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BOUNDARY BEHAVIOR OF THE FAST DIFFUSION EQUATION

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ABSTRACT. The fast diffusion equation $\Delta v^m = v_t$, $0 < m < 1$, is a degenerate nonlinear parabolic equation of which the existence of a unique continuous weak solution has been established. In this paper we are going to obtain a Lipschitz growth rate of the solution at the boundary of Ω and estimate that in terms of the various data.

I. INTRODUCTION

Let Ω be a smooth domain in $\mathbf{R}^N$ which is either bounded or unbounded. The nonlinear diffusion equation

$$\Delta v(x, t)^m = v_t(x, t), \qquad x \in \Omega, \; t \geq 0, \; m > 0, \tag{1}$$

has been studied by many authors; for a detailed survey concerning this equation, one can refer to [3]. The case $m > 1$ is known as the porous media equation. The regularity of the porous media equation has been well studied by a lot of authors. In this paper, we are interested in the case $0 < m < 1$ in which case (1) is known as the fast diffusion equation and Ω is bounded. More precisely, we are interested in the problem

$$\begin{cases} \Delta v^m(x, t) = v_t(x, t), \qquad 0 < m < 1, \; x \in \Omega, \; t \geq 0, \\ v|_{\partial\Omega} = 0, \\ v(x, 0) = v_0(x), \\ v_0 \geq 0, \quad v_0 \not\equiv 0, \quad v_0 \in L^\infty(\Omega). \end{cases} \tag{2}$$

Alternatively, if we let $m = 1/(q-1)$ for $q > 2$, we have

$$\begin{cases} (u^{q-1})_t - \Delta u = 0, \\ u|_{\partial\Omega} = 0, \\ u(x, 0) = u_0(x) = v_0(x)^{1/(q-1)} \in L^\infty(\Omega). \end{cases} \tag{3}$$

The main difference between the fast diffusion case and the slow diffusion case (as the porous media equation is often called) is that, for the former one, there exists a finite extinction time T^* such that the solution of (3) satisfies $u(x, t) \equiv 0$ for $t \geq T^*$. The existence of T^* has been studied by D. Diaz [6], E. Sabanina [17], Crandall and Benilan [1] and Herro and Vazquez [9].

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As far as regularity is concerned, Paul Sacks [18] and E. Di Benedetto [7] proved the existence of a bounded continuous weak solution. Of course, as the uniqueness of (2.3) has already been established, one may simply say that the weak solution of (2) is bounded and continuous. Furthermore, in [7], Di Benedetto proved that just like the porous media case, there exists a modulus of continuity for the solution of the fast diffusion case on $\overline{\Omega} \times [\delta, T]$ for $0 < \delta < T$.

For $0 < t < T^*$, the regularity is actually better. E. Sabanina in [17] proved, for the $N = 1$ case, the existence of a positive classical solution of (2). Her result is improved by the author in [12]; in this paper it is proved that the weak solution of (3) is in fact a positive classical solution on $Q_{T^*} = \Omega \times (0, T^*)$ and is continuous up to the lateral surface $\partial\Omega \times (0, T^*)$ attaining zero boundary value there. In the same paper, the author has also derived a linear upper bound on the decay rate of $u(x, t)$ on $\partial\Omega \times (0, T^*)$ (cf. Theorem 2 of this paper).

The purpose of this paper is to investigate the boundary behavior of u. Indeed, using the fact that u is strictly positive in the interior $\Omega \times [\delta, T^* - \delta]$ where $0 < \delta < T^*$, the author has proved the existence of a linear lower bound on u at the boundary of Ω. More precisely, we have

$$u(x, t) \geq C\, d(x, \partial\Omega), \qquad (x, t) \in \Omega \times [\delta, T^* - \delta]$$

for some constant $C > 0$. Hence, together with the upper bound of Theorem 2, we have established the Lipschitz growth rate of u at the boundary of Ω. Furthermore, we will also obtain an explicit estimate on this lower bound which depends on the data N, Ω, q, δ and $\|u_0\|_{L^\infty(\Omega)}$ for the case $N \leq 2$ or $2 < q < 2N/(N-2)$ if $N > 2$. The latter is being done through a combination of Moser–Krylov technique [11, 14, 15]. We now summarize these in the main theorem of this paper (which is equivalent to Theorem 4 in §IV when Ω is convex).

Main Theorem. *Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary and $u(x, t)$ be the solution of (3). We have the lower bound*

$$u(x, t) \geq C\, d(x, \partial\Omega), \qquad (x, t) \in \Omega \times [\delta, T^* - \delta] \tag{4}$$

for some constant $C > 0$ and $C \to 0$ as $\delta \to 0$.

Furthermore, for $N \leq 2$ or $2 < q < 2N/(N-2)$ if $N > 2$, C can be estimated in terms of the data N, q, Ω, δ and $\|u_0\|_{L^\infty(\Omega)}$ (the dependence of C on the data is very explicit as can be seen in the proof of Theorem 4; please refer to §§III and IV for the details).

Finally, one should note that a Lipschitz growth rate as such can be very helpful in understanding other deeper aspects of the boundary behavior of u and is very useful in obtaining boundary estimates. Indeed, as an immediate consequence, one can use the Lipschitz rate to obtain a boundary modulus of continuity of the Hölder type of u (i.e., the modulus of continuity is up to the lateral surface $\partial\Omega \times (0, T^*)$). This approach however only works for $2 < q < 3$

when $N \leq 2$ and for $2 < q < 2N/(N-2) \wedge 3$ when $N > 2$ (cf. Remark 1). At this point, let us remark that regularity results of this kind may also be obtained in a more general setting through adaptation of the available techniques in the recent literature [8] (e.g., for the whole range $2 < q < \infty$ and for more general inhomogeneous boundary data). But through the Lipschitz growth rate, the Hölder estimate can be obtained in a rather straightforward manner by employing the scaling technique of Chiarenza–Cerapioni [4] together with the estimates in Theorems 2 and 4; and the estimate obtained is more precise for this particular range (cf. Remarks 6). Basically, one only needs to show that, because of the Lipschitz growth rate, u^{2-q} belongs to a certain A_p class at the boundary of Ω and from where we can invoke the results in [4] (note that u^{2-q} corresponds to the coefficients a_{ij} in [4] and for the definition of the A_p classes, please refer to the preliminaries).

This paper is divided into four sections and an appendix. §§I and II are devoted to the introduction and the preliminaries of this paper respectively. §III is devoted to the establishment of a interior lower bound for the solution. This interior lower bound is estimated in terms of the data and the distance from the boundary for the cases, $N \leq 2$ or $2 < q < 2N/(N-2)$ if $N > 2$. Finally in §IV, we establish the linear lower bound at the boundary through a Hopf-type argument and this, together with the results from [12] (Theorem 2 in the preliminaries), would give us a Lipschitz growth rate of the solution at the boundary of Ω. The appendix is for the proof of Proposition III, which is a technical lemma.

II. Preliminaries

1. The weak solution. Let

$$\overset{\circ}{V}(Q_T) = L^\infty(0, T|L^2(\Omega)) \cap L^2(0, T|\overset{\circ}{W}{}_2^1(\Omega))$$

with

$$\|u\|_{V^2(Q_T)} = \sqrt{\sup_{t\in|0,T|} \|u(\cdot, t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q_T)}}, \qquad u \in \overset{\circ}{V}_2(Q_T),$$

where $\overset{\circ}{W}{}^l_p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^l_p(\Omega)$.

$$W_2^{1.1}(Q_T) = \{u \in L^2(Q_T) | \nabla u, u_t \in L^2(Q_T)\}$$

with

$$\|u\|_{W_2^{1.1}(Q_T)} = \sqrt{\|u\|_{L^2(Q_T)} + \|u_t\|^2_{L^2(Q_T)} + \|\nabla u\|^2_{L^2(Q_T)}}$$

and $\overset{\circ}{W}{}_2^{1.1}(Q_T)$ the closure of the set of smooth functions on Q_T which vanish on $\partial\Omega \times [0, T]$.

We then have the following definition.

Definition. Suppose the initial data u_0 is nonnegative and bounded; let

$$E = \{\psi \in \overset{\circ}{W}{}_2^{1.1}(\Omega) | \psi(x, T) = 0 \text{ a.e. } x \in \Omega\}.$$

An element u of $\overset{\circ}{V}_2(Q_T)$ is a weak solution of (2) if $u^{q-1} \in L^2(Q_T)$ and u satisfies

$$\iint_{Q_T} u^{q-1}\psi_t \, dx\, dt - \iint_{Q_T} \nabla u \cdot \nabla \psi \, dx\, dt + \int_\Omega u_0(x)^{q-1}\psi(x, 0)\, dx = 0$$

for every $\psi \in E$. Moreover the weak solution is unique as shown in [13].

2. Interior and boundary regularity of the weak solution. In [12, 13, 2], it has been shown that if $2 < q$ for $N \leq 2$, or $2 < q < 2N/(N-2)$ for $N > 2$, then there is a finite extinction time T^* such that the following theorem holds.

Theorem 1. *Let $u(x, t)$ be the unique weak solution of (3) where $u_0 \in L^\infty(\Omega)$, $u_0 \not\equiv 0$ and $u_0 \geq 0$. Then $u(x, t)$ is a positive bounded classical solution of (3) in Q_{T^*} where T^* is the finite extinction time. More precisely, we have*

(i) $u \in C^{2.1}(Q_{T^*}) \cap L^\infty(Q_{T^*})$ *and* $u > 0$ *in* Q_{T^*}.

(ii) $(u^{q-1})_t - \Delta u = 0$.

Furthermore, we have the estimates

(iii)

$$(T^* - t)^{1/(q-2)} \leq C_1(q, N, \Omega)\left[\int_\Omega u^q(t)\, dx\right]^{1/q}$$

and hence $T^* \leq C_1^{q-2}\|u_0\|^{q-2}_{L^q(\Omega)}$.

(iv) $T^* \geq \int_\Omega u_0^q/(q-1)\int_\Omega |\nabla u_0|^2$ *provided* $u_0 \in \overset{\circ}{W}{}^1_2(\Omega) \cap L^\infty(\Omega)$.

Theorem 2. *Let $u(x, t)$ be the unique weak solution as above (which is in fact classical as proved in the last theorem) and let R_1 be the radius of the "exterior sphere condition" for $\partial\Omega$ (i.e., for every $x_0 \in \partial\Omega$, there exists a ball $B_{R_1}(x^*)$ of radius R_1, uniformly on x_0, and center x^* with $B_{R_1}(x^*) \cap \overline{\Omega} = \{x_0\}$). Then for $t > 0$, $(x_0, t) \in \partial\Omega \times (0, \infty)$ and $(x, t) \in \Omega \times (0, \infty)$, we have*

$$0 \leq u(x, t) \leq C_2(N, q, R_1, \Omega, \|u_0\|_{L^\infty(\Omega)})\frac{|x - x_0|}{t}.$$

(One may refer to [12] for an explicit expression of C_2.)

3. Regularizing effect on $u(x, t)$. According to Theorem 4 of [5], together with the fact that $u(x, t)$ is a classical solution, the following result is immediate.

Theorem 3. *Let $u(x, t)$ be the unique classical solution, as above; the following estimate holds:*

$$-t(u(t)^{q-1})_t \geq -\left(\frac{q-1}{q-2}\right) u(t)^{q-1}.$$

Alternatively,

$$-\Delta u(t) + \left(\frac{q-1}{q-2}\right)\frac{u(t)^{q-1}}{t} \geq 0$$

for $t > 0$.

4. Estimates from linear parabolic theory. The following is extracted from the paper of Krylov and Safonov [11]; since we are applying their results in a special case, we will only present their results in a more narrow setting to avoid unnecessary complications. Consider the parabolic operator,

$$L = \partial/\partial t - a(x, t)\Delta,$$

where $a(x, t)$ is bounded and continuous and satisfies

$$\tilde{\delta} \le a(x, t) \le 1/\tilde{\delta} \quad \text{for some } \tilde{\delta} \in (0, 1].$$

Furthermore, let

$$Q_1 = B_1(0) \times (0, 1), \qquad Q_{x_0, t_0}(R) = B_R(x_0) \times (t_0 - R^2, t_0)$$

and

$$\begin{aligned} &U^+(Q_1) = \{u, \text{ smooth and positive } |\exists L \text{ as above and } Lu \ge 0 \text{ on } Q_1\}, \\ &U_\beta = U^+(Q_1) \cap \{u| |Q_1 \cap \{u(x, t) \ge 1\}| \ge \beta |Q_1|\}, \qquad \beta \in (0, 1], \\ &\gamma(\beta) = \inf_{u \in U_\beta} \{u(x, t)| |x| \le \tfrac{1}{2}\}. \end{aligned}$$

Then we have the following results:

Proposition (I.1). *Assuming* U_β *is nonempty, we have*
(i) $\gamma(\beta) > 0$ *for* $\beta > 0$,
(ii) $\gamma(\beta) \uparrow$ *as* $\beta \uparrow$,
(iii) $\gamma(\beta) \uparrow$ *as* $\tilde{\delta} \uparrow$.
((iii) *is obvious as can be seen by the way in which* $\gamma(\beta)$ *is being defined*).
Furthermore, for $\varepsilon > 0$, $\beta \in (0, 1]$, $R \in (0, 1]$ *and* $u \in U^+(Q_{x_0, t_0}(R))$ *with* $Q_{x_0, t_0}(R) \subset Q_T$, *we have*
(iv) *If*

$$|Q_{x_0, t_0}(R) \cap \{(x, t)|u(x, t) \ge \varepsilon\}| \ge \beta |Q_{x_0, t_0}(R)|$$

then

$$u(x, t_0) \ge \varepsilon\gamma(\beta) \quad \text{for } |x - x_0| \le R/2.$$

From now on, we will denote $\gamma(\beta)$ by $\gamma(N, \tilde{\delta}, \beta)$ to indicate the dependence of γ on the data N, $\tilde{\delta}$, and β and we emphasize that γ is monotonic increasing with respect to both $\tilde{\delta}$ and β.

From Proposition (I.1) we can easily derive proposition (I.2) as follows (as was done in [18]).

Proposition (I.2). *Let* $z(x, t)$ *be smooth and such that*
(i) $z \le M$ *in* $Q_{x_0, t_0}(R)$,
(ii) $z_t - a(x, t)\Delta z \le 0$ *s.t.*

$$\tilde{\delta} \le a(x, t) \le 1/\tilde{\delta} \quad \text{for some } \tilde{\delta} \in (0, 1],$$

(iii) $|Q_{x_0, t_0}(R) \cap \{z \leq M/2\}| \geq \beta |Q_{x_0, t_0}(R)|$, $\beta \in (0, 1]$.

Then,

$$z(x, t) \leq M - \frac{M\gamma(N, \tilde{\delta}, \beta/2)}{2} + \frac{R^2}{\tilde{\delta}} \quad in\ Q_{x_0, t_0}(K_N \beta R),$$

where

$$K_N = \min\left\{\frac{1}{5}, \inf_{\theta \in (0, 1]} \left(\frac{1 - (1 - \theta/2)^{2/(N+2)}}{\theta^2}\right)\right\} > 0.$$

5. The A_p classes for $p > 1$. A nonnegative measurable and integrable function $\omega(x)$ is said to be in A_p if there exists a constant C_0, the A_p constant of $\omega(x)$, such that

$$C_0 = \sup_C \left[\frac{1}{|c|} \int_c \omega(x)\, dx \left(\frac{1}{|c|} \int_c (\omega(x))^{-1/(p-1)}\, dx\right)^{p-1}\right] < +\infty,$$

where the supremum is taken over all cubes c in $\mathbf{R}^N$.

Before we go on to the main part of the paper, let us first remark that all the constants m, δ, δ_1, K_N, D, d_1, R_1, R_2, C_0, C_1, C_2, C_3, C_4, and C_5 have the same (consistent) meaning thoughout the whole paper. Furthermore, all the constants R_1, R_2, R_3, C_0, C_1, $\ldots$, C_5 vary with finite bounds provided the data vary within finite bounds.

III. An interior lower bound of $u(x, t)$

In this paper, we are looking for boundary estimates of the form

$$C\, d(x, \partial\Omega) \leq u(x, t) \leq C'\, d(x, \partial\Omega) \quad \text{for some } C, C' > 0.$$

We will only do that for the case when Ω is convex; the general case then follows analogously with minor modifications (cf. Remark 4 at the end of §IV). The first step in this direction is to establish a positive lower bounded at the interior of $\Omega \times [\delta, T^* - \delta]$ where $T^* > \delta > 0$. This will be carried out in three steps in this section.

(i) Consider $\Omega_t = \{(x, t) | x \in \Omega\}$ where $t \in [\delta, T^* - \delta]$; according to (iii) of Theorem 1,

$$(T^* - t)^{1/(q-2)} \leq C_1(q, N, \Omega) \left[\int_\Omega u(t)^q\, dx\right]^{1/q}$$

for $N \leq 2$ and for $2 < q < 2N/(N-2)$ if $N > 2$. This implies that, for this range of q,

$$\max_{x \in \Omega} u(x, t) \geq M(t) = \frac{(T^* - t)^{1/(q-2)}}{C_1 |\Omega|^{1/q}}.$$

Furthermore, suppose that the maximum is being attained at some interior point (x_0, t); then, according to the boundary estimate from Theorem 2,

$$\begin{aligned} d(x_0, \partial\Omega) &> \frac{tM(t)}{C_2} \geq \left(\frac{\delta}{C_2}\right) M(t) \geq \left(\frac{\delta}{C_2}\right) M(t) \\ &\geq \left(\frac{\delta}{C_2}\right) \frac{\delta^{\frac{1}{q-2}}}{C_1 |\Omega|^{1/q}} \geq \frac{\delta^{\frac{q-1}{q-2}}}{C_1 C_2 |\Omega|^{1/q}} \end{aligned}$$

From now on, we will denote the quantity $\delta^{(q-1)/(q-2)}/C_1 C_2 |\Omega|^{1/q}$ by $d_1 = d_1(\delta, C_1, C_2, \Omega)$.

(ii) Next, we establish a lower bound on $u(x, t)$ in a neighborhood of (x_0, t) where the maximum is being attained. To do so, we simply adopt the same method as in [18] for the fast diffusion case with minor modifications in a process of iteration so as to get a more explicit estimate for this particular case. However, to avoid the tedious details at this moment, we will simply list the technical machineries required and state the result in Proposition III without giving a proof in this section (as a proof may be found in the appendix).

At this point, we need to introduce the following two propositions which were both proved in [18].

Proposition (II.1). *Assume that* $Q_{x_0, t_0}(R) \subset Q_T$ *and* $|u| \leq M$ *in* $Q_{x_0, t_0}(R)$ *where* $0 < M \leq \|u_0\|_{L^\infty(\Omega)} \vee \|u_0\|^{q-1}_{L^\infty(\Omega)}$; *then if*

$$\frac{1}{|Q_{x_0, t_0}(R)|} \iint_{Q_{x_0, t_0}(R)} (M - u)\, dx\, dt \leq \widehat{H}(M),$$

where

$$\widehat{H}(M) = \left[C(N, \|u_0\|_{L^\infty(\Omega)}) M^{N+4} \left(\frac{1}{4} \wedge \frac{M^{q-2}}{2^{q+2}}\right)^{(N+2)/2} \right] \wedge \left(\frac{M}{2}\right),$$

we have

$$u \geq \frac{M}{2} \quad \text{in } Q_{x_0, t_0}\left(\frac{R}{2}\right).$$

Here $a \vee b = \max\{a, b\}$ *and* $a \wedge b = \min\{a, b\}$. *Moreover, it is obvious that*

$$\widehat{H}(M) \leq M/2 \quad \text{and} \quad \widehat{H}(M) \uparrow \text{ as } M \uparrow .$$

The expression for $\widehat{H}(M)$ is worked out in more detail (than in the original proof of [18]) for the fast diffusion case. We postpone the work here; those who are interested can simply follow along the lines of the original proof of this proposition in [18] and adapt it for the fast diffusion case.

From this proposition, the following proposition is now immediate.

Proposition (II.2). *Let* $H(M) = \widehat{H}(M)/(2C + 1)$ *where* $C = \|u_0\|_{L^\infty(\Omega)} \vee \|u_0\|^{q-1}_{L^\infty(\Omega)}$; *then* $H(M) \leq \widehat{H}(M) \leq M/2$ *as well as* $H(M) < M/4C \leq \frac{1}{4}$

for $M \in [0, \|u_0\|_{L^\infty(\Omega)}]$. *Now, if* $Q_{x_0, t_0}(R) \subset Q_T$, $|u| \le M$ *in* $Q_{x_0, t_0}(R)$ *and*

$$u(x_1, t_1) \le \frac{M}{2} \quad \text{for some } (x_1, t_1) \in Q_{x_0, t_0}\left(\frac{R}{2}\right),$$

we have

$$|Q_{x_0, t_0}(R) \cap \{u \le M - H(M)\}| \ge H(M)|Q_{x_0, t_0}(R)|.$$

From the above propositions, together with Proposition (I.2), we can derive the following estimate through an iteration process as in [18].

Proposition (III). *Let* $d_1 = \delta^{(q-1)/(q-2)}/C_1 C_2 |\Omega|^{1/q}$ *as before. Then for* $t \in [\delta, T^* - \delta]$, *at a point* (x_0, t) *where* $\max_{x\in\Omega} u(x, t)$ *is attained, we have*

$$u(x, t) \ge M(t)/2 \quad \text{on } Q_{x_0, t}(R)$$

for

$$R \le \left[\frac{d_1}{4} \wedge \sqrt{\frac{H(M(t)/2)}{4} \gamma\left(N, (q-1)\left(\frac{M(t)}{4}\right)^{q-2}, \frac{H(M(t)/2)}{2}\right)(q-1)\left(\frac{M(t)}{4}\right)^{q-2}}\right] \times \left(K_N H\left(\frac{M(t)}{2}\right)\right)^n,$$

$$n = \left\langle \left(\|u_0\|_{L^\infty(\Omega)} - \frac{M(t)}{2}\right) \Big/ \left(\frac{H(M(t))}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{M(t)}{2}\right)^{q-2}, \frac{H(M(t))}{2}\right)\frac{H(M(t))}{4}\right]\right)\right\rangle + 1.$$

(Here $\langle z \rangle$ stands for the greatest integer value function of z and $\gamma(N, \tilde{\delta}, \beta)$ is the same as in Propositions (I.1) and (I.2).) The proof of Proposition (III) will be given in the appendix.

We note that for $t \in [\delta, T^* - \delta]$,

$$M(t) = \frac{(T^* - t)^{1/(q-2)}}{C_1 |\Omega|^{1/q}} \ge \delta_1 = \frac{\delta^{1/(q-2)}}{C_1 |\Omega|^{1/q}}.$$

Here $\delta_1 = \delta_1(\delta, C_1, \Omega)$ where $C_1 = C_1(q, \Omega, N)$ is the constant from Theorem 1.

Thus, for $t \in [\delta, T^* - \delta]$ and $u(x_0, t) = \max_{x\in\Omega} u(t)$,

$$u(x, t) \ge M(t)/2 \quad \text{on } Q_{x_0, t}(R) \text{ for } R \le R_2,$$

where

$$R_2 = \left[\frac{d_1}{4} \wedge \sqrt{\frac{H(\delta_1/2)}{4} \gamma\left(N, (q-1)\left(\frac{\delta_1}{4}\right)^{q-2}, \frac{H(\delta_1/2)}{2}\right)(q-1)\left(\frac{\delta_1}{4}\right)^{q-2}}\right] \times \left(K_N H\left(\frac{\delta_1}{2}\right)\right)^n,$$

$$n = \left\langle \left(\|u_0\|_{L^\infty(\Omega)} - \frac{\delta_1}{2}\right) \Big/ \left(\frac{H(\delta_1)}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{\delta_1}{2}\right)^{q-2}, \frac{H(\delta_1)}{2}\right)\frac{H(\delta_1)}{4}\right]\right)\right\rangle + 1.$$

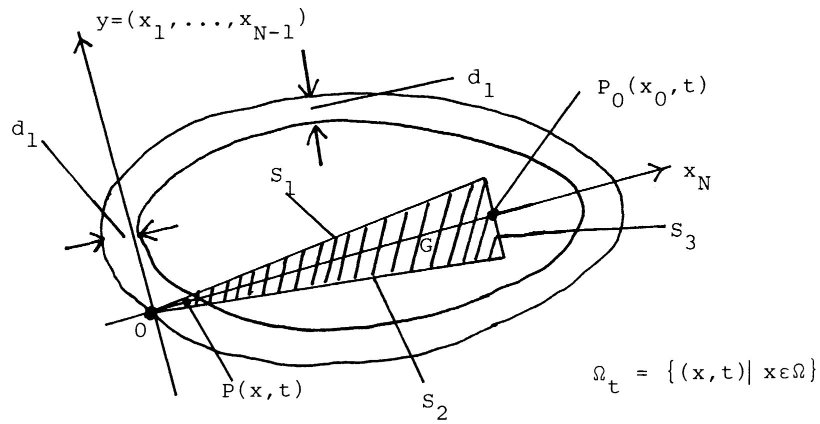

FIGURE (I)

This is due to the monotonic increasing property of γ and H. Here

$$R_2 = R_2(N, q, d_1, \|u_0\|_{L^\infty(\Omega)}, \delta_1, \Omega, C_1, C_2),$$

where d_1 and δ_1 are as above and C_1, C_2 are the constants from Theorems 1 and 2, respectively.

(iii) Now consider any point

$$P = (x, t) \in \Omega \times [\delta, T^* - \delta]$$

with $d(x, \partial\Omega) > 0$; we construct the conic region G as shown in Figure (I) where $P_0 = (x_0, t)$ is a point at which $\max_{x\in\Omega} u(t)$ is being attained and without the loss of generality, we set up our coordinate system at the point O where $\overline{P_0P}$ intersects $\partial\Omega$

Let the conic region G be described by the equation

$$x_N \geq \alpha\sqrt{x_1^2 + \cdots + x_N^2} \quad \text{for some } \alpha \in (0, 1)$$

with S_1, S_2, and S_3 forming its boundary. We construct the corresponding comparison function

$$\omega(x) = x_N^n(x_N - \alpha\sqrt{x_1^2 + \cdots + x_N^2})$$

with n and α to be chosen latter.

By straightforward computations, we have

$$\begin{aligned}\frac{\partial^2\omega}{\partial x_N^2} &= n(n+1)x_N^{n-1} - \alpha n(n-1)x_N^{n-2}\sqrt{x_1^2+\cdots+x_N^2}\\ &\quad - \frac{\alpha(2n+1)x_N^n}{\sqrt{x_1^2+\cdots+x_N^2}} + \frac{\alpha x_N^{n+2}}{\left(\sqrt{x_1^2+\cdots+x_N^2}\right)^3}\\ &\geq x_N^{n-2}[n(n+1)x_N - n(n-1)x_N - (2n+1)\alpha x_N]\\ &= x_N^{n-1}[2n-2n\alpha-\alpha] = x_N^{n-1}[2n(1-\alpha)-\alpha],\\ \frac{\partial^2\omega}{\partial x_1^2} &= \frac{-\alpha x_N^n}{\sqrt{x_1^2+\cdots+x_N^2}} + \frac{\alpha x_1^2 x_N^n}{\left(\sqrt{x_1^2+\cdots+x_N^2}\right)^3}.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta\omega &\geq x_N^{n-1}[2n(1-\alpha)-\alpha] - \frac{\alpha(N-1)x_N^n}{\sqrt{x_1^2+\cdots+x_N^2}}\\ &\geq x_N^{n-1}[2n(1-\alpha)-N\alpha].\end{aligned}$$

On choosing

$$2n(1-\alpha) = 2N\alpha \quad \text{or} \quad n = N\alpha/(1-\alpha),$$

we have

$$\Delta\omega \geq N\alpha x_N^{n-1}, \qquad n = N\alpha/(1-\alpha).$$

Next, we considered $\underline{u} = C_3\omega$ with $C_3 > 0$ to be chosen later and we set

$$\alpha = \frac{D}{\sqrt{D^2+R_2^2}}, \quad D \text{ being the diameter of } \Omega.$$

With this choice, $S_3 \subset B_{x_0}(R_2)$ and

$$\begin{aligned}-\Delta\underline{u} + \left(\frac{q-1}{q-2}\right)\frac{\underline{u}^{q-1}}{t} &\leq -C_3 N\alpha x_N^{n-1} + \left(\frac{q-1}{q-2}\right)\frac{C_3^{q-1}}{t}(x_N^{n+1})^{q-1}\\ &\leq C_3 x_N^{n-1}\left[\left(\frac{q-1}{q-2}\right)\frac{C_3^{q-2}}{t}x_N^{(n+1)(q-2)} - N\alpha\right]\\ &\leq C_3 x_N^{n-1}\left[\left(\frac{q-1}{q-2}\right)\frac{C_3^{q-2}}{\delta}D^{(n+1)(q-2)} - N\alpha\right] < 0,\end{aligned}$$

provided

$$C_3 \leq \left[\left(\frac{q-2}{q-1}\right)\left(\frac{N\alpha\delta}{D^{(n+1)(q-2)}}\right)\right]^{1/(q-2)}.$$

Furthermore, on $S_3 \subset B_{x_0}(R_2)$, by (ii), $u(t)$ is bounded from below by $M(t)/2$, which is in turn bounded from below by $\delta_1/2$ ($\delta_1 = \delta^{1/(q-2)}/C_1|\Omega|^{1/q}$).

Therefore, we can choose

$$C_3 = \left[\left(\frac{q-2}{q-1}\right)\right]\left(\frac{N\alpha\delta}{D^{(N+1)(q-2)}}\right)^{1/(q-2)} \wedge \left[\frac{\delta_1}{2}\right], \qquad \alpha = \frac{D}{\sqrt{D^2+R_2^2}}.$$

Here $C_3 = C_3(N, q, D, C_1, \delta_1, R_2)$. With this choice,

$$\underline{u}|_{S_1} = \underline{u}|_{S_2} = 0 \le u, \qquad \underline{u}|_{S_3} \le u|_{S_3}.$$

Finally, by invoking the regularizing effect in Theorem 3,

$$-\Delta u(t) + \left(\frac{q-1}{q-2}\right)\frac{u^{q-1}}{t} \ge -\Delta\underline{u} + \left(\frac{q-1}{q-2}\right)\frac{\underline{u}^{q-1}}{t} \qquad (q > 2).$$

We conclude by the maximum principle that

$$u(x, t) \ge u(x) \quad \text{on } G \times [\delta, T^* - \delta],$$

which implies

$$u(x, t) \ge C_3 x_N^{n+1}(1-\alpha).$$

But $x_N = |OP|$; thus

$$u(x, t) \ge C_3(1-\alpha)\, d(x, \partial\Omega)^{N\alpha/(1-\alpha)+1}, \qquad \alpha = \frac{D}{\sqrt{D^2+R_2^2}}.$$

The above estimate is good for all (x, t) which belong to the region $\Omega \times [\delta, T^* - \delta]$. However, at this point we still have not proved the linear growth of u at $\partial\Omega$ which is to be established in the next step.

IV. The establishment of the linear lower bound at the boundary

In this section, we will eventually establish a Lipschitz growth rate for $u(x, t)$ on $\partial\Omega$. Since an upper bound is already provided by Theorem 2, all we need is a linear lower bound. Before we go on, we need to introduce more terminology and constructions concerning the geometry of Ω.

Definition. We say that $\partial\Omega$ satisfies the interior sphere condition with radius R if for every $x_0 \in \partial\Omega$, we can construct a ball $B_{x^*}(R)$, $x^* \in \Omega$ and $R > 0$ uniformly on x_0 such that

$$B_{x^*}(R) \subset \Omega \quad \text{and} \quad \partial B_{x^*}(R) \cap \partial\Omega = \{x_0\}.$$

Let R_3 be the radius of the interior sphere condition of $\partial\Omega$; without loss of generality we will assume $R_3 \le d_1$. For any $x_0 \in \partial\Omega$, we can construct the region K uniformly with respect to x_0 as in Figure (II).

Let K_1 and K_2 denote, respectively, the two parts of the boundary of the region K as shown. It is obvious that, for $x \in K_1$,

$$d(x, \partial\Omega) \ge d\left(x, \partial B_{(x^*+x_0)/2}\left(\frac{R_3}{2}\right)\right) \ge \left(1 - \frac{\sqrt{15}}{4}\right) R_3.$$

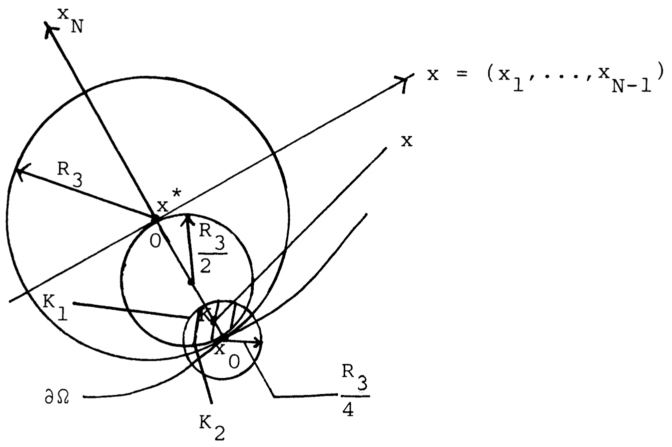

FIGURE (II)

Consider $(x, t) \in \Omega \times [\delta, T^* - \delta]$ and let $x_0 \in \partial\Omega$ be such that

$$d(x, \partial\Omega) \le R_3/4, \qquad |x - x_0| = d(x, \partial\Omega).$$

Furthermore, let x^* be the center of the interior sphere of radius R_3 with respect to x_0 as shown.

To get a linear lower bound on $u(x, t)$, all we need is a simple Hopf-type argument. First of all, by the estimate of the first step,

$$u(x, t)|_{K_1} \ge C_3(1-\alpha)\left[\left(\frac{1-\sqrt{15}}{4}\right) R_3\right]^{N\alpha/(1-\alpha)+1}, \qquad \alpha = \frac{D}{\sqrt{D^2 + R_2^2}}.$$

From now on, we will let C_4 denote

$$C_3(1-\alpha)\left[\left(1 - \frac{\sqrt{15}}{4}\right) R_3\right]^{N\alpha/(1-\alpha)+1}.$$

Obviously $C_4 = C_4(C_3, R_3, N, D, R_2)$ where D is the diameter of Ω, R_3 is the radius of the interior sphere condition, and R_2 and C_3 are the constants from (ii) and (iii) of the first step respectively.

Without loss of generality, we can set up the coordinate system as in Figure (II) with the origin at x^*. We construct the following comparison function,

$$p(x) = \varepsilon[e^{-\theta r^2} - e^{-\theta R_3^2}], \qquad r = |x - x^*| = \sqrt{x_1^2 + \cdots + x_N^2}$$

with ε, $\alpha > 0$ to be chosen later. We have

$$\Delta p = \theta\varepsilon e^{-\theta r^2}[4\theta r - 2N] \geq e^{-\theta R_3^2}\theta\varepsilon\left[4\theta\left(\frac{R_3}{2}\right)^2 - 2N\right]$$

$$\geq \theta\varepsilon e^{-\theta R_3^2}[\theta R_3^2 - 2N] \geq \frac{8N^2}{R_3^2}\varepsilon e^{-4N},$$

where we have chosen $\theta = 4N/R_3^2$.

On choosing ε to be sufficiently small, e.g.,

$$\varepsilon \leq \left[\left(\frac{q-2}{q-1}\right)\left(\frac{4N^2}{R_3^2}\right)\delta e^{-4N}\right]^{1/(q-2)},$$

we have

$$-\Delta p + \left(\frac{q-1}{q-2}\right)\frac{p^{q-1}}{t} \leq \frac{-8N^2}{R_3^2}\varepsilon e^{-4N} + \left(\frac{q-1}{q-2}\right)\frac{\varepsilon^{q-1}}{\delta}$$

$$= \varepsilon\left[\left(\frac{q-1}{q-2}\right)\left(\frac{\varepsilon^{q-2}}{\delta}\right) - \frac{8N^2}{R_3^2}e^{-4N}\right] \leq 0,$$

for $t \geq \delta$. Furthermore,

$$p(x)|_{K_2} = 0, \qquad p(x)|_{K_1} \leq u(x, t)|_{K_1}$$

provided ε also satisfies $\varepsilon \leq C_4$. Hence, let

$$\varepsilon = C_4 \wedge \left[\left(\frac{q-2}{q-1}\right)\left(\frac{4N^2}{R_3^2}\right)\delta e^{-4N}\right]^{1/(q-2)}.$$

We can again conclude, by the regularizing effect of Theorem 3, that

$$u(x, t) \geq \left\{C_4 \wedge \left[\left(\frac{q-2}{q-1}\right)\left(\frac{4N^2}{R_3^2}\right)\delta e^{-4N}\right]^{1/(q-2)}\right\}[e^{-4Nr^2/R_3^2} - e^{-4N}].$$

This estimate holds for all (x, t) such that

$$d(x, \partial\Omega) \leq R_3/4, \qquad t \in [\delta, T^* - \delta].$$

Finally to see that the lower bound we obtained indicates a linear lower bound, we just observe that along the path from x_0 to x^*, if x is any point such that

$$|x - x_0| = d(x, \partial\Omega) \leq R_3/4,$$

we have

$$R_3/2 \leq r = |x - x^*| \leq R_3.$$

From the previous estimate,

$$u(x, t) \geq \left\{C_4 \wedge \left[\left(\frac{q-2}{q-1}\right)\left(\frac{4N^2}{R_3^2}\right)\delta e^{-4N}\right]^{1/(q-2)}\right\}[e^{-4Nr^2/R_3^2} - e^{-4N}]$$

$$\geq \left\{C_4 \wedge \left[\left(\frac{q-2}{q-1}\right)\left(\frac{4N^2}{R_3^2}\right)\delta e^{-4N}\right]^{1/(q-2)}\right\}\left(\frac{2N}{R_3}\right)e^{-4N}\,d(x, \partial\Omega).$$

The last statement is a consequence of the fact that for

$$f(r) = [e^{-\theta r^2} - e^{-\theta R^2}], \quad \text{where } \theta > 0, \; R/2 \le r \le R$$
$$(\theta = 4N/R_3^2 \text{ and } R = R_3 \text{ in our case})$$

we have

$$f(r) \ge \theta(R/2)e^{-\theta R^2}(R-r)$$

and furthermore,

$$R_3 - r = |x_0 - x^*| - |x - x^*| = |x_0 - x| = d(x, \partial\Omega).$$

Summarizing, we have obtained the following theorem.

Theorem 4. *Let* $N \le 2$ *or* $2 < q < 2N/(N-2)$ *if* $N > 2$. *For* Ω *convex and every* (x, t) *which belongs to* $\Omega \times [\delta, T^* - \delta]$, *such that* $d(x, \partial\Omega) \le R_3/4$, *we have*

$$u(x, t) \ge C_5 \, d(x, \partial\Omega),$$

where $C_5 = C_5(N, q, \delta, C_4, R_3)$ *is the constant*

$$\left\{ C_4 \wedge \left[\left(\frac{q-2}{q-1}\right) \left(\frac{4N^2}{R_3^2}\right) \delta e^{-4N} \right]^{1/(q-2)} \right\} \left(\frac{2N}{R_3}\right) e^{-4N},$$

R_3 *being the radius of the interior sphere condition and* C_4 *the constant defined previously. Moreover, this estimate is only good for* $N \le 2$ *or* $2 < q < 2N/(N-2)$ *when* $N > 2$ *as mentioned before.*

At this point let us remark that the Lipschitz growth rate for $u(x, t)$ is true on $\partial\Omega$ for any $q > 2$. However, when q falls beyond the rate $(2, 2N/(N-2))$ when $N > 2$, we do not know how to estimate the interior lower bound in terms of the data as in the first step of this section, and consequently, we do not have the above estimate. Also the convexity of Ω is not necessary as mentioned at the beginning of §III (cf. Remark 4 below).

Remarks. 1. For $N \ge 7$ and $3 > q > 2N/(N-2)$, the Lipschitz growth rate still implies the Hölder continuity on $\overline{\Omega} \times [\delta, T^* - \delta]$; but in this case, since we do not know how to estimate the lower bound in the interior in terms of the data, consequently we do not know how to estimate the modulus of continuity.

2. It is quite obvious that this work can be extended to a wider class of nonlinearities

$$\beta(u)_t - \Delta u = 0, \quad \beta \in C^1[0, \infty), \quad \beta(0) = 0$$

where $\beta(s)$ resembles the structure of s^{q-1}, e.g., the structure in [12].

3. It is not necessary that $u_0 \in L^\infty(\Omega)$ because, due to the regularizing effect, $u(t)$ becomes bounded for $t > 0$ even if u_0 is simply $L^{q-1}(\Omega)$ for $2 < q < (2N-2)/(N-2)$ (cf. [19]).

4. Obviously, the assumption that Ω is convex is not a necessity. For example, if $\partial\Omega$ is sufficiently smooth, we can assume that there exists $R > 0$ and $n \in Z^+$, both depending on Ω, with $\Omega_R = \{x | d(x, \partial\Omega) > R\}$ such that for every $x_0 \in \Omega - \Omega_R$ and $x_0' \in \Omega_R$, we can always connect them by a zigzag path consisting of at most n connected line segments in Ω of which the intermediate joining points $x_1, \dots, x_{n-1}$ all lie in Ω_R.

5. It should be noted that the Lipschitz rate obtained in §III is sharp as motivated by the separable solutions of the fast diffusion equation where the spatial part of the separable solutions have the same kind of growth at $\partial\Omega$. More precisely, by a separable solution we are referring to a solution of the form $S(x)T(t)$, where $S(x)$ and $T(t)$ satisfy respectively

$$-\Delta S = (q-1)S^{q-1}, \quad S|_{\partial\Omega} = 0, \quad S > 0 \quad \text{in } \Omega,$$
$$T(t) = [(q-2)(T^* - t)]^{1/(q-2)}.$$

For the corresponding elliptic theory concerning the existence and uniqueness of $S(x)$, one may refer to standard references like [10, 16].

6. It should be noted that the results from [4] do not immediately give a modulus of continuity of the Hölder type because the Harnack-type estimate in [4] is obtained through a family of cylinders whose dimensions are different from point to point depending on the degeneracy of the equation near the point (i.e., it is not the traditional type of family of cylinders which is being used). As a result, the lack of uniformity of the dimensions of the family of cylinders makes it impossible to obtain Hölder-type estimates from the Harnack inequality obtained in [4]. In the case of the fast diffusion equation, it is the positivity and the linear growth of the solution $u(x, t)$ at the boundary which makes everything work. Roughly speaking, it is due to the fact that the linear growth leads to a more uniform family of cylinders which is much better than that one would generally expect and from where we can recover the Hölder continuity result.

Appendix: The proof of Proposition (III)

Proposition (III). *Let* $d_1 = \frac{\delta^{(q-1)/(q-2)}}{C_1 C_2 |\Omega|^{1/q}}$ *as in* (i) *of* §IV. *Then for* $t \in [\delta, T^* - \delta]$, *at a point* (x_0, t) *where* $\max_{x\in\Omega} u(x, t)$ *is attained, we have*

$$u(x, t) \geq \frac{M(t)}{2} \quad \text{on } Q_{x_0, t}(R) \text{ where } M(t) = \frac{(T^* - t)^{1/(q-2)}}{C_1 |\Omega|^{1/q}}$$

and

$$R \leq \left[\frac{d_1}{4} \wedge \sqrt{\frac{H(M(t)/2)}{4}\gamma\left((N, (q-1)\left(\frac{M(t)}{4}\right)^{q-2}, \frac{H(M(t)/2)}{2}\right)(q-1)\left(\frac{M(t)}{4}\right)^{q-2}}\right]$$
$$\times (K_N H(M(t)/2))^n,$$

$$wn = \left\langle \left(\|u_0\|_{L^\infty(\Omega)} - \frac{M(t)}{2} \right) \Big/ \left(\frac{H(M(t))}{2} \wedge \left[\gamma \left((N, (q-1) \left(\frac{M(t)}{2} \right)^{q-2}, \frac{H(M(t))}{2} \right) \frac{H(M(t))}{4} \right] \right) \right\rangle + 1$$

(*$\langle s \rangle$ being the greatest integer value function of s*).

Proof. Suppose $\max_{x\in\Omega} u(x, t)$ is attained at (x_0, t), by (i) of §III,

$$u(x_0, t) = \max_{x\in\Omega} u(x, t) \geq M(t) = \frac{(T^* - t)^{1/(q-2)}}{C_1 |\Omega|^{1/q}}.$$

and

$$d(x_0, \partial\Omega) \geq d_1.$$

We now prove the above result through a process of iterations that is adopted from the iteration process in the original paper [18] for the fast diffusion case; the original iteration process is implicit and does not provide a closed form estimate for R. We will divide the proof into three steps.

(i) Let $\gamma(N, \tilde{\delta}, \beta)$ and ε be the same as in Propositions (I.1), (I.2) of the preliminaries and $H(\cdot)$, the same as in Proposition (II.2) of §III. We construct sequences of quantities

$$(M_k, R_k), \quad \{\varepsilon_k, \tilde{\delta}_k, \beta_k\}, \quad M_k \downarrow 0, \quad R_k \downarrow 0$$

and

$$\varepsilon_k = \varepsilon(M_k), \quad \tilde{\delta}_k = \tilde{\delta}(M_k), \quad \beta_k = \beta(M_k).$$

To start with, let

$$M_1 = \|u_0\|_{L^\infty(\Omega)}, \quad \varepsilon(M_1) = M_1 - H(M_1), \quad \beta(M_1) = H(M_1),$$
$$\tilde{\delta}(M_1) = (q-1)\varepsilon(M_1)^{q-2}.$$

Furthermore, define

$$R_1 = \left(\frac{d_1}{2} \right) \wedge \sqrt{\frac{H(M_1)}{4} \gamma \left(N, (q-1) \left(\frac{M_1}{2} \right)^{q-2}, \frac{H(M_1)}{2} \right) (q-1) \left(\frac{M_1}{2} \right)^{q-2}}$$
$$\leq \left(\frac{d_1}{2} \right) \wedge \sqrt{\frac{H(M_1)}{4} \gamma \left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2} \right) \tilde{\delta}(M_1)}.$$

This is because

$$\varepsilon(M_1) = M_1 - H(M_1) \geq M_1 - \frac{M_1}{2} = \frac{M_1}{2},$$

and

$$\tilde{\delta}(M_1) = (q-1)\varepsilon(M_1)^{q-2} \geq (q-1) \left(\frac{M_1}{2} \right)^{q-2}$$

as well as the fact that $\gamma(N, \tilde{\delta}, \beta)$ is $\uparrow$ with respect to $\tilde{\delta}$ and β.

Next, let

$$M_2 = M_1 - \frac{H(M_1)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right) \frac{H(M_1)}{4}\right]$$

and as before,

$$\varepsilon(M_2) = M_2 - H(M_2), \quad \beta(M_2) = H(M_2), \quad \tilde{\delta}(M_2) = (q-1)\varepsilon(M_2)^{q-2}.$$

Furthermore, define

$$R_2 = \left[\left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M_2)}{4}\gamma\left(N, (q-1)\left(\frac{M_2}{2}\right)^{q-2}, \frac{H(M_2)}{2}\right)(q-1)\left(\frac{M_1}{2}\right)^{q-2}}\right] K_N H(M_2),$$

$$\leq \left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M_2)}{4}\gamma\left(N, \tilde{\delta}(M_2), \frac{H(M_2)}{2}\right)\tilde{\delta}(M_2)} \wedge K_N H(M_1) R_1.$$

This is because

$$K_N \leq \frac{1}{\sqrt{5}}, \quad M_2 < M_1, \quad H(M) < \frac{1}{4} \wedge \frac{M}{2} \quad \text{for } M \in (0, \|u_0\|_{L^\infty(\Omega)})$$

as well as the monotonic property of H and γ.

Hence, we can define $\{\varepsilon_k, \tilde{\delta}_k, \beta_k\}$ along with (M_k, R_k) where M_1 and R_1 are as in step 1 and

$$M_{k+1} = M_k - \frac{H(M_k)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_k), \frac{H(M_k)}{2}\right) \frac{H(M_k)}{4}\right],$$

$$R_{k+1} = \left[\left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M_{k+1})}{4}\gamma\left(N, (q-1)\left(\frac{M_{k+1}}{2}\right)^{q-2}, \frac{H(M_{k+1})}{2}\right)(q-1)\left(\frac{M_{k+1}}{2}\right)^{q-2}}\right] \times (K_N H(M_{k+1}))^k,$$

$$\varepsilon_k = \varepsilon(M_k), \quad \tilde{\delta}_k = \tilde{\delta}(M_k), \quad \beta_k = H(M_k).$$

Furthermore,

$$R_{k+1} \leq \left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M_{k+1})}{4}\gamma\left(N, \tilde{\delta}(M_{k+1}), \frac{H(M_{k+1})}{2}\right)\tilde{\delta}(M_{k+1})} \wedge K_N H(M_k) R_k.$$

(ii) Before we begin the process of iteration, we first introduce the following linearization of $u(x, t)$ as in [18]. Indeed, let $g_{\varepsilon,n}(s)$ be such that

$$g_{\varepsilon,n} \geq 0, \quad g''_{\varepsilon,n} \geq 0, \quad 0 \leq g'_{\varepsilon,n} \leq 1$$

and

$$[s - \varepsilon - \tfrac{1}{n}]^+ \leq g_{\varepsilon,n}(s) \leq [s - \varepsilon]^+.$$

Let $g_{\varepsilon,n}(u(x, t)) = z(x, t)$. Then on $\{u \leq \varepsilon\}$

$$z_t = \nabla z = \Delta z = 0,$$

so that

$$\begin{aligned}(q-1)(u \vee \varepsilon)^{q-2} z_t - \Delta z &= (q-1)u^{q-2} z_t - \Delta z \\ &= (q-1)u^{q-2} g'_{\varepsilon,n}(u) u_t - \nabla(g'_{\varepsilon,n}(u)\nabla u) \\ &\leq g'_{\varepsilon,n}(u)[(q-1)u^{q-2}u_t - \Delta u] = 0.\end{aligned}$$

On letting $a(x, t) = 1/(q-1)(u \vee \varepsilon)^{q-2}$, the fast diffusion equation (2) is transformed into the parabolic inequality:

$$\text{(7)} \qquad \begin{cases} z_t - a(x, t)\Delta z \le 0, \\ (q-1)\varepsilon^{q-2} \le a(x, t) \le 1/(q-1)\varepsilon^{q-2}. \end{cases}$$

We can therefore identify the $\tilde{\delta}$ in Propositions (I.1), (I.2) as $(q-1)\varepsilon^{q-2}$.

We now begin the iteration process as follows. Obviously,

$$u \le M_1 = \|u_0\|_{L^\infty(\Omega)} \quad \text{in } Q_{x_0, t}(R_1).$$

If $u(x, t) \ge M_1/2$ for

$$\begin{aligned} (x, t) \in Q_{x_0, t}(R_2) &\subset Q_{x_0, t}(K_N H(M_1) R_1) \\ &\subset \overline{Q_{x_0, t}(R_1/2)} \end{aligned}$$

we are done. If not,

$$\exists (x', t') \in \overline{Q_{x_0, t}(R_1/2)} \quad \text{such that } u(x', t') < M_1/2.$$

By Proposition (II.2), we have

$$|Q_{x_0, t}(R_1) \cap \{u \le M_1 - H(M_1)\}| \ge H(M_1) |Q_{x_0, t}(R_1)|.$$

Let $z(x, t) = g_{\varepsilon, n}(u(x, t))$ (as in the above linearization) with $\varepsilon = \varepsilon_1$, $n \in Z^+$. Then,

$$z(x, t) \le [u(x, t) - \varepsilon_1]^+ \le M_1 - (M_1 - H(M_1)) = H(M_1).$$

Furthermore,

$$\begin{aligned} |Q_{x_0, t}(R_1) \cap \{z = 0\}| &\le |Q_{x_0, t}(R_1) \cap \{u(x, t) \le \varepsilon_1\}| \\ &= |Q_{x_0, t}(R_1) \cap \{u \le M_1 - H(M_1)\}| \\ &\ge H(M_1) |Q_{x_0, t}(R_1)|. \end{aligned}$$

Hence,

$$|Q_{x_0, t}(R_1) \cap \{z \le H(M_1)/2\}| \ge |Q_{x_0, t}(R_1) \cap \{z = 0\}| \ge H(M_1) |Q_{x_0, t}(R_1)|.$$

On identifying $\tilde{\delta}$ with $\tilde{\delta}_1$, β with $\beta_1 = H(M_1)$ and invoking Proposition

(I.2), we have

$$z(x,t) \le H(M_1) - \frac{H(M_1)}{2}\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right) + \frac{R_1^2}{\tilde{\delta}(M_1)} \quad \text{in } Q_{x_0,t}(K_N H(M_1) R_1)$$

$$\Rightarrow z(x,t) \le H(M_1) - \frac{H(M_1)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right)\frac{H(M_1)}{4}\right] \quad \text{in } Q_{x_0,t}(R_2)$$

$$\begin{aligned}\Rightarrow u(x,t) &\le \left[u(x,t) - \varepsilon_1 + \frac{1}{n}\right]^+ + \varepsilon_1 + \frac{1}{n}\\ &\le H(M_1) - \frac{H(M_1)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right)\frac{H(M_1)}{4}\right]\\ &\quad + M_1 - H(M_1) + \frac{1}{n}\\ &\le M_1 - \frac{H(M_1)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right)\frac{H(M_1)}{4}\right] + \frac{1}{n} \quad \text{in } Q_{x_0,t}(R_2)\end{aligned}$$

$$\Rightarrow u(x,t) \le M_2 = M_1 - \frac{H(M_1)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_1), \frac{H(M_1)}{2}\right)\frac{H(M_1)}{4}\right] \quad \text{in } Q_{x_0,t}(R_2), \text{ as } n \in Z^+ \text{ is arbitrary.}$$

We keep repeating this process unit we come down to the stage where

$$M_{n+1} \le u(x_0, t) \le M_n, \quad \text{for some } n \in Z^+.$$

We must then have

$$u(x,t) \ge M_n/2 \quad \text{in } Q_{x_0,t}(R_{n+1}),$$

for, if not, the previous inductive argument could be carried forward one further step so as to yield

$$u(x,t) < M_{n+1} \quad \text{in } Q_{x_0,t}(R_{n+1}),$$

which is a contradiction.

(iii) Finally, we use $u(x_0, t) = \max_{x\in\Omega} u(x,t) \ge M(t)$ (cf. (i) of §IV) to get estimates on R_{n+1} and n.

To get a lower bounded on R_{n+1}, we note that at the nth stage,

$$\begin{aligned}M_{n+1} &= M_n - \frac{H(M_n)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_n), \frac{H(M_n)}{2}\right)\frac{H(M_n)}{4}\right]\\ &\ge M_n - \frac{H(M_n)}{2} \ge M_n - \frac{M_n}{4} = \frac{3}{4}M_n \quad \left(\text{as } H(M) < \frac{M}{2}\right).\end{aligned}$$

But

$$M_n \ge u(x_0, t) \ge M(t) \Rightarrow M_{n+1} \ge \frac{3}{4}M(t) \ge \frac{M(t)}{2}.$$

Hence, together with the monotonic increasing property of H and γ, we have

$$R_{n+1} = \left[\left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M_{n+1})}{2}\gamma\left(N, (q-1)\left(\frac{M_{n+1}}{2}\right)^{q-2}, \frac{H(M_{n+1})}{2}\right)(q-1)\left(\frac{M_{n+1}}{2}\right)^{q-2}}\right] \times [K_N H(M_{n+1})]^n$$

$$\geq \left[\left(\frac{d_1}{2}\right) \wedge \sqrt{\frac{H(M(t)/2)}{2}\gamma\left(N, (q-1)\left(\frac{M(t)}{4}\right)^{q-2}, \frac{H(M(t)/2)}{2}\right)(q-1)\left(\frac{M(t)}{4}\right)^{q-2}}\right] \times \left(K_N H\left(\frac{M(t)}{2}\right)\right)^n .$$

To get an upper bound on the integer n, let us first recall that

$$M_{k+1} = M_k - \frac{H(M_k)}{2} \wedge \left[\gamma\left(N, \tilde{\delta}(M_k), \frac{H(M_k)}{2}\right)\frac{H(M_k)}{4}\right],$$

where

$$M_k \geq u(x_0, t) \geq M(t), \qquad k = 1, \dots, n.$$

This implies, for $k = 1, \dots, n$, that

$$\tilde{\delta}(M_k) \geq (q-1)\left(\frac{M_k}{2}\right)^{q-2} \geq (q-1)\left(\frac{M(t)}{2}\right)^{q-2},$$

$$H(M_k) \geq H(M(t))$$

and hence,

$$M_k - M_{k+1} \geq \frac{H(M(t))}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{M(t)}{2}\right)^{q-2}, \frac{H(M(t))}{2}\right)\frac{H(M(t))}{4}\right].$$

Summing up the above inequalities, for $k = 1, \dots, n$, we have

$$\|u_0\|_{L^\infty(\Omega)} - M_{n+1} \geq n\left\{\frac{H(M(t))}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{M(t)}{2}\right)^{q-2}, \frac{H(M(t))}{2}\right)\frac{H(M(t))}{4}\right]\right\}.$$

But $M_{n+1} \geq M(t)/2$; therefore

$$n \leq \left(\|u_0\|_{L^\infty(\Omega)} - \frac{M(t)}{2}\right) \Big/ \left(\frac{H(M(t))}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{M(t)}{2}\right)^{q-2}, \frac{H(M(t))}{2}\right)\frac{H(M(t))}{4}\right]\right).$$

We can, therefore, simply choose n to be

$$\left\langle\left(\|u_0\|_{L^\infty(\Omega)} - \frac{M(t)}{2}\right) \Big/ \left(\frac{H(M(t))}{2} \wedge \left[\gamma\left(N, (q-1)\left(\frac{M(t)}{2}\right)^{q-2}, \frac{H(M(t))}{2}\right), \frac{H(M(t))}{4}\right]\right)\right\rangle + 1$$

whence the result.

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