TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 317, Number 2, February 1990

EQUIVARIANT BP-COHOMOLOGY FOR FINITE GROUPS

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Dedicated to Professor T. Yamanoshita on the occasion of his 60th birthday

ABSTRACT. The Brown-Peterson cohomology rings of classifying spaces of finite groups are studied, considering relations to the other generalized cohomology theories. In particular, $\mathrm{BP}^*(M)$ are computed for minimal nonabelian p-groups M. As an application, we give a necessary condition for the existence of nonabelian p-subgroups of compact Lie groups.

INTRODUCTION

The topology of classifying space BG for a finite group G is important in algebraic topology. Given generalized cohomology theory $h^*(-)$, $h^*(BG)$ plays the central role, e.g., cohomology of a group, completion of the representation ring and the Burnside ring when h is the ordinary cohomology, the complex K-theory, and the stable cohomotopy theory, respectively. Recently, the Morava K-theory of BG has been studied by Hopkins, Kuhn, and Ravenel [20]. For simplicity, let us denote $k^*(BG)$ by $k^*(G)$.

In this paper, we study the Brown-Peterson cohomology $\mathrm{BP}^*(G)$ for a prime p and the related cohomology $k^*(G)$ with the coefficient $k^* = \mathrm{BP}^*/(\mathrm{Ideal}\, S)$, where S is a set of generators in BP^*.

Landweber showed [3] that $\mathrm{BP}^*(Z/p^r)$ is a flat BP^*-module and for an abelian group A, $\mathrm{BP}^*(A)$ is given by the tensor product of $\mathrm{BP}^*(Z/p^r)$. For nonabelian p-groups, when $|G| = p^3$, $\mathrm{BP}^*(G)$ is determined by Tezuka-Yagita [11] and some relations to the other cohomology theories are given by $\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} Z_{(p)} = H^{\mathrm{even}}(G)$ and $K(n)^*(G) = K(n)^* \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(G)$.

Consider the map induced from restrictions

$$r\colon k^*(G) \to \mathrm{Lim\,inv}\ k^*(A),$$

$A \subset G$, conjugacy classes of abelian groups. Ravenel conjectured that for $k = \mathrm{BP}$, r is an isomorphism [8]. Unfortunately, this does not hold, however, we show that for $k = \mathrm{BP}(-;Z/p)$, r is an F-isomorphism by using Quillen's argument, which showed that F-isomorphy for $k = HZ/p$, the ordinary mod p cohomology [6]. Moreover, we show that $\rho : \mathrm{BP}^*(G)_{\mathrm{BP}^*} Z/p \to H^*(G; Z/p)$ is F-epic.

Received by the editors December 28, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55N22; Secondary 57R77.
Key words and phrases. Classifying space BG, cohomology of groups, BP-theory.

We will compute $\mathrm{BP}^*(M)$ for M, minimal nonabelian p-groups. Then $\mathrm{BP}^*(M)$ is a flat BP^*-module and the map r is injective for $k^* = \mathrm{BP}^*$. Moreover, if G is a group whose p-Sylow subgroup is a direct product of minimal nonabelian p-groups and abelian groups, then r is injective and $\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \cong P(n)^*(G)$, $\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} K(n) \cong K(n)^*(G)$.

In the last section, to see that $\mathrm{BP}^*(G)$ is useful, we will study the existence of nonabelian p-subgroups of compact Lie groups. For example, we prove that if G is a compact Lie group such that $H^*(G)_{(p)} = \bigwedge(x_1, \dots, x_n)$ and G contains nonabelian p-groups as subgroups, then p divides $|x_i| + 1$ for some i.

This work builds on the joint work with M. Tezuka and a letter from D. Ravenel. The author thanks them.

1. Cohomology theories

Let $\mathrm{BP}^*(-)$ be the Brown-Peterson cohomology theory with the coefficient $\mathrm{BP}^* = Z_{(p)}[v_1, \dots]$, $|v_i| = -2p^i + 2$ for a prime p. Given a set of generators $S = (v_{i1}, \dots, v_{in}, \dots)$, (here $v_0 = p$), by using Baas-Sullivan theory [2, 13], we can construct cohomology theory $\mathrm{BP}(S)^*(-)$ with the coefficient

$$\begin{aligned}(1.1)\qquad \mathrm{BP}(S)^* = \mathrm{BP}^*/(\mathrm{Ideal}\, S) &= Z_{(p)}[v_j \mid j \neq i_k] \quad \text{if } p \notin S, \\ &= Z/p[v_j \mid j \neq i_k] \quad \text{if } p \in S.\end{aligned}$$

The cohomology $\mathrm{BP}(S)^*(-)$ has a good multiplication, and if $p \geq 3$ it is commutative. A useful result of this theory is the following Sullivan-Bockstein exact sequence; that is, if v_n is not contained in S, then

$$(1.2)\qquad \mathrm{BP}(S)^*(X) \xrightarrow{v_n} \mathrm{BP}(S)^*(X) \xrightarrow{\rho} \mathrm{BP}(S, v_n)^*(X) \xrightarrow{\delta} \mathrm{BP}(S)^{*+2p^n-1}(x)$$

is exact, where v_n is a map of multiplying by v_n, ρ is the natural induced map, and δ is the v_n-Bockstein boundary map (for details see [2, 13]).

The examples of $\mathrm{BP}(S)^*(-)$ are

$$\begin{aligned}P(n)^* &= \mathrm{BP}(p, v_1, \dots, v_{n-1})^* = Z/p[v_n, v_{n+1}, \dots], \\ k(n)^* &= \mathrm{BP}(p, \dots, v_n, \dots)^* = Z/p[v_n], \qquad K(n)^* = [v_n^{-1}] \cdot k(n)^*, \\ \mathrm{BP}\langle n\rangle &= \mathrm{BP}(v_{n+1}, \dots)^* = Z_{(p)}[v_1, \dots, v_n], \\ HZ^*_{(p)} &= \mathrm{BP}(v_1, \dots)^* = Z_{(p)}, \qquad HZ/p^* = Z/p.\end{aligned}$$

In this paper we consider these cohomology theories $\mathrm{BP}(S)^*(-)$. For simplicity of notation, we write it as $k^*(-)$ and denote by $\#(k)$ the cardinal number of the set $(p, v_1, \dots) - S$; that is, $\#(k) = n$ if $k^* = Z/p[v_{i1}, \dots, v_{in}]$ or $k^* = Z_{(p)}[v_{i1}, \dots, v_{in-1}]$.

Let G be a compact Lie group and BG be its classifying space. Let PG be a contractible free G-space. Then the equivariant cohomology of a G-space X is defined by

$$(1.3)\qquad K_G^*(X) = k^*(PG \times_G X) \quad \text{and} \quad k_G^*(pt) = k_G^* = k^*(G) = k^*(BG).$$

2. Cohomology of abelian groups

Consider the homomorphism

$$m\colon S^1 \times S^1 \to S^1$$

defined by $m(x,y) = x + y$ identifying $S^1 = R/Z$. The induced map of classifying spaces

$$m\colon BS^1 \times BS^1 \to BS^1, \qquad BS^1 \cong CP^\infty, \tag{2.1}$$

is the usual product map induced from the tensor bundle. Take two-dimensional elements u, u_1, u_2 so that $k^*(CP^\infty \times CP^\infty) \cong k^*[[u_1, u_2]]$ and $k^*(CP^\infty) \cong k^*[[u]]$. Then the map from (2.1)

$$m^*(u) = \sum a_{ij} u_1^i u_2^j = u_1 +_k u_2$$

defines the formal group law [2, 7].

The formal group law for BP^*-theory is the universal group law for group laws over rings which are $Z_{(p)}$-modules. It is well known that

$$u_1 +_{\mathrm{BP}} u_2 = u_1 + u_2 + v_1 \sum_{0<i<p} \frac{1}{p}\binom{p}{i} u_1^i u_2^{p-i} + \cdots, \tag{2.2}$$

$$[p](u) = pu + v_1 u^p + \cdots + v_n u^{p^n} + \cdots \tag{2.3}$$

where $[p](u)$ is the pth sum $u +_{\mathrm{BP}} \cdots +_{\mathrm{BP}} u$. Given an $m \times m$-matrix $C = (c_{ij})$ over Z, it induces a map

$$\begin{aligned} &C\colon S^1 \times \cdots \times S^1 \to S^1 \times \cdots \times S^1, \\ &C^*\colon k^*[[u_1, \dots, u_n]] \to k^*[[u_1, \dots, u_n]], \quad \text{and} \\ &C^*(u_i) = \sum_k [c_{ji}](u_j). \end{aligned} \tag{2.4}$$

In particular, the short exact sequence

$$0 \to Z/p^r \to S^1 \to S^1 \to 0$$

induces the map of fiber spaces $S^1 \to BZ/p^r \to BS^1$. This follows the Gysin exact sequence and we have

$$k^*(Z/p^r) \cong k^*[[u]]/[p^r](u). \tag{2.5}$$

Landweber showed that the Künneth formula holds for BP^*-cohomology of abelian groups; that is,

$$\begin{aligned} \mathrm{BP}^*\left(\bigoplus^{s} Z/p^{r_i}\right) &\cong \bigotimes_{\mathrm{BP}^*}^{s} \mathrm{BP}^*(Z/p^{r_i}) \\ &\cong \mathrm{BP}^*[[u_1, \dots, u_s]]/([p^{r_1}](u_1), \dots, [p^{r_s}](u_s)). \end{aligned} \tag{2.6}$$

We always assume that $\otimes$ means the complete tensor product. Of course, the Künneth formula of this type does not hold for $k = HZ(p)$. By using Stretch's argument [10], we prove the Künneth formula for smaller p-rank groups.

Lemma 2.7. *Let A be an abelian p-group with* $\operatorname{rank}_p A \leq \#(k)$. *Then* $k^*(A) \cong \bigotimes_{k^*} k^*(Z/p^{r_i})$.

Proof. From the Sullivan-Bockstein exact sequence and (2.6), it is easily seen that $\rho : \mathrm{BP}^*(A) \to k^*(A)$ is epic if and only if $k^*(A) \cong \otimes_{k^*} k^*(Z/p^r)$. Hence we need only prove the result for the case $\operatorname{rank}_p A = \#(k)$.

Let $\operatorname{rank}_p A = n$ and $k^* = Z/p[v_{i1}, \dots, v_{in}]$. Let us write $s_i = [p^{r_i}](u_i)$, $S_n = (s_1, \dots, s_n)$, and $k^*[U_n] = k^*[[u_1, \dots, u_n]]$. Then we need to prove that S_n is regular in $k^*[U_n]$. Assume by induction that S_{n-1} is regular in $k'^*[U_{n-1}]$ for all k' with $\#(k') = n-1$ and $k'^0 = Z/p$.

Suppose the regularity does not hold, namely, there is $a \in k^*[U_n]$ such that $as_n \in \operatorname{Ideal} S_{n-1}$ but $a \notin \operatorname{Ideal} S_{n-1}$. Let us write

$$s_n = v_{i1}^l u_n^{l'} + \cdots \quad \text{and} \quad a = a_1 u_n^j + a_2 u_n^{j+1} + \cdots$$

with $a_1 \notin \operatorname{Ideal} S_{n-1}$. Since

$$as_n = a_1 v_{i1}^l u_n^{l''} + au_n^{l''+1} + \cdots \in \operatorname{Ideal} S_{n-1},$$

we have $a_1 v_{i1}^l \in \operatorname{Ideal} S_{n-1}$. Therefore there is $b \in k^*[U_{n-1}]$ such that $b \notin \operatorname{Ideal} S_{n-1}$ but $v_{i1} b \in \operatorname{Ideal} S_{n-1}$.

Let $k'^* = Z/p[v_{i2}, \dots, v_{in}]$. Then by the inductive assumption, S_{n-1} is regular in $k'[U_{n-1}]$. Therefore $K = k'^*[U_{n-1}] \otimes \bigwedge(e_1, \dots, e_{n-1})$, $de_i = s_i$, is a Koszul complex and it is an acyclic complex. Let us write $v_{i1} b = \sum^{n-1} \mu_i s_i$. Then $d(\sum \mu_i e_i) = 0$ in K because $v_{i1} = 0$ in k'^*. By the exactness of K, we can take $c_{ij} \in k'[U_{n-1}]$ with

$$\sum \mu_i e_i = d\left(\sum c_{ij} e_i e_j\right).$$

Hence we get in $k^*[U_n]$

$$\mu_i = \sum c_{ij} s_j + v_{i1} l_i, \qquad c_{ij} = -c_{ij},$$
$$v_{i1} b = \sum c_{ij} s_i s_j + \sum v_{i1} l_i s_i = \sum v_{i1} l_i s_i.$$

Therefore $b = \sum l_i s_i \in \operatorname{Ideal} S_{n-1}$. This is a contradiction. Hence we prove the theorem when $k^0 = Z/p$.

When $k^0 = Z_{(p)}$, we can prove the theorem by similar arguments, taking $s_i = p^{r_i} u + \cdots$ and $k'^* = Z/p[v_{i1}, \dots, v_{in-1}]$. Q.E.D.

Remark 2.8. For $k^* = k(n)^*$, we can easily see that

$$k(n)^*(Z/p \oplus Z/p) \cong k(n)^*[[y_1, y_2]]/([p](y_1), [p](y_2)) \oplus Z/p[y_1, y_2]\alpha$$

where $\rho(\alpha) = Q_n(x_1 x_2)$ in $H^*(Z/p \oplus Z/p; Z/p)$ and Q_i is the Milnor exterior operation, $Q_0(x_i) = y_i$. Moreover, when $\#(k) = n$, we see that $k^*(\bigoplus^{n+1} Z/p)$

$\not\cong \bigotimes_{k^*} k^*(Z/p)$ because there is an element α such that $Q_{i1}\cdots Q_{in}(x_1\cdots x_{n+1}) = \rho(\alpha)$ or $Q_{i1}\cdots Q_{in-1}Q_0(x_1\cdots x_{n+1}) = \rho(\alpha)$.

3. The restriction homomorphism

Restriction map $i_A^*\colon k^*(G) \to k^*(A)$ for all conjugacy classes of abelian subgroups A of G induce the map

$$r : k^*(G) \to \operatorname{Lim\,inv} k^*(A), \tag{3.1}$$

$A \subset G$, conjugacy classes of abelian p-groups. We will show that (3.1) is an F-isomorphism if $\#(k) \geq \operatorname{rank}_p G$ and $k^0 = Z/p$. A ring homomorphism $f\colon A \to B$ is said to be an F-isomorphism if $\operatorname{Ker} f \subset \sqrt{0}$ (nilpotent elements) and for all $b \in B$ there is i such that $b^{p^i} \in \operatorname{Image} f$. Quillen proved the F-isomorphism of r for $k = HZ/p$ and the conjugacy classes of elementary abelian p-groups eA [6]. However, $\operatorname{Ker} i^*$ of the restriction map $i^*\colon k^*(Z/p^2) \to k^*(Z/p)$ is not nilpotent for $\#(k) \geq 2$, and we consider all abelian p-groups. Most arguments of this section are k^*-theory versions of Quillen's arguments [5].

Lemma 3.2. *Let X be a compact manifold and G act on X smoothly. If $u \in k_G^*(X)$ restricts to zero on each orbit of X, then u is nilpotent.*

Proof. The k^*-theory version of Lemma 3.9 in [5].

Theorem 3.3. *The kernel of r in (3.1) is nilpotent.*

Proof. Let $\rho\colon G \hookrightarrow U$ be a unitary representation and T be a maximal torus of U. Consider the map of equivariant cohomologies

$$k^*(G) \cong k_G^*(pt) \underset{pr^*}{\to} k_G^*(U/T) \underset{i^*}{\to} k_G^*(Gx) \cong k_G^*(G/A) \cong k^*(A). \tag{3.4}$$

The orbits of G on the flag manifold U/T are of the form G/A where A is an abelian group as it is conjugated in U to a subgroup of T.

Assume that $u \in k^*(G)$ and $u \mid A = 0$. The image $pr^*(u)$ restricts to zero on each orbit of U/T, hence it is nilpotent by Lemma 3.2. But the map pr^* is injective. Indeed, since $k^*(-)$ is complex oriented, the Leray-Hirsch theorem holds; that is, $k^*(BG) \to k^*(P(\rho^*\xi))$ is injective where $P(\rho^*\xi)$ is a U/S^1-bundle induced from $B\rho : BG \to BU$ of the universal bundle ξ. Q.E.D.

Let H be a subgroup of G such that $[G;H] = m$. Let Σ^m be the symmetric group of m letters. Then there is the inclusion

$$\Phi\colon G \hookrightarrow \Sigma^m \wr H = \Sigma^m \ltimes (H \times \cdots \times H). \tag{3.5}$$

Consider the inclusion map of classifying spaces

$$Bi\colon (BH)^m \hookrightarrow P\Sigma^m \times_{\Sigma^m} (BH)^m = B(\Sigma^m \ltimes H). \tag{3.6}$$

Denote by $i_!$ the Gysin map of Bi, constructed by Quillen in [7] and for $k = \mathrm{BP}(S)$ in [13].

$$i_!\colon k^*((BH)^m) \to k^*(B(\Sigma^m \ltimes H)). \tag{3.7}$$

Remark 3.8. Let BG^N be an N-dimensional skeleton of BG. By dimensional reason of the spectral sequence $H^*(BG^n, k^*) \Rightarrow k^*(BG^N)$, we can easily prove

$$\text{(3.9)} \qquad \operatorname{Lim}_{N\to\infty} k^*(BG^N) = k^*(BG).$$

Here BG^N is a finite complex since G is a finite group. The Gysin map defined above is defined only on finite complexes; however, we can extend this to BG by (3.9).

Define the Evens norm $N[H \hookrightarrow G]\colon k^*(H) \to k(G)$ by

$$\text{(3.10)} \qquad N[H \hookrightarrow G](x) = \Phi^*(i_! x^m).$$

Then we can show that this norm has the following properties by the arguments of Evens for $k^* = HZ_{(p)}$ in §6 in [1], namely transitivity, naturality, multiplicative property, and double coset formula.

Lemma 3.11 ((2.1) in [5]). *If $u \in k^*(G')$ is such that $u \mid G'' = 1$ for all $G'' \subsetneqq G'$, then we have*

$$N[G' \hookrightarrow G](u) \mid K = \begin{cases} 1 & \text{if } G' \not\twoheadrightarrow K, \\ \prod_{g\in I} i_g^* u & \text{if } K = G', \end{cases}$$

where I is the set of cosets representative for G' in the normalizer $N_G(G')$, the notation $G' \not\twoheadrightarrow K$ means G' is not conjugate to a subgroup of K, and i_g^* is the conjugation map by g.

Let $A = \bigoplus^n Z/p^{r_i}$ and $k^*(A) \cong \bigotimes_{k^*} k^*[[u_i]]/([p^{r_i}](u_i))$. Define an element $e_A \in k^*(A)$ by

$$\text{(3.12)} \qquad e_A = \prod_{0\neq(\lambda_1,\dots,\lambda_n)\in A} ([\lambda_1](u_1) +_k \cdots +_k [\lambda_n](u_n)).$$

The element is unique except for multiplying units. It is immediate that if $A' \not\supseteq A$ then $e_A \mid A' = 0$.

Lemma 3.13 (Lemma 2.4 in [5]). *Let $\#(k) \geq \operatorname{rank}_p G$ and $k^0 = Z/p$. If $[N_G(A); A] = qh$, $q = p^s$ and $(p, h) = 1$, then there is $v_A \in k^*(G)$ such that*

$$v_A \mid A' = \begin{cases} 0 & \text{if } A \not\twoheadrightarrow A', \\ e_A^q & \text{if } A = A'. \end{cases}$$

Moreover if $y \in k^*(A)$ is invariant under $N_G(A)$, then there is an $\alpha(y)$ in $k^*(G)$ with $\alpha(y)|A = y^q e_A^q$.

Proof. Set $z = N[A \hookrightarrow G](1 + e_A)$. Then from (3.11) and the property of e_A, we have

$$\begin{aligned} z \mid A &= (1 + e_A)^{qh} = (1 + e_A^q)^h \\ &= 1 + he_A^q + \text{terms of higher degree}. \end{aligned}$$

Taking $1/h$ times the homogeneous component of z of degree e_A^q, we have v_A. By taking $z = N[A \hookrightarrow G](1 + e_A y)$ we have $\alpha(y)$. Q.E.D.

Theorem 3.14. *Let* $\#(k) \geq \mathrm{rank}_p\, G$ *and* $k^0 = Z/p$. *Then* r *in* (3.1) *is an* F*-isomorphism.*

Proof. Given $0 \neq (\lambda_1, \ldots, \lambda_m) = p^s(\lambda_1', \ldots, \lambda_m')$, $A = \bigoplus^m Z/p^{r_i}$ with $\lambda_1' \neq 0 \mod p$ for some i, take M to be a matrix such that the 1st column is $(\lambda_1', \ldots, \lambda_m')$ and M induces an automorphism of A. Then from (2.4) the kernel of the map in k^*-theory induced from

$$Z/p^s \times Z/p^{r_2} \times \cdots \times Z/p^{r_m} \hookrightarrow A \xrightarrow{M^{-1}} A$$

is the ideal generated by the following element in $k^*(A)$:

$$[\lambda_1](\mu_1) +_k \cdots +_k [\lambda_m](u_m).$$

Therefore if $x \in k^*(A)$ satisfies $x \mid A' = 0$ for all $A' \subsetneqq A$, then $x^{p^r} \in \mathrm{Ideal}\, e_A$.

Given $x \in \mathrm{Lim\,inv}\; k^*(A)$ in (3.1), there is an abelian group A such that $x \mid A' = 0$ for all $A' \subsetneqq A$ and $0 \neq x \mid A \in k^*(A)^{N_G(A)}$. Then $x^{p^s} \mid A = e_A \alpha$ and $\alpha \in k^*(A)^{N_G(A)}$ since $e_A \in k^*(A)^{\mathrm{Aut}(A)}$ from the definition of e_A. By Lemma 3.13, we have completed the proof. Q.E.D.

4. Relation to $H^*(G; Z/p)$

In [11], we see that when G is an abelian p-group or $|G| = p^3$, there is an isomorphism

$$(\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} Z_{(p)})/\sqrt{0} \cong H^*(G)/\sqrt{0}.$$

We consider some extensions of this fact. Restriction maps to elementary abelian p-groups eA of G induce the map

$$k^*(G) \xrightarrow{r'} \mathrm{Lim\,inv}\; k^*(eA) \overset{j}{\hookrightarrow} \prod k^*(eA) \tag{4.1}$$

$eA \hookrightarrow G$, conjugacy classes of elementary abelian p-groups. Let

$$J = (ir')^{-1}(\mathrm{Ideal}(p, v_1, \ldots)).$$

Of course, $k^*(G)/J$ is a quotient algebra of $k^*(G) \otimes_{k^*} Z/p$.

Theorem 4.2. *If* $\#(k) \geq \mathrm{rank}_p\, G$, *then there is an* F*-isomorphism* $\rho/J \colon k^*(G)/J \hookrightarrow H^*(G; Z/p)/\sqrt{0}$.

Proof. By the definition of J, there is an injection

$$k^*(G)/J \hookrightarrow \prod k^*(eA) \otimes_{k^*} Z/p \cong \prod H^*(eA; Z/p)/\sqrt{0}.$$

Hence $k^*(G)/J \to_{\rho r'/J} \mathrm{Lim\,inv}\, H^*(eA; Z/p)/\sqrt{0}$ is injective. By the same arguments as in the proof of Theorem 3.14, r' is F-epic. We show

$$\rho \colon k^*(eA)^{W_G(A)} \to H^*(eA; Z/p)^{W_G(A)}$$

is also F-epic. Indeed, given $w \in W_G(A)$ and $\alpha \in H^*(eA;Z/p)^{\langle w\rangle}$, $|w| = p^k p'$, $(p,p') = 1$, we take $\tilde\alpha \in k^*(eA)$ with $\rho(\tilde\alpha) = \alpha$ and

$$\tilde\alpha_w = \frac{1}{p'}\sum_{j=0}^{p'-1} w^{*^{jp^k}}\left(\prod_{i=0}^{p^k-1} w^{*^{ip'}}\tilde\alpha\right)$$

so that $w^*\tilde\alpha_w = \tilde\alpha_w$ and $\rho(\tilde\alpha_w) = \alpha^{p^r}$. Therefore we can prove $\rho r'/J$ is an F-isomorphism by the arguments similar to Theorem 3.14.

Quillen's main theorem [6] says that $H^*(G;Z/p) \to \lim\operatorname{inv} H^*(eA;Z/p)$ is an F-isomorphism. Hence we have the theorem. Q.E.D.

Corollary 4.3. *There is an F-isomorphism*

$$\rho\colon k(n)^*(G) \otimes_{k(n)^*} Z/p \hookrightarrow H^*(G;Z/p).$$

Proof. Since there is cohomology theory k^* such that $\rho : k^*(-) \to k(n)^*(-)$ and $\#(k) = \infty$, the map is F-epic from Theorem 4.2. By the Sullivan-Bockstein exact sequence, it is also injective. Q.E.D.

5. Relation to the Morava K-theory

Recall $P(n)^* = \mathrm{BP}^*/(p, v_1, \ldots, v_{n-1}) \cong Z/p[v_n, \ldots]$. We see in [11] that when G is an abelian p-group or $|G| = p^3$, there is an isomorphism

$$\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \cong P(n)^*(G) \quad \text{for all } n \geq 1. \tag{5.1}$$

By the Sullivan-Bockstein exact sequence, (5.1) is equivalent to

$$v_n : \mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \to \mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \tag{5.2}$$

is injective for each $n \geq 0$.

The Landweber exact functor theorem [4] says that if $\mathrm{BP}^*(G)$ satisfies (5.2), then $\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} -$ is an exact functor for finite $\mathrm{BP}^*(\mathrm{BP})$ modules. Moreover, $P(n)^*(G) \otimes_{P(n)^*} -$ is also an exact functor from [12].

Theorem 5.3. *If a p-Sylow subgroup P of G is a direct product of groups which satisfy* (5.2), *then we have*

$$\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \cong P(n)^*(G),$$
$$\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} K(n)^* \cong K(n)^*(G).$$

Proof. Let $P = P_1 \oplus \cdots \oplus P_s$ and P_i satisfies (5.2). By the exact functor theorem for $P(n)^*$-theory

$$P(n)^*(- \wedge \mathrm{BP}_i) \cong P(n)^*(-) \otimes_{P(n)^*} P(n)^*(P_i)$$

because both are cohomology theories with the same coefficient. Hence $P(n)^*(P) \cong \bigotimes_{P(n)^*} P(n)^*(P_i)$. Therefore we have

$$P(n)^*(P) \cong \left(\bigotimes_{\mathrm{BP}^*} \mathrm{BP}^*(P_i)\right) \otimes_{\mathrm{BP}^*} P(n)^* \cong \mathrm{BP}^*(P) \otimes_{\mathrm{BP}^*} P(n)^*.$$

Hence P satisfies (5.1) and so (5.2). Since $P(n)^*(G) \hookrightarrow P(n)^*(P)$, multiplying v_n in $P(n)^*(G)$ is injective. By the Conner-Floyd type theorem, $P(n)^*(-) \otimes_{P(n)^*} K(n)^* \cong K(n)^*(-)$, and we have the theorem. Q.E.D.

6. Minimal nonabelian p-groups

For an odd prime p, the minimal nonabelian p-groups are of two types (Redéi [9])

Type 1. $G_1 = \langle a, b;\ a^{p^\alpha} = b^{p^\beta} = 1,\ [a,b] = a^{p^{\alpha-1}} \rangle$,

Type 2. $G_2 = \langle a, b, c;\ a^{p^\alpha} = b^{p^\beta} = c^p = 1,\ [a,b] = c,\ [c,a] = [c,b] = 1 \rangle$.

When $p = 2$, there is an isomorphism $G_1\ (\alpha = 2, \beta = 1) \cong G_2\ (\alpha = 1, \beta = 1)$ and we need to add another type

$$Q = \langle a, b;\ a^4 = 1,\ a^2 = b^2 = [a,b] \rangle .$$

For each of the above types, there is an exact sequence

$$1 \to C \to G \to Z/p \oplus Z/p \to 1 \tag{6.1}$$

where C is the center of G and is isomorphic to $\langle a^p, b^p \rangle$ for Type 1 and $\langle c, a^p, b^p \rangle$ for Type 2. The induced Hochschild-Serre spectral sequence is

$$\begin{aligned} E_2^{*,*} &= H^*(Z/p \oplus Z/p;\ \mathrm{BP}^*(C)) \\ &\cong \widetilde{Z}/p[y_1, y_2] \otimes \textstyle\bigwedge(\alpha) \otimes \mathrm{BP}^*(C) \Rightarrow \mathrm{BP}^*(G) \end{aligned} \tag{6.2}$$

where $\widetilde{Z}/p[a]$ means $Z[a]/(pa)$ and $|y_1| = |y_2| = 2$ and $|\alpha| = 3$. Let us write

$$\begin{aligned} \mathrm{BP}^*(C) &= \mathrm{BP}^*[[u, u_1, u_2]]/([p](u), [p^{\alpha-1}](u_1), [p^{\beta-1}](u_2)) \quad \text{or} \\ &= \mathrm{BP}^*[[u_1, u_2]]/([p^\alpha](u_1), [p^{\beta-1}](u_2)). \end{aligned}$$

We will compute the spectra sequence (6.2).

Type 2. Consider the quotient map q

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & Z/p \oplus Z/p & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \| & & \\ 1 & \longrightarrow & \langle c \rangle & \longrightarrow & G/\langle a^p, b^p \rangle & \longrightarrow & Z/p \oplus Z/p & \longrightarrow & 1. \end{array}$$

The spectral sequence $\widetilde{E}_r^{*,*}$ induced from the lower exact sequence is known from [11]. The differentials are

$$d_3 u = \alpha, \tag{6.4}$$

$$d_{2p-1} u^{p-1}\alpha = y_1^p y_2 - y_1 y_2^p, \tag{6.5}$$

and we get

(6.6)
$$\begin{aligned} \widetilde{E}_{2p}^{*,*} \cong \widetilde{E}_\infty^{*,*} &\cong \mathrm{BP}^* \otimes (Z\{pu, \dots, pu^{p-1}\}) \oplus Z/p[[y_1, y_2]]/(y_1^p y_2 - y_1 y_2^p) \\ &\quad \otimes Z[[u^p]]/([p](u)). \end{aligned}$$

There are splitting maps $\langle b\rangle \rightleftarrows G$, $\langle a\rangle \rightleftarrows G$ which induced that u_1 and u_2 are permanent cycles. Since $\mathrm{BP}^*(\langle a^p, b^p\rangle)$ is a flat BP^*-module for $\mathrm{BP}^*(\mathrm{BP})$-modules, we have

(6.7) $$E_r^{*,*} \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(\langle a^p, b^p\rangle) \cong E_r^{*,*}.$$

In particular, we get

(6.8) $$E_\infty^{*,*} \cong (6.6) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*[[u_1, u_2]]/([p^{\alpha-1}](u_1), [p^{\beta-1}](u_2)).$$

Type 1 case. First, we consider the case $\beta = 1$ and denote by $\widetilde{E}_r^{*,*}$ the induced spectral sequence from (6.1). Consider also the spectral sequence converging to $H^*(G_1; Z)$. Since

$$H^2(G_1; Z) \cong \mathrm{Hom}(G_1; Q/Z) \cong Z/p^{\alpha-1}, \qquad \beta = 1,$$

we get $d_3 u = \alpha$. Moreover, considering the spectral sequence converging to $H^*(G_1; Z/p)$, we also have $d_{2p-1}u^{p-1}\alpha = y_1^p y_2 - y_1 y_2^p$. Similar results hold in BP^*-theory. Therefore $E_\infty^{*,*} \cong E_{2p}^{*,*} \cong (6.6)$. The flatness of $\mathrm{BP}^*(\langle b^p\rangle)$ implies $E_r^{*,*} \cong (6.6) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(\langle b^p\rangle)$,

(6.9) $$E_\infty^{*,*} \cong (6.6) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*[[u_2]]/([p^{\beta-1}](u_2)).$$

Corollary 6.10. *For G_1 or G_2, the image of the map*

$$j\colon \mathrm{BP}^*(G) \to \mathrm{BP}^*(\langle c\rangle) \otimes_{\mathrm{BP}^*} Z/p \cong Z/p[u]$$

is $\mathrm{Im}\, j = Z/p[u^p]$.

Remark 6.11. For G_2, $\alpha \ge 2$, $\beta \ge 2$, the image of the map

$$j\colon H^*(G_2) \to H^*(\langle c\rangle) \cong \widetilde{Z}/p[u]$$

is $\mathrm{Im}\, j = \mathrm{Ideal}\, u^2$. (See [14, 19] for more details.) We explain here the difference of spectral sequences for $H^*(G_2)$ and $\mathrm{BP}^*(G_2)$. Consider the spectral sequence from (6.1) for $\alpha = 2$ and $\beta = 2$:

$$E_2^{*,*} = H^*(Z/p \oplus Z/p; H^*(C)) \Rightarrow H^*(G),$$
$$E_2^{*,*}(Z/p) = H^*(Z/p \oplus Z/p; H^*(C; Z/p)) \Rightarrow H^*(G; Z/p).$$

Then $E_2^{*,0}(Z/p) = Z/p[y_1, y_2] \otimes \Lambda(x_1, x_2)$, $E_2^{0,*}(Z/p) = Z/p[u_1, u_2, u_3] \otimes \Lambda(z_1, z_2, z_3)$, and $E_2^{*,*'}(Z/p) \cong E_2^{*,0}(Z/p) \otimes E_2^{0,*'}(Z/p)$. The integral parts are $E_2^{*,0} \cong \widetilde{Z}/p[y_1, y_2] \otimes \Lambda(\beta(x_1 x_2))$ where β is the Bockstein operation, $E_2^{0,*} \cong \widetilde{Z}/p[u_1, u_2, u_3] \otimes \{1, \beta(z_i z_j), \beta(z_1 z_2 z_3)\}$, and $E_2^{*,*'} \cong E_2^{*,0} \otimes E_2^{0,*'}(Z/p)$ for $* > 0$. The first differentials are

$$d_2 z_1 = y_1, \qquad d_2 z_2 = y_2, \quad \text{and} \quad d_2 z_3 = x_1 x_2.$$

Then $d_3 u_3 = \beta(x_1 x_2) \ne 0$. However, $d_3 u_3^2 = 0$. Indeed,

$$d_2(x_2 \beta(z_1 z_3) - x_1 \beta(z_2 z_3)) = \beta(x_1 x_2) u_3,$$

which is also the image $d_3 u_3^2$. Therefore we see that u_3^2 is a permanent cycle.

Remark 6.12. From Corollary 6.10 and Remark 6.11, the map ρ/J in Theorem 4.2 is not epic for G_2, $\beta \ge 2$, $\alpha \ge 2$.

Theorem 6.13. *If a p-Sylow subgroup P of G is a direct product of minimal nonabelian p-groups and abelian p-groups, then*

$$\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} P(n)^* \cong P(n)^*(G),$$
$$\mathrm{BP}^*(G) \otimes_{\mathrm{BP}^*} K(n)^* \cong K(n)^*(G).$$

Proof. From Theorem 5.3, we need to prove (5.1) only in the cases G_1 and G_2. We will prove here the case G_2 only; the other case is proved by a similar argument. Consider the exact sequence

$$1 \to \langle a^p, b^p \rangle \to G_2 \to E \to 1,$$

where $E = G_2$ $(\alpha = 1, \beta = 1)$, and the induced spectral sequences

$$E_2^{*,*} = H^*(E; P(n)^*(\langle a^p, b^p \rangle)) \cong H^*(E; P(n)^*) \otimes_{P(n)^*} P(n)^*(\langle a^p, b^p \rangle)$$
$$\Rightarrow P(n)^*(G_2),$$
$$\widetilde{E}_2^{*,*} = H^*(E; P(n)) \Rightarrow P(n)^*(E).$$

Since all elements in $P(n)^*(\langle a^p, b^p \rangle)$ are permanent, we have $E_r^{*,*} \cong \widetilde{E}_r^{*,*} \otimes_{P(n)^*} P(n)^*(\langle a^p, b^p \rangle)$ by flatness and naturality. In particular, $\widetilde{E}_\infty^{*,*}$ is generated by even-dimensional elements; so is $E_\infty^{*,*}$. Hence G_2 satisfies (5.2) because if $v_n \colon P(n-1)^*(G_2) \to P(n-1)^*(G_2)$ is not injective, then $P(n)^{\mathrm{odd}}(G_2) \neq 0$ by the Sullivan-Bockstein spectral sequence. Q.E.D.

Theorem 6.14. *If a p-Sylow subgroup p of G is a direct product of minimal nonabelian p-groups and abelian groups, then r in* (3.1) *is injective for $k^* = \mathrm{BP}^*$ or $P(n)^*$.*

Proof. We need to prove the case $G = G_1$ or G_2. We will prove the case $k^* = \mathrm{BP}^*$ and the other cases are proved similarly. Let A be a maximal abelian p-subgroup of G. Let us denote by $E_r^{*,*}(A)$ the spectral sequence induced from

$$1 \to C \to A \to Z/p \to 1.$$

Then the spectral sequence collapses and we have

$$E_\infty^{*,*}(A) \cong \mathrm{BP}^*(C) \otimes \widetilde{Z}/p[y].$$

Let us denote by $E_r^{*,*}(G)$ the spectral sequence induced from (6.1). Then we know $E_\infty^{*,*}(G)$ by (6.6), (6.8), and (6.9).

We will prove that for each $x \in E_\infty^{*,*}(G)$, there is an abelian subgroup A with $i^*(x) \neq 0$ in $E_\infty^{*,*}(A)$, $i \colon A \hookrightarrow G$.

When $x \in E_\infty^{0,*}(G)$, this is obvious since $E_\infty^{0,*}(G) \hookrightarrow E_\infty^{0,*}(A)$.

We will prove that $i^*(x) \neq 0$ for $x \in E_\infty^{r,*}(G)$, r positive. Suppose x is written such that $x = ay_1^s + y_2 c$ where $a \neq 0$ in $\mathrm{BP}^*/p[[u^p]]/([p](u))$. Then $x \mid \langle a \rangle = ay_1^s \neq 0$. Suppose x is written so that

$$x = ay_1^s(\lambda_1 y_1^{p-1} y_2 + \lambda_2 y_1^{p-2} y_2^2 + \cdots + \lambda_{p-1} y_1 y_2^{p-1}) + b(y_1^{s+p} y_2 + \cdots) + \cdots.$$

If $A/C \cong \langle ab^{\mu}\rangle$, we take a two-dimensional element y so that $i^*(y_1) = y$ and $i^*(y_2) = \mu y$. Hence

$$i^*(x)(\lambda_1\mu + \lambda_2\mu^2 + \cdots + \lambda_{p-1}\mu^{p-1})ay^{s+p} + b'.$$

Since $0 = 1 - \mu^{p-1} = (1-\mu)(1+\mu+\cdots+\mu^{p-2})$, if $i^*(x) = 0$ for all $\mu \neq 1$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_{p-1} = 1$. But when $\mu = 1$, $i^*(x) = (p-1)ay^{s+p} + b' \neq 0$. Q.E.D.

Proposition 6.15. *For each minimal nonabelian p-group G, the restriction maps* $\mathrm{BP}^*(G) \to \mathrm{BP}^*(A)^{W_G(A)}$ *are epic for all maximal abelian subgroups A.*

Proof. Each maximal abelian subgroup of G is isomorphic to $\langle ab^{\mu}, c, b^p\rangle$ or $\langle b, c\rangle$ (Type 1 case $c = a^p$). We will prove the map is epic for the case $A = \langle a, c, b^p\rangle$ and Type 2. The other cases are proved similarly.

The map induced from the conjugation on b is given by

$$b^*u = u +_{\mathrm{BP}} [p^{\alpha-1}](y_1), \qquad b^*y_1 = y_1, \qquad b^*y_2 = y_2.$$

We can prove that

$$\mathrm{BP}^*(A)^{\langle b\rangle} \cong \frac{\mathrm{BP}^*(\{1, Nu, \ldots, Nu^{p-1}\} \otimes [[U, y_1, y_2]])}{([p](u), [p^{\alpha-1}](y_1), [p^{\beta-1}](y_2))}$$

where

$$Nu^s = \sum b^{i*}u^s = pu^s + \cdots,$$
$$U = \prod b^{i*}u = u^p + p^{\alpha-1}y_1^p + \cdots.$$

Therefore, from (6.6) and (6.8) we show the epimorphism.

The invariant is computed, for example, as follows. Let

$$x = (u^s + a_1u^{s+1} + \cdots)y_1^k + by_1^{k+1} + \cdots \quad \text{in } \mathrm{BP}^*(\langle a, c\rangle),$$

with $s \neq 0 \mod p$ and $a_i \in \mathrm{BP}^*[[y]]/[p^{\alpha}](y)$. Then

$$b^*x = ((u +_{\mathrm{BP}} [p^{\alpha-1}]y_1)^s + a_1(u +_{\mathrm{BP}} [p^{\alpha-1}]y_1)^{s+1} + \cdots)y_1^k + \cdots,$$

$$\begin{aligned}(1-b^*)x &\equiv p^{\alpha-1}((su^{s-1}y_1 + v_1su^{p+s-2}y_1 + \cdots) + a_1(s+1)u^sy_1 + \cdots)y_1^k \\ &\equiv sv_1^{\alpha-1}u^{(p-1)(\alpha-1)+s-1}y_1^{k+1} \quad \mod(p^{\alpha}, y_1^{k+2}, u^{(p-1)(\alpha-1)+s}),\end{aligned}$$

which is nonzero. Q.E.D.

Ravenel conjectured that r in (3.1) is isomorphic for $k^* = \mathrm{BP}$. However this does not correct. Suppose $p \geq 3$ and $G = G_2$ $(\alpha = \beta = 1)$. Let $A^{\mu} = \langle ab^{\mu}, c\rangle$ and $A^p = \langle b, c\rangle$ be the maximal abelian subgroups in G. By the arguments similar to the proof of Proposition 6.15, there is an element $\tilde{y}_{\mu} \in \mathrm{BP}^2(A^{\mu})^{W_G(A_{\mu})}$ such that $\tilde{y}_{\mu}|\langle ab^{\mu}\rangle \neq 0 \mod(p, v_1, \ldots)$ and $\hat{y}_{\mu}|\langle c\rangle = 0$. Consider the element

$$y = (0, \tilde{y}_1, 0, 0, \ldots, 0) \in \mathrm{BP}^*(A^1)^W \times \mathrm{BP}^*(A^1)^W \times \cdots \times \mathrm{BP}^*(A^p)^W,$$

which is in $\operatorname{Lim} \mathrm{BP}^*(A)$ since $A^\mu \cap A^\lambda = \langle c \rangle$ for $\mu \neq \lambda$. Recall that [11] $\mathrm{BP}^*(G)/(p, v_1, \dots)$ is generated by y_1 and y_2 with $y_1|A^0 = \tilde{y}_0$, $y_1|A^p = 0$, $y_2|A^0 = 0$ and $y_2|A^p = \tilde{y}_p$. Hence there is no two-dimensional element y in $\mathrm{BP}^*(G)$ such that $y|A^1 = \tilde{y}_1$ and $y|A^\mu = 0$ for all $\mu \neq 1$.

7. Applications; nonabelian p-subgroup

In this section we consider the existence of nonabelian p-subgroups of topological groups by using Corollary 6.10.

Theorem 7.1. *Let G be a compact group such that $H^*(BG)_{(p)}$ is finitely generated as a ring and ρ: $\mathrm{BP}^*(BG) \to H^*(BG)/(p, \sqrt{0})$ is epic. If G contains nonabelian p-subgroups, then there is a ring generator $x \in H^*(G)/(p, \sqrt{0})$ with $2p \mid |x|$.*

Proof. Let P be a minimal nonabelian p-subgroup and $D \cong Z/p$ be the subgroup generated by c for Type 2 and $a^{p^{\alpha-1}}$ for Type 1. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{BP}^*(BG) & \xrightarrow[i^*_{\mathrm{BP}}]{} & \mathrm{BP}^*(P) & \xrightarrow[j^*_{\mathrm{BP}}]{} & \mathrm{BP}^*(D) \\
\big\downarrow{\scriptstyle \rho_G} & & \big\downarrow{\scriptstyle \rho_P} & & \big\downarrow{\scriptstyle \rho_D} \\
H^*(BG)/(p, \sqrt{0}) & \xrightarrow[i_{*H}]{} & \mathrm{BP}^*(P)/(p, \sqrt{0}) & \xrightarrow[j^*_H]{} & H^*(D)/(p) \cong Z/p[u].
\end{array}$$

From Corollary 6.10, $\operatorname{Im}(\rho_D j_{\mathrm{BP}}) = Z/p[u^p]$. Hence

$$\operatorname{Im}(j^*_H i^*_H \rho_G) = \operatorname{Im}(\rho_D j^*_{\mathrm{BP}} i^*_{\mathrm{BP}}) \subset Z/p[u^p].$$

Since ρ_G is epic, $\operatorname{Im}(j^*_H i^*_H) \subset Z/p[u^p]$. From Quillen's main theorem of equivariant cohomology [6], $j^*_H i^*_H \neq 0$ for some $* > 0$. Therefore there is a ring generator $x \in H^*(G)/(p, \sqrt{0})$ such that $j^*_H i^*_H(x) = u^{ps}$. Q.E.D.

Corollary 7.2. *Let G be a compact Lie group containing nonabelian p-subgroups.*

(1) *If $H^*(G)_{(p)} \cong \bigwedge(x_1, \dots, x_n)$, then there is i with $2p \mid |x_i| + 1$.*

(2) *If $H^*(BG)/(p, \sqrt{0})$ is generated by c_{i_s}, $1 \le s \le n$, i_s th Chern classes of some representations, then there is s such that $2p \mid i_s$.*

Remark 7.3. (1) of the above corollary is an immediate consequence of a result of Borel-Serre [15] and its converse also holds. Let $P \subset G$ be a p-group. By [15], we may (after conjugation) assume $P \subset N(T)$, the normalizer of a maximal torus T. If P is nonabelian, then $P \not\subset T$ and p divides the order $|W|$ of the Wyle group $W = N(T)/T$. Since $|W| = \prod(|x_i| + 1)/2$, we have (1).

Conversely, if $p \mid |W|$, then $N(T)$ contains nonabelian p-subgroups. The extension $T \to N(T) \to W$ defines an element of $H^2(W, T) \cong 0$ or $Z/2 \oplus \cdots \oplus Z/2$ [18]. So an element of order p in W lifts to an element x of order

p (, 2 or 4 for $p = 2$). Let $V \subset T$ be the set of solutions of $t^p = 1$ ($t^4 = 1$ for $p = 2$). Since x acts nontrivially on T, x also acts nontrivially on V. If we consider V as a vector space over F_p, then the action on it is given by a Jordan decomposition, so we can find a subspace of dimension 2 on which the action is given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This means that there is a subgroup of Type 2 $\alpha = \beta = 1$ (nonabelian group of order $\le 4^3$ for $p = 2$).

This remark is due to J. F. Adams. The author is grateful to Professor Adams for his kind comments.

Example 7.4. The cohomologies of simply connected simple Lie groups are known and the cohomologies of some cases of their classifying spaces are known. For example,

$$H^*(BSU(n)) \cong Z[y_4, \dots, y_{2n-1}],$$
$$H^*(BE_7)_{(p)} \cong Z_{(p)}[y_i \mid i = 4, 12, 16, 20, 24, 28, 36] \quad \text{for } p \ge 5.$$

Thus $SU(n)$, (, $\mathrm{Sp}(n)$, $\mathrm{SO}(2n+1)$) contains nonabelian p-subgroup if and only if $p \le n$. The exceptional Lie group G_2 (resp. F_4, E_6, E_7, E_8) contains nonabelian p-subgroups if and only if $p \le 3$ (resp. ≤ 3, ≤ 5, ≤ 7, ≤ 7).

Let $G(F_q)$ be the F_q-rational points of the universal Chevally group of the reductive complex Lie group type G. Let $q = p^s$ and $l \ne p$. Then $H^*(BG; Z/l) \cong H^*(BG(\overline{F}_p); Z/l)$ where $\overline{F}_p$ is the algebraic closure of F_p [16]. the cohomology of the F_q-rational points is computed by considering the coinvariant under the Frobenius-Adams operation σ_p. Let r be the smallest number such that $q^r = 1 \bmod l$. Quillen showed $H^*(\mathrm{GL}_n(F_q))/(l, \sqrt{0}) \cong Z/l[c_r, c_{2r}, \dots, c_{r[n/r]}]$; in this case we get $\sigma_q c_i = q^i c_i$. Hence if $\mathrm{GL}_n(F_q)$ contains nonabelian l-subgroups, then $lr \le n$. Exceptional Lie group types are computed by Kleinerman [17]. For example, in the case $G = E_7$, $\sigma_q y_i = q^{i/2} y_i$. Hence $H^*(BE_7(F_q))/(l, \sqrt{0}) \cong Z/l[y_i \mid 2l|i]$. Therefore we see that if $E_7(F_q)$ contains nonabelian 5-subgroups (resp. 7-subgroups), then $r = 1, 2, 5, 10$ (resp. $r = 1, 2, 7, 14$). When $l = p$, exceptional Lie types always contain nonabelian l-subgroups, since they contain $\mathrm{SL}_3(F_p)$.

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