant, whereas $(LM(A))^* = RM(A)$. Moreover, $LM(A)$ and $RM(A)$ are Banach algebras. The best behaved class is $M(A)$ which is a C^*-algebra. It is clear that $M(A) = LM(A) \cap RM(A)$ and that $LM(A) + RM(A) \subset QM(A)$. The question was raised by Akemann and Pedersen [1] in 1973 whether $QM(A) = LM(A) + RM(A)$. McKennon [20] gave a nonseparable counterexample in 1978. Recently, L. Brown showed [6] that even when A is stable and separable, $QM(A)$ may not equal $LM(A) + RM(A)$.

In this paper, we give exact conditions for $QM(A) = LM(A) + RM(A)$ and for $QM(A) \neq LM(A) + RM(A)$.

Definition 1.3. A topological space X is scattered if every closed subset of X has a relatively isolated point.

Definition 1.4. Let X be a scattered topological space. We define $X_{[0]} = X$, $X_{[1]} = X \backslash \{\text{isolated points of } X\}$. If $X_{[\alpha]}$ is defined for some ordinal number α, define $X_{[\alpha+1]} = X_{[\alpha]} \backslash \{\text{isolated points of } X_{[\alpha]}\}$, if β is a limit ordinal, define $X_{[\beta]} = \bigcap_{\alpha<\beta} X_{[\alpha]}$.

Definition 1.5. Let X be a scattered topological space. We define $\lambda(X) = \alpha$, if α is the least ordinal such that $X_{[\alpha]}$ is discrete. Since X is scattered, $\lambda(X)$ is well defined.

Definition 1.6. Let $Y_1 = \{0, 1/n, n = 1, 2, \ldots\}$, a subset of $[0, 1]$ with the usual topology, and let Y_2 be the one-point compactification of the disjoint union of countably many copies of Y_1. If Y_α is defined for some ordinal number α, define $Y_{\alpha+1}$ as the one-point compactification of the disjoint union of countably many copies of Y_α. If β is a limit ordinal, define Y_β as the one-point compactification of the disjoint union of Y_α, $\alpha < \beta$. We also define $Z_\alpha^{(m)}$ to be the union of m disjoint copies of Y_α.

Theorem 1.7 [17] (or see [19, Theorem 1.9]). *Let X be a countable, compact Huasdorff space with $\lambda(X) = \alpha \geq 1$ and assume that $X_{[\alpha]}$ consists of n points. Then X is homeomorphic to $Z_\alpha^{(n)}$.*

Let $\{X, A(t), \mathscr{F}\}$ be a continuous field of C^*-algebras with X a locally compact Hausdorff space. Let $A = C_0(X, A(t), \mathscr{F})$ be the set of all continuous cross sections of $\{X, A(t), \mathscr{F}\}$ vanishing at infinity. Then A is a C^*-algebra.

We say a bounded cross section x in the bundle

$$\{X, LM(A(t))\} \quad (\{X, RM(A(t))\}, \{X, QM(A(t))\})$$

is left-strictly (right-strictly, quasi-strictly) continuous at t_0, if for every $a \in \mathscr{F}$, xa (ax, axa) is continuous at t_0. We denote by $C^b(X, LM(A(t))_{\text{L.S.}}, \mathscr{F})$

($C^b(X, \mathrm{RM}(A(t))_{\mathrm{R.S.}}, \mathscr{F})$, $C^b(X, \mathrm{QM}(A(t))_{\mathrm{Q.S.}}, \mathscr{F})$) the set of all bounded left-strictly (right-strictly, quasi-strictly) continuous cross sections in

$$\{X, \mathrm{LM}(A(t))\} \quad (\{X, \mathrm{RM}(A(t))\}, \{X, \mathrm{QM}(A(t))\}).$$

Let $A = C_0(X, A(t), \mathscr{F})$. Exactly as in [2, 3.3] we obtain

Theorem 1.8 (see [19, §1.3] also).

$$C^b(X, \mathrm{LM}(A(t))_{\mathrm{L.S.}}, \mathscr{F}) = \mathrm{LM}(A);$$
$$C^b(X, \mathrm{RM}(A(t))_{\mathrm{R.S.}}, \mathscr{F}) = \mathrm{RM}(A);$$
$$C^b(X, \mathrm{QM}(A(t))_{\mathrm{Q.S.}}, \mathscr{F}) = \mathrm{QM}(A).$$

2. A criterion for $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$

Let A be a σ-unital C^*-algebra, and a a strictly positive element, $0 < a \le 1$. For each n let f_n be a continuous function such that $f_n(t) = 1$ if $t \ge 1/n$, $f_n(t) = 0$ if $0 \le t \le 1/(n+1)$ and f_n is linear in $[1/(n+1), 1/n]$. Define $e_n = f_n(a)$. Then $\{e_n\}$ is an approximate identity for A. Moreover $e_m e_n = e_n e_m = e_n$, if $m > n$.

Lemma 2.1. *Let A be a σ-unital C^*-algebra and $\{e_n\}$ an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Suppose that $y \in \mathrm{QM}(A)$, then $y \in \mathrm{LM}(A)$ if and only if there exists an increasing sequence $\{n_k\}$ of nonnegative integers such that*

$$\sum_{k=1}^{\infty} \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right)$$

converges in norm to an element of A where $e_{n_0} = 0$.

Proof. Assume that $y \in \mathrm{LM}(A)$. For every m, $ye_m \in A$. Hence there is m' such that $\|(1 - e_{m'})ye_m\| < 1/2^m$. Therefore we can recursively define $n_1 < n_2 < \cdots$ so that

$$\left\|\left(1 - e_{n_{k+1}}\right) y e_{n_k}\right\| < \frac{1}{2^k}.$$

This implies that $\sum_{k=1}^{\infty}(1 - e_{n_{k+1}})y(e_{n_k} - e_{n_{k-1}})$ is norm convergent to an element in A.

For the converse, let $z = y - \sum_{k=1}^{\infty}(1 - e_{n_{k+1}})y(e_{n_k} - e_{n_{k-1}})$. For fixed n, let m be the least integer such that $n_m > n$. Then

$$\begin{aligned} ze_n &= ye_n - \sum_{k=1}^{m} \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) e_n \\ &= \sum_{k=1}^{m} \left[y \left(e_{n_k} - e_{n_{k-1}}\right) e_n - \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) e_n\right] \\ &= \sum_{k=1}^{m} e_{n_{k+l}} y \left(e_{n_k} - e_{n_{k-1}}\right) e_n . \end{aligned}$$

Since $y \in \mathrm{QM}(A)$, we conclude that $ze_n \in A$, for all n. Hence $z \in \mathrm{LM}(A)$. It follows that

$$y = z + \sum_{k=1}^{\infty} \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) \in \mathrm{LM}(A).$$

Lemma 2.2. *Let A be a σ-unital C^*-algebra and $\{e_n\}$ an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Suppose that $x_n \in \mathrm{QM}(A)$ with $\|x_n\| \le M$ for some M, j is an integer and $0 < \alpha \le 1$. Then*

$$\sum_{n=1}^{\infty} (e_{n+j+1} - e_{n+j})^{\alpha} x_n (e_n - e_{n-1})^{\alpha}$$

converges strictly.

Proof. Let P_s be the range projection of $(e_s - e_{s-1})$ and

$$y_s = (e_{s+j+1} - e_{s+j})^{\alpha} x_s (e_s - e_{s-1})^{\alpha}.$$

Clearly $P_s \cdot P_{s+2+i} = 0$ for $i = 0, 1, 2, \ldots$. Suppose that $A \subset B(H)$ and $f \in H$. Then

$$\Big\| \sum_{\substack{s=2k \\ s \le N}} y_s f \Big\|^2 = \Big\| \sum_{\substack{s=2k \\ s \le N}} P_{s+j+1} y_s P_s f \Big\|^2 = \sum_{\substack{s=2k \\ s \le N}} \|P_{s+j+1} y_s P_s f\|^2 \le M^2 \|f\|^2$$

for all N. Similarly

$$\Big\| \sum_{\substack{s=2k+1 \\ s \le N}} y_s f \Big\|^2 \le M^2 \|f\|^2 \quad \text{for all } N.$$

So $\{\| \sum_{n=1}^{N} y_n \|\}$ is bounded. For fixed m, if $N > m+1$, then

$$e_m \sum_{n=N}^{N+k} y_n = \sum_{n=N}^{N+k} y_n e_m = 0$$

for every k. Hence $\sum_{n=1}^{\infty} y_n$ converges strictly.

Theorem 2.3. *Let A be a σ-unital C^*-algebra and $\{e_n\}$ and approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. Then $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ if and only if for every $x \in \mathrm{QM}(A)_{\mathrm{s.a.}}$, there exists an increasing sequence $\{n_k\}$ of nonnegative integers such that*

$$\sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) x \left(e_{n_k} - e_{n_{k-1}}\right)$$

converges strictly $(e_{n_0} = 0)$.

Proof. Let $x \in \mathrm{QM}(A)_{\mathrm{s.a.}}$ and $n_1 < n_2 < \cdots$ be chosen such that $\sum_{k=1}^{\infty}(1-e_{n_k})x(e_{n_k}-e_{n_{k-1}})$ converges strictly. Let $x_k = (1-e_{n_k})x(e_{n_k}-e_{n_{k-1}})$. Since $\sum_{k=1}^{N} x_k \in \mathrm{RM}(A)$ for all N, we conclude that $\sum_{k=1}^{\infty} x_k \in \mathrm{RM}(A)$. For a fixed m, suppose that k_0 is the least integer such that $n_{k_0} > m$. Then

$$\left(x - \sum_{k=1}^{\infty} x_k\right) e_m = xe_m - \sum_{k=1}^{k_0} x_k e_m$$
$$= \sum_{k=1}^{k_0} e_{n_k} x \left(e_{n_k} - e_{n_{k-1}}\right) e_m \in A.$$

Hence $x - \sum_{k=1}^{\infty} x_k \in \mathrm{LM}(A)$. This implies that $\mathrm{QM}(A)_{\mathrm{s.a.}} \subset \mathrm{LM}(A) + \mathrm{RM}(A)$, and hence $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Next assume that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. Equivalently, $\mathrm{QM}(A)_{\mathrm{s.a.}} = \mathrm{Re}\,\mathrm{LM}(A)$. Let $x \in \mathrm{QM}(A)_{\mathrm{s.a.}}$. Thus there is $y \in \mathrm{LM}(A)$ such that $x = y + y^*$. By Lemma 2.1, we can choose $n_1 < n_2 < \cdots$ such that the elements $y_k = (1-e_{n_{k+1}})y(e_{n_k}-e_{n_{k-1}})$ satisfy $\|y_k\| < 2^{-k}$, whence $\sum_{k=1}^{\infty} y_k \in A$. By Lemma 2.2 $\sum_{k=1}^{\infty}(e_{n_{k+1}} - e_{n_k})y(e_{n_k} - e_{n_{k-1}})$ converges strictly. Hence $\sum_{k=1}^{\infty}(1 - e_{n_k})y(e_{n_k} - e_{n_{k-1}})$ converges strictly.

Let

$$y_{kj} = \left(e_{n_k} - e_{n_{k-1}}\right) e_{n_{j+1}} y \left(e_{n_j} - e_{n_{j-1}}\right),$$
$$y_k^{(1)} = \left(e_{n_k} - e_{n_{k-1}}\right) e_{n_{k+1}} y \left(1 - e_{n_k}\right) \left(e_{n_k} - e_{n_{k-1}}\right)$$

and

$$y_k^{(2)} = \left(e_{n_k} - e_{n_{k-1}}\right) e_{n_{k+2}} y \left(1 - e_{n_k}\right) \left(e_{n_{k+1}} - e_{n_k}\right).$$

Then by Lemma 2.2,

$$\sum_{k=1}^{\infty} y_k^{(1)}, \qquad \sum_{k=1}^{\infty} y_k^{(2)} \quad \text{and} \quad \sum_{k=1}^{\infty}\sum_{j=1}^{k+1} y_{kj} = \sum_{k=1}^{\infty}\sum_{j=k+2}^{k+1} y_{kj}$$

converge strictly. Since

$$\sum_{k=1}^{\infty}\sum_{j=k+2}^{\infty} y_{kj} + \sum_{k=1}^{\infty}\sum_{j=1}^{k+1} y_{kj} = \sum_{j=1}^{\infty} e_{n_{j+1}} y \left(e_{n_j} - e_{n_{j-1}}\right) = y - \sum_{j=1}^{\infty} y_j.$$

We conclude that $\sum_{k=1}^{\infty}\sum_{j=k+2}^{\infty} y_{kj}$ converges strictly. Thus

$$\sum_{k=1}^{\infty} \left(e_{n_k} - e_{n_{k-1}}\right) \left(y - \sum_{j=1}^{\infty} y_j\right) \left(1 - e_{n_k}\right)$$
$$= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left(e_{n_k} - e_{n_{k-1}}\right) e_{n_{j+1}} y \left(e_{n_j} - e_{n_{j-1}}\right) \left(1 - e_{n_k}\right)$$
$$= \sum_{k=1}^{\infty}\sum_{j=k+2}^{\infty} y_{kj} + \sum_{k=1}^{\infty} y_k^{(1)} = \sum_{k=1}^{\infty} y_k^{(2)}$$

converges strictly. So

$$\sum_{k=1}^{\infty}\left(1-e_{n_k}\right)\left(y^*-\sum_{j=1}^{\infty}y_j^*\right)\left(e_{n_k}-e_{n_{k-1}}\right)$$

converges strictly. Since $(1-e_{n_k})(e_{n_j}-e_{n_{j-1}})=0$ if $k>j$ and

$$(1-e_{n_{j+1}})(e_{n_k}-e_{n_{k-1}})=0, \quad \text{if } j\geq k,$$

we have

$$\sum_{k=1}^{\infty}\left(1-e_{n_k}\right)\left(\sum_{j=1}^{\infty}y_j\right)*\left(e_{n_k}-e_{n_{k-1}}\right)=0.$$

Finally, since $x=y+y^*$, $\sum_{k=1}^{\infty}x_k$ converges strictly. This completes the proof.

3. Lifting and hereditary properties

Considering the problem $\mathrm{QM}(A)=\mathrm{LM}(A)+\mathrm{RM}(A)$, one may ask the following questions:

(i) If I is an ideal of A such that $\mathrm{QM}(A/I)=\mathrm{LM}(A/I)+\mathrm{RM}(A/I)$ and $\mathrm{QM}(I)=\mathrm{LM}(I)+\mathrm{RM}(I)$, does it follow that $\mathrm{QM}(A)=\mathrm{RM}(A)+\mathrm{LM}(A)$?

(ii) If $\mathrm{QM}(A)=\mathrm{LM}(A)+\mathrm{RM}(A)$, does it follow tht $\mathrm{QM}(B)=\mathrm{LM}(B)+\mathrm{RM}(B)$ for B in A?

In this section, we shall show that (i) has a positive answer under a suitable assumption on A, and for some special B's, (ii) also has a positive answer. However, in general (ii) has a negative answer, as we shall see in Example 8.2.

Theorem 3.1. *Let A be a σ-unital C^*-algebra and B a C^*-subalgebra of A such that the hereditary C^*-subalgebra generated by B is A itself. If* $\mathrm{QM}(A)=\mathrm{LM}(A)+\mathrm{RM}(A)$, *then* $\mathrm{QM}(B)=\mathrm{LM}(B)+\mathrm{RM}(B)$.

Proof. Let $\widetilde{A}$ and $\widetilde{B}$ be C^*-algebras obtained by adding identities to A and B. Since the hereditary C^*-subalgebra generated by B is A itself, B contains a strictly positive element of A, say a. It follows that A and B share a common approximate identity $\{e_n\}$ satisfying $e_ne_m=e_me_n=e_n$, if $m>n$. Since e_n converges weakly to the identity of A and the identity of B in A^{**}, $\widetilde{A}$ and $\widetilde{B}$ have the same identity. Thus

$$\mathrm{QM}(B)_{\text{s.a.}}=[(\widetilde{B}_{\text{s.a.}})^m]^-\cap[(\widetilde{B}_{\text{s.a.}})_m]^-\subset[(\widetilde{A}_{\text{s.a.}})^m]^-\cap[(\widetilde{A}_{\text{s.a.}})_m]^-=\mathrm{QM}(A)_{\text{s.a.}}$$

(see [1, Theorem 4.1]). Since A and B have the same approximate identity $\{e_n\}$, we can apply Theorem 2.3 to obtain the desired conclusion.

Let A be a C^*-algebra and I a closed ideal. We shall denote $\mathrm{M}(A)\cap I^{**}$, $\mathrm{LM}(A)\cap I^{**}$, $\mathrm{RM}(A)\cap I^{**}$ and $\mathrm{QM}(A)\cap I^{**}$ by $\mathrm{M}(A,I)$, $\mathrm{LM}(A,I)$, $\mathrm{RM}(A,I)$ and $\mathrm{QM}(A,I)$, respectively. If $x\in\mathrm{M}(A,I)$, and $a\in A$, one can see that ax, $xa\in I$. Moreover, if $x\in\mathrm{LM}(A,I)$, $xa\in I$, etc.

Lemma 3.2. *Let A be a σ-unital C^*-algebra and I an ideal of A. Then* $\mathrm{QM}(A, I) = \mathrm{LM}(A, I) + \mathrm{RM}(A, I)$ *if one of the following holds*:

(i) $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ *or*

(ii) $\mathrm{QM}(I) = \mathrm{LM}(I) + \mathrm{RM}(I)$.

Proof. Let $\{e_n\}$ be an approximate identity of A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$, and $\{u_\lambda\}$ be an approximate identity for I. Let $x \in \mathrm{QM}(A, I)$. If we define $x_{ij} = (e_i - e_{i-1})^{1/2} x (e_j - e_{j-1})^{1/2}$ $(e_0 = 0)$, then $x_{ij} \in A \cap I^{**} = I$. There is a subsequence $\{u_n\}$ of $\{u_\lambda\}$ such that

$$\|(1 - u_i)x_{ij}\| < 1/2^{i+j}, \qquad j \le i,\ i = 1, 2, \dots,$$

and

$$\|x_{ij}(1 - u_i)\| < 1/2^{i+j}, \qquad i \le j,\ j = 1, 2, \dots.$$

Define $u = \sum_{i=1}^{\infty} (e_i - e_{i-1})^{1/2} u_i (e_i - e_i)^{1/2}$. By Lemma 2.2, $u \in I^{**}$, it is then easy to check that $u \in \mathrm{M}(A, I)$. Define $a_{ji} = (1 - e_{j+1})(e_i - e_{i-1})^{1/2}$, $b_j^{(1)} = (e_j - e_{j-1})^{1+1/2}$, $b_j^{(2)} = (e_{j+1} - e_j)^{1/2}(e_j - e_{j-1})$ and $b_j^{(3)} = (e_{j-1} - e_{j-2})^{1/2}(e_j - e_{j-1})$. Since $a_{ji} = 0$ if $i < j + 1$, we have

$$(1 - e_{j+1})[(1 - u)xu](e_j - e_{j-1}) = \sum_{i=j+1} a_{ji}(1 - u_i)x_{ij}u_j b_j^{(1)} + \sum_{i=j+1} a_{ji}(1 - u_i)x_{ij+1}u_{j+1} b_j^{(2)} + \sum_{i=j+1} a_{ji}(1 - u_i)x_{ij-1}u_{j-1} b_j^{(3)}.$$

Thus

$$\|(1 - e_{j+1})(1 - u)xu(e_j - e_{j-1})\| < 1/2^{j-2}.$$

This implies

$$\sum_{j=1}^{\infty} \|(1 - e_{j+1})(1 - u)xu(e_j - e_{j-1})\| < \infty.$$

By Lemma 2.1, $(1 - u)xu \in \mathrm{LM}(A)$. Similarly, $ux(1 - u) \in \mathrm{RM}(A)$ and $(1-u)x(1-u) \in \mathrm{LM}(A) \cap \mathrm{RM}(A)$. For every $a \in A$, $(1-u)xu \cdot a \in A \cap I^{**} = I$, $a \cdot ux(1 - u) \in I$ and $a(1 - u)x(1 - u)$, $(1 - u)x(1 - u)a \in I$. So $(1 - u)xu \in \mathrm{LM}(A, I)$, $ux(1 - u) \in \mathrm{RM}(A, I)$ and $(1 - u)x(1 - u) \in \mathrm{M}(A, I)$.

Now we need only show that $uxu \in \mathrm{LM}(A, I) + \mathrm{RM}(A, I)$.

(i) If $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$, there are $y_1 \in \mathrm{LM}(A)$ and $z_1 \in \mathrm{RM}(A)$ such that $x = y_1 + z_1$, so $uxu = uz_1u + uy_1u$. Since $u \in \mathrm{M}(A, I)$, $y_1u, ua \in I$ for every $a \in A$. Hence $uy_1u \in \mathrm{LM}(A, I)$. Similarly, $uz_1u \in \mathrm{RM}(A, I)$.

(ii) If $\mathrm{QM}(I) = \mathrm{LM}(I) + \mathrm{RM}(I)$, there are $y_2 \in \mathrm{LM}(I)$ and $z_2 \in \mathrm{RM}(I)$ such that $x = y_2 + z_2$, so $uxu = uy_2u + uz_2u$. One can easily check, as above, that $uy_2u \in \mathrm{LM}(A, I)$, $uz_2u \in \mathrm{RM}(A, I)$. This completes the proof.

At this point, one may ask whether $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ implies $\mathrm{QM}(I) = \mathrm{LM}(I) + \mathrm{RM}(I)$. This turns out to be false, as we shall see in Example 8.1. However, we have the following "lifting" theorem.

Theorem 3.3. *Let I be an ideal of a σ-unital C^*-algebra A, and suppose that* $\mathrm{QM}(A/I) = \mathrm{LM}(A/I) + \mathrm{RM}(A/I)$ *and* $\mathrm{QM}(I) = \mathrm{LM}(I) + \mathrm{RM}(I)$. *Then*

$$\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A).$$

Proof. Let $\phi\colon A \to A/I$ be the canonical homomorphsim and take $x \in \mathrm{QM}(A)$. So there is $\bar{y} \in \mathrm{LM}(A/I)$ and $\bar{z} \in \mathrm{RM}(A/I)$ such that $\phi^{**}(x) = \bar{y} + \bar{z}$, where ϕ^{**} is the extension of ϕ to A^{**}. It follows from [6, 4.13] that there are $y_1 \in \mathrm{LM}(A)$, $z_1 \in \mathrm{RM}(A)$ such that $\phi^{**}(y_1) = \bar{y}$ and $\phi^{**}(z_1) = \bar{z}_1$. Thus $\phi^{**}(x - z_1 - y_1) = 0$. So we may assume that $x \in \ker\phi^{**} \cap \mathrm{QM}(A)$, hence $x \in \mathrm{QM}(A, I)$. By Lemma 3.2 $x \in \mathrm{LM}(A, I) + \mathrm{RM}(A, I) \subset \mathrm{LM}(A) + \mathrm{RM}(A)$.

Let K be the C^*-algebra of all compact operators on l^2.

Corollary 3.4. *Let A be a C^*-algebra such that* $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$. *Then* $\mathrm{QM}(\widetilde{A} \otimes K) = \mathrm{LM}(\widetilde{A} \otimes K) + \mathrm{RM}(\widetilde{A} \otimes K)$.

Theorem 3.5. *Let A be a C^*-algebra such that* $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$ *and let B be a σ-unital C^*-subalgebra of A such that* $\mathrm{QM}(B) = \mathrm{M}(B)$. *Then*

$$\mathrm{QM}(B \otimes K) = \mathrm{LM}(B \otimes K) + \mathrm{RM}(B \otimes K).$$

Proof. By Corollary 3.4 we may assume that A has an identity. Take $x \in \mathrm{QM}(B \otimes K)_{\mathrm{s.a.}}$ and let $\{e_{ij}\}$ be a set of matrix units for K. Then x can be identified with an infinite matrix (a_{ij}) which represents a bounded operator, where a_{ij} is defined by $(1 \otimes e_{ii})x(1 \otimes e_{jj}) = a_{ij} \otimes e_{ij}$. Clearly each $a_{ij} \in \mathrm{QM}(B) = \mathrm{M}(B)$.

Let $\{u_n\}$ be an approximate identity of B which is quasi-central for $\mathrm{M}(B)$, i.e.

$$\lim \|u_n b - b u_n\| = 0 \quad \text{for all } b \in \mathrm{M}(B).$$

For every i, we have an integer n_i such that

$$\max_{k,j \le i} \left[\left\|\left(1 - u_{n_i}\right) u_k\right\| \cdot \left\|a_{ij}\right\|\right] < 1/2^{2i+1}$$

and

$$\left\|u_{n_i} a_{ij} - a_{ij} u_{n_i}\right\| < 1/2^{i+j}, \qquad i \ge j.$$

Let $w = (b_{ij})$, where $b_{ii} = u_{n_i}$, $b_{ij} = 0$, if $i \neq j$. Clearly, w is bounded and so is $wx = (u_{n_i} a_{ij})$. Since $a_{ij} \in \mathrm{M}(B)$, $u_{n_i} a_{ij} \in B \subset A$. We may view wx as an element in $\mathrm{QM}(A \otimes K)$. It follows from [6, 4.20] or Theorem 2.3 that there exist $n_1 < n_2 < \cdots$ such that $L(wx)$ is bounded, where

$$L(wx) = \sum_{k=1}^{\infty} \left(1 - f_{n_k}\right) wx \left(f_{n_k} - f_{n_{k-1}}\right)$$

and $f_n = \sum_{i=1}^{n} 1 \otimes e_{ii}$. Let $\sigma = -L(wx) + L(wx)^*$ and $y = x + \sigma$. Then y is bounded and $\operatorname{Re} y = x$. Let $L(y)' = (c_{ij})$, where $c_{ij} = a_{ij}(1 - u_{n_i})$, if there is $k > l$ such that $n_{k-1} < i \le n_k$, $n_{l-1} < j \le n_l$, and $c_{ij} = 0$ otherwise. Then

$L(y) - L(y)' = (d_{ij})$ where $d_{ij} = a_{ij}u_{n_i} - u_{n_i}a_{ij}$ if there is $k > l$ such that $n_{k-1} < i \leq n_k$, $n_{l-1} < j \leq n_l$ and $d_{ij} = 0$ otherwise. Since $\|d_{ij}\| < 1/2^{i+j}$, $d_{ij} \in B$, we see that $L(y) - L(y)' \in B \otimes K$. For every k,

$$L(y)' \sum_{i=1}^{k} u_k \otimes e_{ii} \in B \otimes K,$$

because

$$\max_{k,j\leq i} \|a_{ij}\| \cdot \left\| \left(1 - u_{n_i}\right) u_k \right\| < 1/2^{2i+1}.$$

Moreover $[y - L(y)] \cdot \sum_{i=1}^{k} u_k \otimes e_{ii} \in B \otimes K$. Hence

$$y \cdot \sum_{i=1}^{k} u_k \otimes e_{ii} \in B \otimes K \quad \text{for all } k.$$

Since $\{\sum_{i=1}^{k} u_k \otimes e_{ii}\}$ forms an approximate identity for $B \otimes K$, we conclude that $y \in \mathrm{LM}(A)$, so $x \in \mathrm{LM}(A) + \mathrm{RM}(A)$.

4. A constant associated with the equation $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$

Definition 4.1. Let A be a C^*-algebra such that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. For every $x \in \mathrm{QM}(A)$, let

$$\alpha(x) = \inf\{\|y\| : x = y + z,\ y \in \mathrm{LM}(A),\ z \in \mathrm{RM}(A)\}.$$

Clearly $\alpha(x) < \infty$. Let $\alpha(A) = \sup_{\|x\|\leq 1} \alpha(x)$. To see that $\alpha(A) < \infty$, we consider the mapping $\phi : \mathrm{LM}(A) \to \mathrm{QM}(A)_{\text{s.a.}}$ defined by $\phi(x) = (x + x^*)/2$. Then ϕ is a bounded real linear map from the real Banach space $\mathrm{LM}(A)$ onto the real Banach space $\mathrm{QM}(A)_{\text{s.a.}}$. By the open mapping theorem, ϕ is open. Thus the image of unit ball of $\mathrm{LM}(A)$ under ϕ contains a ball around the origin. It follows that $\alpha(A) < \infty$.

The following is an immediate consequence of Theorem 1.8.

Proposition 4.2. *Let A_n be C^*-algebras satisfying $\mathrm{QM}(A_n) = \mathrm{LM}(A_n) + \mathrm{RM}(A_n)$ and $\alpha(A_n) < c$, for some $c > 0$. Then*

$$\mathrm{QM}(\Sigma \oplus A_n) = \mathrm{LM}(\Sigma \oplus A_n) + \mathrm{RM}(\Sigma \oplus A_n)$$

and $\sup_n \alpha(A_n) \leq \alpha(\Sigma \oplus A_n) \leq c$.

Lemma 4.3. *Suppose that A is a σ-unital C^*-algebra such that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. If $\{e_n\}$ is an approximate identity satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$, and $x \in \mathrm{QM}(A)$ with $x \leq 1$, then for every $\varepsilon > 0$, there is $n_1 < n_2 < \cdots$ such that*

$$\left\| \sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) x \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \leq 7\alpha(A) + 5 + \varepsilon.$$

Proof. Let $x = y + z$, where $y \in \mathrm{LM}(A)$, $z \in \mathrm{RM}(A)$ and $\|y\| \leq \alpha(A) + (1/21)\varepsilon$.

As in the proof of Lemma 2.1, there exist $n_1 < n_2 < \cdots$ such that

$$\sum_{k=1}^{\infty} \left\| \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) \right\| < \frac{\varepsilon}{3}$$

and

$$\sum_{k=1}^{\infty} \left\| \left(e_{n_k} - e_{n_{k-1}}\right) z \left(1 - e_{n_{k+1}}\right) \right\| < \frac{\varepsilon}{3}.$$

Define $z_{ik} = (e_{n_i} - e_{n_{i-1}}) z (e_{n_k} - e_{n_{k-1}})$. We have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) z \left(e_{n_k} - e_{n_{k-1}}\right) \right\| &= \left\| z - \sum_{k=1}^{\infty} e_{n_k} z \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \\ &\leq \|z\| + \left\| \sum_{k=1}^{\infty} \sum_{i=1}^{k} z_{ik} \right\| \\ &\leq \|z\| + \left\| \sum_{k=1}^{\infty} \sum_{i=1}^{k-2} z_{ik} \right\| + \left\| \sum_{k=1}^{\infty} z_{k-1k} \right\| + \left\| \sum_{k=1}^{\infty} z_{kk} \right\|. \end{aligned}$$

For every large N

$$\begin{aligned} \left\| \sum_{k=1}^{N} \sum_{i=1}^{k-2} z_{ik} \right\| &= \left\| \sum_{i=1}^{N-2} \sum_{k=i+2}^{N} z_{ik} \right\| \\ &= \left\| \sum_{i=1}^{N-2} \left(e_{n_i} - e_{n_{i-1}}\right) z \left(e_{n_N} - e_{n_{i+1}}\right) \right\| < \frac{\varepsilon}{3}. \end{aligned}$$

Thus, by the proof of Lemma 2.2,

$$\left\| \sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) z \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \leq 5\|z\| + \frac{\varepsilon}{3}.$$

Now we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) x \left(e_{n_k} - e_{n_{k-1}}\right) \right\| &\leq \left\| \sum_{k=1}^{\infty} \left(1 - e_{n_{k+1}}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \\ &\quad + \left\| \sum_{k=1}^{\infty} \left(e_{n_{k+1}} - e_{n_k}\right) y \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \\ &\quad + \left\| \sum_{k=1}^{\infty} \left(1 - e_{n_k}\right) z \left(e_{n_k} - e_{n_{k-1}}\right) \right\| \\ &< \frac{\varepsilon}{3} + 2\|y\| + 5\|z\| + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + 2\alpha(A) + \frac{2}{21}\varepsilon + 5\alpha(A) + \frac{5}{21}\varepsilon + 5 + \frac{\varepsilon}{3} \\ &= 7\alpha(A) + 5 + \varepsilon. \end{aligned}$$

Remark 4.4. From the proof of Lemmas 2.2 and 4.3, we find that if the e_n's are projections, we will have

$$\left\|\sum\left(1-e_{n_k}\right)x\left(e_{n_k}-e_{n_{k-1}}\right)\right\|\leq 4\alpha(A)+3+\varepsilon.$$

Lemma 4.5. *Let B be a C^*-algebra which has an approximate identity consisting of countably many projections. Suppose that $\mathrm{QM}(B)=\mathrm{M}(B)$ and $A=C(X)\otimes B$, where X is homeomorphic to one of the spaces Y_n described in Definition 1.6. Then*

$$\mathrm{QM}(A)=\mathrm{LM}(A)+\mathrm{RM}(A)\quad \textit{and}\quad \alpha(A)\leq\sqrt{n}.$$

Proof. Let $x\in\mathrm{QM}(A)=C(X,\mathrm{QM}(B)_{\mathrm{Q.S.}})$ and $\|x\|\leq 1$. Let $\{e_k\}$ be an approximate identity of B consisting of projections. We use induction.

(1) If $n=1$, $Y_1=\{0\}\cup\{1/m\}_{m=1}^{\infty}$.

For every k, there is an N such that whenever $m\geq N$

$$\|e_k(x(0)-x(1/m))e_k\|<1/k.$$

There is an integer m_0 such that for every $m\geq m_0$, there is a largest integer k_m such that

$$\left\|e_{k_m}(x(0)-x(1/l))e_{k_m}\right\|\leq 1/k_m\quad\text{for all } l\geq m.$$

(In the trivial case $x(1/m)=x(0)$ for all $m\geq m^*$, for some m^*, define $e_{k_m}=e_m$.) Hence $k_m\to\infty$, as $m\to\infty$ and $k_{m+1}\geq k_m$. Define $u(1/m)=e_1$ if $m<m_0$, $u(1/m)=e_{k_m}$ and $y(0)=x(0)$, $y(1/m)=u(1/m)x(1/m)$, $z(0)=0$ and $z(1/m)=(1-u(1/m))x(1/m)$. Then $x=y+z$. It is easy to check that for every a and $b\in A$, $y(1/m)a\to y(0)a$ and $bz(1/m)\to bz(0)$. So $y\in\mathrm{LM}(A)$, $z\in\mathrm{RM}(A)$ and

$$\|y\|=\|ux\|\leq 1=\sqrt{1}.$$

(2) Next we assume that Lemma 4.5 is true for all integers less than n. In particular, we can choose $y\in\mathrm{LM}(A)$ such that $\|y\|\leq\sqrt{k}$, where $k<n$.

Notice that Y_n is the one-point compactification of the disjoint union of Z_i, where each Z_i is homeomorphic to Y_{n-1}.

Let $x_i(t)=x(t)|_{Z_i}$. There is an integer i_0 such that for every $i\geq i_0$, there is a largest integer m_i such that $\|e_{m_i}[x(\infty)-x_i(t)]e_{m_i}\|\leq 1/m_i$ for $t\in Z_i$. (In the case that $x_i(t)\equiv x(\infty)$ for all $i\geq i_0$, for some i_0, we define $e_{m_i}=e_i$.) Hence $m_i\to\infty$, as $i\to\infty$ and $m_{i+1}\geq m_i$. By the induction assumption, there are $y_i\in C(Z_i,\mathrm{LM}(B)_{\mathrm{L.S.}})$ and $z_i\in C(Z_i,\mathrm{RM}(B)_{\mathrm{R.S.}})$ such that $x_i=y_i+z_i$ and $\|y_i\|\leq\sqrt{n-1}$.

Define $y(t)=e_{m_i}x_i(t)+(1-e_{m_i})y_i(t)(1-e_{m_i})$ if $t\in Z_i$, $y(\infty)=x(\infty)$, $z(t)=(1-e_{m_i})z_i(t)+(1-e_{m_i})y_ie_{m_i}$, if $t\in Z_i$ and $z(\infty)=0$. Clearly, $x=y+z$ and $y(t)|_{z_i}\in C(Z_i,\mathrm{LM}(B)_{\mathrm{L.S.}})$ and $z(t)|_{Z_i}\in C(Z_i,\mathrm{RM}(B)_{\mathrm{R.S.}})$. Similarly to (1), one can check that $y(t)\in C(Y_n,\mathrm{LM}(B)_{\mathrm{L.S.}})=\mathrm{LM}(A)$ and

$z(t) \in C(Y_n, \mathrm{RM}(B)_{\mathrm{R.S.}}) = \mathrm{RM}(A)$. Hence $x \in \mathrm{LM}(A) + \mathrm{RM}(A)$. Moreover, let B act on a Hilbert space H and $f \in H$,

$$\|y(t)f\|^2 = \left\|e_{m_i}x_i(t)f\right\|^2 + \left\|\left(1-e_{m_i}\right)y_i(t)\left(1-e_{m_i}\right)f\right\|^2$$
$$\leq \|f\|^2 + (n-1)\|f\|^2 = n\|f\|^2.$$

This implies that $\|y\| \leq \sqrt{n}$.

Lemma 4.6. *Let* $A = C_0(X, A(t), A)$ *be a separable* C^**-algebra, where* X *is a countable, locally compact Hausdorff space with* $\lambda(X) < \infty$ *and* $A(t)$ *are* C^**-algebras such that* $\mathrm{QM}(A(t)) = \mathrm{M}(A(t))$. *Then* $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Proof. Let $I_k = \{f \in A, f(t) = 0, \text{ if } t \in X_{[k]}\}$ (cf. Definition 1.4). By Proposition 4.2, and Theorem 3.3, we can easily prove the lemma by induction.

Remark 4.6. Define $f(e^{i\theta}) = i\theta/\pi$ $(-\pi < \theta \leq \pi)$ and let a_n $(n \in \mathbf{Z})$ be its Fourier coefficients. Then $a_n = (-1)^{n+1}/n\pi$ $(n \neq 0)$, $a_0 = 0$. But $\sum_{n\in\mathbf{Z}} a_{|n|}e^{in\theta}$ is the Fourier series of the L^2 function $2\pi^{-1}\log|1+e^{i\theta}|$ which is not in $L^\infty(T)$. (T denoting the unit circle.) Thus the matrix (a_{i-j}) represents an operator on l^2 of norm 1. But the lower triangle of the matrix is not bounded. Let $L_n = (b_{ij})$, where $b_{ij} = \sqrt{-1}a_{i-j}$ if $i \geq j$ and $i \leq n$, $b_{ij} = 0$ if $i > n$, or $j > i$.

Let $g \in l^2$, $g = (d_j)$, $d_j = (-1)^j 1/\sqrt{n}^{1/2}$, $j \leq n$, $d_j = 0$, $j > n$. Then $\|g\|_2 = 1$ and

$$\|L_n g\|^2 = \frac{1}{\pi^2 n}\sum_{k=1}^{n}\left(\sum_{j=1}^{k-1}\frac{1}{j}\right)^2 \geq \frac{1}{\pi^2 n}\sum_{k=1}^{n}(\log K)^2$$
$$\geq \frac{1}{\pi^2 n}[n\log n(\log n - 2)] \geq \frac{1}{4\pi^2}(\log n)^2$$

if n is large enough $(n > 15)$. We conclude that $\|L_n\| \geq (1/2\pi)\log n$ when n is large.

Lemma 4.7. *Let* $A_n = C(Y_n) \otimes K$. *Then*

$$\frac{1}{9\pi}\log n \leq \alpha(A_n) \leq \sqrt{n}$$

when n *is large enough* $(n > 20)$.

Proof. It follows from Lemma 4.5 that $\mathrm{QM}(A_n) = \mathrm{LM}(A_n) + \mathrm{RM}(A_n)$ and $\alpha(A_n) \leq \sqrt{n}$.

For every sequence $\{n_k\}$, $n_1 < n_2 < \cdots$, define the operator $\alpha(\{n_k\}) = (t_{ij})$ where

$$t_{ij} = \begin{cases} \sqrt{-1}(a_{k-l}), & \text{if } i = n_k,\ j = n_l, \\ 0 & \text{otherwise} \end{cases}$$

and $a_n = (-1)^{n+1}/n\pi$, $n \neq 0$, $n \in \mathbf{Z}$, $a_0 = 0$.

Then $\alpha(\{n_k\})$ is selfadjoint and $\|\alpha(\{n_k\})\| \leq 1$. For every n, define $\alpha_n(\{n_k\}) = (t'_{ij})$, where

$$t'_{ij} = \begin{cases} \sqrt{-1}(a_{k-l}), & \text{if } i = n_k,\ j = n_l \text{ and } k, l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_n = \{a_n(\{n_k\}),\ \{n_k\} \subset \mathbf{N}\}$, so that each S_n is countable $(n \in \mathbf{N})$. We claim that the cluster points of S_n are

$$\{\alpha_j(\{n_k\}),\ j \leq n-1,\ \{n_k\} \subset N\} = \bigcup_{j=0}^{n-1} S_j$$

(in the weak operator topology).

Let $p_m = (\varepsilon_{ij})$ where $\varepsilon_{ii} = 1$, if $i \leq m$, $\varepsilon_{ij} = 0$, if $i \neq j$ or $i > m$.

Let $\beta \in \bigcup_{j=0}^{n-1} S_j$, say $\beta = \alpha_j(\{n_k\})$ for some $j \leq n-1$ and $\{n_k\} \subset \mathbf{N}$. Define $n_k^{(s)} = n_k$ if $k \leq j$, $n_k^{(s)} = n_k + k + s$, if $k > j$. For every m, if $s > m$, $p_m[\beta - \alpha_n(\{n_k^{(s)}\})]p_m = 0$. This implies that $\alpha_n(\{n_k^{(s)}\}) \to \beta$ weakly as $s \to \infty$.

Next let $\alpha_n(\{m_k^{(s)}\}) \to \beta$ weakly as $s \to \infty$. Since for every i there are only finitely many different elements $p_i\alpha_n(\{m_k^{(s)}\})p_i$, we see that

$$p_i[\alpha_n(m_k^{(s)}) - \beta]p_i = 0,$$

when s is large. Thus we conclude that

$$\beta \in \{\alpha_j(\{n_k\}),\ j \leq n-1,\ \{n_k\} \subset N\} = \bigcup_{j=0}^{n-1} S_j$$

unless $\alpha_n\{n_k^{(s)}\} = \beta$ for $s \geq s_0$ for some s_0. This establishes the claim.

By induction, we have $(S_n)^{\bar{w}} = \bigcup_{j=1}^{n} S_j$ (where "$\bar{w}$" means the weak closure) and $(S_n)^{\bar{w}}_{[i]} = \bigcup_{j=0}^{n-i} S_j$. Hence $\lambda(S_n^{\bar{w}}) = n$. It is also clear, by a similar argument as the above, that every sequence of S_n has a convergent subsequence. So $S_n^{\bar{w}}$ is compact.

The weak operator topology on bounded subsets of $B(l^2)$ coincides with the quasi-strict topology $B(l^2) = \mathrm{QM}(K)$, so we define a continuous mapping F_n from Y_n onto $S_n^{\bar{w}}$, then $F_n \in C(Y_n, \mathrm{QM}(K)_{\mathrm{Q.S.}}) = \mathrm{QM}(A_n)$. The existence of F_n comes from Theorem 1.7. Now let $\{e_{ij}\}$ be a set of matrix units for K and $f_m = \sum_{i=1}^{m} 1 \otimes e_{ii}$ (f_m can be identified with a constant function: $Y_n \to P_m \in K$). Then $\{f_m\}$ forms an approximate identity for A. For every $n_1 < n_2 < \cdots$, we see clearly, by the construction of $S_n^{\bar{w}}$, that

$$\left\| \sum_{k=1}^{\infty} \left(1 - f_{n_k}\right) F_n \left(f_{n_k} - f_{n_{k-1}}\right) \right\| \geq \|L_n(\alpha)\| \geq \frac{1}{2\pi} \log n,$$

if n is large enough $(n > 15)$.

Combining this and Remark 4.4, we have that $\alpha(A_N) \geq \frac{1}{9\pi}\log n$, if n is large enough $(n > 20)$.

We shall see in Example 8.1 that $\mathrm{M}(A/I) = \mathrm{QM}(A/I)$ and $\mathrm{M}(I) = \mathrm{QM}(I)$ does not imply $\mathrm{QM}(A) = \mathrm{M}(A)$ even for very simple algebras. Theorem 3.3 shows that if moreover A is σ-unital, we do have $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. Hence the only significance of the following proposition is the estimate of $\alpha(A)$.

Proposition 4.8. *Let A be a σ-unital C^*-algebra and I an ideal of A. Suppose $\mathrm{M}(A/I) = \mathrm{QM}(A/I)$ and $\mathrm{M}(I) = \mathrm{QM}(I)$. Then for every $x \in \mathrm{QM}(A)$, there are $y \in \mathrm{LM}(A)$ and $z \in \mathrm{RM}(A)$ such that $x = y + z$, $\|y\| \leq \|x\|$ and $\|z\| \leq \|x\|$, hence $\alpha(A) \leq 1$.*

Proof. Let $\phi : A \to A/I$ be the canonical homomorphism, ϕ^{**} be its extension to A^{**}. So $\phi(x) \in \mathrm{QM}(A/I) = \mathrm{M}(A/I)$. By [26], there is $x' \in \mathrm{M}(A)$ such that $\phi(x') = \phi(x)$ and $x_1 = x - x' \in \ker\phi^{**} \cap \mathrm{QM}(A)$. Thus $x_1 \in \mathrm{QM}(A, I)$. Let $\{e_i\}$ be an approximate identity for A satisfying $e_ie_j = e_ie_j = e_i$, if $i < j$, and put $x_{ij} = (e_i - e_{i-i})^{1/2}x(e_j - e_{j-j})^{1/2}$. Then $x_{ij} \in I$. Suppose that $\{u_\lambda\}$ is an approximate identity for I. There is a subsequence $\{u_{\lambda_n}\}$ of $\{u_\lambda\}$ such that

$$\left\|\left(1 - u_{\lambda_i}\right) x_{ij}\right\| < \frac{1}{2^{i+j}}, \qquad i \geq j,\ i = 1, 2, \ldots.$$

If $u = \sum_{i=1}^{\infty}(e_i - e_{i-1})^{1/2}u_{\lambda_i}(e_i - e_{i-1})^{1/2}$, then $u \in \mathrm{M}(A, I)$. As in the proof of Lemma 3.2 we have

$$\sum_i \left\|(1 - e_{i+1})(1 - u)x_1(e_i - e_{i-1})\right\| < \infty.$$

Hence $(1 - u)x_1 \in \mathrm{LM}(A)$ by Lemma 2.1.

Let $y = (1 - u)x = (1 - u)x' + (1 - u)x_1$ and put $z = ux = ux_1 + ux'$. Since $u \in \mathrm{M}(A, I)$, we see that $u \in \mathrm{LM}(A)$ and $ux' \in \mathrm{M}(A)$. Since $\mathrm{M}(I) = \mathrm{QM}(I)$, $x_1 \in \mathrm{M}(I)$. For every $a \in A$, $au \in I$, so $aux_1 \in I \subset A$. This implies that $ux_1 \in \mathrm{RM}(A)$. Hence $z \in \mathrm{RM}(A)$ since $0 \leq u \leq 1$. We have $\|y\| = \|(1 - u)x\| \leq \|x\|$ and $\|z\| = \|ux\| \leq \|x\|$. Thus $\alpha(A) \leq 1$.

5. The spectrum of an element in a scattered C^*-algebra

In this section, we shall discuss the relationship between the spectrum of a single element in a scattered C^*-algebra A and the spectrum of the algebra A.

Jensen [13] defined a C^*-algebra to be scattered if every state on the algebra is atomic. He showed [14] that a C^*-algebra is scattered if and only if it is type I and has scattered spectrum $\widehat{A}$. He also showed [14] that a C^*-algebra is scattered if and only if it has a composition series with elementary quotients.

We recall that a C^*-algebra A is AF (approximately finite-dimensional) if for each $\varepsilon > 0$ and $a_1, a_2, \ldots, a_n \in A$ there is a C^*-subalgebra B of A and $b_1, b_2, \ldots, b_n \in B$ such that B is of finite dimension and $\|a_i - b_i\| < \varepsilon$, for all $i = 1, 2, \ldots, n$.

Lemma 5.1. *Every scattered C^*-algebra A is AF.*

Proof. Supose that A has a series of ideals $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_\alpha \subset \cdots \subset I_\lambda = A$, where each $I_{\beta+1}/I_\beta$ is an elementary C^*-algebra and $I_\alpha = (\bigcup_{\beta<\alpha} I_\beta)^-$ for each limit ordinal α. We prove the lemma by induction on λ. Assume Lemma 5.1 is true for all $\lambda < \lambda_0$.

If λ_0 is not a limit ordinal, $I_{\lambda_0}/I_{\lambda_0-1}$ is an elementary C^*-algebra, hence $I_{\lambda_0}/I_{\lambda_0-1}$ is an AF-algebra. By the induction hypothesis I_{λ_0-1} is also an AF-algebra. It follows from [11] that A is an AF-algebra.

If λ_0 is a limit ordinal, A is the norm closure of $\bigcup_{\lambda<\lambda_0} I_\lambda$. For each $\varepsilon > 0$ and $a_1, a_2, \dots, a_n \in A$, there is $\lambda < \lambda_0$ and $b_1', b_2', \dots, b_n' \in I_\lambda$ such that

$$\|a_i - b_i\| < \varepsilon/2, \qquad i = 1, 2, \dots, n.$$

Since, by hypothesis, I_λ is an AF-algebra, there is a C^*-subalgebra B of I_λ and $b_1, b_2, \dots, b_n \in B$ such that B is of finite dimension and

$$\|b_i' - b_i\| < \varepsilon/2, \qquad i = 1, 2, \dots, n.$$

Hence $\|b_i - a_i\| < \varepsilon$, $i = 1, 2, \dots, n$. So A is an AF-algebra.

Lemma 5.2. *Let A be a scattered C^*-algebra. If $\lambda(\widehat{A}) = \alpha$, then for every $a \in A_{\text{s.a.}}$, we have $\lambda[\sigma(a)] \le \alpha + 1$. If α is a limit ordinal and $\widehat{A}_{[\alpha]} = \varnothing$, then $\lambda[\sigma(a)] \le \alpha$.*

Proof. Let $I_i = \{x \in A; \pi(x) = 0, \ \forall \pi \in \widehat{A}_{[i]}\}_{i\le\alpha}$. Suppose that $\alpha \in A_{\text{s.a.}}$. Let B be the C^*-algebra generated by a. Define $J_i = B \cap I_i$. Clearly, since I_{i+1}/I_i and A/I_α are dual C^*-algebras [12, 4.7.20], so are J_{i+1}/J_i and B/J_α. Thus $\widehat{B}$ is the union of closed subsets X_i satisfying $X_i \supset X_{i+1}$, $X_{i+1} \subset (X_i)_{[1]}$ and $X_{i+1} \backslash X_i$ is discrete, $i < \alpha$. If $\widehat{A}_{[\alpha]} = \varnothing$, $A = I_\alpha$. Hence $X_\alpha = \varnothing$. Since $B = C_0(\widehat{B})$, it is clear that $\lambda(\widehat{B}) \le \alpha$ and if $\widehat{A}_{[\alpha]} = \varnothing$, $\widehat{B}_{[\alpha]} = \varnothing$. Thus $\lambda(\sigma(a)) \le \alpha + 1$ and if $\widehat{A}_{[\alpha]} = \varnothing$, $\lambda(\sigma(a)) \le \alpha$.

Lemma 5.3. *Let A be a scattered C^*-algebra. Suppose that $\lambda(\widehat{A}) = \alpha$, $I_\beta = \{x \in A; \pi(x) = 0, \pi \in \widehat{A}_{[\beta]}\}$. Then $I_{\beta+1}/I_\beta$ is of infinite dimension, if $\beta < \alpha$.*

Proof. We shall use the facts that A is of type I and $\beta + 1 \le \alpha$.

Let $J_{\beta+1} = I_{\beta+1}/I_\beta$. If $\widehat{J}_{\beta+1}$ is an infinite set, the result is clear. We may assume therefore that $\widehat{J}_{\beta+1} = \{\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m\}$. Let π_i be an irreducible representation of A corresponding to $\tilde{\pi}_i$. We have $\widehat{A}_{[\beta]} = \bigcup_{i=1}^m \{\pi_i\}^-$. Since $\widehat{A}_{[\beta+1]} \ne \varnothing$, there is $\pi \in \widehat{A}_{[\beta+1]}$ and hence there is $i \le m$ such that $\ker \pi_i \subset \ker \pi$. This implies that $\pi_i(A)$ must be infinite dimensional. Hence $\pi_i(I_{\beta+1}) \supset K(H_{\pi_i})$ (the compact operators on H_{π_i}), where $\dim H_\pi = \infty$. Since $\pi_i(I_\beta) = 0$, we conclude that $J_{\beta+1}$ is of infinite dimension.

Theorem 5.4. *Let A be a scattered C^*-algebra with $\lambda(\widehat{A}) = \alpha$. Then*

(i) *For every $a \in A_{\text{s.a.}}$ $\lambda(\sigma(a)) \le \alpha + 1$.*

(ii) *If* α *is not a limit ordinal, there is* $a \in A_{\text{s.a.}}$ *such that* $\lambda(\sigma(a)) \geq \alpha$.

(iii) *If* α *is not a limit ordinal, then there is* $a \in A_{\text{s.a.}}$ *such that* $\lambda(\sigma(a)) = \alpha + 1$ *if and only if* A/I_α *is of infinite dimension, where* $I_\alpha = \{x \in A; \pi(x) = 0, \pi \in \widehat{A}_{[\alpha]}\}$.

(iv) *If* α *is a limit ordinal and* A/I_α *is of finite dimension (or zero), then for every* $a \in A_{\text{s.a.}}$, $\lambda(\sigma(a)) \leq \alpha$. *Moreover, for every* $\beta < \alpha$, *there is* $a \in A_{\text{s.a.}}$ *such that* $\lambda(\sigma(a)) > \beta$.

(v) *If* α *is a limit ordinal such that* $\alpha = \lim \beta_n$ $(\beta_n < \alpha)$ *and* A/I_α *is of infinite dimension, then there is* $a \in A_{\text{s.a.}}$ *such that* $\lambda(\sigma(a)) \geq \alpha$.

Proof. We shall use induction.

Assume the theorem is true for all $\beta < \alpha$.

(i) is the same as Lemma 5.4.

(ii) If α is not a limit ordinal, by Lemma 5.5, $I_\alpha / I_{\alpha-1}$ is of infinite dimension. By the induction hypothesis for (iii), there is $a \in I_\alpha$ such that a is selfadjoint and $\lambda(\sigma(a)) \geq (\alpha - 1) + 1 = \alpha$.

(iii) If A/I_α is of finite dimension, $a \in A_{\text{s.a.}}$, then there is a polynomial $p(t)$ $(p \neq 0)$ such that $p(a) \in I_\alpha$. By the induction hypothesis $\lambda(\sigma(p(a))) \leq \alpha$, since $\lambda(\widehat{I}_\alpha) = \alpha - 1$. By the spectral mapping theorem, one sees easily that $\lambda(\sigma(a)) \leq \alpha$.

If A/I_α is of infinite dimension, there is a sequence of mutually orthogonal projections $\bar{p}_n \in A/I_\alpha$, $\bar{p}_n \neq 0$. Let $\phi : A \to A/I_\alpha$ be the canonical homomorphism. Since I_α is an AF-algebra, by the projection lifting theorem [4], there is $p_1 \in A$ such that $\phi(p_1) = \bar{p}_1$. Using the projection lifting theorem on $(1 - p_1)A(1 - p_1)/I_\alpha \cap (1 - p_1)A(1 - p_1) \cong (1 - \bar{p}_1)(A/I)(1 - \bar{p}_1)$, and continuing, we construct a sequence of mutually orthogonal projections $\{p_n\} \subset A$ such that $\pi(p_n) = \bar{p}_n$. Since there is $\pi \in \widehat{A}_{[\alpha]}$ such that $\pi(p_n) \neq 0$, we have $\pi \in \widehat{A} \setminus \text{hull}(p_n A p_n)$. It follows from the fact that $p_n A p_n$ is a hereditary C^*-subalgebra of A that $(p_n A p_n)^\wedge$ is homeomorphic to $\widehat{A} \setminus \text{hull}(p_n A p_n)$. Since $(\widehat{A} \setminus \text{hull}(p_n A p_n))$ is open and $\widehat{A}_{[\alpha]} \cap (\widehat{A} \setminus \text{hull}(p_n A p_n)) \neq \varnothing$, $\lambda((p_n A p_n)^\wedge) = \alpha$. By (ii), there are $a_n \in p_n A p_n$, $a_n = a_n^*$, $\|a_n\| \leq 1$ and $\lambda(\sigma(a_n)) \geq \alpha$. Taking a_n^2, if necessary, we may assume that $0 \leq a_n \leq 1$. Define

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n}(p_n + a_n);$$

then a is selfadjoint and $\lambda(\sigma(a)) = \alpha + 1$.

(iv) Assume that α is a limit ordinal and A/I_α is of finite dimension. If $a \in I_\alpha$, $a = a^*$, then by Lemma 5.2 $\lambda(\sigma(a)) \leq \alpha$. For every $a \in A_{\text{s.a.}}$, there is a polynomial $p(t)$ $(p(t) \neq 0)$ such that $p(a) \in I_\alpha$. Hence $\lambda(\sigma(p(a))) \leq \alpha$. By the spectral mapping theorem, one can see easily that $\lambda(\sigma(a)) \leq \alpha$. For each $\beta < \alpha$, consider $I_{\beta+1} \subset A$. By the induction hypothesis, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) \geq \beta$.

(v) If α is a limit ordinal such that $\alpha = \lim \beta_n$, $\beta_n < \alpha$, and A/I_α is of infinite dimension, then, as in the proof of (iii), A contains a sequence of mutually orthogonal projections $\{q_n\}$ such that $\lambda[(q_nAq_n)^\wedge] = \alpha$. By (iv), there are $a_n \in q_nAq_n$, $0 \le a_n \le 1$ such that, $\lambda(\sigma(a_n)) \ge \beta_n$. Define

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n}(q_n + a_n).$$

Clearly $a \in A_{\text{s.a.}}$ and $\lambda(\sigma(a)) \ge \alpha$.

The proof is complete.

6. Quasi-multipliers of stable C^*-algebras

Lemma 6.1. *Let A be a separable scattered C^*-algebra with $\lambda(\widehat{A}) < \infty$. Then* $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Proof. Let $I_i = \{a \in A, \pi(a) = 0, \forall \pi \in \widehat{A}_{[i]}\}$. Then $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset A$, $n = \lambda(\widehat{A})$, and I_i/I_{i-1} and A/I_n are separable dual C^*-algebras. Since A and I_i are σ-unital and $\mathrm{M}(I_i/I_{i-1}) = \mathrm{QM}(I_i/I_{i-1})$, $\mathrm{M}(A/I_n) = \mathrm{QM}(A/I_n)$, by Theorem 3.3 and induction; $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Corollary 6.2. *Let A be a separable scattered C^*-algebra with $\lambda(\widehat{A}) < \infty$. Then* $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$.

Theorem 6.3. *Let A be a separable C^*-algebra. Then* $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$ *if and only if A is scattered and $\lambda(\widehat{A}) < \infty$.*

Proof. By Corollary 6.2, we need only show the "only if" part. So we assume that $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$. It follows from Corollary 3.4 that we may assume that A has an identity. It follows from [6, 4.23] that A is scattered. If $\lambda(\widehat{A})$ is not finite, by Theorem 5.4, for every integer $m > 0$, there is $a \in A_{\text{s.a.}}$ such that $\lambda(\sigma(a)) = m$. Let B be the C^*-algebra generated by a and 1. It follows from the proof of Proposition 2.4 that $\mathrm{QM}(B \otimes K) \subset \mathrm{QM}(A \otimes K)$ and $A \otimes K$ and $B \otimes K$ share a common approximate identity $f_n = \sum_{i=1}^{n} 1 \otimes e_{ij}$, where $\{e_{ij}\}$ is a set matrix units for K. By Lemma 2.16, there is $F \in \mathrm{QM}(B\otimes K) \subset \mathrm{QM}(A\otimes K)$ such that for every $\{n_k\}$, $n_1 < n_2 < \cdots$,

$$\left\| \sum_{k=1}^{\infty} \left(1 - f_{n_k}\right) F \left(f_{n_k} - f_{n_{k-1}}\right) \right\| \ge \frac{1}{2\pi} \log m,$$

if m is large enough. It follows from Lemma 4.3 and Remark 4.4 that

$$\alpha(A \otimes K) \ge \frac{1}{9\pi} \log m$$

for m large enough. Hence $\alpha(A \otimes K) = \infty$, a contradiction.

Corollary 6.4. *Let A be a separable C^*-algebra. Then* $\mathrm{QM}(A \otimes K) = \mathrm{LM}(A \otimes K) + \mathrm{RM}(A \otimes K)$ *if and only if there is an integer $m > 0$ such that for every $a \in A_{\text{s.a.}}$, $\sigma(a)$ is countable and $\lambda(\sigma(a)) \le m$.*

Proof. It is an immediate consequence of Theorem 5.1, [14, Theorem 2.2] and Theorem 6.2.

Corollary 6.5. *Let A be a separable stable C^*-algebra. Then* $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ *if and only if A is scattered and $\lambda(\widehat{A}) < \infty$.*

7. C^*-ALGEBRAS WITH FINITE DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

In this section we shall consider C^*-algebras whose irreducible representations are finite dimensional. Let M_n denote the C^*-algebra of all complex $n \times n$ matrices. If A is a C^*-algebra whose irreducible representations are finite dimensional and $\widehat{A}$ is Hausdorff, then by [9, Theorem 10.54], $A = C_0(\widehat{A}, M_{n(t)}, A)$. If $A = C_0(\widehat{A}, M_{n(t)}, A)$ is locally trivial, one can easily show by Theorem 1.3 that $\mathrm{QM}(A) = \mathrm{M}(A)$. However, even if $\widehat{A}$ is countable and Hausdorff, $\mathrm{QM}(A) \neq \mathrm{LM}(A) + \mathrm{RM}(A)$, in general.

Proposition 7.1. *There is a C^*-algebra A such that all of its irreducible representations are finite dimensional, $\widehat{A}$ is a countable locally compact Hausdorff space, and* $\mathrm{QM}(A) \neq \mathrm{LM}(A) + \mathrm{RM}(A)$.

Proof. Keep the notations in the proof of Lemma 4.5. Let $P^{(n)}(t)$ be the range projection of $F_n(t)$. By the proof of Lemma 4.5, it is clear that $P^{(n)}(t)$ is a weakly continuous mapping from Y_n to K. Since $P^{(n)}(t)$ is bounded, we conclude that $P^{(n)}(t) \in \mathrm{QM}(C(Y_n, K))$.

Let X be the disjoint union of Y_n, $n = 1, 2, \dots$. Define

$$B_0 = \{x \in C_0(X, K) : x(t) = P^{(n)}(t)x(t)P^{(n)}(t); \forall t \in Y_n\}.$$

Clearly, B_0 is a *-algebra. Let $M_n(t) = P^{(n)}(t)KP^{(n)}(t)$. Then each $M_n(t)$ is isomorphic to some M_k. We define $A = C_0(X, M_n(t), B_0)$. A is a C^*-algebra all of whose irreducible representations are of finite dimension and $\widehat{A} = X$, a countable, locally compact Hausdorff space. Define

$$q_k(t) = P^{(n)}(t) \sum_{i=1}^{k} 1 \otimes e_{ii} P^{(n)}(t), \quad \text{if } k \geq m(n) \text{ and } t \in Y_n,$$

and $q_k(t) = 0$, if $k < m(n)$ and $t \in Y_n$, where $m(n)$ is the largest integer such that

$$\|L_{m(n)}(\alpha)\| \leq [\log(n+1)]^{1/4}.$$

Since $m(n) \to \infty$, $q_k(t) \in A$. Moreover, $\{q_k(t)\}$ forms an approximate identity for A.

Define $F(t) = F_n(t)$, if $t \in Y_n$, $n = 1, 2, \dots$, so that $F(t) \in \mathrm{QM}(A)$. By the proof of Lemma 4.5 we have for every $\{n_k\} \subset \mathbf{N}$, if n is large, that

$$\begin{aligned} &\left\|\sum \left(1 - q_{n_k}\right) F \left(q_{n_k} - q_{n_{k-1}}\right)\right\| \\ &\geq \frac{1}{2\pi} \log n - [\log(n+1)]^{1/4} \qquad (\to \infty, \text{ as } n \to \infty). \end{aligned}$$

Thus $F \notin \mathrm{LM}(A) + \mathrm{RM}(A)$.

Theorem 7.2. *Let A be a σ-unital C^*-algebra whose dimensions of irreducible representations are bounded by an integer n. Then*

$$\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A).$$

Proof. We shall use induction on n.

Assume that Theorem 7.2 is true for all $n \leq k$. Let $n = k+1$ and $I = \{x \in A : \pi(x) = 0, \text{ if } \dim \pi \leq k\}$. By [21, 4.4.10], I is an ideal. Moreover, I is a homogeneous C^*-algebra of order $n = k+1$. So I arises from a locally trivial M_{k+1}-bundle [12]. Hence $\mathrm{QM}(I) = \mathrm{M}(I)$. Now A/I is a σ-unital C^*-algebra whose irreducible representations have dimensions bounded by k. By the induction hypothesis, $\mathrm{QM}(A/I) = \mathrm{LM}(A/I) + \mathrm{RM}(A/I)$. It follows from Theorem 3.3 that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Akemann and Shultz showed in [3] that a type I C^*-algebra is perfect if and only if every convergent sequence in $\widehat{A}$ converges to at most a countable number of points. So the algebras in Proposition 7.1 and Theorem 7.2 are perfect. We shall produce an imperfect C^*-algebra A, such that all of its irreducible representations are finite dimensional and $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Example 7.3. Let H be a separable infinite dimensional Hilbert space and $\{H_n\}$ a sequence of mutually orthogonal, infinite dimensional subspaces. Let e_n be the projection corresponding to H_n. There are sequences of finite rank projections $\{p_i^n\}$ together with a collection $\{q_\sigma^n\}$ of infinite rank projections indexed by binary strings σ of 0's and 1's such that

(i) $\sum p_i^n = e_n$ for each n,
(ii) $p_i^n q_\sigma^n = q_\sigma^n p_i^n$ for all i, σ and n,
(iii) $q_0^n + q_1^n = e_n$ for each n,
(iv) $q_{\sigma_0}^n + q_{\sigma_1}^n = q_\sigma^n(e_n - p_m^n)$ for all σ, where $m = |\sigma|$ (see [3, Proposition 3.14]).

Let I be the C^*-algebra of all compact operators on H which commute with $\{p_i^{(n)}\}$. Let A be the C^*-algebra generated by I and by the set of projections $\{q_\sigma^n\}$.

We claim that A is an imperfect, separable C^*-algebra all of whose irreducible representations are finite dimensional (and without identity). Clearly I is an ideal of A. Moreover, I is the restricted directed sum of finite dimensional ideals of A. Since the q_σ^n's commute with each other, A/I is abelian. It follows that every irreducible representation of A is finite dimensional. By [3, Proposition 3.14], A is not perfect. By Theorem 3.3, $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

8. Examples

Example 8.1. $\mathrm{QM}(A/I) = \mathrm{M}(A/I)$ and $\mathrm{QM}(I) = \mathrm{M}(I)$, but $\mathrm{QM}(A) \neq \mathrm{M}(A)$.

Let A be the C^*-algebra of convergent sequences in M_2 with limits of the form $\left[\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then it is easy to see that $\mathrm{QM}(A)$ consists of those bounded sequences $\{x_n\}_{n=1}^{\infty}$ in M_2 such that $(x_n)_{11} \to (x_\infty)_{11}$, whereas $\mathrm{M}(A)$ consists of those bounded sequences $\{x_n\}_{n=1}^{\infty}$ in M_2 such that $(x_n)_{11} \to (x_\infty)_{11}$, $(x_n)_{21} \to 0$ and $(x_n)_{12} \to 0$. Thus $\mathrm{QM}(A) \neq \mathrm{M}(A)$.

Let I be the ideal of A consisting of sequences $\{x_n\}_{n=1}^{\infty}$ in M_2 such that $x_n \to 0$. Then $\mathrm{QM}(I) = \mathrm{M}(I)$. Since A/I is one dimensional, $\mathrm{QM}(A/I) = \mathrm{M}(A/I)$.

Example 8.2. $\mathrm{QM}(A) \neq \mathrm{M}(A)$, $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ but $\mathrm{QM}(I) \neq \mathrm{LM}(I) + \mathrm{RM}(I)$.

Let x be a countable compact Huasdorff space with $\lambda(x) = \omega$, where ω is the first limit ordinal. Let $\{e_{ij}\}$ be a set of matrix units for K.

Suppose $B_0 = C(x) \otimes K$, $B = \widetilde{B}_0$. Let A be the C^*-algebra of convergent sequences in B with limits in $C(x) \otimes e_{11}$. We identify $x \in B_0$ with an infinite matrix (a_{ij}), where $a_{ij} \in C(x)$ is defined by $(1 \otimes e_{ii}) x (1 \otimes e_{jj}) = a_{ij} \otimes e_{ij}$. Identifying the identity of $\widetilde{B}_0$ with the identity matrix, we can identify elements of $\widetilde{B}_0$ with some infinite matrices. It is easy to check (by Theorem 1.4, for example) that $\mathrm{QM}(A)$ consists of these bounded sequences $\{(a_{ij}^{(n)})\}_{n=1}^{\infty}$ in B such that $a_{11}^{(n)} \to a_{11}^{\infty}$ and $\mathrm{M}(A)$ consists of those bounded sequences $\{(a_{ij}^{(n)})\}_{n=1}^{\infty}$ in B such that $a_{11}^{(n)} \to a_{11}^{\infty}$ and $a_{ij}^{(n)} \to 0$, if $i \cdot j \neq 1$, clearly $\mathrm{QM}(A) \neq \mathrm{M}(A)$. It follows from Lemma 4.6 that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$, since B has an identity and $C(x) \otimes e_{11}$ is abelian. Let

$$I = \{\{(a_{ij}^{(n)})\}_{n=1}^{\infty} : (a_{ij}^{(n)}) = 0,\ \text{if } n \neq 1, (a_{ij}^{(n)}) \in B_0\}.$$

Clearly I is an ideal of A. It follows from Theorem 6.3 that $\mathrm{QM}(I) \neq \mathrm{LM}(I) + \mathrm{RM}(I)$, since $I \cong C(x) \otimes K$.

Example 8.3. There is a separable antiliminal C^*-algebra A such that $\mathrm{QM}(A) \neq \mathrm{M}(A)$, but $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

Let B be the nonelementary separable matroid C^*-algebra with identity obtained as the inductive limit of the following

$$M_{m(1)} \xrightarrow{g_1} M_{m(2)} \xrightarrow{g_2} M_{m(3)} \xrightarrow{g_3} \cdots$$

where $g_i(x) = x \otimes p$ and $\dim p = m(2)/m(1)$ (see [10]). Let A_0 be the C^*-subalgebra of B generated by the elements a such that $a \in M_{m(k)}$ for some k, $a = (a_{ij})$, $a_{ij} = 0$, if $ij \neq 1$. Let A be the C^*-algebra of convergent sequences $\{a(n)\}$ in B with limits in A_0.

(1) A is an antiliminal C^*-algebra. Let I be a nontrivial ideal of A and $I(k) = \{a(k) : a \in I\}$. There is a smallest integer k_0 such that $I(k_0) \neq \{0\}$. Clearly, $I(k_0)$ is an ideal of B. Since B is simple (see [10]), $I(k_0) = B$. Suppose $I_0 = \{a \in I : a(k_0) = 0\}$. Then I_0 is an ideal of I. Moreover $I/I_0 \cong I(k_0) = B$. Thus I is not liminal. So A is an antiliminal C^*-algebra.

(2) $\mathrm{QM}(A) \neq \mathrm{M}(A)$. Let x be the sequence such that $x(n) \in M_{m(k)}$ for some k and each n, moreover $(x(n))_{ij} = 1$ for all i, $j \leq m(k)$, and $x(\infty) = (a_{ij}^{\infty})$, where $a_{11}^{\infty} = 1$, $a_{ij}^{\infty} = 0$, $ij \neq 1$. As in Example 8.1 and Example 8.2, one can easily check that $x \in \mathrm{QM}(A)$, but $x \notin \mathrm{M}(A)$.

(3) $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$. Since B has an identity, $\mathrm{M}(B) = \mathrm{QM}(B) = B$. Moreover A_0 is abelian, so $\mathrm{M}(A_0) = \mathrm{QM}(A_0)$. It follows from Lemma 4.6 that $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$.

9. The density of $\mathrm{LM}(A) + \mathrm{RM}(A)$ in $\mathrm{QM}(A)$

We know that $\mathrm{QM}(A) \neq \mathrm{LM}(A)+\mathrm{RM}(A)$, in general. But is $\mathrm{LM}(A)+\mathrm{RM}(A)$ dense in $\mathrm{QM}(A)$ in a suitable topology? (See [6, 7.2].)

Example 9.1. $\mathrm{LM}(A) + \mathrm{RM}(A)$ may not be norm closed.

Let X be the one-point compactification of the disjoint union of Y_n, $n = 1, 2, \ldots$. Let $A = C(X, K)$. Use the same notations in the proof of Theorem 6.3. Define

$$F(t) = F_n(t)/\alpha(A_n)^{1/2}, \quad \text{if } t \in Y_n, \qquad F(\infty) = 0.$$

As in the proof of Lemma 4.7, we see that $F \in \mathrm{QM}(A)$, but $F \notin \mathrm{LM}(A) + \mathrm{RM}(A)$. Let $G_m(t) = F(t)$, if $t \in Y_n$, $n \leq m$, $G_m(t) = 0$, if $t \in Y_n$, $n > m$. Clearly $G_m \in \mathrm{LM}(A) + \mathrm{RM}(A)$ and $\|G_m(t) - F(t)\| \leq 1/\alpha(A_m)^{1/2} \to 0$, as $m \to \infty$. Hence $\mathrm{LM}(A) + \mathrm{RM}(A)$ is not norm closed.

Proposition 9.2. *Let X be the disjoint union of Y_n, $n = 1, 2, \ldots$, and take $A = C_0(X, K)$. Then $\mathrm{LM}(A) + \mathrm{RM}(A)$ is not norm dense in $\mathrm{QM}(A)$.*

Proof. Let $A_n = C(Y_n, K) \cong C(Y_n) \otimes K$. Take $x^{(n)} \in \mathrm{QM}(A_n)$ such that $\|x^{(n)}\| \leq 1$ and $\alpha(x^{(n)}) \geq \alpha(C(Y_n, K)) - 1/n$. Define $x(t) = x^{(n)}(t)$ if $t \in Y_n$. Assume that $u = y + z$, such that $y \in \mathrm{LM}(A)$, $z \in \mathrm{RM}(A)$ and

$$\|x - u\| < 1/16.$$

Suppose $u = u^{(n)}(t)$, $t \in Y_n$, $y = y^{(n)}(t)$, $t \in Y_n$ and $z = z^{(n)}(t)$, $t \in Y_n$, $n = 1, 2, \ldots$. Choose an integer N such that

$$\alpha(A_N) \geq \max(16, 16a),$$

where $a = \max(\|y\|, \|z\|)$. Suppose $x^{(N)} = y_1^{(N)} + z_1^{(N)}$ and $x^{(N)} - u^{(N)} = y_2^{(N)} + z_2^{(N)}$ such that $y_1^{(N)}, y_2^{(N)} \in \mathrm{LM}(A)$, $z_1^{(N)} \in \mathrm{RM}(A)$ and $\|y_2^{(N)}\| \leq (1/16)(\alpha(A_N) + 1/21)$.

Let $\{e_n\}$ be an approximate identity for A satisfying $e_m e_n = e_n e_m = e_n$, if $m > n$. By the proof of Lemma 2.1 and Theorem 2.3, there exists $n_1 < n_2 < \cdots$ such that

$$\left\| \sum_{k=1}^{\infty} (1 - e_{n_{k+1}}) y (e_{n_k} - e_{n_{k-1}}) \right\| < \frac{1}{12},$$

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_{k+1}})z^*(e_{n_k}-e_{n_{k-1}})\right\| < \frac{1}{12},$$

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_{k+1}})y_i^{(N)}(e_{n_k}-e_{n_{k-1}})|_{Y_N}\right\| < \frac{1}{12},$$

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_{k+1}})(z_i^{(N)})^*(e_{n_k}-e_{n_{k-1}})|_{Y_N}\right\| < \frac{1}{12},$$

$i = 1, 2$, and

$$\sum_{k=1}^{\infty}(1-e_{n_k})x^{(N)}(e_{n_k}-e_{n_{k-1}})|_{Y_N} \in \mathrm{RM}(A_N).$$

Thus

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_k})x^{(N)}(e_{n_k}-e_{n_{k-1}})|_{Y_N}\right\| \geq \alpha(A_N) - 1 - \frac{1}{N}.$$

By the proof of Lemma 4.3,

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_k})u(e_{n_k}-e_{n_{k-1}})\right\| \leq 7a + 4,$$

and

$$\left\|\sum_{k=1}^{\infty}(1-e_{n_k})(x^{(N)}-u^{(N)})(e_{n_k}-e_{n_{k-1}})|_{Y_N}\right\| \leq \frac{1}{16}(7\alpha(A_N)+6) \leq \frac{1}{2}\alpha(A_N).$$

But

$$\frac{1}{2}\alpha(A_N) + 7a + 4 < \alpha(A_N) - 1 - \frac{1}{n}.$$

A contradiction. Hence

$$\|x - u\| \geq 1/16.$$

Theorem 9.3. *Let A be a C^*-algebra. Then* $\mathrm{LM}(A)+\mathrm{RM}(A)$ *is strictly dense in* $\mathrm{QM}(A)$. *Moreover, for every $x \in \mathrm{QM}(A)$, there is a net $\{x_\lambda\} \subset \mathrm{LM}(A)+\mathrm{RM}(A)$ such that $\|x_\lambda\| \leq 2\|x\|$ and $x_\lambda \to x$ strictly. If A is σ-unital, $\{x_\lambda\}$ can be taken as a sequence.*

Proof. Take $x \in \mathrm{QM}(A)$ with $\|x\| \leq 1$. Let $\{e_\lambda\}$ be an approximate identity for A. Define $x_\lambda = e_\lambda x(1-e_\lambda) + xe_\lambda$. Clearly $e_\lambda x(1-e_\lambda) \in \mathrm{LM}(A)$, $xe_\lambda \in \mathrm{RM}(A)$.

For every $\varepsilon > 0$ and $a \in A$, there is λ_0 such that if $\lambda \geq \lambda_0$, then $\|a(1-e_\lambda)\| < \varepsilon/2$ and $\|(1-e_\lambda)a\| < \varepsilon/2$. Thus

$$\begin{aligned}\|a(x_\lambda - x)\| &= \|ae_\lambda x(1-e_\lambda) - ax(1-e_\lambda)\| \\ &\leq \|ae_\lambda - a\|\,\|x(1-e_\lambda)\| < \varepsilon/2 < \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|(x_\lambda - x)a\| &= \|e_\lambda x(1-e_\lambda)a + xe_\lambda a - xa\| \\ &\leq \|e_\lambda x\|\,\|(1-e_\lambda)a\| + \|x\|\,\|(e_\lambda - 1)a\| < \varepsilon.\end{aligned}$$

Moreover $\|x_\lambda\| \leq 2$. If A is σ-unital, $\{e_\lambda\}$ can be taken as a sequence, so $\{x_\lambda\}$ is a sequence.

Let X be the disjoint union of Y_n, $n = 1, 2, \ldots$, and take $A = C_0(X) \otimes K$. It follows from Theorem 6.3 that $\mathrm{QM}(A) \neq \mathrm{LM}(A) + \mathrm{RM}(A)$. However, for every $x \in \mathrm{QM}(A)$, if we define $x_n(t) = x(t)$ for $t \in Y_m$ and $m \leq n$, $x_n(t) = 0$ for $t \in Y_m$ and $m > n$, then $x_n \in \mathrm{LM}(A) + \mathrm{RM}(A)$ (Lemma 4.7), and $\|\pi^{**}(x_n) - \pi^{**}(x)\| \to 0$ uniformly on every compact subset of $\widehat{A}$, with $\|x_n\| \leq \|x\|$. This type of density is stronger than the strict density considered in Theorem 9.3. Indeed, if $a \in A$, then $C = \{\pi \in \widehat{A}, \|\pi(a)\| \geq \varepsilon\}$ is a compact subset of $\widehat{A}$. Thus there is N such that

$$\|\pi(a)[\pi^{**}(x_n) - \pi^{**}(x)]\| < \varepsilon, \qquad \pi \in C,$$

and

$$\|\pi(a)[\pi^{**}(x_n) - \pi^{**}(x)]\| < \varepsilon \cdot 2\|x\|,$$

if $\pi \in \widehat{A} \backslash C$. From these inequalities, we see that $x_n \to x$ strictly. The construction of x_n depends largely on the fact that $\widehat{A}$ is Hausdorff. If X is a countable locally (quasi-) compact space with $\lambda(X) \leq \infty$, we say X satisfies condition (C), if for every $t \in X \backslash X_{[\infty]}$ there is an open set O_t such that $t \in O_t$ and $\overline{O}_t \cap X_{[k]} = \emptyset$ for some k. Clearly, if X is Hausdorff, then X satisfies condition (C). If each point in $X \backslash X_{[\infty]}$ has a clopen neighborhood, then X also satisfies condition (C).

Theorem 9.4. *Let A be a separable C^*-algebra with countable spectrum $\widehat{A}$ and $\widehat{A}_{[\infty]} = \emptyset$. If $\widehat{A}$ satisfies condition (C), then for every $x \in \mathrm{QM}(A)$, there is a sequence $\{y_n\} \subset \mathrm{LM}(A) + \mathrm{RM}(A)$ such that $\|y_n\| \leq 3\|x\|$ and $\pi^{**}(y_n) = \pi^{**}(x)$ eventually on every compact subset of $\widehat{A}$.*

Proof. Take $x \in \mathrm{QM}(A)$ with $\|x\| \leq 1$. Put $I_n = \{a \in A\colon \pi(a) = 0, \forall \pi \in \widehat{A}_{[n]}\}$, $n = 1, 2, \ldots$. Let $\{e_i\}$ be an approximate identity for A and $\{p_m^n\}_{m=1}^\infty$ be an approximate identity for I_n. Define

$$x_{ij} = (e_i - e_{i-1})^{1/2} x (e_j - e_{j-1})^{1/2}.$$

Thus $x_{ij} \in A$, and since the norm closure $\bigcup_n I_n$ is A, we can find $\{p_i\} \subset \{p_m^n, m, n = 1, 2, \ldots\}$ satisfying:

$$\|x_{ij}(1 - p_j)\| < \frac{1}{2^{i+j}}, \qquad i \leq j,$$

and

$$\|(1 - p_i)x_{ij}\| < \frac{1}{2^{i+j}}, \qquad j \leq i.$$

Define $p = \sum_{i=1}^\infty (e_i - e_{i-1})^{1/2} p_i (e_i - e_{i-1})^{1/2}$. Clearly $p \in \mathrm{M}(A)$. By Lemma 2.1, we see that $(1 - p)xp + x(1 - p) \in \mathrm{LM}(A) + \mathrm{RM}(A)$ (as in the proof of Lemma 3.2). Without loss of generality, we may assume that $p_i \in I_i$.

Let $\widehat{A} = \{\pi_1, \pi_2, \ldots\}$. Fix n, and let O_n be an open set of $\widehat{A}$ such that $\pi_1, \pi_2, \ldots, \pi_n \in O_n$ and $\overline{O}_n \cap \widehat{A}_{[k]} = \varnothing$ for some k. This is possible since $\widehat{A}$ satisfies condition (C). Moreover, we may assume that $O_n \subset O_{n+1}$.

Let $J_n = \{a \in A; \pi(a) = 0, \forall \pi \in \overline{O}_n\}$. Clearly, if ϕ_n is the canonical homomorphism from A to A/J_n, then $\phi_n(I_k) = \phi(A)$. Let q_i be an element in I_k such that $\|q_i\| \leq 1$ and $\phi_n(q_i) = \phi_n(p_i)$. Thus $\pi(q_i) = \pi(p_i)$ if $\pi \in \overline{O}_n$. Define

$$q^{(n)} = \sum_{i=1}^{\infty} (e_i - e_{i-1})^{1/2} q_i (e_i - e_{i-1})^{1/2}.$$

Then $q^{(n)} \in \mathrm{M}(A, I_k)$. Put $z_n = q^{(n)} x q^{(n)}$. Then $z_n \in \mathrm{QM}(A, I_k)$. It follows from Lemma 6.1 that $\mathrm{QM}(I_k) = \mathrm{LM}(I_k) + \mathrm{RM}(I_k)$. By Lemma 3.2, $z_n \in \mathrm{LM}(A, I_k) + \mathrm{RM}(A, I_k) \subset \mathrm{LM}(A) + \mathrm{RM}(A)$. Define $y_n = (1-p)xp + x(1-p) + z_n$. Clearly $y_n \in \mathrm{LM}(A) + \mathrm{RM}(A)$. Moreover, $\|y_n\| \leq 3\|x\|$.

Let S be a compact subset of $\widehat{A}$, $S = \{\pi_1, \pi_2, \ldots\}$. We have $\bigcup_n O_n \supset S$. Thus there are $n_1, n_2, \ldots, n_m$ such that $\bigcup_{j=1}^m O_{n_j} \supset S$. Since $O_n \subset O_{n+1}$, there is an integer N, such that $O_N \supset S$. If $n \geq N$, $\pi^{**}(z_n) = \pi^{**}(pxp)$ for $\pi \in \overline{O}_N$. Thus $\|\pi^{**}(y_n) - \pi^{**}(x)\| = 0$ if $\pi \in S$.

Theorem 9.5. *Let A be a separable C^*-algebra of type* I. *Suppose that there is an integer N such that for every $\pi \in \widehat{A}$, the closure $\{\pi\}^-$ of $\{\pi\}$ is countable and $\lambda(\{\pi\}^-) \leq N$. Then for every $x \in \mathrm{QM}(A)$, there is a bounded net $\{x_\alpha\} \subset \mathrm{LM}(A) + \mathrm{RM}(A)$ such that for every $\pi \in \widehat{A}$*

$$\lim \|\pi^{**}(x_\alpha) - \pi^{**}(x)\| = 0$$

and $x_\alpha \to x$ strictly.

Proof. Let Γ be the family of finite subsets of $\widehat{A}$. Fix $\alpha \in \Gamma$. Then α^- is countable. Moreover, $\lambda(\alpha^-) \leq \max\{\lambda(\{\pi\}^-), \pi \in \alpha\} \leq N$.

We may assume that $\|x\| \leq 1$. Let $J_\alpha = \bigcap_{\pi \in \alpha} \ker \pi$. Then $(A/J_\alpha)^\wedge$ is countable and $\lambda[(A/J_\alpha)^\wedge] \leq N$. Let $\phi : A \to A/J_\alpha$ be the canonical homomorphism from A to A/J_α. It follows from the proof of Lemma 6.1 that there are $\bar{y}'_\alpha \in \mathrm{LM}(A/J_\alpha)$, $\bar{y}''_\alpha \in \mathrm{RM}(A/J_\alpha)$ such that $\phi(x) = \bar{y}'_\alpha + \bar{y}''_\alpha$, $\|\bar{y}'_\alpha\| \leq 3^N$ and $\|\bar{y}''_\alpha\| \leq 3^N$. It follows from [6] that there are $y_\alpha \in \mathrm{LM}(A) + \mathrm{RM}(A)$ such that $\phi(y_\alpha) = \bar{y}'_\alpha + \bar{y}''_\alpha = \phi(x)$ and $\|y_\alpha\| \leq 2 \cdot 3^N$. Let $z_\alpha = x - y_\alpha$, then $\|z_\alpha\| \leq 2 \cdot 3^N + 1$. Suppose that $\{e_n\}$ is an approximate identity for A. Define

$$u_\alpha = e_{|\alpha|} z_\alpha (1 - e_{|\alpha|}) + z_\alpha e_{|\alpha|}$$

and $x_\alpha = y_\alpha + u_\alpha$. Clearly $x_\alpha \in \mathrm{LM}(A) + \mathrm{RM}(A)$ and $\|x_\alpha\| \leq 4 \cdot 3^N + 2$. It is easy to check that

$$\|\pi^{**}(x_\alpha) - \pi^{**}(x)\| \to 0$$

for every $\pi \in \widehat{A}$. Moreover, since $x - x_\alpha = z_\alpha - u_\alpha$, by the proof of Theorem 9.3, we have $x_\alpha \to x$ strictly.

Corollary 9.6. *Let A be a separable liminal C^*-algebra. Then for every $x \in \mathrm{QM}(A)$, there is a bounded net $\{x_\lambda\} \subset \mathrm{LM}(A) + \mathrm{RM}(A)$ such that for every $\pi \in \widehat{A}$*

$$\lim \|\pi^{**}(x_\lambda) - \pi^{**}(x)\| = 0$$

and $x_\alpha \to x$ strictly.

Proof. $\widehat{A}$ is a T_1 space.

Note. The problem $\mathrm{QM}(A) = \mathrm{LM}(A) + \mathrm{RM}(A)$ for simple C^*-algebras has been studied and the results will appear elsewhere.

Acknowledgments. The author wishes to express his sincere thanks to his thesis advisor Professor Lawrence G. Brown. The author would also like to thank Professor Gert K. Pedersen for a critical reading of an earlier version of the paper.

References

1. C. A. Akemann and G. K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J. **40** (1973), 785–795.
2. C. A. Akemann, G. K. Pedersen and J. Tomiyama, *Multipliers of C^*-algebra*, J. Funct. Anal. **13** (1973), 277–301.
3. C. A. Akemann and F. Shultz, *Perfect C^*-algebras*, Mem. Amer. Math. Soc., vol. 55, no. 326 (1985).
4. L. G. Brown, *Extensions of AF algebras: The projection lifting problem*, Proc. Sympos. Pure Math., vol. 38, Part I, Amer. Math. Soc., Providence, R.I., 1982.
5. ——, *Stable isomorphisms of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), 335–348.
6. ——, *Closed hereditary C^*-algebras and the structure of quasi-multipliers*, preprints.
7. ——, *Semicontinuity and multipliers of C^*-algebras*, preprint.
8. R. Busby, *Doubly centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132** (1968), 79–99.
9. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
10. ——, *On some C^*-algebras considered by Glimm*, J. Funct. Anal. **1** (1967), 182–203.
11. E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., no. 46, Amer. Math. Soc., Providence, R.I., 1980.
12. J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280.
13. H. E. Jensen, *Scattered C^*-algebras*, Math. Scand. **41** (1977), 308–314.
14. ——, *Scattered C^*-algebras*. II, Math. Scand. **43** (1978), 308–310.
15. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. **14** (1964), 299–320.
16. ——, *A counterexample in the perturbation theory of C^*-algebras*, Canad. Math. Bull. **25** (1982), 311–316.
17. S. Mazurkiewicz and W. Sierpinski, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. **1** (1920), 21.
18. A. J. Lazar, *On scattered C^*-algebras*, preprint.
19. H. Lin, *The structure of quasi-multipliers of C^*-algebras*, Ph.D. thesis, Purdue Univ., 1986.
20. K. McKennon, *The quasi-multiplier conjecture*, Proc. Amer. Math. Soc. **72** (1978), 258–260.

21. G. K. Pedersen, *Applications of weak* semi-continuity in C^*-algebra theory*, Duke Math. J. **39** (1973), 437–450.

22. ——, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

23 ——, *SAW^*-algebras and corona C^*-algebras, contributions to non-commutative topology*, J. Operator Theory **15** (1986), 15–32.

Department of Mathematics, University of California, Santa Barbara, California 93106

Current address: Department of Mathematics, East China Normal University, 3663 Zhongshan Road (N), Shanghai 200062, China