TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 321, Number 1, September 1990

THE NONSTANDARD TREATMENT OF HILBERT'S FIFTH PROBLEM

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ABSTRACT. We give a nonstandard proof that every locally Euclidean group is a Lie group. The heart of the proof is a strong nonstandard variant of Gleason's lemma for a class of groups that includes all locally Euclidean groups.

The following is usually considered as the positive answer to Hilbert's fifth problem.

Theorem 0. *Every locally Euclidean group is a Lie group.*

The theorem was proved in 1952 by Montgomery, Żippin, and Gleason, and it can be found in [5]. Another version of the proof is given in [3].

In a 1976 symposium on Hilbert's problems, specialists on each problem discussed the solution of each particular problem. Dealing with the fifth problem [10, pp. 142–145], Yang says, "Since the proof of the theorem is very complicated and technical it is impossible for us to sketch it here. Besides it is certainly a challenge to find a simpler proof."

Using nonstandard analysis, we are able to present a much *simplified* proof. It differs from the proofs in [3] and [5] about as much as they differ from each other. It certainly is possible for us to sketch it here, and we shall do so following the introduction of the notation.

0.1. *From now on, let G be a Hausdorff, locally compact topological group.* The letters V, W, V_i, etc. will be reserved to denote a (standard) neighborhood of the identity e in G.

We shall deal with the following three classes of groups:

0.2. *An NSS group* is a group with no small subgroups. That is, there is some V such that the only subgroup contained in V is $\{e\}$.

0.3. *An NCSC group* is a group with no small *connected* subgroups. That is, there is some V such that the only connected subgroup contained in V is $\{e\}$.

0.4. *Locally Euclidean* groups—groups that have a neighborhood of e that is homeomorphic to an open set in a Euclidean space E_n.

Received by the editors November 7, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03H05, 22D05.

A central notion in the discussion is

0.5. *OPS*—one parametric subgroup—is a (continuous) additive homomorphism

$$X: R \to G.$$

We call "one parametric subgroup" both the homomorphism and its image. Thus we use notations like $X \subseteq G$.

0.6. Our plan is to show:

0.6.1 Every locally Euclidean group is NSS.

0.6.2 Every NSS group is a Lie group.

To avoid discussion of the analytic nature of a Lie group, we show 0.6.3 instead of 0.6.2. This assumes, somewhat arbitrarily, that it is standard Lie group theory to show that 0.6.3 describes a Lie group.

0.6.3 Every NSS group has a subgroup C that is locally *isomorphic* to a finite-dimensional vector space, such that G/C is a matrix group.

Inspired by Van Der Dries [9], I add the proof of two more facts that are used in the actual definition of a Lie structure on NSS groups (as outlined in [3]):

0.6.4 In NSS groups the OPS form a finite-dimensional vector space with an operation of addition that satisfies

$$\lim_{n\to\infty}\left(X\left(\frac{1}{n}\right)\cdot Y\left(\frac{1}{n}\right)\right)^n = (X+Y)(1).$$

0.6.5 every NSS group has a neighborhood W that is ruled by OPS; i.e., every point of W lies on some OPS.

0.7. The logical components of the proof are as follows. *For NSS groups*:

0.7.1 The OPS form a finite-dimensional vector space M over R.

0.7.2 There is a homomorphism T of G into the group of regular matrices (linear transformations) of M, called the *adjoint representation*,

$$T_g(X) = gXg^{-1} \quad \text{where } gXg^{-1}(a) = g(X(a))g^{-1}.$$

0.7.3 The kernel C of T is locally isomorphic to a space of OPS over G, by the *exponential map* $E(X) \to X(1)$.

0.7.4 There is a neighborhood of e covered by OPS.

These results are complemented by

0.7.5 Every locally connected NSCS group is NSS.

0.7.6 Every locally Euclidean group is (locally connected and) NSCS.

0.8. The actual proof does not go linearly through 0.7. It is divided as follows.

§1 develops the framework in nonstandard analysis:

0.8.1 "Regular infinitesimals" generate OPS.

0.8.2 "Singular infinitesimals" generate a net of small connected subgroups.

0.8.3 Therefore NSCS groups (and eventually locally Euclidean groups) are *regular*—they do not have singular elements.

0.8.4 Thus there are many OPS, and if G is NSS they form a locally compact topological space.

§§2 and 3 provide the main tools that we call Gleason's lemmas. They include the following results:

0.8.5 Multiplication of small elements is almost commutative.

0.8.6 Given a set A of small elements, you cannot move away from the identity faster by using products in A than by using powers of a single element in A.

These results have a precise formulation in nonstandard analysis (3.4.3 and 3.1). They are proved by looking at elements of G as operators on the space C of continuous functions from G to R (with compact support).

0.8.7 Having 0.8.5 and hoping for 0.6.4, it is almost clear how to turn the space of OPS into a linear vector space which is finite-dimensional if G is NSS, since it is locally compact. This is the start of §4. Then 0.7.2 comes easily and it is shown that:

0.8.8 Small elements lie on some OPS because if α is not on some OPS then by compactness of the neighborhood in M you can get a point on some OPS that is closest to α. But by the abundance of OPS proved in 0.8.4, there is no such closest point.

0.8.9 It now follows from 0.8.8 that the kernel of T is commutative and the map $X \to X(1)$ is a local isomorphism between a subgroup of M and C. This concludes the proof of 0.7.1 to 0.7.4.

§5 gives three results of Gleason's lemmas to be used for the completion of the proof.

In §6 it is shown that NSCS groups have a subgroup H that includes all the nonregular infinitesimals such that G/H is NSS. This follows naturally from Gleason's 0.8.6. A little more technique goes into showing that G is locally isomorphic to G/H; thus we have 0.7.5.

Finally, in §7 we do the computation that shows that locally Euclidean group are NSCS, which completes the proof (0.7.6).

0.9. There are no equicontinuous sequence of functions in our work, nor inverse limits. Gleason's lemmas are proved much more easily, in a stronger form, and for quite general groups. Their results follow very smoothly and one should compare our Theorem 4.10 with Kaplansky's §9 [3, pp. 120–130], or our Theorem 5.1 with the equivalent §10 [pp. 131–138]. We do not need the Peter Weyl theorem (except perhaps to prove 7.2; see the remark there) and we do not use the analysis of the group GL_n (beyond what is implied in 0.6.3). We do not separate the metric case and there is no part of the proof that would become easier in the metric case.

0.10. The proof is written for people who speak the nonstandard analysis language. The proof takes place in an enlargement of a universe that contains everything that is needed. It may be that at one or two points in the proof we use the kind of overspill that requires the enlargement to be somewhat saturated.

0.11. Nonstandard aspects of Hilbert's fifth problem were investigated around 1975. Bate [1] and Singer [8] discussed Theorem 0, proving some of its aspects in nonstandard analysis and giving the nonstandard equivalents of others. In particular, Singer used infinitesimals to construct OPS and identified the equivalent form 3.1 of Gleason's lemma. At about the same time I gave a nonstandard proof for the commutative NSS case [2, p. 570].

In 1981 Van Der Dries lectured at Yale University and apparently gave a complete nonstandard treatment of the problem. I have seen the first few sections of the notes and it seems to me that he follows the standard treatment more closely.

I would like to thank C. Ward Henson, who brought to my attention the works by Singer and Van Der Dries.

Much credit should go to A. Robinson. Not just on general grounds: he was the first to construct OPS by nonstandard analysis, at least in Lie groups. He also proved that Theorem 3.4 holds in Lie groups [7].

Over 20 years ago, I asked a young logician to be the supervisor of my M.Sc. thesis. M. Machover suggested nonstandard analysis of Hilbert's fifth problem. This was the first I had ever heard of nonstandard analysis or of this particular problem.

So here, somewhat belatedly, I submit to M. Machover what can be thought of as the last chapter of the thesis.

1. Infinitesimals as parameters

1.1. We define *order of magnitude* of * natural numbers. For $n, m \in {}^*N$:

1.1.1 $m \in O(n)$ if $kn \geq m$ for some (finite) $k \in N$, i.e., if $m = ln + n_1$ for some $l \in N$ and $n_1 < n$; in other words, if m/n is finite.

1.1.2 $m \in o(n)$ if $km < n$ for all $k \in N$; in other words, if m/n is infinitesimal.

1.1.3 $m \equiv n$ if $m \in O(m) - o(n)$.

1.1.4 Clearly, for all $n, m \in {}^*N$: $n \in o(m)$ or $n \equiv m$, or $m \in o(n)$.

1.1.5 We shall also use the symbol ∞ and $o(\infty) = {}^*N$.

1.2. We define order of infinitesimality of elements in μ, the monad of the identity in G.

1.2.1 $\beta \in O(\alpha)$ if $(\forall i \leq n\ \alpha^i \in \mu)$ implies $\beta^n \in \mu$. That is, powers of β stay in μ at least as long as powers of α do.

1.2.2 $\beta \in o(\alpha)$ if $\beta \in O(\alpha)$ and $\alpha \notin O(\beta)$.

1.2.3 $\beta \equiv \alpha$ if $\beta \in O(\alpha) - o(\alpha)$.

1.2.4 Clearly, for all $\alpha, \beta \in \mu$: $\alpha \in o(\beta)$ or $\alpha \equiv \beta$ or $\beta \in o(\alpha)$.

1.3. Regular singular and degenerate elements. $o(e) = \emptyset$ and we ignore it. Elements of $O(e)$ will be called *degenerate*. Thus

1.3.1 α is degenerate iff $\alpha^n \in \mu$ for all $n \in {}^*N$; in other words, if the *group generated by α is infinitesimal.

Elements that are not degenerate have powers outside of μ. It turns out that the basic distinction in this work is between *regular* elements that have a first order of magnitude outside μ and elements that are *singular*—without such power.

1.3.2 α is regular if for some n_0

$$\alpha^{n_0} \notin \mu \quad \text{and} \quad \alpha^n \in \mu \quad \text{for all } n \in o(n_0).$$

1.3.3 We say that n_0 *is regular* for α or that $\langle \alpha, n_0 \rangle$ is regular.

1.3.4 α is singular if α is not degenerate and for every n_0 if $\alpha^{n_0} \notin \mu$ then $\alpha^n \notin \mu$ for some $n \in o(n_0)$.

1.4. It is easy to see that if $\langle \alpha, n_0 \rangle$ is regular and $n \equiv n_0$ then $\langle \alpha, n \rangle$ is regular. On the other hand, if n and n_0 are regular for α then $n \equiv n_0$, because $n \in o(n_0)$ and $n_0 \in o(n)$ are excluded by 1.3.2. Therefore *the regular power* of a regular α is well defined up to equivalence.

1.5. If V is a standard neighborhood we denote by $L(\alpha, V)$ the first n in ${}^*N$ such that $\alpha^n \notin {}^*V$. $L(\alpha, V) = \infty$ if $\alpha^n \in {}^*V$ for all $n \in {}^*N$.

1.5.1 If $L(\alpha, V)$ is regular for α we say that the *neighborhood V is regular* for α.

1.5.2 If $\langle \alpha, n_0 \rangle$ is regular then for some neighborhood V we have $\alpha^{n_0} \notin V$. Therefore $L(\alpha, V) \leq n_0$ and yet $L(\alpha, V) \notin o(n_0)$. Hence $L(\alpha, V) \equiv n_0$. Therefore the regular number n_0 for α can be chosen as $L(\alpha, V)$ for some V.

1.5.3 Finally, if V is regular for α then so is every (standard) neighborhood $W \subset V$.

1.6. Summary. α is either *degenerate* with all powers in μ, or *regular* with a first power (up to order of magnitude) outside μ, or else singular. If α is regular this first power can be realized by some $L(\alpha, V)$.

1.7. Regular singular and degenerate sets. A similar classification also holds for *internal sets.*

1.7.1 Q is degenerate if the *group generated by Q is infinitesimal.

1.7.2 $L(Q, V)$ is the smallest n such that a product $\alpha_1, \dots, \alpha_n$ of (not necessarily different) elements of Q is outside V. $L(Q, V) = \infty$ if $Q^\infty \subseteq V$ (for some V this may happen even if Q is not degenerate).

1.7.3 V *is regular* for Q if $L(Q, V) < \infty$ and for all $n \in o(L(Q, V))$ we have $Q^n \subseteq \mu$. In this case we also say that Q is regular.

1.7.4 If Q is neither degenerate nor regular then Q is singular.

1.7.5 Finally: A group G is regular *if it has no singular set* (*or element*).

1.8 *Remark.* It has not been unheard of for a nonstandard analyst to invent unnecessary terminology to toy around with. This is not the case here. Our classification is essential for the new look at the solution to the "fifth problem."

1.9 Theorem. *Let Q be a symmetric internal nondegenerate set of infinitesimals and let V be a standard neighborhood with compact closure that is not regular* *for Q. We define*

$$G(Q,V)=\left\{{}^0\!\left(\prod_{i=1}^{n}\alpha_i\right) \parallel n\in o(L(Q,V)) \text{ and } \alpha_i\in Q,\ i=1,\dots,n\right\}.$$

Then

1.9.1 $G(Q,V)$ *is a nontrivial (standard) group;*
1.9.2 $\overline{G(Q,V)}\subseteq\overline{V}$;
1.9.3 $\overline{G(Q,V)}$ *is connected.*

Proof. 1.9.1 and 1.9.2 are immediate from the definitions (that $G(Q,V)$ is not trivial uses compactness as well).

To prove 1.9.3, assume that $\overline{G(Q,V)}$ is the union of two disjoint closed sets A and B. We shall prove that one of them is empty. Now $\overline{V}\supseteq\overline{G(Q,V)}$ and we continue our discussion in $\overline{V}$ and its relative topology. Since $\overline{V}$ is compact it is normal and A and B can be replaced by two open disjoint sets W_1 and W_2 such that $G(Q,V)\subseteq W_1\cup W_2$. Without loss of generality $e\in W_1$. We shall prove that every $g\in G(Q,V)$ is in W_1 and therefore no element of $\overline{G(Q,V)}$ is in W_2.

Assume that $g\in G(Q,V)\cap W_2$ and $g={}^0(\prod^{n_1}\alpha_i)$ with $n_1\in o(n_0)$ where $n_0=L(Q,V)$. By definition of $G(Q,V)$, $\prod\alpha_i$ is in ${}^*\overline{V}$ and since W_2 is open in $\overline{V}$ we have $\prod\alpha_i\in{}^*W_2$. Therefore, we can choose a shortest product of elements from Q that is in ${}^*W_2$. That is, $s=\prod_{i=1}^{n}\beta_i$ with $s\in{}^*W_2$ and $s'=\prod_{i=1}^{n-1}\beta_1$ satisfies $s'\notin{}^*W_2$. Clearly $s\sim s'$ and $n\in o(n_0)$. It follows that ${}^0(s)=g\in G(Q,V)$. However, $g\notin W_1$, or else $s\in{}^*W_1$ and $g\notin W_2$ or else $s'\in{}^*W_2$. Contradiction.

Remark. It is a typical nonstandard exercise to show that if the universe is somewhat saturated then $G(Q,V)=\overline{G(Q,V)}$, but we shall not need this.

1.10 Corollaries.

1.10.1 *If V contains no connected subgroup (i.e., G is NSCS) then G has no singular element or set and V is regular for all the regular elements in μ.*

1.10.2 *If G is not regular then to every neighborhood V there corresponds a nontrivial connected closed group G_V such that*

(a) $G_V\subseteq\overline{V}$,
(b) $V\subseteq W\to G_V\subseteq G_W$.

Proof. Immediate from 1.9 (with $G_V=\overline{G(Q,V)}$), where Q is any internal symmetric singular set.

1.11. We recall that a *one parametric subgroup* of G is a continuous, additive, homomorphism from the reals:

$$X: R\to G.$$

We shall use the abbreviation OPS and at times we shall not distinguish between an OPS and its range in G.

Theorem 1.12 is the analog of 1.9 for regular elements. It shows how regular elements generate OPS. For this reason Singer calls them *parameters* in [8].

1.12 Theorem. *Let $\alpha \in \mu$ and n_0 be regular for α. Then there is a nontrivial OPS $X: R \to G$ such that if we denote by α_s the point $X(1/n_0)$ then $a^n \sim \alpha_s^n$ for all $n \in O(n_0)$.*

1.12.0 *We denote this OPS by $X_{\langle \alpha, n_0 \rangle}$ and say that it is generated by $\langle \alpha, n_0 \rangle$.*

Proof.

1.12.1 Note that n_0 is regular means that if $n/n_0 \sim 0$ then $\alpha^n \in \mu$.

More generally,

1.12.2 If $n/n_0 \sim m/n_0$ then $(n-m)/n_0 \sim 0$ and $\alpha^{n-m} \in \mu$. Thus $\alpha^n \sim \alpha^m$.

We describe a standard function X from R to G:

Given $t \in R$, choose some $n \in {}^*N$ such that $n/n_0 \sim t$ (we assume that t is positive). Then $n \in O(n_0)$ and α^n is near standard; put $X(t) = {}^0(\alpha^n)$. By 1.12.2 X is well defined.

Now given $s, t \in R^+$ choose n and m such that $n/n_0 \sim s$ and $m/n_0 \sim t$. Then $\alpha^n \sim X(s)$ and $\alpha^m \sim X(t)$, and $\alpha^n \cdot \alpha^m \sim X(s) \cdot X(t)$. But since $(n+m)/n_0 \sim s + t$ we also have

$$\alpha^n \cdot \alpha^m = \alpha^{n+m} \sim X(s+t).$$

Thus $X(s+t) \sim X(s) \cdot X(t)$, which means equality for standard group elements. This shows that X is a homomorphism.

X is also continuous: Given an arbitrary neighborhood W of e, we pick some symmetric W_1 such that $\overline{W}_1 \subseteq W$ and we look at the set

$$\{\delta \in {}^*R | \ n/n_0 < \delta \to \alpha^n \in W_1\}.$$

By 1.12.1 this set contains every infinitesimal δ. By overspill it also contains a standard δ_0. We show that X takes $(-\delta_0, \delta_0)$ into W. Indeed, if $0 \le t < \delta_0$ we can find some n such that $t \sim n/n_0 < \delta_0$ so that $\alpha^n \in W_1$ and $X(t) = {}^0(\alpha^n) \in \overline{W}_1 \subseteq W$.

Therefore, X is an OPS. And if $n \in O(n_0)$ then $n/n_0 \sim t$ for some standard t and we have $\alpha_s^n = X(n/n_0) \sim X(t)$, by continuity of X. But also $\alpha^n \sim X(t)$ by the definition of X.

It is a trivial nonstandard analysis exercise, and will not be proved, that

1.13 Theorem. *G is NSS iff G has no degenerate elements.*

We can strengthen it.

1.14 Theorem. *Let G be NSS. Then*:

1.14.1 *All the elements (and sets) in μ are regular.*

1.14.2 *If G has no OPS then G is discrete.*

Proof.

1.14.1 No degenerate sets by 1.13 and no singular sets by 1.10.

1.14.2 If G has no OPS then by 1.12 it has no regular elements either, which means $\mu(e) = \{e\}$.

1.15. The one parametric subgroups are functions from R to G so that they form a topological space with the compact open topology. *The space of* OPS *will be denoted by* M. A base for the neighborhoods of O, the trivial OPS, are sets of the form

$$D_V = \{X \mid X(t) \in V \text{ for all } 0 \leq t \leq 1\}$$

where V ranges over the open neighborhoods of e.

It is easy nonstandard analysis to see that in this topology:

1.15.1 $X \sim O$ iff $X(t) \sim O$ for all finite $t \in {}^*R$;
iff $X(t) \sim O$ for all $0 \leq t < 1$.

Similarly, if $Y \in M$ and $X \in {}^*M$, then

1.15.2 $X \sim Y$ iff $X(t) \sim Y(t)$ for all finite $t \in {}^*R$.

There is a natural map called the exponential map $E: M \to G$ defined by $E(X) = X(1)$. Clearly, E is continuous.

1.15.3 If $\overline{V}$ is compact, symmetric, and contains no groups, $X \in {}^*D_V$, and $X(\alpha) \in \mu$ for some $0 \leq \alpha \leq 1$, then $X([0, \alpha]) \subset \mu$.

This is so because for β, $\beta_1 \leq \alpha$ we have

$${}^0(X(\beta)) \cdot {}^0(X(\beta_1)) = {}^0(X(\alpha)) \cdot {}^0(X(\beta + \beta_1 - \alpha)) \in \overline{V}$$

so that $\{{}^0(X(\beta)) \mid 0 \leq \beta \leq \alpha\}$ forms a group in $\overline{V}$.

In particular,

1.15.4 If $X \in {}^*D_V$ and $E(X) \in \mu$ then $X \in \mu(O)$.

1.16. The following lemma will prove later that if G is NSS then M is locally compact.

Lemma. *Let $\overline{V}$ be compact with no subgroup. Then the following set is a compact neighborhood of O in M*:

$$D_{\overline{V}} = \{X \mid X([0, 1]) \subseteq \overline{V}\}.$$

Proof. $D_{\overline{V}}$ is a neighborhood as it contains D_V. We shall prove that every $X \in {}^*D_{\overline{V}}$ is near standard.

Let X be given.

1.16.1 $X(\mu(0)) \leq \mu(e)$.

Else $X(s) \sim a \neq e$ for some $s \sim 0$ by 1.15.4. But then for all $n \in N$ $X(ns) \sim a^n$ and $a^n \in \overline{V}$ because $ns < 1$ and $X([0, 1]) \subseteq \overline{V}$. Therefore a generates a group in $\overline{V}$, contradiction.

1.16.2 If $t \sim s \in {}^*[0, 1]$ then $X(t) \sim X(s)$. If $\tau = s - t$ then $X(s) = X(t)X(\tau) \sim X(t) \cdot e$, by (1.16.1).

1.16.3 Now choose an infinite n_0 and $\alpha = X(1/n_0)$. Then by 1.16.1 $\alpha^n \in \mu$ for $n \in o(n_0)$ and by 1.15.4 n_0 is regular for α unless $X \sim 0$. Therefore, by Theorem 1.12, $\langle \alpha, n_0 \rangle$ defines an OPS Y and for each $t \in {}^*[0, 1]$ if we choose n such that $n/n_0 \sim t$, we get

$$X(t) \sim X(n/n_0) = \alpha^n \sim Y(n/n_0) \sim Y(t)$$

by 1.16.2, the choice of α, the definition of Y, and the continuity of Y, respectively.

Hence $X \sim Y$ in the compact open topology and $D_{\overline{V}}$ is compact. □

2. Gleason's lemmas—technical form

2.1. Let G be any locally compact topological group and let V be an open symmetric neighborhood such that V^4 has a compact closure. Let Q be an internal symmetric set (usually infinitesimal but not necessarily). And let $L(Q, V) = n_0$ (see 1.7).

2.2. We start by defining an internal step function that measures how easily x is obtained as a product of elements of Q.

2.2.1 $\Delta(e) = 0$.

2.2.1 $\Delta(x) = 1$ if x is not a *product of less than n_0 elements of Q.

2.2.3 $\Delta(x) = n/n_0$ if x is the product of (not necessarily different) less than n_0 elements of Q and n is the length of the smallest such product.

Then

2.2.4 $0 \le \Delta(x) \le 1$;

2.2.5 $\Delta(e) = 0$ and $\Delta(x) = 1$ if $x \notin V$;

2.2.6 $|\Delta(\alpha x) - \Delta(x)| \le 1/n_0$ for $\alpha \in Q$ and $x \in G$ (remember that Q is symmetric).

2.3. Let C be the space of real continuous functions on G, with compact support. On this space we have the norm

2.3.1 $\|f\| = \mathrm{Sup}\{|f(x)| \mid x \in G\}$.

Elements of G act on this space, as left translation.

2.3.2 $(af)(b) = f(a^{-1}b)$.

2.3.3 Then $\|af\| = \|f\|$ and 2.2.6 can be rewritten as

2.3.4 $\|\alpha \cdot \Delta - \Delta\| \le 1/n_0$ for $\alpha \in Q$ (actually Δ is not in C but this will be corrected in 2.4).

2.3.5 Since $\|(abf - f)\| \le \|abf - af\| + \|af - f\|$ we get by 2.3.3 and 2.3.2 $\|abf - f\| \le \|af - f\| + \|bf - f\|$.

2.4. Next we turn Δ into a *continuous function by taking into account the behavior of Δ in the neighborhood of x.

Let $t(x)$ be a continuous function granted by Urisohn's lemma that satisfies

2.4.1 $t(e) = 0$, $t(x) = 1$ for $x \notin V$.

Then $t(y^{-1}x)$ gives a notion of closeness between y and x. We define

2.4.2 $\theta_1(x) = \inf_{y \in {}^*G}(\Delta(y) + t(y^{-1}x))$ and we replace θ_1 by θ,

2.4.3 $\theta(x) = 1 - \theta_1(x)$.

To obtain a function with a compact support,

2.4.4 $0 \le \theta(x) \le 1$;

2.4.5 $\theta(e) = 1$, $\theta(x) = 0$ if $x \notin V^2$;
2.4.6 $\alpha \in Q \to |\theta(\alpha x) - \theta(x)| < 1/n_0$ for all $x \in {}^*G$;
2.4.7 $\alpha \in Q \to \|\alpha\theta - \theta\| \leq 1/n_0$.
It is easy to see that whether Q is standard or not, we have the rule
2.4.8 $x \sim y \to \theta(x) \sim \theta(y)$.

This means that θ is S-continuous. But more than that—since this is true for every standard Q, we conclude that for every standard Q the appropriate θ is continuous, so that for every internal Q (by transfer)
2.4.9 $\theta(x)$ is *continuous.
(This argument can be replaced by an ${}^*\varepsilon$, ${}^*\delta$ argument.)

Property 2.4.9 is usually useless in nonstandard analysis. Here it is central because it enables us to * integrate θ.

2.5. Finally we smooth $\theta(x)$ to obtain $\varphi(x)$.

We start with (standard) Haar measure normalized so that $\int f(x)\,dx = 1$ for some Urisohn function f which is one on $\overline{V}^2$ and zero on $G - V^3$. Such a function exists since $\overline{V}^4$ is compact and hence also normal.

Now $0 \leq \theta(x) \leq f(x)$ so that
2.5.1 $0 \leq \int \theta(x)\,dx \leq 1$.

On the other hand, $\theta(x) \sim 1$ for $x \sim e$ (by 2.4.8) so that on some standard neighborhood W, $\theta^2(x) \geq \frac{1}{2}$. And we conclude for further reference that
2.5.2 $M = \int \theta^2(x)\,dx \geq \frac{1}{2}m(W)$.
We now define
2.5.3 $\varphi(x) = \int \theta(xu)\theta(u)\,du$ and we still have, by 2.5.2,
2.5.4 $\varphi(e) = M \gg 0$. (Here $M \gg 0$ means M is not infinitesimal.)
2.5.5 $\varphi(x) = 0$ if $x \notin V^4$ (as $\theta(x) = 0$ if $x \notin V^2$).
2.5.6 $\alpha \in Q \to |\varphi(\alpha x) - \varphi(x)| \leq 1/n_0$ or
2.5.7 $\|\alpha\varphi - \varphi\| \leq 1/n_0$ and
2.5.8 φ is S-continuous (and bounded by 1).
Formulas 2.5.6 and 2.5.8 follow from 2.4.6 and 2.4.8 as

$$|\varphi(\alpha x) - \varphi(x)| \leq \int |\theta(\alpha xu) - \theta(xu)|\theta(u)\,du \leq \|\alpha\theta - \theta\| \int \theta(u)\,du \leq \|\alpha\theta - \theta\|.$$

We now come to the important properties of φ.

2.6 Lemma. *If $\alpha \in Q$ and $n \leq n_0 = L(Q, V)$ then $n(\alpha\varphi - \varphi)$ is bounded and S-continuous.*

Proof. $\|\alpha\varphi - \varphi\| \leq 1/n_0$ so that $\|n(\alpha\varphi - \varphi)\|$ is bounded by 1.

Assume now that $x \sim x_0$; then
2.6.1 $(\alpha\varphi - \varphi)(x) = \int[\theta(\alpha xu) - \theta(xu)]\theta(u)\,du$;
2.6.2 $(\alpha\varphi - \varphi)(x_0) = \int[\theta(\alpha x_0 u) - \theta(x_0 u)]\theta(u)\,du$.
The Haar integral is left invariant and we can replace u in 2.6.2 by $x_0^{-1}xu$:
2.6.3 $(\alpha\varphi - \varphi)(x_0) = \int[\theta(\alpha xu) - \theta(xu)]\theta(x_0^{-1}xu)\,du$.

Taking the difference between 2.6.1 and 2.6.3, we get
2.6.4

$$[n(\alpha\varphi - \varphi)](x_0) - [n(\alpha\varphi - \varphi)](x) = n \int [\theta(\alpha x u) - \theta(xu)][\theta(x_0^{-1} xu) - \theta(u)]\, du.$$

But $[\theta(x_0^{-1}xu) - \theta(u)]$ is infinitesimal by 2.4.8, so that $\|\theta(x_0^{-1}xu) - \theta(u)\| = \delta \sim 0$ and $\|\theta(\alpha xu) - \theta(xu)\| \leq 1/n_0$ by 2.4.6. Thus we have

2.6.5 $[n(\alpha\varphi - \varphi)](x_0) - [n(\alpha\varphi - \varphi)](x) \leq n_0(1/n_0) \cdot \delta \cdot m(V^4) \sim 0$. Q.E.D.

2.6.6 Corollary. *If $\alpha \in Q$ and $n \in O(L(Q, V))$ then for every ε there is some standard neighborhood W of e such that for $\beta \in W$*

$$\|\beta \cdot n(\alpha\varphi - \varphi) - n(\alpha\varphi - \varphi)\| < \varepsilon.$$

Proof. It is clear that Lemma 2.6 holds for $n \in O(L(Q, V))$ and not just $n \leq L(Q, V)$. 2.6.6 is just the $\varepsilon - \delta$ version of S-continuity.

2.7 Lemma. *If $\alpha \in Q$ and $n_1 \in O(L(Q, V))$ then for every standard ε there is a neighborhood W of e such that if $\alpha^i \in W$ for all $i \leq n$ then*

$$\|(n_1/n)(\alpha^n\varphi - \varphi) - n_1(\alpha\varphi - \varphi)\| < \varepsilon.$$

Proof. Write $n_1(\alpha^n\varphi - \varphi)$ telescopically:

2.71 $n_1(\alpha^n\varphi - \varphi) = \sum_{i=1}^{n-1} \alpha^i n_1(\alpha\varphi - \varphi)$.

Next, think of $n \cdot n_1(\alpha\varphi - \varphi)$ as the sum of n equal terms. Then

2.7.2 $n_1(\alpha^n\varphi - \varphi) - nn_1(\alpha\varphi - \varphi) = \sum_{i=1}^{n-1}(\alpha^i n_1(\alpha\varphi - \varphi) - n_1(\alpha\varphi - \varphi))$.

By Corollary 2.6.6, given ε we can find a W such that if $\alpha^i \in W$ for $i < n$, then

$$\|\alpha^i n_1(\alpha\varphi - \varphi) - n_1(\alpha\varphi - \varphi)\| \leq \varepsilon.$$

Hence, by 2.7.2

$$\|n_1(\alpha^n\varphi - \varphi) - n \cdot n_1(\alpha\varphi - \varphi)\| \leq n \cdot \varepsilon$$

and dividing by n we get the required result.

2.8 Corollary. *If $\alpha \in Q$ and $\alpha^i \in \mu$ for $i \leq n_1 \in O(L(Q, V))$ then*

$$\|n_1(\alpha\varphi - \varphi)\| \sim 0.$$

Proof. For *every* standard ε and for $n = n_1$ the condition of 2.7 holds, and we get

$$\|1(\alpha^{n_1}\varphi - \varphi) - n_1(\alpha\varphi - \varphi)\| < \varepsilon;$$

therefore

$$\|(\alpha^{n_1}\varphi - \varphi) - n_1(\alpha\varphi - \varphi)\| \sim 0.$$

But φ is S-continuous and $\|\alpha^{n_1}\varphi - \varphi\| \sim 0$ and hence also

$$\|n_1(\alpha\varphi - \varphi)\| \sim 0.$$

3. Gleason's lemmas—monadic form

In this section we present the nonstandard equivalents of Gleason's lemmas. These results hold in all regular groups (see 1.7) and not just in NSS groups.

3.1 Lemma. *Let G be a regular group. For some $k \in {}^*N$ let $\alpha_1, \dots, \alpha_k$ be given, not necessarily different from each other. If $\alpha_i^n \in \mu$ for every $i, n \leq k$ then $\prod_{i \leq k} \alpha_i \in \mu$.*

Proof. Assume that $\beta = \prod^k \alpha_i \notin \mu$ and let $Q = \{\alpha_1, \dots, \alpha_k, \alpha_1^{-1}, \dots, \alpha_k^{-1}\}$. As G has no singular sets we may assume that k is regular for Q, replacing β by another product from Q if necessary.

Now choose V with $\beta \notin V^4$ and let $n_0 = L(Q, V)$. Note that since k is regular for Q we have

3.1.1 $k \in O(L(Q, V))$.

We construct Gleason's function φ for Q and V, and since $\beta \notin V^4$ we have

3.1.2 $\varphi(\beta) = 0$.

Also, $\varphi(e)$ is not infinitesimal (2.5.4), so that

3.1.3 $\|\beta\varphi - \varphi\|$ is not infinitesimal.

On the other hand,

3.1.4 $\|\beta\varphi - \varphi\| \leq \sum^k \|\alpha_i\varphi - \varphi\| \leq k\|\alpha_{i_0}\varphi - \varphi\| \sim 0$, where the first inequality is 2.3.5, the second is obtained by choosing the maximal summand, and the third is 2.8. This contradicts 3.1.3.

3.2 Lemma. *If $\overline{V}$ is compact and for all $n \leq n_0$, $\alpha^n \in V$, and if for all $n \leq n_0$, $\gamma^n \in \mu$ then $(\alpha\gamma)^n \sim \alpha^n$ for all $n \leq n_0$.*

Proof. $(\alpha\gamma)^n = (\prod_{i=1}^n \gamma_i)\alpha^n$ where $\gamma_i = \alpha^i\gamma\alpha^{-i}$. For all $i \leq n$, α^i is near standard and $\gamma_i \in \mu$. Moreover $\gamma_i^j = \alpha^i\gamma^j\alpha^{-i} \in \mu$. We now apply Lemma 3.1 to $\gamma_1 \cdots \gamma_n$ and get $\prod \gamma_i \in \mu$.

3.3 Lemma. *Let G be a regular group and $\langle \alpha, n_1 \rangle$ regular. If $\alpha^n \sim \alpha_1^n$ for all $n \leq n_1$, then $(\alpha^{-1}\alpha)^n \in \mu$ for all $n \leq n_1$.*

Proof. From the assumption we get that $\langle \alpha_1, n_1 \rangle$ is also regular. If we define $Q = \{\alpha, \alpha^{-1}, \alpha_1, \alpha_1^{-1}\}$, then by Lemma 3.1 $\langle Q, n_1 \rangle$ is also regular.

3.3.1 Assume now that $(\alpha^{-1}\alpha_1)^{n_0} \notin \mu$, for some $n_0 \leq n_1$. Then also $n_0 \equiv n_1$ (else $n_0 \in o(n_1)$ and $(\alpha^{-1}\alpha_1)^{n_0} \in \mu$).

Choose some V such that V^4 is with compact closure and $(\alpha^{-1}\alpha_1)^{n_0} \notin V^4$. Let $n_2 = L(Q, V)$; then also $n_2 \equiv n_0$ and for some finite k, $k \cdot n_2 > n_0$. We shall see that this leads to contradiction, so that 3.3.1 is impossible.

We construct Gleason's φ for Q and V, with $n_2 = L(Q, V)$. Then since $a = (\alpha^{-1}\alpha_1)^{n_0} \notin V^4$ we have $\varphi(a) = 0$ and $\varphi(e) = \varepsilon$ not near 0 (2.5.4, 2.5.5), so that

3.3.2 $\varepsilon \leq \|(\alpha^{-1}\alpha_1)^{n_0}\varphi - \varphi\| \leq \|n_0(\alpha\varphi - \alpha_1\varphi)\| = \|n_0(\alpha\varphi - \varphi) - n_0(\alpha_1\varphi - \varphi)\|$ (by invariance of norm and trivial manipulation).

We invoke Lemma 2.7 to show that the rightmost term is less than ε. There is some W such that if $\alpha^i \in W$ and $\alpha_1^i \in W$ for all $i \le n$ then

$$\|(n_0/n)(\alpha^n\varphi - \varphi) - n_0(\alpha\varphi - \varphi)\| < \varepsilon/3; \tag{3.3.3}$$
$$\|(n_0/n)(\alpha_1^n\varphi - \varphi) - n_0(\alpha_1\varphi - \varphi)\| < \varepsilon/3.$$

We have $L(\alpha, W) \equiv n_0$ by regularity of n_1 and therefore n_0, so that n can be chosen satisfying 3.3.3 and n_0/n finite. Therefore

3.3.4 $\|(n_0/n)(\alpha^n\varphi - \varphi) - (n_0/n)(\alpha_1^n\varphi - \varphi)\| = (n_0/n)\|\alpha^n\varphi - \alpha_1^n\varphi\| \sim 0$ as $\alpha_1^n \sim \alpha^n$ and φ is S-continuous.

Putting 3.3.3 and 3.3.4 together, we get

$$\|n_0(\alpha\varphi - \varphi) - n_0(\alpha_1\varphi - \varphi)\| < \varepsilon,$$

contradicting 3.3.2.

Gleason's lemmas yield the following structure theorem for the group of infinitesimals:

3.4 Theorem. *Let G be a regular group and $\alpha \in \mu$ nondegenerate. Then*

3.4.1 *$O(\alpha)$ and $o(\alpha)$ are (external) groups (see 1.2);*

3.4.2 *$O(\alpha)$ and $o(\alpha)$ are normal in μ;*

3.4.3 *for $\beta \in \mu$, $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in o(\alpha)$;*

3.4.4 *$O(\alpha)/o(\alpha)$ is commutative.*

Proof.

3.4.1 If $\gamma^i \in \mu$ and $\beta^i \in \mu$ for all $i \le n$, then the same is true for all $i \le 2n$. It now follows from 3.1 that if $\gamma, \beta \in O(\alpha)$, then $\alpha \cdot \beta \in O(\alpha)$. (Similarly for $o(a)$.)

3.4.2 If $\gamma \in O(\alpha)$ and $\beta \in \mu$ then for all $n \in {}^*N$, $\gamma^n \sim \beta\gamma^n\beta^{-1} = (\beta\gamma\beta^{-1})^n$. Therefore if $\gamma^n \in \mu$ also $(\beta\gamma\beta^{-1})^n \in \mu$, and if $\gamma \in O(\alpha)$ also $\beta\gamma\beta^{-1} \in O(\alpha)$. (Similarly for $o(\alpha)$.)

3.4.3 α is not degenerate and by regularity of G, $\alpha^{n_0} \notin \mu$ with some n_0 regular for α.

As in 3.4.2, $\alpha^n \sim (\beta\alpha\beta^{-1})^n$ for all $n \le n_0$ and by 3.3 $(\alpha(\beta\alpha\beta^{-1})^{-1})^n \in \mu$ for all $n \le n_0$. This shows that $\alpha\beta\alpha^{-1}\beta^{-1} \in o(\alpha)$.

3.4.4 By 3.4.3 the commutator group $(O(\alpha))' \subseteq o(a)$.

The following strengthening of 3.2 and 3.3 is very useful:

3.5 Lemma. *Let G be regular. If $\alpha^i \in \mu$ and $\beta^i \in \mu$ for all $i \le n$, then $(\alpha\beta)^n = \alpha^n\beta^n \cdot \delta$, with $\delta \in o(\beta^n)$.*

Proof. Note that if γ is a product of at most $2n$ elements chosen from $\{\alpha, \beta\}$ then γ is infinitesimal by 3.1 and $[\beta, \alpha] \in o(\beta)$ by 3.4.3. We shall define by internal induction a sequence $\delta_1, \ldots, \delta_n$ such that $(\alpha\beta)^n = \alpha^n\beta^n\delta_n \cdots \delta_1$, with $\delta_i \in o(\beta)$ for $i = 1, \ldots, n$. For $i = 1$, $\alpha\beta(\alpha\beta)^{n-1} = \alpha(\alpha\beta)^{n-1}\beta\delta_1$. Assume that $(\alpha\beta)^n = \alpha^i(\alpha\beta)^{n-i}\beta^i(\delta_i \cdots \delta_1)$; then this equals

$$\alpha^{i+1}\beta(\alpha\beta)^{n-i-1}\beta^i(\delta_i \cdots \delta_1)$$

and therefore $\alpha^{i+1}(\alpha\beta)^{n-i-1}\beta^{i+1}\delta_{i+1}(\delta_i\cdots\delta_1)$.

It remains to show that if $\delta = \delta_n\cdots\delta_1$ then $\delta \in o(\beta^n)$. Let k be regular for β^n. We note that kn is regular for β. Indeed $\beta^{kn} \notin \mu$, and if $s \in o(kn)$ then $s = ln + n_1$ with $n_1 < n$ and $\beta^s = \beta^{ln}\cdot\beta^{n_1} \sim \beta^{ln}$. $s, n_1 \in o(kn)$ so that $ln \in o(kn)$ and $l \in o(k)$. Hence $(\beta^n)^l \in \mu$. This proves that $\beta^s \in \mu$. Thus kn is regular for β. Now if $\delta \notin o(\beta^n)$ then $\delta^k \notin \mu$ and by 3.1 $\delta_i^{kn} \notin \mu$ for some $i \leq n$. But $\delta_i \in o(\beta)$ and kn is regular for β, a contradiction.

4. NSS groups and Lie groups

In this section we assume always that G is NSS. 4.1–4.5 prove that the space M of OPS is a finite-dimensional vector space. 4.6–4.8 identify a kernel C for which G/C is a matrix group, and 4.12 shows that C is locally E_n. Along the way we prove in 4.9, 4.10 that there is a neighborhood of G ruled by OPS.

4.1 Theorem. *Let G be regular and $\langle\alpha, n_0\rangle$ regular. Then the map*

$$S: M \to O(\alpha)/o(\alpha)$$

defined by $S(X) = [X(1/n_0)]$ is well defined, one to one, and onto. (*Note that M is the standard spaces of OPS.*)

Proof.

4.1.1 $\alpha_X = X(1/n_0)$ is indeed in $O(\alpha)$: for $n \in o(n_0)$, $(\alpha_X)^n = X(n/n_0) \sim e$ by continuity of X.

4.1.2 S is one to one: If $\alpha_0 = X(1/n_0)$ and $\beta_0 = Y(1/n_0)$ are in the same class then $\alpha_0^n \sim \beta_0^n$ for all $n \leq n_0$ by 3.2. But then the standard maps X and Y almost agree on each n/n_0. It follows that $X(t) = Y(t)$ for all $0 \leq t \leq 1$.

4.1.3 S is onto: If $\beta \in O(\alpha) - o(\alpha)$ then n_0 is regular and $\langle\beta, n_0\rangle$ define an OPS X (1.12). Now if $\beta_1 = X(1/n_0)$ then $\beta_1^n \sim \beta^n$ for $n \leq n_0$ and by 3.3 $\beta_1^{-1}\beta \in o(\beta)$.

4.2. Note that the map S depends on n_0.

From the proof we have

$$S: X \to [X(1/n_0)], \qquad S^{-1}: [\beta] \to X_{\langle\beta, n_0\rangle}.$$

We now use the correspondence S to define an operation of addition on M by:

4.2.1 $X + Y = S^{-1}(S(X)\cdot S(Y))$.

Recalling from 1.12 how $X_{\langle\beta', n_0\rangle}$ was constructed, we have for $n/n_0 \sim t$

4.2.2 $(X+Y)(t) \sim [X(1/n_0)\cdot Y(1/n_0)]^n$.

Of course $O(\alpha)/o(\alpha)$ is isomorphic with the commutative group M just defined.

4.3. Although the map S depends on the choice of $n_0 \in {}^*N$, the definition of the sum of OPS does not.

Lemma. *If n_0 and n_1 in $^*N - N$ are used to define $+_{n_0}$ and $+_{n_1}$ as in 4.2.1 and 4.2.2, then $X +_{n_0} Y = X +_{n_1} Y$.*

Proof. Choose some n so that $n_0 \in o(n)$ and $n_1 \in o(n)$. We shall prove the lemma for n_0 and n (and similarly for n_1 and n).

4.3.1 Choose k such that $k/n \leq 1/n_0 < (k+1)/n$ and put

$$\alpha = X(1/n), \qquad \beta = X(1/n_0),$$
$$\alpha_1 = Y(1/n), \qquad \beta_1 = Y(1/n_0).$$

Then if $m/n_0 \sim t$ we have

4.3.2 $(X +_{n_0} Y)(t) \sim (\beta \cdot \beta_1)^m \sim (\alpha^k \alpha_1^k)^m$ because $[\beta] = [\alpha^k]$ and $[\beta_1] = [\alpha_1^k]$ in $O(\beta)/o(\beta)$, by 4.3.1. Hence $[\beta\beta_1] = [\alpha^k \alpha_1^k]$.

We can now continue the estimates by 3.5.

4.3.2 $(\alpha^k \cdot \alpha_1^k)^m \sim (\alpha\alpha_1)^{k \cdot m} \sim (X +_n Y)(t)$ since $km/n \sim m/n_0 \sim t$.

Incidentally, we get immediately the standard corollary:

4.4 Corollary.

$$\lim_{n \to \infty} (X(1/n) \cdot Y(1/n))^n = (X+Y)(1).$$

Proof. Using the usual nonstandard characterization of convergence of sequences, we must show

$$(X(1/n_0) \cdot Y(1/n_0))^{n_0} \sim (X+Y)(1)$$

for all nonstandard $n_0 \in {}^*N$.

But this is just 4.2.2, in view of the freedom that Lemma 4.3 gives us in choosing n_0.

4.5 Theorem. *The space M of OPS with this addition and the natural scalar multiplication is a finite-dimensional vector space.*

Proof. Multiplication by scalars is defined by $(cX)(t) = c(X(t))$. It is easy to see that it is continuous, and it remains to show that addition is continuous and that the axioms of linear spaces hold. M is then finite-dimensional by [4] because it is locally compact, by 1.16.

We show that addition is continuous (at O). Given a neighborhood D_V of O in M; we replace it by D_W where $\overline{W} \subseteq V$. Let v be any infinitesimal neighborhood; then by 3.1 $^*G \models \forall\alpha\, \forall\beta\, \forall n[\forall i \leq n(\alpha^i \in v \wedge \beta^i \in v) \to (\alpha\beta)^n \in W]$. The same holds in G with $\exists v$ added to the beginning. Let U be such a v in G; then it is easy to check that $D_U + D_U \subseteq D_V$.

As for the linear axioms, the nontrivial one is $r(X+Y) = rX + rY$. Assume first that $r = k \in N$ and $Z = X + Y$. Put $X(1/n_0) = \alpha$, $Y(1/n_0) = \beta$, $Z(1/n_0) = \gamma$ and $(kX)(1/n_0) = \alpha^k$, $(kY)(1/n_0) = \beta^k$, $(kZ)(1/n_0) = \gamma^k$.

Since multiplication in $O(\alpha)/o(\alpha)$ is commutative we have

$$[\alpha^k][\beta^k] = [(\alpha\beta)^k] = [\gamma^k].$$

But $\alpha^k = X(k/n_0) = (kX)(1/n_0)$ and similarly for Y and Z. Therefore, the left-hand side corresponds to $kX + kY$ and the right side to kZ.

Next, if $X_1 = (1/k)X$, $Y_1 = (1/k)Y$, and $Z_1 = X_1 + Y_1$, then by what we have just seen $kZ_1 = X_1 + Y_1$, hence $Z_1 = (1/k)(X+Y)$, or $(1/k)(X+Y) = (1/k)X + (1/k)Y$.

Putting the two together, we get the statement for rational r and, by continuity of multiplication by scalar, also for real r.

4.6. Let G be NSS and M the finite-dimensional vector space of OPS. Let $\mathrm{GL}(M)$ be the group of invertible linear transformations of M.

To every $g \in G$ we relate an operation $T_g: M \to M$ by defining

$$(T_g X)(t) = gX(t)g^{-1}.$$

4.7 Theorem. *The map T which takes g to T_g is a continuous homomorphism of G into* $\mathrm{GL}(M)$.

Proof. Fix α and n_0 to identify M with $O(\alpha)/o(\alpha)$. It is easy to see that with this identification $T_g([\alpha]) = [g\alpha g^{-1}]$ so that

$$T_g([\alpha][\beta]) = [g\alpha\beta g^{-1}] = [g\alpha g^{-1} g\beta g^{-1}] = T_g([\alpha])T_g([\beta]).$$

Therefore $T_g(X+Y) = T_g(X) + T_g(Y)$.

Similarly, it is easy to see from the definition of $T_g(rX)$ that it equals $rT_g(X)$. Therefore T_g is a linear transformation.

Clearly $T_e = 1$ and $T_{gh} = T_g \circ T_h$. It follows that T_g is invertible for every g and therefore $T: G \to \mathrm{GL}(M)$, and T is a homomorphism.

Finally, T is continuous: If $\alpha \sim e$ then $[T_\alpha(X)](t) \sim X(t)$ for all t, hence $T_\alpha(X) \sim X$ for all X and therefore $T_\alpha \sim 1$. Thus, T is continuous at e and therefore everywhere.

4.8. Let C be the kernel of the map T in 4.7. Then T/C is isomorphic to a group of matrices (the isomorphism into $\mathrm{GL}(M)$ is continuous but not necessarily open). C is the closed subgroup

$$C = \{g \mid g \cdot X(t) = X(t) \cdot g, \quad \forall t \in R, \ \forall X \in M\}$$

and it remains to be seen that C is locally isomorphic to a finite-dimensional vector space. This could be done without the use of Gleason's lemmas [2, p. 570]. But since we promised 4.10 anyway, we might as well use it also for C.

4.9 Lemma. *Let X and Y be infinitesimal OPS. Then $X(1) \cdot Y(1) = (X+Y)(1) \cdot \delta$, with $\delta \in o(Y(1))$.*

Proof. X is infinitesimal means that $X([0, 1]) \subseteq \mu$. Let $\alpha = X(1)$, $\beta = Y(1)$, and $\gamma = (X+Y)(1)$. Choose V with no subgroups and let n_1 be arbitrary with $L(\beta, V) \in o(n_1)$. Then the following set is *open:

$$w = \{z \mid L(z, V) > n_1\},$$

and $w \subseteq o(\beta)$. By 4.4 there is some n_0 with $(X(1/n_0)Y(1/n_0))^{n_0}$ in the neighborhood $\gamma \cdot w$ of γ. Therefore $(X(1/n_0)Y(1/n_0))^{n_0} = \gamma\delta_1$ with

$\delta_1 \in o(\beta)$. By 3.5 $(X(1/n_0)Y(1/n_0))^{n_0} = X(1)\cdot Y(1)\cdot \delta_2$ with $\delta_2 \in o(Y(1))$. Thus $\gamma\delta_1 = \alpha\beta\delta_2$ and $\alpha\beta = \gamma\delta_1\delta_2^{-1}$.

We recall that $D_{\overline{V}}$ is the set of OPS such that $X([0, 1]) \subseteq \overline{V}$. It is a compact neighborhood of O in M if $\overline{V}$ is compact with no subgroups (1.16). The exponential map $E(X) = X(1)$ is continuous.

4.10 Theorem. *Let $\overline{V}$ be compact and symmetric with no subgroups. Then $E(D_{\overline{V}})$ contains a neighborhood of e in G.*

Proof. $K = E(D_{\overline{V}})$ is compact. We show that ${}^*K \supseteq \mu$. Let $\alpha \in \mu$ and assume for contradiction that $\alpha \notin {}^*K$. We look in K for some α_1 which is as close to α as possible, i.e., so that $L(\alpha_1^{-1}\alpha, V)$ is maximal.

Assume that there is such an α_1. Then $\alpha = \alpha_1\beta$ with $L(\beta, V) = n_0$ maximal. Clearly, n_0 is infinite and β is infinitesimal or else we would have had to choose e instead of α_1. By 1.12 and 3.3 there is some $\beta_s \in K$ such that $\beta = \beta_s\gamma$ with $\gamma \in o(\beta)$. Hence $\alpha = \alpha_1\beta_s\gamma$. By 4.9, $\alpha_1\beta_s = \alpha_2\delta$ with $\alpha_2 \in K$ and $\delta \in o(\beta)$. Therefore $\alpha = \alpha_2(\delta\gamma)$ with $(\delta\gamma) \in o(\beta)$. This contradicts the choice of α.

Therefore, there is no maximal n_0 and for every $n \in {}^*N$ there is some α_n such that $L(\alpha_n^{-1}\alpha, V) > n$. This yields an internal sequence α_n that has some *limit point α_0 in K. It is easy to see that $L(\alpha_0^{-1}\alpha, V) = \infty$ and by 1.13 $\alpha_0^{-1}\alpha = e$. Therefore $\alpha \in K$.

We shall also need the following:

4.11 Lemma. (a) *There is a neighborhood $W \subseteq V$ such that every $a \in W$ has at most one root in V.*

(b) *For this W the map $X \to X(1)$ from D_W to W is one to one.*

Proof. (a) Choose $w \subseteq \mu$ an infinitesimal neighborhood and assume that $\alpha^2 = \beta^2 \in w$, where $\alpha, \beta \in V$. Then $({}^0(\alpha))^2 = e$ so that $\alpha, \beta \in \mu$ as V has no subgroup.

Since $\alpha^2 = \beta^2$ we have

$$\gamma = \alpha\beta^{-1} = \alpha^{-1}\beta$$

and for every $n \in {}^*N$

$$\gamma^n = (\alpha\beta^{-1})^n \sim \beta^{-1}(\alpha\beta^{-1})^n\alpha = (\beta^{-1}\alpha)^{n+1} \sim (\beta^{-1}\alpha)^n = \gamma^{-n}.$$

And if we put $n_0 = L(\gamma, V)$ then $e \neq {}^0(\gamma^{n_0}) = c \in \overline{V}$ and $c^2 = e$, and we obtained a subgroup in $\overline{V}$.

(b) $X(1) = Y(1)$ implies $X(1/2^n) = Y(1/2^n)$ for all $n \in N$ by uniqueness of roots, $X(k/2^n) = Y(k/2^n)$ because X and Y are homomorphisms, and $X(t) = Y(t)$ because they are continuous.

4.12 Theorem. *If G is NSS then G contains a normal subgroup C that is locally isomorphic to some E_n and such that G/C is continuously embedded in some GL_m.*

Proof. It remains to show that the kernel C of the adjoint representation is locally isomorphic to some E_n (see 4.8). C is closed in G and hence locally compact NSS. By 4.10 there is a neighborhood W in C covered by OPS. By the definition of C (see 4.8, again) W is in the center of C. Therefore C has an open center and replacing C by it we assume that C is commutative.

But for commutative groups the sum $(X+Y)(t)$ is just $X(t)\cdot Y(t)$ and the exponential map $E\colon M_C \to C$ is a homomorphism. We show that E is a local isomorphism, i.e., that E is one to one from μ to μ (this natural characterization of local isomorphism is proved in [6]).

$E(\mu(0)) \subseteq \mu(e)$ as E is continuous, E is one to one on some D_W, by 4.11, and in particular on $\mu(0)$.

$E(D_V) \supseteq \mu$ by 4.10. By 1.15.4, if $\alpha \in \mu$ and $X(1)=\alpha$ for some $X \in D_V$, then $X \in \mu(0)$. Therefore $E(\mu) \supseteq \mu$. □

4.13 *Remark.* It is also easy to see that the map $E(X)$ is a coordinate system by 4.5, 4.10, and 4.11(b). However, we do not have a direct way of proving that multiplication is analytic in this system.

5. Further corollaries of Gleason's lemmas

Three more facts will be needed in order to reduce the general case to the NSS case. The setting is still a regular group so that Gleason's lemmas apply.

5.1 Theorem. *Let G be regular. Given a neighborhood V, there are some neighborhood W and a compact group H such that $H \subseteq V$ and H contains all the groups that are in W.*

Proof. Let V be given. Choose w an infinitesimal neighborhood and let L be the *group generated by all the subgroups that are in w. By 3.1 L is an infinitesimal group. Clearly $H=\overline{L}$ is also in V.

5.2 Corollary. *If G is regular and has no OPS, then G is totally disconnected. That is, the topology has a base of open groups.*

Proof. Given V, choose (standard) W and H as in 5.1. By 1.12 μ has no regular elements and, by assumption, no singular elements. Therefore, every $\alpha \in \mu$ *generates an infinitesimal subgroup and is therefore in $^*H$. Thus $\mu \subseteq {}^*H$ and H is open.

5.3 Lemma. *Let G be NSCS, $f\colon G \to H$ an open homomorphism, and $\beta \in \mu(H)$ with $\langle \beta, n_0\rangle$ regular.*

Then β "can be lifted"—there is some $\alpha \in \mu(G)$ such that $\langle \alpha, n_0\rangle$ is regular and $f(\alpha)=\beta$.

Proof. Let $W \subseteq H$ be open with $\beta^{n_0} \notin W$. Let $V \subseteq G$ be such that $f(\overline{V}) \subseteq W$ and $\overline{V}$ is compact and contains no connected subgroups. Let A be $f^{-1}(\beta)\cap\overline{V}$. A is not empty because f is open and $f(V) \supseteq \mu_H$. Also, if $\alpha \in A$ then $f(\alpha^{n_0}) = \beta^{n_0} \notin W$; therefore $\alpha^{n_0} \notin V$ and we conclude that $L(\alpha, V)$ is bounded on A by n_0. Now choose in A an α_0 with $L(\alpha, V)$ maximal on A.

We may assume that $\alpha_0 \in \mu(G)$ or else replace it by $\alpha_0 a^{-1}$ where $a = {}^0(\alpha_0)$, as $f(a) = e$.

$L(\alpha, V) \le n_0$ and $L(\alpha, V)$ is regular for α by 1.10.1 and it remains to show that it is impossible that $L(\alpha, V) \in o(n_0)$. Indeed, if so then

5.3.1 $f(\alpha^n) = \beta^n \in \mu(H)$ for $n \le L(\alpha, V)$ as n_0 is regular for β.

Now let X be the OPS constructed from α and $L(\alpha, V)$ in 1.12; then $f(X([0, 1])) = e$ by 5.3.1. But then $X \subseteq \operatorname{Ker} f$ and if $\alpha_S = X(1/L(\alpha, V))$ then $f(\alpha_S) = e$, $f(\alpha_S^{-1}\alpha_0) = \beta$ and $\alpha_S^{-1}\alpha_0 \in o(\alpha_0)$ by 3.3. But then $L(\alpha_S^{-1}\alpha_0, V) > L(\alpha_0, V)$, contradicting the choice of α_0.

5.4 Corollary. *If G is NSCS and $f\colon G \to H$ is an open homomorphism, then every OPS Y in H can be lifted to G. That is, there is some OPS $X \subseteq G$ such that $f \circ X = Y$.*

Proof. If Y is not trivial then $Y(t) \ne e$ for some t and we assume that $Y(1) \ne e$. Let $\beta = Y(1/n_0)$. n_0 is regular for β as Y is continuous. By 5.3, n_0 is regular for some $\alpha \in \mu(G)$ with $f(\alpha) = \beta$.

It is now immediate from the construction of X in 1.12 with α and n_0 that $f \circ X = Y$.

6. Locally connected NSCS groups

Theorem. *Let G be a locally connected group, with a neighborhood V_0 that does not contain a nontrivial connected subgroup. Then:*

6.1. *V_0 contains a totally disconnected compact subgroup H that contains all the degenerate elements in μ and none of the regular elements.*

6.2. *G has an open subgroup G_1 in which H is normal.*

6.3. *G_1/H is NSS.*

6.4. *The natural map $T\colon G_1 \to G_1/H$ is a local isomorphism. Hence, every locally connected NSCS group is NSS.*

Proof.

6.1. By 1.10.1 V_0 is regular for all the elements in μ that are not degenerate. We may assume that $\overline{V}_0$ is compact and choose H_1 and W as in Theorem 5.1. Then ${}^*H_1$ contains all the degenerate elements since they generate a subgroup in ${}^*W$, and ${}^*H_1$ contains no regular element because they have a power outside V_0. Also, since H_1 contains no regular elements it is totally disconnected by 5.2, and we replace H_1 by an even better H—an open (in H_1) subgroup H contained in the open set (in H_1) $W \cap H_1$.

6.2. H is open in H_1 so that $H = H_1 \cap W_1$ with $W_1 \subseteq W$ open in G. H is compact so that the following is open:

$$U = \{a \in W_1 \mid a^{-1}Ha \subseteq W_1\}.$$

However, if $a^{-1}Ha \subseteq W_1 \subseteq W$, then by choice of H_1 and W, $a^{-1}Ha \subseteq H_1$ so that $a^{-1}Ha \subseteq H_1 \cap W_1 = H$. This shows that the normalizer G_1 of H includes the open set U and is therefore open.

6.3. For $\alpha \in \mu_{G_1}$ if α is degenerate then $\alpha \in {}^*H$ and $T\alpha = e$. If α is regular then for some n_0, $\alpha^{n_0} \notin {}^*V_0$. Let W be such that $W \cdot H \subseteq V$; then $T(\alpha^{n_0}) \notin T(W)$ so that $(T\alpha)^{n_0} \notin \mu$. But $\alpha^n \in \mu(G)$ for $n \in o(n_0)$ so that $(T\alpha)^n \in \mu$ for $n \in o(n_0)$. Therefore $T\alpha$ is regular. Hence G_1/H has only regular infinitesimals.

6.4. G_1/H is NSS. We choose $\overline{V}$ in G_1/H compact with no subgroup and uniqueness of root (4.11). We choose W in G_1 connected such that $T(W)$ is covered by OPS of D_V (4.10).

For every $x \in W$, $T(x)$ lies on some OPS Y in G_1/H and Y can be raised to X on G_1 by (5.4). Therefore for every $x \in W$ there is some $h \in H$ such that $x_1 = xh$ lies on some OPS in G_1.

We shall see for every $x \in W$ there is such a unique h_x and that the function $x \to h_x$ is continuous from W into H.

6.4.1 Uniqueness: Assume that $T(X(1)) = T(Y(1)) = Tx$. By uniqueness of root $T(X(1/2^n)) = T(Y(1/2^n))$ and there is an infinite n_0 such that

$$T(X(1/n_0)) - T(Y(1/n_0))$$

and therefore $X(1/n_0) \cdot Y(-1/n_0) = h \in {}^*H$ and $h \in \mu$ by continuity of X and Y. Therefore h is degenerate by 6.1. But then $h^n \in \mu$ for all $n \in {}^*N$ and $[X(1/n_0)Y(-1/n_0)]^n \sim e$ for all $n \in {}^*N$. this proves that the standard OPS $X - Y$ is O (4.2.2 and 4.3).

6.4.2 Continuity: Assume that $x \sim a$ in ${}^*V$. We shall see that $h_x \sim h_a$. By compactness of H, $h_x \sim h$ for some standard h, and by definition $Y(1) = x \cdot h_x$ for some $Y \in {}^*D_{\overline{V}}$. ${}^*D_{\overline{V}}$ is compact so that $Y \sim X$ for some standard OPS $X \in D_{\overline{V}}$. By continuity of exponentiation $Y(1) \sim X(1)$ hence $X(1) = {}^0(xh_x) = a \cdot h$. Therefore $h = h_a$.

6.4.3 Now W is connected and $h_e = e$. Therefore the image of W is the component of e which is $\{e\}$. This shows that $h_x = e$ and $x \cdot e$ is on some OPS for every $x \in W$.

But we have just seen in the proof of 6.4.1 that for points on OPS T is one to one, so that T is one to one from W into G_1/H. Therefore it is a local isomorphism, as it is always open and on a neighborhood in G_1/H.

7. Locally Euclidean groups

Locally Euclidean groups are locally compact and locally connected and we shall complete the paper by showing that they are also NSCS and therefore NSS by 6. The proof is technical and in two stages: 7.4 shows that they are regular and therefore consequences of Gleason's lemmas can be used to show that they are NSCS. 7.5 then completes the proof.

7.1. A locally Euclidean group has a neighborhood of e that is connected, has a compact closure, and does not contain homeomorphic images of cubes with arbitrary high dimension.

The proof that G is regular will depend on some OPS construction. It is the weakest point of this paper that we were unable to come up with a direct proof of the following:

7.2 Lemma. *Every compact connected nontrivial topological group contains nontrivial OPS.*

Proof. [5, p. 105].

7.3 Lemma.

7.3.1 *Let H be a closed subgroup of G and $X: R \to G$ an OPS. Then either $X(R) \subseteq H$ or there is some neighborhood D of O in R such that $X(D) \cap H = \{e\}$.*

7.3.2 *Every nontrivial OPS X has a neighborhood D of O in R on which X is one to one.*

Proof.

7.3.1 If no such D exists we can get a sequence $t_n \to O$ with $X(t_n) \in H$. But then $A = \{m \cdot t_n | m, n \in N\}$ is dense in R^+ and $X(A) \subseteq H$, and $X(R^+) \subseteq \overline{H} = H$.

7.3.2 This is 7.3.1 with $H = \{e\}$.

7.4 Theorem. *If G is locally Euclidean then G is regular.*

Proof. Assume that μ contains a singular set Q. By 1.10.2 we get a collection of groups

$$\{\overline{G(Q, V)} | \overline{V} \text{ is compact}\}.$$

We shall prove by induction on n that each of these groups contains a cube of dimension n, contradicting 7.1. For $n = 0$ there is nothing to prove. Assume now that the statement holds for some n. By 7.2 $\overline{G(Q, V)}$ contains a nontrivial OPS $X(t)$. Since X is not trivial $X(R) \nsubseteq \overline{W}$ for some W and in particular $X(R) \nsubseteq G(Q, W)$. By 7.3.1 there is some $D \subseteq R$ around O with

7.4.1 $X(D^2) \cap \overline{G(Q, W)} = \{e\}$.

By induction assumption there is some cube $Y: I^n \to \overline{G(Q, W)}$. We define $Z: D \times I^n \to \overline{G(Q, V)}$ by $Z(t, s) = X(t) \cdot Y(s)$. Then Z is one to one because if $X(t)Y(s) = X(t_1)Y(s_1)$ then $X(t - t_1) = Y(s_1)(Y(s))^{-1}$. Now $t - t_1 \in D^2$ and $Y(s_1)(Y(s))^{-1} \in \overline{G(Q, W)}$ and by 7.4.1 $t = t_1$ and by induction assumption $s = s_1$.

It is easy now to normalize Z to obtain $Z_1: I^{n+1} \to \overline{G(Q, V)}$.

Thus Gleason's lemmas and their corollaries apply to locally Euclidean groups.

7.5 Theorem. *If G is locally Euclidean then G is NSCS.*

Proof. Assume that every W in G contains a connected (compact) group and therefore an OPS. For any $W = W_0$ construct by induction $\langle X_n, V_n, W_n \rangle$ such that X_n is an OPS in W_{n-1} and V_n does not contain X_n but contains all the subgroups in $\overline{W}_n$ (by 5.1). Therefore for arbitrary n and $k \leq n$, X_k is not in the group generated by $X_{k+1}, \ldots, X_n$ $(k = 1, \ldots, n-1)$.

This shows, again by 7.3.1, that the function $C(t_1, \ldots, t_n) = X_1(t_1) \cdot X_2(t_2) \cdots X_n(t_n)$ is locally one to one (starting from n and working back). Therefore W contains cubes of arbitrarily high dimension. Q.E.D.

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