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SOME LARGE DEVIATION RESULTS FOR DYNAMICAL SYSTEMS

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ABSTRACT. We prove some large deviation estimates for continuous maps of compact metric spaces and apply them to attractors in differentiable dynamics, rate of escape problems, and to shift spaces.

INTRODUCTION

Consider a discrete time dynamical system generated by a self-map $f: X \circlearrowleft$ of some domain X. Let m be a reference measure on X, and let $\varphi: X \to \mathbf{R}$ be an observable. Suppose that $\frac{1}{n}\sum_{i=1}^{n-1} \varphi \circ f^i$ converges to some constant $\overline{\varphi}$ m-a.e. This paper is concerned with the rate of convergence of this time average. More precisely, let $\delta > 0$ denote the accepted margin of error, and let

$$B_n = \left\{ x \in X: \left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i x - \overline{\varphi} \right| > \delta \right\}.$$

Then $mB_n \to 0$ as $n \to \infty$. We wish to know if $mB_n \approx e^{\alpha n}$ for some α, or at least if we can find α and β so that $e^{\beta n} \lesssim mB_n \lesssim e^{\alpha n}$. We are particularly interested in exponential convergence, i.e., $\alpha < 0$.

A different situation, but one that involves the same set of ideas, is the following: Let f be a continuous map or flow, and let $\Lambda \subset X$ be an invariant set that is not an attractor. Because of the invariance of Λ, if a point is near Λ then its next few iterates are not likely to be far away. We are interested in the rate of escape from a neighborhood of Λ. In the case of a flow, this rate also measures the capacity of Λ as a barrier to transport. More precisely, we let U be a neighborhood of Λ, define

$$C_n = \{x \in U: x, fx, \ldots, f^n x \in U\}$$

and ask if $mC_n \approx e^{\alpha n}$.

Large deviation questions have been successfully dealt with for various stochastic processes (see e.g., [E, S, V]). In the case of dynamical systems, one does not expect nice, explicit rate functions in general, especially when the trajectories do not have good statistical properties. One can, however, ask how

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the exponents discussed above are related to the dynamical characteristics of the system, and when to expect exponential convergence. In this paper we attempt to explore these questions, and to do so in a slightly more general setting than that of some of the existing results.

Our large deviation estimates are proved for continuous self-maps of compact metric spaces. The applications we have in mind are to attractors in differentiable systems with Lebesgue as the reference measure, to rate of escape problems in differentiable dynamics, and to shift spaces. We obtain as almost immediate corollaries to our main theorem some known results. They include large deviation principles for Axiom A attractors and for certain Gibbs states on the 1-dimensional lattice. (See [B1, BRa, D, OP1, OP2].)

This paper is divided into two parts. Part I contains the statements and discussions of all our results along with some of the shorter proofs. Other proofs, particularly those involving estimates of a more technical nature, are postponed to Part II.

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Part I. Statements and Discussions of Results

Part I is divided into four sections. In §A we present our results in a general setting, while §§B, C, and D are devoted to the application of these results to three different situations.

A. Main theorem

In this section X is a compact metric space, $f: X \circlearrowleft$ is a continuous map of X into itself, and m is a finite Borel measure on X. We think of m as our reference measure. Let $\mathscr{M}$ denote the set of f-invariant Borel probability measures on X and $\mathscr{M}_e$ the ergodic elements of $\mathscr{M}$.

Deviation functions, when they exist, are often given in terms of 'relative entropies'. (See e.g., [E, DV, Or].) We formulate a version of this: Let

$$V(x, n, \varepsilon) = \{y \in X : d(f^i x, f^i y) < \varepsilon, 0 \le i < n\}.$$

We define

$$h_m(f, x) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} -\frac{1}{n} \log m V(x, n, \varepsilon)$$

and

$$h_m(f; \nu) = \nu\text{-ess sup}\, h_m(f, x).$$

Note that m is not necessarily f-invariant, and that for $\nu \in \mathscr{M}_e$, we have for ν-a.e. x, $h_\nu(f, x) = h_\nu(f)$, the usual metric entropy of f with respect to ν. The topological entropy of f is denoted by $h_{\text{top}}(f)$.

Sometimes more uniform bounds for $h_m(f, x)$ are needed. For that purpose we introduce two sets of functions: $\mathscr{V}^+$ and $\mathscr{V}^-$. First let $C(X, \mathbf{R})$ denote the space of continuous real valued functions on X. For $\varphi \in C(X, \mathbf{R})$, write

$S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ f^i$. We define

$$\mathscr{V}^+ = \{\xi \in C(X, \mathbf{R}) : \exists C, \varepsilon > 0 \text{ s.t. } \forall x \in X \text{ and } \forall n \geq 0, \quad mV(x, n, \varepsilon) \leq Ce^{-S_n\xi(x)}\}$$

and

$$\mathscr{V}^- = \{\xi \in C(x, \mathbf{R}) : \exists \text{ arbitrarily small } \varepsilon > 0 \text{ and } C = C(\varepsilon) \text{ s.t. } \forall x \in X \text{ and } \forall n \geq 0, \; mV(x, n, \varepsilon) \geq Ce^{-S_n\xi(x)}\}.$$

Finally, for $\varphi \in C(X, \mathbf{R})$ and $E \subset \mathbf{R}$, we write

$$\overline{R}(\varphi, E) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log m \left\{ \frac{1}{n} S_n\varphi \in E \right\}$$

and

$$\underline{R}(\varphi, E) = \underline{\lim_{n\to\infty}} \frac{1}{n} \log m \left\{ \frac{1}{n} S_n\varphi \in E \right\}.$$

All the results in this paper are derived from the following theorem.

Theorem 1. *Assume $h_{\text{top}}(f) < \infty$. Then for every $\varphi \in C(X, \mathbf{R})$ and $c \in \mathbf{R}$, the following hold*:

(1) $\underline{R}(\varphi, (c, \infty)) \geq \sup\{h_\nu(f) - h_m(f; \nu) : \nu \in \mathscr{M}_e, \int \varphi \, d\nu > c\}$.

(2) For $\xi \in \mathscr{V}^+$, we have

$$\overline{R}(\varphi, [c, \infty)) \leq \sup\left\{ h_\nu(f) - \int \xi \, d\nu : \nu \in \mathscr{M}, \int \varphi \, d\nu \geq c \right\}.$$

(3) Assume f satisfies specification (to be defined below). Then for $\xi \in \mathscr{V}^-$, we have

$$\underline{R}(\varphi, (c, \infty)) \geq \sup\left\{ h_\nu(f) - \int \xi \, d\nu : \nu \in \mathscr{M}, \int \varphi \, d\nu > c \right\}.$$

Definition 1. f satisfies *specification* if for every $\theta > 0$, $\exists p = p(\theta) \in \mathbf{Z}^+$ s.t. given any k points $x_1, \dots, x_k \in X$, $n_1, \dots, n_k \in \mathbf{Z}^+$, and $p_1, \dots, p_{k-1} \geq p(\theta)$, $\exists x \in X$ s.t.

$$\begin{aligned}
d(f^i x, f^i x_1) &< \theta, & 0 &\leq i < n_1, \\
d(f^{n_1+p_1+i} x, f^i x_2) &< \theta, & 0 &\leq i < n_2, \\
&\vdots \\
d(f^{n_1+\cdots+n_{k-1}+p_1+\cdots+p_{k-1}+i} x, f^i x_k) &< \theta, & 0 &\leq i < n_k.
\end{aligned}$$

Discussion. (i) To prove the lower bound in (1) all we have to do is to observe that for $\nu \in \mathscr{M}_e$ with $\int \varphi \, d\nu > c$, most ν-typical points satisfy $\frac{1}{n} S_n\varphi x > c$ for large n. For these points, $mV(x, n, \varepsilon) > e^{-nh_m(f;\nu)}$ and there are roughly $e^{nh_\nu(f)}$ disjoint $V(x, n, \varepsilon)$'s. This is a sketch of the proof of (1).

(ii) An upper estimate on the "bad set" is much harder to come by since we now have to control simultaneously all parts of X. Varying rates of decay of $mV(x, n, \varepsilon)$ for different x are no longer tolerated; hence we use $\xi \in \mathscr{V}^+$. Inequality (2) follows from a variational principle.

(iii) Note that if we assume $\int \xi \, d\nu = h_m(f; \nu)$, then (1) and (3) differ primarily in that in (1) we take supremum over *ergodic* measures. It is easy to construct examples to show that in general,

$$\sup\left\{h_\nu(f) - \int \xi \, d\nu : \nu \in \mathscr{M}, \int \varphi \, d\nu \geq c\right\}$$
$$\neq \sup\left\{h_\nu(f) - \int \xi \, d\nu : \nu \in \mathscr{M}_e, \int \varphi \, d\nu \geq c\right\},$$

and that (1) is false if $\mathscr{M}_e$ is replaced by $\mathscr{M}$.

(iv) Suppose we try to carry out the argument in (1) using nonergodic measures, say for $\nu = \frac{1}{2}(\nu_1 + \nu_2)$, $\nu_1, \nu_2 \in \mathscr{M}_e$. Since $\int \varphi \, d\nu > c$, all we know is that if x_1 is a ν_1-typical point and x_2 is ν_2-typical, then

$$\frac{1}{2n}(S_n\varphi x_1 + S_n \varphi x_2) > c$$

for n large. This suggests that if we can 'glue' the orbit segments $[x_1, fx_1, \ldots, f^n x_1]$ and $[x_2, fx_2, \ldots, f^n x_2]$ together, then counting the concatenated orbits, of which there are roughly $e^{nh_{\nu_1} + nh_{\nu_2}} = e^{2nh_\nu}$, the argument in (i) will probably go through. Specification allows us to glue arbitrary orbit segments together to form one orbit. We have just given a sketch of the proof of (3).

(v) As will be evident in the formal proof of Theorem 1, we only need to glue certain orbit segments together and in certain order. So the full force of specification is not at all needed for (2).

Many of our results can easily be formulated in the space of measures without the continuous observable φ. We give an example of such a formulation: Let $\widetilde{\mathscr{M}}$ denote the set of Borel probability measures on X and let $d_{\widetilde{\mathscr{M}}}$ be any one of the standard metrics on $\widetilde{\mathscr{M}}$ compatible with its weak topology. The Dirac measure at x is written δ_x.

Corollary 2 (Corollary to Theorem 1(1)). *Suppose that m-a.e. $x \in X$ is μ-generic, i.e., $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \to \mu$, for some $\mu \in \mathscr{M}$. Then for every $\delta > 0$,*

$$\varliminf_{n\to\infty} \frac{1}{n} \log m \left\{x \in X : d_{\widetilde{\mathscr{M}}}\left(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}, \mu\right) > \delta\right\}$$
$$\geq \sup\{h_\nu(f) - h_m(f; \nu) : \nu \in \mathscr{M}_e, d_{\widetilde{\mathscr{M}}}(\nu, \mu) > \delta\}.$$

Both Theorem 1 and Corollary 1 are proved in Part II.

B. Attractors in differentiable dynamical systems

Let M be a d-dimensional C^∞ Riemannian manifold without boundary and consider a C^2 diffeomorphism f of M onto itself. We let $d(\cdot, \cdot)$ denote the metric on M induced by its Riemannian structure and let m be its Riemannian measure.

Definition 2. A compact f-invariant set $\Lambda \subset M$ is called an attractor if $\exists$ an open set $U \supset \Lambda$, $\overline{U}$ compact, s.t. $f\overline{U} \subset U$ and $\bigcap_{n\geq 0} f^n U = \Lambda$.

Included in this definition is the case $\Lambda = U = M$ when M is a compact manifold. Throughout §B let $\Lambda \subset U$ be an attractor. We will apply Theorem 1 with $X = \overline{U}$, $f = f|\overline{U}$ and $m = m|\overline{U}$. Here $\mathscr{M} = \mathscr{M}(\overline{U}) = \mathscr{M}(\Lambda)$ and the same is true for $\mathscr{M}_e$.

Let $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$ denote the Lyapunov exponents of f at x whenever they are defined. For $\nu \in \mathscr{M}_e$, let $\lambda_\nu = \int \sum \lambda_i^+ \, d\nu$ where $a^+ = \max(a, 0)$.

Proposition 1. *Let* $\nu \in \mathscr{M}_e$. *Then* $h_m(f; \nu) \leq \lambda_\nu$.

There are various notions of "natural" invariant measures associated with attractors. We recall the definitions of two of them.

Definition 3. (a) $\mu \in \mathscr{M}$ is called a Sinai-Bowen-Ruelle (SBR) measure if m-a.e. $x \in U$ is μ-generic.

(b) $\mu \in \mathscr{M}$ is called an *equilibrium state* if $h_\mu(f) = \lambda_\mu$.

For attractors in general, it is not known if SBR measures or equilibrium states exist, or if they coincide when they do exist. These questions are better understood for certain classes of attractors.

Definition 4. (a) A compact invariant set $\Gamma \subset M$ is said to be *uniformly partially hyperbolic* (abbreviated *uph*) if $\exists \lambda > 0$ and a splitting of the tangent bundle over Γ into continuous subbundles $E^u \oplus E^{cs}$ s.t. the dimensions of E^u and E^{cs} are constant on Λ and $\forall \varepsilon > 0$, $\exists N \in \mathbf{Z}^+$ s.t. $\forall x \in \Gamma$ and $\forall n \geq N$,

$$v \in E^u(x) \Rightarrow |Df_x^n v| \geq e^{\lambda n}|v|,$$
$$v \in E^{cs}(x) \Rightarrow |Df_x^n v| \leq e^{\varepsilon n}|v|.$$

(b) $\Gamma \subset M$ is called *uniformly hyperbolic* if it is uph and $Df|E^{cs}$ is uniformly strictly contracting.

(c) An attractor Λ is called an *Axiom* A *attractor* if it is uniformly hyperbolic and $f|\Lambda$ has a dense orbit.

The corresponding objects for flows are defined analogously. In particular, if Γ is a uniformly hyperbolic set for the flow f^t, then it is a uph set for the time-one-map f^1 of f^t.

Theorem 2. *Let* $\Lambda \subset U$ *be an attractor.*

(1) *If* $\mu \in \mathscr{M}$ *is SBR, then* $\forall \varphi \in C(\overline{U}, \mathbf{R})$ *and* $\delta > 0$,

$$\varliminf_{n\to\infty} \frac{1}{n} \log m \left\{ \left| \frac{1}{n} S_n \varphi - \int \varphi \, d\mu \right| > \delta \right\}$$
$$\geq \sup \left\{ h_\nu(f) - \lambda_\nu : \nu \in \mathscr{M}_e, \left| \int \varphi \, d\nu - \int \varphi \, d\mu \right| > \delta \right\}.$$

(2) *If* Λ *is uph and* μ *is its unique equilibrium state, then*

$$\varlimsup_{n\to\infty} \frac{1}{n} \log m \left\{ \left| \frac{1}{n} S_n\varphi - \int \varphi \, d\mu \right| \geq \delta \right\} < 0 .$$

It follows from this that μ *is SBR.*

Theorem 2(1) follows immediately from Proposition 2 and Theorem 1(1). Proposition 1 and Theorem 2(2) are proved in Part II.

Discussion. (i) The two parts of this theorem taken together carry the following message: Roughly speaking, (2) says that if Λ has some uniform expansion and a unique candidate for SBR, then it *is* SBR and $\frac{1}{n}S_n\varphi$ converges exponentially (meaning the "bad set" dies exponentially fast). In examples with nonuniformly hyperbolic behavior, there often are invariant sets on which the expanding subbundle is "degenerate". (In pseudo-Anosov diffeomorphisms, for instance, this happens on a finite set of points.) If these invariant sets support an invariant measure ν with $\lambda_\nu = 0$, then (1) says that there exist test functions φ for which $\frac{1}{n}S_n\varphi$ converges no faster than subexponentially, if at all.

(ii) It has been proved that uph attractors always have equilibrium states [PS]. For attractors of Axiom A diffeomorphisms or flows, equilibrium states are known to be unique and to coincide with SBR [B1, BRu].

Next we specialize to Axiom A attractors, but first we recall a definition from large deviation theory.

Definition 5. Given a dynamical system $f: X \circlearrowleft$ with reference measure m and observable $\varphi \in C(X, \mathbf{R})$ as defined in §A, we say that $\frac{1}{n}S_n\varphi$ satisfies a *large deviation principle* with *rate function* $k: X \to [0, \infty]$ if

1. k is lower semicontinuous;
2. for every open set $E \subset \mathbf{R}$,
$$\underline{R}(\varphi, E) \geq -\inf\{k(s), s \in E\};$$
3. for every closed set $E \subset \mathbf{R}$,
$$\overline{R}(\varphi, E) \leq -\inf\{k(s), s \in E\}.$$

For more information we refer the reader to [Or].

Theorem 3. *Let* $\Lambda \subset U$ *be an Axiom* A *attractor. Then* $\frac{1}{n}S_n\varphi$ *satisfies a large deviation principle with rate function*

$$k(s) = -\sup\left\{ h_\nu(f) - \lambda_\nu : \nu \in \mathscr{M}, \int \varphi \, d\nu = s \right\}.$$

Clearly $k(s) = 0$ *iff* $s = \int \varphi \, d\mu$ *where* μ *is the unique equilibrium state.*

The Anosov case of Theorem 3 was first proved in [OP1 and OP2], parts of which rely on [DV]. Let us see how it follows from Theorem 1:

Proof of Theorem 3. Let j be the number of components in the spectral decomposition of $f|\Lambda$. Since it suffices to prove the theorem for f^j, we may as well assume that $f|\Lambda$ is topologically mixing. Let $\xi = \log|\operatorname{Jac}(Df|E^u)|$. Then $\xi \in \mathscr{V}^+ \cap \mathscr{V}^-$. (Use the fact that stable manifolds of Λ foliate a neighborhood of Λ together with the estimate in [B1, p. 95], or do as in §D of Part II.) It follows from the shadowing and topological mixing properties of $f|\Lambda$ that it satisfies specification. (This is also proved in [B2].) Upper and lower bounds for $\overline{R}(\varphi, [c, \infty))$ and $\underline{R}(\varphi, (c, \infty))$ respectively are therefore given by Theorem 1. To complete the proof we need to show that k as defined is the rate function. That is straightforward, provided that we verify the upper semicontinuity of $\nu \mapsto h_\nu(f) - \lambda_\nu$. Since f is Axiom A, $\nu \mapsto \lambda_\nu$ is continuous; $\nu \mapsto h_\nu(f)$ is usc because $h_\nu(f) = h_\nu(f, \mathscr{P})$ for any partition $\mathscr{P}$ with diameter less than the expansive constant of f. □

We do not anticipate any difficulties in proving the corresponding results for Axiom A flows, but will not claim that here since certain modifications have to be made to obtain the flow version of Theorem 1(3). The proof of the upper bound goes through without change.

C. Rates of escape from neighborhoods of invariant sets

All notations are as in §B except that $\Lambda \subset U$ is not necessarily an attractor in this section.

Definition 6. A compact invariant set $\Lambda \subset M$ is said to be *maximal* in some compact neighborhood U of Λ if $\Lambda = \bigcap_{n \in \mathbf{Z}} f^n \overline{U}$.

Throughout this section, $\Lambda \subset U$ will be a maximal invariant set, and $\mathscr{M} = \mathscr{M}(\overline{U}) = \mathscr{M}(\Lambda)$. We define

$$\underline{Q} = \varliminf_{n \to \infty} \frac{1}{n} \log m\{x \in \overline{U} \colon f^i x \in \overline{U}, 0 \le i \le n\}$$

and

$$\overline{Q} = \varlimsup_{n \to \infty} \frac{1}{n} \log m\{x \in \overline{U} \colon f^i x \in \overline{U}, 0 \le i \le n\},$$

Theorem 4. (1) $\underline{Q} \ge \sup\{h_\nu(f) - \lambda_\nu \colon \nu \in \mathscr{M}\}$.

(2) *If* Λ *is uph, then*

$$\underline{Q} = \overline{Q} = \sup\{h_\nu(f) - \lambda_\nu \colon \nu \in \mathscr{M}\}.$$

The Axiom A case of this theorem was proved in [B1].

Discussion. (i) The proof of Theorem 4 is identical to that of Theorem 2(1), (2) together with the volume estimates in §D of Part II and will be omitted. Notice that for the lower bound, it is immaterial whether we take supremum over measure in $\mathscr{M}$ or in $\mathscr{M}_e$, because for every $\nu \in \mathscr{M}$ we can write

$$h_\nu(f) - \lambda_\nu = \int [h_\tau(f) - \lambda_\tau] \, d\pi(\tau)$$

for some Borel probability measure π on $\mathscr{M}_e$.

(ii) To see that (2) is not valid without some conditions on Λ, consider the time-one-map of the "figure 8" flow: $\Lambda \subset \mathbf{R}^2$ consists of a saddle p with $|\det(Df_p^1)| < 1$ together with its separatrices all of which are homoclinic orbits. Clearly $\underline{Q} = \overline{Q} = 0$ since Λ is an attractor, but $h_\nu(f) > \lambda_\nu$ for the unique element $\nu \in \mathscr{M}$.

D. Shift spaces

Let $\{1, \ldots, s\}$ be a finite alphabet and let $\Sigma = \prod_{-\infty}^{\infty}\{1, \ldots, s\}$ or $\Sigma^+ = \prod_0^{\infty}\{1, \ldots, s\}$ be endowed with the product topology and any one of the standard metrics. Let $\sigma: \Sigma \circlearrowleft$ or $\sigma: \Sigma^+ \circlearrowleft$ be the shift operator, meaning that if $\underline{x} \in \Sigma$ then $(\sigma\underline{x})_n = (\underline{x})_{n+1}$, where $(\underline{x})_n = x_n$ denotes the nth coordinate of $\underline{x}$. If $A = (A_{ij})$ is an $s \times s$ matrix whose entries are either 0 or 1, then $\sum_A = \{\underline{x} \in \Sigma: A_{x_i x_i+1} = 1\ \forall i\}$ and $\sigma|\Sigma_A: \Sigma_A \circlearrowleft$ is called the *subshift of finite type* (*ssft*) defined by A. Σ_A^+ is defined analogously.

In this section we apply the results of §A to shift spaces with $f = \sigma$. Both of the results here are known; the first is well known. We include them only to show how they can be obtained as corollaries of Theorem 1.

Before proceeding further, let us observe that shift spaces have certain advantages over arbitrary metric spaces. First, specification is automatic on full shifts, and is satisfied on Σ_A or Σ_A^+ if for some $n \in \mathbf{Z}^+$, all the entries of A^n are strictly positive. (We abbreviate this as $A^n > 0$.) Also, $\nu \mapsto h_\nu(\sigma)$ is always upper semicontinuous.

(1) Markov chains with a finite number of states. Let $\underline{p} = (p_1, \ldots, p_s)$ be a probability vector with $p_i > 0\ \forall i$ and let $P = (P_{ij})$ be an $s \times s$ stochastic matrix. Let $m \in \mathscr{M}(\Sigma^+)$ denote the joint distribution of the Markov chain with initial distribution $\underline{p}$ and stationary transition probabilities (P_{ij}). Assume that P is irreducible.

Theorem 5. *Consider $\sigma: \Sigma^+ \circlearrowleft$ with reference measure m. Then for every $\varphi \in C(\Sigma^+, \mathbf{R})$, $\frac{1}{n}S_n\varphi$ has a large deviation principle with rate function*

$$k(s) = -\sup\left\{h_\nu(\sigma) + \int \log P_{x_0 x_1}\, d\nu(\underline{x}) : \nu \in \mathscr{M}(\Sigma_A^+),\ \int \varphi\, d\nu = s\right\}$$

where A *is the $s \times s$ matrix with*

$$A_{ij} = \begin{cases} 1 & \text{if } P_{ij} > 0, \\ 0 & \text{if } P_{ij} = 0. \end{cases}$$

Moreover, $k(s_0) = 0$ for a unique s_0.

See [E or DV] for far more elaborate results.

Proof of Theorem 5. First we restrict ourselves to the ssft σ: $\Sigma_A^+ \circlearrowleft$ where A is as above. Note that $\operatorname{supp}(m) = \Sigma_A^+$ and that $\forall \underline{x} \in \Sigma_A^+$, $P_{x_0 x_1} > 0$. Let $\xi(x) = -\log P_{x_0 x_1}$. Since $m\{x_0 = i_0, \ldots, x_{n+1} = i_{n+1}\} = p_{i_0} e^{-S_n\xi(\underline{x})}$, we have

$\xi \in \mathscr{V}^+ \cap \mathscr{V}^-$. By considering a power of σ if necessary we may assume that $A^n > 0$ for some n, so that parts (2) and (3) of Theorem 1 apply to give the rate function as claimed. The last assertion follows from the uniqueness of equilibrium state on Σ_A^+ for the function $-\xi$. □

(2) Gibbs State. Consider a physical system on a 1-dimensional lattice where each site can be one of a finite number of states labelled $1, \dots, s$. Suppose that the energy due to state i occurring some place is given by $\Phi_0(i)$ and the energy of interaction due to states i_1 and i_2 occuring j sites apart is given by $\Phi_2(j; i_1, i_2)$. We further assume that

$$\|\Phi_2\|_j \overset{\text{def}}{=} \max_{(i_1, i_2)} |\Phi_2(j; i_1, i_2)|$$

satisfies $\sum_j j\|\Phi_2\|_j < \infty$. It is known that for such a system, the Gibbs state m on Σ is unique and has the following characterization:

Let $\Psi: \Sigma \to \mathbf{R}$ be defined by

$$\Psi(\underline{x}) = -\beta \left\{ \Phi_0(x_0) + \frac{1}{2} \sum_{j \neq 0} \Phi_2(|j|; x_0 x_j) \right\}.$$

Then there are constants $C_1, C_2 > 0$ s.t.

$$C_1 \leq \frac{m\{x_0 = i_0, \dots, x_n = i_n\}}{e^{-Pn + S_n \Psi \underline{z}}} \leq C_2$$

where $\underline{z}$ is any element in the cylinder $\{x_0 = i_o, \dots, x_n = i_n\}$ and P is a constant called "pressure". In fact,

$$P = \sup_{\nu \in \mathscr{M}(\Sigma)} \left(h_\nu(\sigma) + \int \Psi \, d\nu \right)$$

and this supremum is attained by a unique element of $\mathscr{M}(\Sigma)$, namely m. (See [B or R] for more details.)

Theorem 6. *Consider the dynamical system $\sigma: \Sigma \circlearrowleft$ with reference measure m equal to the Gibbs state as discussed above. Let $\varphi \in C(\Sigma, \mathbf{R})$. Then $\frac{1}{n} S_n \varphi$ has a large deviation principle with rate function*

$$k(s) = \inf \left\{ P - \left(h_\nu(\sigma) + \int \Psi \, d\nu \right) : \nu \in \mathscr{M}(\Sigma), \int \varphi \, d\nu = s \right\}.$$

Moreover, $k(s) = 0$ iff $s = \int \varphi \, dm$.

Proof. Set $\xi(\underline{x}) = \Psi \underline{x} - P$. Since $\xi \in \mathscr{V}^+ \cap \mathscr{V}^-$, Theorem 1 applies. □

Theorem 6 holds (with the same proof) if Σ is replaced by Σ_A for irreducible A. I first learned of this result in [OP1]. In [D] there is an independent proof. Their methods are quite different than ours. See also [C, FO and O1] for some more general large deviation results for Gibbs states on $\mathbf{Z}^d$.

Part II. Proofs

A. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1(1). Pick $\nu \in \mathscr{M}_e$ with $\int \varphi\, d\nu > c$ and let $B_n = \{x \in X: \frac{1}{n} S_n \varphi x > c\}$. We need to show that

$$\varliminf_{n\to\infty} \frac{1}{n} \log mB_n \geq h_\nu(f) - h_m(f;\nu).$$

First let $\delta = \frac{1}{2}(\int \varphi\, d\nu - c)$ and choose $\varepsilon_0 > 0$ s.t. $d(x, y) < \varepsilon_0 \Rightarrow |\varphi x - \varphi y| < \delta$. This implies that if $\frac{1}{n} S_n \varphi x > c + \delta$, then $V(x, n, \varepsilon_0) \subset B_n$.

Let $\gamma > 0$ be an arbitrarily small number. Let $N(n, \varepsilon, b)$ be the minimum number of points needed to (n, ε)-span a set of ν-measure b. Choose ε_1, $0 < \varepsilon_1 \leq \varepsilon_0$, s.t. $\forall \varepsilon < \varepsilon_1$,

$$\varliminf \frac{1}{n} \log N\left(n, 4\varepsilon, \frac{1}{2}\right) \geq h_\nu(f) - \gamma,$$

and choose ε_2, $0 < \varepsilon_2 \leq \varepsilon_1$, s.t.

$$\nu\left\{x:\ \varlimsup_{n\to\infty} -\frac{1}{n} \log mV(x, n, \varepsilon_2) \leq h_m(f;\nu) + \gamma\right\} > \frac{2}{3}.$$

(We have implicitly assumed that $h_m(f;\nu) < \infty$ here. If $h_m(f;\nu) = \infty$, then there is nothing to prove since $h_\nu(f) \leq h_{\mathrm{top}}(f) < \infty$.)

Now we choose a measurable set $\Gamma \subset X$ with $\nu\Gamma \geq \frac{1}{2}$ and a positive integer N s.t. $\forall x \in \Gamma$ and $\forall n \geq N$, we have

(i) $\frac{1}{n} S_n \varphi x > c + \delta$ and

(ii) $mV(x, n, \varepsilon_2) \geq e^{-(h_m(f;\nu) + 2\gamma)n}$.

For each n, let $\mathscr{E}_n$ be a maximal set of $(n, 2\varepsilon_2)$-separated points contained in Γ. Then $\bigcap_{x\in\mathscr{E}_n} V(x, 4\varepsilon_2, n) \supset \Gamma$ by maximality of $\mathscr{E}_n$ and so $|\mathscr{E}_n| \geq N(n, 4\varepsilon_2, \frac{1}{2})$. Also, $x \neq y \in \mathscr{E}_n \Rightarrow V(x, n, \varepsilon_2) \cap V(y, n, \varepsilon_2) = \varphi$. Thus

$$\begin{aligned}\varliminf \frac{1}{n} \log mB_n &\geq \varliminf \frac{1}{n} \log \sum_{x\in\mathscr{E}_n} mV(x, n, \varepsilon_2)\\ &\geq \varliminf \frac{1}{n} \log |\mathscr{E}_n| e^{-(h_m(f;\nu)+2\gamma)n}\end{aligned}$$

which gives the desired estimate. □

Proof of Theorem 1(2). Let $B_n = \{x \in X: \frac{1}{n} S_n \varphi x \geq c\}$ and let $\xi \in \mathscr{V}^+$. We need to produce a $\nu \in \mathscr{M}$ with $\int \varphi\, d\nu \geq c$ s.t.

$$\varlimsup \frac{1}{n} \log mB_n \leq h_\nu(f) - \int \xi\, d\nu.$$

Let C and ε be s.t. $\forall x \in X$, $mV(x, n, \varepsilon) \leq Ce^{-S_n\xi x}$. For each n, we let $\mathscr{E}_n$ be a maximal (n, ε)-separated set contained in B_n. Define probability measures

$$\sigma_n = \frac{1}{Z_n} \sum_{x \in \mathscr{E}_n} e^{-S_n \xi x \delta_x}$$

where

$$Z_n = \sum_{x \in \mathscr{E}_n} e^{-S_n \xi x}$$

and

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ f^{-i} .$$

Let ν be a weak limit of μ_n. Clearly $\nu \in \mathscr{M}$. The usual proof of the variational principle for pressure (see e.g. [W, p. 220]) gives

$$\overline{\lim} \frac{1}{n} \log Z_n \leq h_\nu(f) - \int \xi \, d\nu .$$

Since μ_n is a linear combination of measures of the form $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$, $x \in \mathscr{E}_n$, and

$$\int \varphi \, d\left(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}\right) = \frac{1}{n} S_n \varphi x \geq c ,$$

we have $\int \varphi \, d\nu \geq c$. Finally since $B_n \subset \bigcup_{x \in \mathscr{E}_n} V(x, n, \varepsilon)$, we have

$$\overline{\lim} \frac{1}{n} \log mB_n \leq \overline{\lim} \frac{1}{n} \sum_{x \in \mathscr{E}_n} mV(x, n, \varepsilon) \leq \overline{\lim} \frac{1}{n} \log Z_n . \square$$

Proof of Theorem 1(3). Fix $\xi \in \mathscr{V}^-$. Let $B_n = \{x \in X : \frac{1}{n} S_n \varphi x > c\}$ and pick an arbitrary $\nu \in \mathscr{M}$ with $\int \varphi \, d\nu > c$. We will prove that

$$\underline{\lim} \frac{1}{n} \log mB_n \geq h_\nu(f) - \int \xi \, d\nu - 4\gamma$$

for any preassigned $\gamma > 0$. The proof comes in two steps. Step I consists of approximating ν by $\mu \in \mathscr{M}$ so that $\mu = \sum a_i \mu_i$, $\sum a_i = 1$, $\mu_i \in \mathscr{M}_e$. Step II is to estimate mB_n in terms of $h_\mu(f) - \int \xi \, d\mu$.

Let $\delta = \frac{1}{4}(\int \varphi \, d\nu - c)$. To choose μ, we pick $\varepsilon_0 > 0$ s.t. $\forall \tau_1, \tau_2 \in \mathscr{M}$,

$$d_{\mathscr{M}}(\tau_1, \tau_2) < \varepsilon_0 \Rightarrow \begin{cases} |\int \varphi \, d\tau_1 - \int \varphi \, d\tau_2| < \delta , \\ |\int \xi \, d\tau_1 - \int \xi \, d\tau_2| < \gamma . \end{cases}$$

Let $\mathscr{P} = \{P_1, \ldots, P_k\}$ be a partition of $\mathscr{M}$ with $\operatorname{diam} \mathscr{P} \leq \varepsilon_0$. Corresponding to $\nu \in \mathscr{M}$, there is a Borel probability measure π on $\mathscr{M}$ s.t. $\pi \mathscr{M}_e = 1$ and $\int \Psi \, d\nu = \int (\int \Psi \, d\tau) d\pi(\tau) \, \forall \Psi \in C(X, \mathbf{R})$. Let $a_i = \pi P_i$ and pick $\mu_i \in P_i \cap \mathscr{M}_e$ s.t. $h_{\mu_i}(f) \geq h_\tau(f) - \gamma$ for π-a.e. $\tau \in P_i$. It is easy to check that $\mu = \sum_{i=1}^k a_i \mu_i$ satisfies $\int \xi \, d\mu > c + 3\delta$ and $h_\mu(f) - \int \xi \, d\mu \geq h_\nu(f) - \int \xi \, d\nu - 2\gamma$.

To obtain a lower bound for mB_n, choose $\varepsilon > 0$ sufficiently small and N sufficiently large that the following hold.

1. $d(x, y) < \varepsilon \Rightarrow |\varphi x - \varphi y| < \delta$,

2. $\forall n \geq N$ and $1 \leq i \leq k$, $\exists$ at least $e^{a_i n(h_{\mu_i} - \gamma)}$ $([a_i n], 4\varepsilon)$-separated points $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$ with the property that $\forall j$,

 (a) $S_{[a_i n]}\xi x_j^{(i)} \leq [a_i n](\int \xi \, d\mu_i + \gamma)$,

 (b) $S_{[a_i n]}\varphi x_j^{(i)} \geq [a_i n](\int \varphi \, d\mu_i - 2\delta)$.

For each k-tuple $(j_1, \ldots, j_k)$, $1 \leq j_i \leq n_i$, let $y = y_{j_1 \cdots j_k} \in X$ be chosen so that it shadows the orbit segments $[x_{j_1}^{(1)}, fx_{j_1}^{(1)}, \ldots, f^{[a_1 n]}x_{j_1}^{(1)}], \ldots,$ and $[x_{j_k}^{(k)}, fx_{j_k}^{(k)}, \ldots, f^{[a_k n]}x_{j_k}^{(k)}]$ up to ε with time lag $p = p(\varepsilon)$ in between. (See the definition of 'specification' in Part I, §A.) Let $\hat{n} = (k-1)p + \sum_i [a_i n]$. Then clearly, $V(y, \hat{n}, \varepsilon) \subset B_{\hat{n}}$ for sufficiently large n, and for $(j_1, \ldots, j_k) \neq (j_1', \ldots, j_k')$, $V(y, \hat{n}, \varepsilon) \cap V(y', \hat{n}, \varepsilon) = \varnothing$. Count up the measures of these sets as in the proof of (1). $\square$

Proof of Corollary 1. Let $\varphi_1, \varphi_2, \ldots$ be a countable dense subset of $C(X, \mathbf{R})$ with $|\varphi_i| = 1$ and define $d_{\widetilde{\mathscr{M}}}$ by

$$d_{\widetilde{\mathscr{M}}}(\nu_1, \nu_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} |\nu_1(\varphi_i) - \nu_2(\varphi_i)|$$

where $\nu(\varphi_i) = \int \varphi_i \, d\nu$. Pick $\nu \in \mathscr{M}_e$ with $d_{\widetilde{\mathscr{M}}}(\nu, \mu) > \delta$, and write $\delta_x^n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}$. We need to show that

$$\underline{\lim} \frac{1}{n} \log m\{x \in X : d_{\widetilde{\mathscr{M}}}(\delta_x^n, \mu) > \delta\} \geq h_\nu(f) - h_m(f; \nu).$$

Define

$$\tilde{\varphi}_i = \begin{cases} \varphi_i & \text{if } \nu(\varphi_i) \geq \mu(\varphi_i), \\ -\varphi_i & \text{if } \nu(\varphi_i) < \mu(\varphi_i). \end{cases}$$

Let $\Psi = \sum_{i=1}^{\infty} \frac{\tilde{\varphi}_i}{2^i}$. Then $\nu(\Psi) - \mu(\Psi) > \delta$. Apply Theorem 2(1) to Ψ. It remains to verify that $\forall x \in X$, if $|\frac{1}{n}S_n\Psi x - \mu(\Psi)| > \delta$, then $d_{\widetilde{\mathscr{M}}}(\delta_x^n, \mu) > \delta$. This is true because

$$\begin{aligned} d_{\widetilde{\mathscr{M}}}(\delta_x^n, \mu) &= \sum_{i=1}^{\infty} \frac{1}{2^i} |\delta_x^n(\tilde{\varphi}_i) - \mu(\tilde{\varphi}_i)| \\ &\geq |\delta_x^n(\Psi) - \mu(\Psi)|. \quad \square \end{aligned}$$

B. Volume estimates

This section is devoted to some geometric estimates near hyperbolic or partially hyperbolic fixed points. They are probably quite obvious to readers familiar with graph transform type estimates. Unless declared otherwise, $|\cdot|$ in this paper refers to the norm that is most natural to the setting in question—assuming there is one. So for instance, for $v \in \mathbf{R}^n$ $|v|$ refers to its Euclidean norm, whereas for $v \in T_xM$, $|v|$ refers to its norm as defined by the Riemannian metric on M.

We fix some notations. Let $R^p(r) = \{x \in \mathbf{R}^p : |x| < r\}$. For the rest of this section, p and $q \in \mathbf{Z}^+$ are fixed and $R(r) = R^p(r) \times R^q(r)$. Consider a map $F: R(r) \to \mathbf{R}^p \times \mathbf{R}^q$. The *graph transform* by F from $R(r)$ to $R(s)$, written $\Gamma = \Gamma_{r,s}(F)$, is defined as follows: Let $g: R^p(r) \to R^q(r)$. We say that Γg is well defined if it is a function from $R^p(s) \to R^q(s)$ satisfying

$$\operatorname{graph}(\Gamma g) = F(\operatorname{graph}(g)) \cap R(s).$$

Other notations we use include: $L(\mathbf{R}^p, \mathbf{R}^q)$ = the space of linear maps from $\mathbf{R}^p$ to $\mathbf{R}^q$, $L(\mathbf{R}^p) = L(\mathbf{R}^p, \mathbf{R}^p)$; for $L \in L(\mathbf{R}^p, \mathbf{R}^q)$, $\|L\| = \max_{|x|=1} |Lx|$ and $m(L) = \min_{|x|=1} |Lx|$. Also, $T(\operatorname{graph}(g))$ refers to the tangent bundle over $\operatorname{graph}(g)$ viewed as a submanifold of $R(r)$.

Lemma 1. *Let p, $q \in \mathbf{Z}^+$ and $\lambda > 0$ be fixed. Then for all sufficiently small $\varepsilon > 0$, $\exists \tau = \tau(p, q, \lambda, \varepsilon) > 0$ and $\sigma = \sigma(p, q, \lambda, \varepsilon) > 0$ s.t. whenever the following scenario occurs, the stated conclusion holds*:

Scenario. $F: R(r) \to \mathbf{R}^p \times \mathbf{R}^q$ is a C^1 diffeomorphism onto its image, r is any positive number, and F satisfies

(a) $F0 = 0$,

(b) $\operatorname{Lip}(F - L) < \tau$ for some $L \in L(\mathbf{R}^{p+q})$ of the form $L = L_1 \oplus L_2$, $L_1 \in L(\mathbf{R}^p)$, $L_2 \in L(\mathbf{R}^q)$, $m(L_1) \geq e^\lambda$, $\|L_2\| \leq e^\varepsilon$.

Conclusion. If $g: R^p(r) \to R^q(r)$ is a C^1 function with $|g0| \leq r/2$ and $|Dg| \leq \sigma$, then $\Gamma g = \Gamma_{r, re^\varepsilon} g$ is well defined and satisfies $|\Gamma g(0)| \leq \frac{r}{2} e^{2\varepsilon}$ and $|D\Gamma g| \leq \sigma$. Moreover,

$$e^{-2\varepsilon} \leq \frac{|\operatorname{Jac}(DF|T(\operatorname{graph}(g)))|}{|\det L_1|} \leq e^{2\varepsilon}.$$

Proof. In this proof, P will always denote a p-dimensional subspace in $\mathbf{R}^{p+q}$. We write $|P| = \sigma$ if $P = \operatorname{graph}(h)$ for some $h \in L(\mathbf{R}^p, \mathbf{R}^q)$ with $\|h\| = \sigma$.

First we choose $\tau_1 > 0$ s.t. if $\widehat{L} \in (\mathbf{R}^{p+q})$ is s.t. $\|L - \widehat{L}\| < \tau_1$ and $|P| \leq 1$, then

$$e^{-\varepsilon} \leq \frac{|\det \widehat{L}|P|}{|\det L|P|} \leq e^\varepsilon. \tag{1}$$

Next we choose $\sigma > 0$ s.t. $\forall P$ with $|P| < \sigma$,

$$e^{-\varepsilon} \leq \frac{|\det L|P|}{|\det L_1|} \leq e^\varepsilon. \tag{2}$$

Now it follows from standard graph transform estimates (see e.g., [HPS]) that given λ, ε, σ, $\exists \tau_2 > 0$ s.t. if $\operatorname{Lip}(F - L) < \tau_2$, then Γg is well defined and has the desired properties.

Take $\tau = \min(\tau_1, \tau_2)$. To check the last assertion, fix $x \in \operatorname{graph}(g)$, write $P = T_x \operatorname{graph}(g)$, let $\widehat{L} = DF_x$ and use (1) and (2). □

Remark. Lemma 1 says that whenever F is sufficiently near a linear map L with some hyperbolicity properties, then F has good graph transform estimates. It is important to note that except for the last assertion in the lemma, these estimates depend only on the *strength of hyperbolicity* of L and not on L itself. Also, we do not require that $DF_0 = L$.

Next we write down exactly how Lemma 1 will be used. Let $r_0, r_1, \dots, r_{n+1}$ be positive numbers and for $0 \le i \le n$, let $F_i \colon R(r_i) \to \mathbf{R}^{p+q}$ be maps fitting the Scenario in Lemma 1, i.e., there exist linear maps $L_i \in L(\mathbf{R}^{p+q})$ with $L_i = L_{i,1} \oplus L_{i,2}$, all satisfying the estimates in Lemma 1 with the same λ and ε etc. We write $F^i = F_i \circ \cdots \circ F_0$. Let Γ_i be the graph transform $\Gamma_{r_i, r_{i+1}}(F_i)$ if it is defined and write $\Gamma^i = \Gamma_i \circ \cdots \circ \Gamma_0$. For the rest of this section m_l will denote l-dimensional Lebesgue measure, be it on $\mathbf{R}^l$ or on some l-dimensional submanifold in $\mathbf{R}^k$, $k \ge l$.

Let

$$U = \{x \in R^p(r_0) \times R^q(\tfrac{1}{4}r_0) \colon F^i x \in R^p(r_{i+1}) \times R^q(\tfrac{1}{4}r_{i+1}),\ i = 1, \dots, n\}.$$

Lemma 2. (1) *Let* $r_i = r_0 e^{-i\varepsilon}$. *Then* $\exists b_1 = b_1(p)$ *and* $k_1 = k_1(p, q)$ *s.t.*

$$m_{p+q} U \ge b_1 r_0^{p+q} \left[\prod_{i=0}^{n} |\det L_{i,1}|\right]^{-1} e^{-k_1 n\varepsilon}.$$

(2) *Let* $r_i = r_0 \forall i$. *Then* $\exists b_2 = b_2(p)$ *and* $k_2 = k_2(p, q)$ *s.t.*

$$m_{p+q} U \le b_2 r_0^{p+q} \left[\prod_{i=0}^{n} |\det L_{i,1}|\right]^{-1} e^{k_2 n\varepsilon}.$$

Proof. For $w \in R^q(r_0)$, let $g^w \colon R^p(r_0) \to R^q(r_0)$ be the constant function $g^w(t) = w$.

(1) Consider $w \in R^q(\frac{1}{4}r_0 e^{-3n\varepsilon})$. Using Lemma 2 we check inductively that for all $i \le n$, $\Gamma^i g^w$ makes sense and $|\Gamma^i g^w(0)| \le \frac{1}{4}r_0 \cdot e^{-\varepsilon i} = \frac{1}{4}r_i$. Let $C_n^w = (F^n)^{-1}\,\text{graph}(\Gamma^n g^w)$. Then

$$\begin{aligned} m_p C_n^w &\ge \min_{w \in C_n^w} (|\operatorname{Jac}(DF^n | \text{graph}(g^w)|^{-1}) \cdot m_p\, \text{graph}(\Gamma^n g^w)) \\ &\ge \tilde{b}_1 \left[\prod_{i=0}^{n} |\det L_{i,1}|\right]^{-1} e^{-2\varepsilon n} r_n^p \quad \text{for some } \tilde{b}_1 = \tilde{b}_1(p) \\ &= \tilde{b}_1 \left[\prod_{i=0}^{n} |\det L_{i,1}|\right]^{-1} e^{-2\varepsilon n} (r_0^{-\varepsilon n})^p. \end{aligned}$$

Integrating over $w \in R^q(\frac{1}{4}r \cdot e^{-3n\varepsilon})$, we get the desired lower bound.

(2) Let $A = \{w \in R^q(\frac{1}{4}r_0) \colon \text{graph}(\Gamma^i g^w) \subset R^p(r_0) \times R^q(\frac{1}{2}r_0),\ i = 1, \dots, n\}$. Then $U \subset \bigcup_{w \in A} (F^n)^{-1}\,\text{graph}(\Gamma^n g^w)$. We estimate $m_p C_n^w$ as above (assuming $\sigma \le 1$) and integrate over $w \in A$. $\square$

C. Proof of Proposition 1

Let $\nu \in \mathscr{M}_e$. We need to show that

$$\lim_{\varepsilon' \to 0} \overline{\lim_{n\to\infty}} -\frac{1}{n} \log mV(x, n, \varepsilon') \leq \lambda_\nu$$

for ν-a.e. x. Our strategy is to make use of the so-called *Lyapunov charts* and to apply Lemma 2(1) to the induced maps between these charts.

Let $\Lambda' \subset \Lambda$ be a measurable set with $\nu\Lambda' = 1$ and such that every $x \in \Lambda'$ is regular in the sense of Oseledec. For $x \in \Lambda'$ let $E^u(x)$ be the subspace of T_xM corresponding to those $\lambda_i(x)$ with $\lambda_i(x) > 0$, and let $E^{cs}(x)$ correspond to those $\lambda_i(x) \leq 0$. Let $\lambda_1 \geq \cdots \geq \lambda_d$ denote the Lyapunov exponents for ν-typical points, and let $p = \dim E^u$, $q = \dim E^{cs}$ ν-a.e. We choose λ s.t. $0 < \lambda < \min\{\lambda_i : \lambda_i > 0\}$, and as in the last section write $R(r) = R^p(r) \times R^q(r)$.

We now describe some changes of coordinates that are standard in nonuniform hyperbolic theory. For more details we refer the reader to [P] or to the appendix of [LY].

Let $\varepsilon > 0$ be an arbitrarily small number that is fixed throughout. Then associated with ν-a.e. x there is a chart $\Phi_x : R(l(x)^{-1}) \to M$ where $l : \Lambda' \to [1, \infty)$ is a measurable function satisfying $l(f^{\pm}x) \leq e^\varepsilon l(x)$ ν-a.e. These charts have the following properties:

(i) Φ_x is a diffeomorphism onto its image; $\Phi_x(0) = x$; $D(\Phi_x)(0)$ takes $\mathbf{R}^p \times \{0\}$ and $\{0\} \times \mathbf{R}^q$ to $E^u(x)$ and $E^{cs}(x)$ respectively.

(ii) Let $\tilde{f}_x$ be the connecting map between the chart at x and that at fx, i.e., $\tilde{f}_x = \Phi_{f_x}^{-1} \circ f \circ \Phi_x$, defined wherever it makes sense. Then

$$|D((\tilde{f}x)(0)v| \geq e^\lambda |v| \quad \forall v \in \mathbf{R}^p \times \{0\},$$
$$|D(\tilde{f}_x)(0)v| \leq e^\varepsilon |v| \quad \forall v \in \{0\} \times \mathbf{R}^q,$$

and

$$e^{-\varepsilon} \leq \frac{|\det(D(\tilde{f}_x)(0)|\mathbf{R}^p \times \{0\})|}{e^{\lambda_\nu}} \leq e^\varepsilon.$$

(iii) $\operatorname{Lip}(D\tilde{f}_x) \leq l(x)$,

(iv) $\forall z, z' \in R(l(x)^{-1})$,

$$K^{-1} d(\Phi_x z, \Phi_x z') \leq |z - z'| \leq l(x) d(\Phi_x z, \Phi_x z')$$

for some universal constant K.

Let $\varepsilon' > 0$ be given. We choose $\tau = \tau(p, q, \lambda, \varepsilon)$ as in Lemma 1. Shrink τ if necessary so that $\tau K \leq \varepsilon'$. By property (iii) above, the following is true for ν-a.e. x: if $|z| \leq \tau l(x)^{-1}$, then $|D(\tilde{f}_x)(z) - D(\tilde{f}_x)(0)| \leq \tau$, so that when restricted to $R(\tau l(x)^{-1})$ we have $\operatorname{Lip}(\tilde{f}_x - D(\tilde{f}_x)(0)) \leq r$.

Now we fix a ν-typical $x \in \Lambda'$ and apply Lemma 2(1) with $r_i = \tau l(x)^{-1} e^{-i\varepsilon}$ and $F_i = \tilde{f}_{f_x^i}$ for $i = 0, 1, \ldots, n$. Let U be as in that lemma. Then in

$R(r_0) = R(\tau l(x)^{-1})$,

$$m_{p+q} U \geq b_1 (\tau l(x)^{-1})^{p+q} e^{n(-\lambda_\nu - \varepsilon)} e^{-k_1 n\varepsilon}$$

for some $b_1 = b_1(p)$ and $k_1 = k_1(p, q)$. It remains to bring these estimates back to the manifold via Φ_x. By property (iv), we have

$$\Phi_x U \subset V(x, n, \tau K) \subset V(x, n, \varepsilon')$$

which gives

$$mV(x, n, \varepsilon') \geq C_x e^{n(-\lambda_n - (k_1+1)\varepsilon)}$$

for some constant C_x depending only on x. □

D. Proof of Theorem 2(2)

Here we assume that $\Lambda \subset U$ is a uph attractor and that μ is its unique equilibrium state. We claim that it suffices to prove the following:

(a) Every $\xi \in C(U, \mathbf{R})$ with $\xi < \log|\mathrm{Jac}(Df|E^u)|$ on Λ is in $\mathscr{V}^+$.

(b) $\nu \mapsto h_\nu(f)$ is upper semicontinuous on $\mathscr{M}$.

To see that it is sufficient to prove (a) and (b) we observe that (a) together with Theorem 1(2) gives the upper bound

$$\overline{R}\left(\varphi, \left(\int \varphi\, d\mu - \delta, \int \varphi\, d\mu + \delta\right)^c\right)$$
$$\leq \sup\left\{h_\nu(f) - \lambda_\nu : \nu \in \mathscr{M}, \left|\int \varphi\, d\mu - \int \varphi\, d\nu\right| \geq \delta\right\}.$$

Suppose the right side of the above inequality was zero. Then $\exists \nu_n \in \mathscr{M}$, $|\int \varphi\, d\nu_n - \int \varphi\, d\mu| \geq \delta$, s.t. $h_{\nu_n}(f) - \lambda_{\nu_n} \uparrow 0$. Let $\hat{\nu}$ be an accumulation point of the ν_n's. Since $\log|\mathrm{Jac}(Df|E^u)|$ is a continuous function, $\lambda_{\nu_n} \to \lambda_{\hat{\nu}}$. But then (b) would imply that $h_{\hat{\nu}}(f) = \lambda_{\hat{\nu}}$, contradicting the uniqueness of equilibrium state. It remains to verify that μ is SBR. Let $\varphi \in C(U, \mathbf{R})$ and define

$$A_{k,n} = \left\{x \in U : \left|\frac{1}{n} S_n \varphi x - \int \varphi\, d\mu\right| \geq \frac{1}{k}\right\}.$$

Then for fixed k, our large deviation estimate tells us that $\sum_{n=1}^\infty m A_{n,k} < \infty$. Using the Borel-Cantelli lemma and letting $k \to \infty$, we have $\frac{1}{n} S_n \varphi x \to \int \varphi\, d\mu$ for m-a.e. $x \in U$.

We now proceed to prove (a). Let ξ be a continuous function on U satisfying $\xi < \log|\mathrm{Jac}(Df|E^u)|$ on Λ. We fix $\lambda >> \varepsilon > 0$ with ε bounded above by something to be explained later. The definition of a uph attractor guarantees that $\exists N \in \mathbf{Z}^+$ s.t. $\forall x \in \Lambda$,

$$v \in E^u(x) \Rightarrow |Df_x^N v| \geq e^\lambda |v|,$$
$$v \in E^{cs}(x) \Rightarrow |Df_x^N v| \leq e^\varepsilon |v|.$$

We will be working exclusively with f^N, so for simplicity let us assume that $N = 1$.

Since we are concerned with the decay of $mV(x, n, \varepsilon)$ for m-a.e. $x \in U$, we must prove volume estimates that apply not only to $x \in \Lambda$ but to all points in at least some neighborhood of Λ. First we extend E^u and E^{cs} continuously to a compact neighborhood U' of Λ—without requiring that these subbundles be Df-invariant off Λ. Then we define charts for $x \in U'$. Let $E_r^u(x) = \{v \in E^u(x) \colon |v| < r\}$, $E_r^{cs}(x) = \{v \in E^{cs}(x) \colon |v| < r\}$ and $E^r(x) = E_r^u(x) \times E_r^{cs}(x)$. We define $\tilde{f}_x \colon E_r(x) \to T_{f_x} M$ by $\hat{f}_x = \exp_{fx}^{-1} \circ f \circ \exp_x$ and fix $r > 0$ small enough that everything makes sense whenever x, $fx \in U'$. Let $p = \dim E^u$, $q = \dim E^{cs}$.

Choose λ' s.t. $0 < \lambda' < \lambda$ and let $\varepsilon' = 2\varepsilon$. Let $\tau = \tau(p, q, \lambda', \varepsilon')$ and $\sigma = \sigma(p, q, \lambda', \varepsilon')$ be chosen as in Lemma 1. (Here E^u and E^{cs} are not perpendicular, but since the angles between them are uniformly bounded away from 0 on U', a slight modification of Lemma 1 holds.) Let $r' > 0$ be small enough that $\forall z \in \Lambda$, $\mathrm{Lip}(\tilde{f}_z - D\tilde{f}_z(0)) > \frac{\tau}{3}$ when restricted to $E_{r'}(z)$.

Lemma 3. *$\exists \delta$, $r_0 > 0$ s.t. $\forall x \in U$, if $\exists z \in \Lambda$ s.t. $d(x, z) < \delta$ and $d(fx, fz) < \delta$, then $\tilde{f}_x | E_{r_0}(x)$ satisfies the scenario in Lemma 1 with respect to $E^u(x) \oplus E^{cs}(x)$, λ', ε' and τ.*

Proof. Given a pair $x \in U$, $z \in \Lambda$ as above, our strategy is to choose two linear isomorphisms $h_1 \colon T_x M \to T_z M$, $h_2 \colon T_{fx} M \to T_{fz} M$ with $h_i E^u = E^u$, $h_i E^{cs} = E^{cs}$, $i = 1, 2$, so that if $\hat{f}_{xz} \colon T_x M \to T_{fx} M$ is given by $\hat{f}_{xz} = h_2^{-1} \circ \tilde{f}_z \circ h_1$, then $\hat{f}_{xz}$ is near $\tilde{f}_x$. We then let $L_x = D\hat{f}_{xz}(0)$ and show that $\mathrm{Lip}(\tilde{f}_x - L_x) < r$.

First choose $\delta > 0$ s.t. whenever $d(x, z)$, $d(fx, fz) < \delta$ then

$$\text{(i)} \qquad \mathrm{Lip}(\hat{f}_{xz} - D\hat{f}_{xz}(0)) < 2\tau/3$$

on E_{r_0} for some $r_0 < r'$. This is possible since $\hat{f}_{xz}$ differs from $\tilde{f}_z$ by two linear changes of coordinates, which can be arranged to perturb things as little as we wish by choosing δ small.

Next we shrink δ if necessary so that if $d(x, z)$, $d(fx, fz) < \delta$ and $L_x \stackrel{\text{def}}{=} D\hat{f}_{xz}(0) = L_{x,1} \oplus L_{x,2}$ respects $E^u(x) \oplus E^{cs}(x)$, then $m(L_{x,1}) \geq e^{\lambda'}$, $\|L_{x,2}\| \leq e^{\varepsilon'}$ and

$$\text{(ii)} \qquad \mathrm{Lip}(\tilde{f}_x - \hat{f}_{x,z}) < \tau/3 \quad \text{on } E_{r_0}(x).$$

Putting (i) and (ii) together we have $\mathrm{Lip}(\tilde{f}_x - L_x) < \tau$ on $E_{r_0}(x)$. This completes the proof of Lemma 3. □

Let $k_2 = k_2(p, q)$ be as in Lemma 2(2). Earlier on we should have assumed that ε is sufficiently small that $\xi < \log|\mathrm{Jac}(Df|E^u)| - 2k_2\varepsilon$ on Λ and we should have chosen δ sufficiently small that L_x in Lemma 3 satisfies $|\det L_{x,1}| \geq e^{\xi(x) + k_2\varepsilon'}$.

Let $U_n = f^n U$, $n = 1, 2, \ldots$. Then $fU_n \subset U_n$. Let $U'' = U_n$ for some sufficiently large n that (i) $U'' \subset U'$ and (ii) $\forall x \in U''$, $\exists z \in \Lambda$ s.t. $d(x, z)$,

$d(fx, fz) < \delta$ where δ is as in Lemma 3. It suffices to estimate $mV(x, n, r_0)$ for $x \in U''$. To do that we apply Lemma 2(2) to

$$E_{r_0}(x) \xrightarrow{\tilde{f}_x} E_{r_0}(fx) \xrightarrow{\tilde{f}_{fx}} \cdots \to E_{r_0}(f^n x)$$

where each $\tilde{f}_{f^i_x}$ is near its corresponding $L_{f^i_x}$. Note that there need not be any $z \in \Lambda$ shadowing the orbit of x, and that the maps $L_{f^i_x}$ are in no way canonical; they serve only to force some estimates on $\tilde{f}_{f^i_x}$. Lemma 2(2) tells us that

$$mV(x, n, r_0) \le b_2 r_0^{p+q} \left[\prod_{i=0}^{n} |\det L_{f^i_x, 1}|\right]^{-1} e^{nk_2\varepsilon'} \le b_2 r_0^{p+q} e^{-S_n\xi(x)}.$$

This completes the proof of (a).

To prove (b) we will show that given $\varepsilon > 0$, $\exists\delta > 0$ s.t. if $\mathscr{P}$ is a partition of Λ with $\operatorname{diam}\mathscr{P} < \delta$, then for every $\nu \in \mathscr{M}$, $h_\nu(f, \mathscr{P}) \ge h_\nu(f) - \varepsilon$. This together with the fact that $\overline{\lim}\, h_{\nu_n} \le h_\nu(f, \mathscr{P})$ for every finite partition $\mathscr{P}$ with $\nu(\partial\mathscr{P}) = 0$ and every convergent sequence $\nu_n \to \nu$ gives the upper semicontinuity of $\nu \mapsto h_\nu(f)$.

As before we assume that for some arbitrary small $b > 0$, we have for all $x \in \Lambda$, $|Df_x v| \ge e^\lambda |v|$ $\forall v \in E^u(x)$ and $|Df_x v| \le e^b |v|$ $\forall v \in E^{cs}(x)$. We will use the charts $\tilde{f}_x: E_r(x) \to E_r(fx)$ defined in this section. The following lemma is standard in stable manifold theory (see e.g., [HPS]):

Lemma 4. *$\exists\delta > 0$ s.t. $\forall x \in \Lambda$,*

$$\widetilde{W}^{cs}_\delta(x) \stackrel{\text{def}}{=} \{y \in E_\delta(x) : \tilde{f}_{f^i_x} \circ \cdots \circ \tilde{f}_x y \in E_\delta(f^i_x)\ \forall i \ge 0\}$$

is the graph of a Lipschitz $g_x: D_x \to E^u_\delta(x)$ for some $D_x \subset E^{cs}_\delta(x)$. Let $W^{cs}_\delta(x) = \exp_x \widetilde{W}^{cs}_\delta(x)$. Then $fW^{cs}_\delta(x) \subset W^{cs}_\delta(fx)$, and for δ sufficiently small, $\operatorname{Lip}(f|W^{cs}_\delta(x)) \le e^{2b}$ $\forall x \in \Lambda$.

Choose $b = \varepsilon(2 \cdot \dim E^{cs})^{-1}$ and let $\mathscr{P}$ be a finite partition with diameter $< \delta$. Then following Theorem 3.5 *of* [B3] *we have $\forall \nu \in \mathscr{M}$,*

$$h_\nu(f) \le h_\nu(f, \mathscr{P}) + h^*_\delta(f)$$

where

$$h^*_\delta(f) = \sup_{x\in\Lambda} h\left(f, \bigcap_{n\ge 0} V(x, n, \delta)\right),$$

$$h(f, A) = \lim_{\gamma\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log r_n(A, \gamma)$$

and $r_n(A, \gamma)$ is the minimum number of points needed to (n, γ)-span A. Here,

$$h\left(f, \bigcap_{n\ge 0} V(x, n, \delta)\right) \le \dim E^{cs} \cdot \left[\sup_{n\ge 0} \log^+ \operatorname{Lip}(f|W^{cs}_\delta(f^n x))\right] < \varepsilon. \quad \square$$

Added in proof. It was brought to our attention after this manuscript was completed that there is some overlap of our results with those in [T].

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