

## RANDOM BLASCHKE PRODUCTS

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**ABSTRACT.** Let  $\{\theta_n(\omega)\}$  be a sequence of independent random variables uniformly distributed on  $[0, 2\pi]$ , and let  $z_n(\omega) = r_n e^{i\theta_n(\omega)}$  for a fixed but arbitrary sequence of radii  $r_n$  satisfying the Blaschke condition  $\sum(1 - r_n) < \infty$ . We show that the random Blaschke product with zeros  $z_n(\omega)$  is almost surely not in the little Bloch space, and we describe necessary conditions and sufficient conditions on the radii  $r_n$  so that  $\{z_n(\omega)\}$  is almost surely an interpolating sequence.

This paper deals with Blaschke products whose zeros are chosen randomly in the following way: for a fixed but arbitrary sequence of numbers  $\{r_n\}$ ,  $0 < r_n < 1$ , that satisfy the Blaschke condition  $\sum(1 - r_n) < \infty$ , choose random angles  $\theta_n(\omega)$  that are independent and uniformly distributed on the interval  $[0, 2\pi]$ . Let  $z_n(\omega) = r_n e^{i\theta_n(\omega)}$ . In §1 it is shown that the random Blaschke product with zeros  $\{z_n(\omega)\}$  is, with probability 1, not in the little Bloch space, partially answering a question raised by D. Sarason. In §2 a necessary condition and a sufficient condition is given on  $\{r_n\}$  so that the random Blaschke product is an interpolating Blaschke product almost surely. These two conditions are much closer together than the conditions that are available if the angles  $\theta_n$  are chosen deterministically. In §3 a proof is given of the theorem of Naftalevič that if  $\{r_n\}$  satisfies the Blaschke condition then one may always choose (nonrandom) angles  $\{\theta_n\}$  so that the sequence  $\{r_n e^{i\theta_n}\}$  is an interpolating sequence. The proof presented is essentially the same as that in [Naf56]; it has been reorganized and additional details are provided. It is included in this paper because the journal containing [Naf56] is rather obscure and is not readily available in the United States.

The results of this paper are contained in the author's doctoral dissertation at The University of Michigan. The author is deeply grateful that he had the opportunity to have Allen Shields as his thesis advisor.

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## 1. RANDOM BLASCHKE PRODUCTS AND THE LITTLE BLOCH SPACE

The Bloch space is the vector space of holomorphic functions on the unit disk of  $\mathbf{C}$  such that  $|f'(z)| = O(1/(1 - |z|))$ . That is,

$$\mathbf{B} = \left\{ f \in \text{Hol}(D) : \sup_{z \in D} (1 - |z|) |f'(z)| < \infty \right\}.$$

Contained in  $\mathbf{B}$  is the "little" Bloch space  $\mathbf{B}_0$  consisting of those holomorphic functions satisfying the corresponding little-o condition:

$$\mathbf{B}_0 = \left\{ f \in \text{Hol}(D) : \lim_{|z| \rightarrow 1} (1 - |z|) |f'(z)| = 0 \right\}.$$

Notice that the limit in the definition above is necessarily uniform in  $\arg z$ .  $\mathbf{B}$  is a Banach space with the norm

$$\|f\| = \sup_{z \in D} (1 - |z|) |f'(z)| + |f(0)|$$

and  $\mathbf{B}_0$  is a closed subspace of  $\mathbf{B}$ .

These spaces play a role in the theory of Bergman spaces similar to that of BMOA and VMOA in the theory of Hardy spaces. (For the definition of BMOA and VMOA see Chapter 6 of [Gar81].) For example, there is a duality pairing such that  $(\mathbf{B}_0)^* = L_a^1$  and  $(L_a^1)^* = \mathbf{B}$ , where  $L_a^1$  is the Bergman space of analytic functions on  $D$  which are absolutely integrable with respect to area measure on  $D$ . This is analogous to the fact that  $\text{VMOA}^* = H^1$  and  $(H^1)^* = \text{BMOA}$ . These facts and others may be found in Anderson's survey article [And84].

It follows from the Cauchy integral formula that every bounded analytic function is in  $\mathbf{B}$ , but one can construct examples that show that  $H^\infty \not\subset \mathbf{B}_0$  and  $\mathbf{B}_0 \not\subset H^\infty$ . A particularly interesting class of examples can be obtained as follows: Let  $f(z) = \sum_{k=0}^\infty a_k z^{n_k}$  be a lacunary power series with  $n_{k+1}/n_k \geq \lambda > 1$ . Then  $f \in \mathbf{B}$  if and only if  $\{a_k\}$  is bounded, and  $f \in \mathbf{B}_0$  if and only if  $a_k \rightarrow 0$  [And84, Theorem 2, p. 5]. But if  $\{a_k\}$  is not square-summable then  $f(z)$  has radial limits almost nowhere; i.e.,  $\{\theta : \lim_{r \rightarrow 1} f(re^{i\theta}) \text{ exists}\}$  has Lebesgue measure 0 [Zyg59, Chapter V, Theorem 6.4, p. 203]. So if one chooses a sequence  $\{a_k\}$  such that  $a_k \rightarrow 0$  but  $\sum a_k^2 = \infty$ , then the corresponding function  $f(z)$  will be in the little Bloch space but not in  $H^p$  for any  $p$ . In fact  $f(z)$  will not even be in the Nevanlinna class.

Those functions in  $H^\infty$  which are also in the little Bloch space are rather special. It is known that  $f \in H^\infty \cap \mathbf{B}_0$  if and only if  $f$  is "constant on parts"; i.e., if  $f$  is regarded as being defined on the maximal ideal space  $\mathcal{M}$  of  $H^\infty$  (by  $f(\lambda) \equiv \lambda(f)$ ,  $\lambda \in \mathcal{M}$ ), then  $f$  is constant on every nontrivial Gleason part in  $\mathcal{M} \setminus D$  (see [Gar81, Chapter X, Problem 11, p. 442]). Because Blaschke products play a role in understanding the structure of  $\mathcal{M}$ , several authors have tried to describe exactly when Blaschke products will be in  $\mathbf{B}_0$ .

Here is an indirect proof that there exists an infinite Blaschke product in the little Bloch space (this argument is from [Sar84]). A continuous function  $g$  on

$[0, 2\pi]$  is in the little- $o$  Zygmund class  $\lambda_*$  if

$$\sup_{\theta \in [0, 2\pi]} |g(\theta + h) - 2g(\theta) + g(\theta - h)| = o(h)$$

as  $h \rightarrow 0$ . There is a corresponding big- $O$  class  $\Lambda_*$ . If  $f(z)$  is in the disk algebra then  $f' \in \mathbf{B}$  iff  $f(e^{i\theta}) \in \Lambda_*$  and  $f' \in \mathbf{B}_0$  iff  $f(e^{i\theta}) \in \lambda_*$  [Dur70, Chapter 5, Theorem 5.3, p.76]. There are constructions of continuous singular measures on  $[0, 2\pi]$  whose distribution functions are in  $\lambda_*$  (see [Kah69, pp. 188–192], for example). One can show that if  $\mu$  is such a singular measure then the singular inner function  $S_\mu(z)$  induced by  $\mu$  is in  $\mathbf{B}_0$  [AFS, Theorem 3, for example]. Using Frostman’s theorem one can then get an infinite Blaschke product in  $\mathbf{B}_0$ . So  $H^\infty \cap \mathbf{B}_0$  contains singular inner functions and Blaschke products.

Because the Blaschke products above were obtained in a nonconstructive manner, one is led to try to describe Blaschke products in  $\mathbf{B}_0$  explicitly in terms of their zeros. The zeros of such a Blaschke product must be well spread around the disk. Gregory Hungerford has shown in his doctoral thesis [Hung] that if  $B(z)$  is an infinite Blaschke product in  $\mathbf{B}_0$  then the set  $S$  of accumulation points of the zeros has Hausdorff dimension 1, proving a conjecture of Tom Wolff. Donald Sarason in [Sar84] asks the following two questions:

- (1) Can one explicitly construct or characterize infinite Blaschke products in  $\mathbf{B}_0$  in terms of their zero sets?
- (2) Will an infinite Blaschke product whose zeros are chosen randomly (in some sense) be in  $\mathbf{B}_0$  almost surely?

Chris Bishop [Bish 90] has characterized the infinite Blaschke products in  $\mathbf{B}_0$  in terms of the measure

$$\sigma_B \equiv \sum_{k : B(z_k)=0} (1 - |z_k|) \delta_{z_k}$$

and using this characterization he has given an explicit construction of a Blaschke product in  $\mathbf{B}_0$  in terms of its zeros. Other constructions of infinite Blaschke products in  $\mathbf{B}_0$  have been given by Ken Stephenson [Ste88] and Carl Sundberg (reference not available).

However, Theorem 1 below shows that if one interprets “random” in a natural sense, then Sarason’s second question has a negative answer.

**Theorem 1.** *Let  $\{r_n\}$  with  $0 \leq r_n < 1$  be a sequence of numbers satisfying the Blaschke condition  $\sum_{n=1}^\infty (1 - r_n) < \infty$ , and let  $\{\theta_n(\omega)\}$  be a sequence of independent random variables which are uniformly distributed on  $[0, 2\pi]$  (in other words,  $\{\theta_n\}$  is a Steinhaus sequence). Then with probability 1, the Blaschke product with zeros  $\{r_n e^{i\theta_n(\omega)}\}$  is not in the little Bloch space. In fact,*

$$\limsup_{n \rightarrow \infty} (1 - |z_n|^2) |B'(z_n)| = 1 \text{ almost surely.}$$

Although the random Blaschke product  $B_\omega(z)$  in the theorem above must fail Bishop’s characterization almost surely, the author does not see how to prove this directly.

The author originally showed that the Blaschke product  $B_\omega(z)$  is not in the little Bloch space by proving that the  $\limsup$  in Theorem 1 is positive almost surely. Walter Rudin subsequently pointed out to the author that his argument could be strengthened to show that the  $\limsup$  actually equals 1 a.s. The author later noticed that this stronger statement is a consequence of the fact that equation (1.2) in the proof below tends to 0 as  $n \rightarrow \infty$ .

The random points  $\{z_n(\omega)\}$  considered in the theorem above have the entire boundary of  $D$  as their set of accumulation points with probability 1, so the zeros are certainly well spread around. One way to see this is by the following argument: Let  $\zeta \in \partial D$  and  $\varepsilon > 0$  be given and let

$$\mathcal{N}(\zeta, \varepsilon) = \{z \in D : |z| > 1 - \varepsilon \text{ and } |\arg z - \arg \zeta| < \varepsilon\pi\}.$$

Then  $P\{\omega : z_n(\omega) \in \mathcal{N}(\zeta, \varepsilon)\} = \varepsilon$  for all  $n$  sufficiently large. So

$$\sum_n P(z_n \in \mathcal{N}(\zeta, \varepsilon)) = \infty.$$

Since the random variables  $z_n(\omega)$  are independent, the converse to the Borel-Cantelli Lemma (see the next section) implies that

$$P(z_n \in \mathcal{N}(\zeta, \varepsilon) \text{ for infinitely many } n) = 1.$$

By taking a countable sequence of  $\varepsilon$ 's tending to 0 one sees that  $\zeta$  is an accumulation point of the sequence  $(z_n)$  almost surely; that is, there is a set  $\Omega_\zeta \subset \Omega$ ,  $P(\Omega_\zeta) = 1$ , such that  $\zeta$  is an accumulation point of  $\{z_n(\omega)\}$  for all  $\omega \in \Omega_\zeta$ . If  $\{\zeta_n\}$  is a countable dense set in  $\partial D$  then for  $\omega \in \bigcap_n \Omega_{\zeta_n}$  the set of accumulation points of  $\{z_n(\omega)\}$  is dense in  $\partial D$ . Since the set of accumulation points is necessarily a closed set, it is all of  $\partial D$  almost surely.

The conclusion of Theorem 1 depends on the particular random model by which the sequence of zeros is chosen. One could also ask what will happen with different random models. The proof of the theorem presented below uses the uniform distribution of the random variables  $\theta_n$  and it is not clear to the author what one could say if the  $\theta_n$ 's had another distribution  $\mu$ . However one does have the following corollary, which says that the same conclusion as in Theorem 1 holds if one randomizes the radii  $\{r_n\}$  independently of the angles  $\{\theta_n\}$ .

**Corollary 1.** *Suppose that  $\{z_n(\omega)\}$  is a sequence of complex-valued random variables that satisfy the following conditions:*

- (1)  $|z_n(\omega)| < 1$  for all  $n$  and  $\sum(1 - |z_n(\omega)|) < \infty$  almost surely.
- (2)  $\{|z_n|\}_{n=1}^\infty$  is independent of  $\{\arg z_k\}_{k=1}^\infty$ .
- (3) The random variables  $\{\arg z_k(\omega)\}$  are independent and uniformly distributed on  $[0, 2\pi]$ .

*Then with probability one the Blaschke product with zeros  $\{z_n(\omega)\}$  is not in the little Bloch space.*

*Proof of Corollary 1.* The random variables  $\{z_n(\omega)\}$  are defined on a probability space  $(\Omega, \mathcal{A}, P)$ . The second hypothesis implies that  $(\Omega, \mathcal{A}, P)$  can be

written as a product probability space

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2, \quad \mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$$

such that for all  $n$ ,  $z_n(\omega) = r_n(\omega_1)e^{i\theta_n(\omega_2)}$  where the  $r_n$ 's are random variables defined on  $(\Omega_1, \mathcal{A}_1, \mathbf{P}_1)$  and the  $\theta_n$ 's are random variables defined on  $(\Omega_2, \mathcal{A}_2, \mathbf{P}_2)$ . The first hypothesis implies that

$$\mathbf{P}_1 \left\{ \omega_1 : \sum (1 - r_n(\omega_1)) < \infty \right\} = 1.$$

Call this set  $\Omega'_1$ . For each fixed  $\omega_1 \in \Omega'_1$ , Theorem 1 implies that

$$\mathbf{P}_2 \{ \omega_2 : B_{(\omega_1, \omega_2)}(z) \text{ is not in the little Bloch space} \} = 1.$$

If  $f(\omega_1, \omega_2)$  is the indicator function (or characteristic function) of the set

$$\{ (\omega_1, \omega_2) : \omega_1 \in \Omega'_1, B_{(\omega_1, \omega_2)}(z) \notin \mathbf{B}_0 \},$$

then by Fubini's Theorem

$$\mathbf{P} \{ \omega : B_\omega(z) \notin \mathbf{B}_0 \} = \int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2) d\mathbf{P}_2 d\mathbf{P}_1 = 1.$$

This proves Corollary 1.  $\square$

*Proof of Theorem 1.* Assume that  $\{r_n\}$  is indexed so that  $r_k \leq r_{k+1}$  for all  $k$ , and let  $B(\omega, z)$  denote the (random) Blaschke product with zeros  $z_n(\omega) = r_n e^{i\theta_n(\omega)}$ :

$$B_\omega(z) = \prod_{k=1}^{\infty} \left( \frac{-|z_k|}{z_k} \right) \left( \frac{z - z_k}{1 - \bar{z}_k z} \right).$$

To show that an infinite Blaschke product with zeros  $(z_n)$  is not in  $\mathbf{B}_0$  it suffices to show that

$$\limsup_{n \rightarrow \infty} (1 - |z_n|^2) |B'(z_n)| > 0.$$

We will show that

$$\limsup_{n \rightarrow \infty} (1 - |z_n|^2) |B'(z_n)| = 1.$$

A direct computation shows that

$$(1 - |z_n|^2) \cdot |B'(z_n)| = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right|$$

So we will prove that

$$\limsup_{n \rightarrow \infty} \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k(\omega) - z_n(\omega)}{1 - \bar{z}_n(\omega) z_k(\omega)} \right| = 1 \quad \text{a.s.},$$

or that

$$\limsup_{n \rightarrow \infty} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \log \left| \frac{z_k(\omega) - z_n(\omega)}{1 - \bar{z}_n(\omega) z_k(\omega)} \right| = 0 \quad \text{a.s.}$$

Let

$$Y_{k,n}(\omega) = \log \left| \frac{z_k(\omega) - z_n(\omega)}{1 - \bar{z}_n(\omega)z_k(\omega)} \right|,$$

$$Y_n(\omega) = \sum_{\substack{k=1 \\ k \neq n}}^{\infty} Y_{k,n}(\omega), \quad Y(\omega) = \limsup_{n \rightarrow \infty} Y_n(\omega).$$

The goal is to show that  $Y(\omega) = 0$  almost surely.

In the lemma below, the symbol  $\mathbf{E}$  denotes expected value—if  $X(\omega)$  is a (real-valued) random variable defined on a probability space  $(\Omega, \mathcal{A}, P)$ ,

$$\mathbf{E}(X) = \int_{\Omega} X(\omega) dP.$$

**Lemma.** For  $k \neq n$ ,  $\mathbf{E}(Y_{k,n}) = \max\{\log r_k, \log r_n\}$ .

*Proof.* Since  $\theta_k(\omega)$  and  $\theta_n(\omega)$  are independent and uniformly distributed on  $[0, 2\pi]$ ,

$$(1.1) \quad \mathbf{E}(Y_{k,n}) = \int_0^{2\pi} \int_0^{2\pi} \log \left| \frac{r_k e^{is} - r_n e^{it}}{1 - (r_n e^{-it})(r_k e^{is})} \right| \frac{ds}{2\pi} \frac{dt}{2\pi}.$$

Let

$$f(z) = \frac{z - z_n}{1 - \bar{z}_n z},$$

where  $z_n = r_n e^{it}$  is fixed. Then  $f$  is holomorphic in the unit disk with one zero at  $z_n$ . By Jensen's formula, if  $r_k \leq r_n$  then

$$\int_{|z|=r_k} \log |f(z)| \frac{d\theta}{2\pi} = \log |f(0)| = \log r_n,$$

while if  $r_k > r_n$  then

$$\begin{aligned} \int_{|z|=r_k} \log |f(z)| \frac{d\theta}{2\pi} &= \log |f(0)| + \log \frac{r_k}{r_n} \\ &= \log r_n + \log r_k - \log r_n \\ &= \log r_k. \end{aligned}$$

Therefore the inner integral in (1.1) is the constant  $\max\{\log r_k, \log r_n\}$ , so

$$\mathbf{E}(Y_{k,n}) = \max\{\log r_k, \log r_n\}. \quad \square$$

Now to prove the theorem, note that since the function  $f(z)$  above is an automorphism of the disk,  $Y_{k,n}(\omega)$  is negative for all  $\omega, k, n$ . So by the Monotone Convergence Theorem and the lemma above,

$$(1.2) \quad \begin{aligned} \mathbf{E}(Y_n) &= \sum_{k=1}^{n-1} \mathbf{E}(Y_{k,n}) + \sum_{k=n+1}^{\infty} \mathbf{E}(Y_{k,n}) \\ &= (n-1) \log r_n + \sum_{k=n+1}^{\infty} \log r_k. \end{aligned}$$

$\sum \log r_k$  converges because of the Blaschke condition and the fact that  $\log r_k \geq -2(1-r_k)$  for  $k$  sufficiently large. If  $\{a_n\}$  is a *monotone* sequence and  $\sum a_n < \infty$  then  $na_n \rightarrow 0$  (to see this sum from  $n$  to  $2n$ ). Thus in equation (1.2)  $\log r_n = o(1/n)$ , and so  $\mathbf{E}(Y_n)$  tends to 0 as  $n \rightarrow \infty$ . Applying Fatou's Lemma to the positive sequence  $-Y_n(\omega)$  one has

$$\begin{aligned} \mathbf{E} \left( \liminf_{n \rightarrow \infty} (-Y_n) \right) &\leq \liminf_{n \rightarrow \infty} \mathbf{E}(-Y_n) \\ &\Rightarrow \mathbf{E} \left( \limsup_{n \rightarrow \infty} (Y_n) \right) \geq \limsup_{n \rightarrow \infty} \mathbf{E}(Y_n) = 0. \end{aligned}$$

So  $\mathbf{E}(Y) \geq 0$ . But  $Y(\omega) \leq 0$  for all  $\omega \in \Omega$  so therefore  $Y(\omega) = 0$  almost surely. It follows that

$$\limsup_{n \rightarrow \infty} (1 - |z_n|^2) |B'(z_n)| = 1 \quad \text{a.s.} \quad \square$$

## 2. RANDOM BLASCHKE PRODUCTS AND INTERPOLATING SEQUENCES

A sequence  $\{z_n\} \subset D$  is said to be an interpolating sequence (for  $H^\infty$ ) if for every  $\{\alpha_n\} \in l^\infty$  there exists some function  $f \in H^\infty$  such that  $f(z_n) = \alpha_n$  for all  $n$ . A Blaschke product is said to be interpolating if its zero set is an interpolating sequence. In addition to their role in interpolation theory, interpolating Blaschke products also play an important role in describing the analytic structure of the maximal ideal space of  $H^\infty$ . A good reference for this material is Chapters 7 and 10 of Garnett's book [Gar81].

Given a sequence  $\{z_n\}$  in the disk, let

$$P_n = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \rho(z_k, z_n).$$

The symbol  $\rho$  above refers to the pseudo-hyperbolic metric in the disk

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|, \quad z, w \in D.$$

Since  $\rho(z, w) < 1$  for points  $z$  and  $w$  in the unit disk,  $0 \leq P_n < 1$  for all  $n$ . If the points  $\{z_n\}$  satisfy the Blaschke condition then  $0 < P_n < 1$  for all  $n$ . L. Carleson has completely described the interpolating sequences in terms of the numbers  $P_n$  [Car58]:

**Carleson's interpolation theorem.** *A sequence  $\{z_n\}$  is interpolating if and only if  $\inf_n P_n > 0$ .*

If a sequence of points satisfies the condition in Carleson's theorem, then most of the points must be far apart in the pseudo-hyperbolic metric. If  $\{z_n\}$  is interpolating, then for  $\delta$  sufficiently small the level set  $|B(z)| = \delta$  of the corresponding Blaschke product is a union of disjoint curves, each surrounding

exactly one point  $z_n$  [Hof67]. For this reason a sequence satisfying  $\inf_n P_n > 0$  is said to be *uniformly separated*.

A sequence  $\{z_n\}$  of points in  $D$  is said to be interpolating for  $H^p$  if the map

$$f \mapsto \{f(z_n)(1 - |z_n|)^{1/p}\}$$

maps  $H^p$  onto  $l^p$ . In [SSh61] H. S. Shapiro and Shields discovered that the same uniform separation condition also characterizes the sequences which are interpolating for  $H^p$ , for any  $p$  with  $1 \leq p < \infty$ . So the Hardy spaces  $H^p$  all have the same interpolating sequences.

There is another characterization of interpolating sequences which is useful. For a proof of this next Proposition see [Gar81, Chapter VII, Theorem 1.1].

**Proposition.** *A sequence  $\{z_n\}$  of points in the disk is interpolating if and only if both*

- (1)  $\{z_n\}$  is weakly separated, in the sense that

$$\inf_{\substack{k, n \\ k \neq n}} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| > 0$$

and

- (1) the measure  $\sigma_{\{z_n\}} \equiv \sum_n (1 - |z_n|) \delta_{z_n}$  is a Carleson measure on  $D$ .

One can produce examples of interpolating sequences fairly easily. If  $\{z_n\}$  is a sequence of points in  $D$  such that  $|z_n| \rightarrow 1$  exponentially fast, in the sense that there is a number  $\lambda < 1$  such that

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} \leq \lambda < 1 \quad \text{for all } n,$$

then  $\{z_n\}$  must be interpolating. If the points  $z_n$  all lie on one radius, then for  $\{z_n\}$  to be interpolating it is both necessary and sufficient that  $|z_n| \rightarrow 1$  exponentially fast. On the other hand, if  $\{r_n\}$  only satisfies the Blaschke condition  $\sum(1 - r_n) < \infty$  then one can always choose angles  $\theta_n$  so that  $\{r_n e^{i\theta_n}\}$  is an interpolating sequence. This result, due to A. G. Naftalevič, is contained in the third section of this paper. One can also use the two conditions in the proposition above to obtain a geometric characterization of interpolating sequences. See the section on "generations" in Garnett's book [Gar81, Chapter VII, §3].

Recall from the previous section that if a Blaschke product is in the little Bloch space it is necessary that  $P_n \rightarrow 0$  as  $n \rightarrow \infty$ . So interpolating Blaschke products are in some sense a long way from being in the little Bloch space. In Theorem 1 above it was shown that if  $\{z_n\}$  is chosen randomly in the manner described then

$$\limsup_{n \rightarrow \infty} P_n = 1 \quad \text{a.s.},$$

so one might ask whether these randomly chosen points are interpolating almost surely. The answer is yes and no, depending on how fast the radii  $r_n$  approach 1.

Given a sequence of radii  $\{r_n\}$  let

$$N_k = \#\{r_j : 1 - 2^{-(k-1)} \leq r_j < 1 - 2^{-k}\},$$

where  $\#S$  denotes the cardinality of the set  $S$ . It is easy to see that the Blaschke condition  $\sum(1 - r_n) < \infty$  is equivalent to the condition  $\sum_{k=1}^{\infty} N_k 2^{-k} < \infty$ .

**Theorem 2.** Let  $z_n(\omega) = r_n e^{i\theta_n(\omega)}$  where  $\{\theta_n\}$  is a Steinhaus sequence. Then

$$P(\{z_n(\omega)\} \text{ is weakly separated}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{if and only if} \\ \sum_{k=1}^{\infty} N_k^2 2^{-k} \begin{pmatrix} < \infty \\ = \infty \end{pmatrix}.$$

**Theorem 3.** If  $\sum N_k 2^{-(1/3)k} < \infty$  then  $\sigma_{\{z_n\}}$  is a Carleson measure almost surely.

In particular, if  $\sum N_k 2^{-(1/3)k} < \infty$  then  $z_n(\omega)$  is interpolating almost surely, and if  $\sum N_k^2 2^{-k} = \infty$  then  $z_n(\omega)$  is not even weakly separated a.s. Either of these situations can occur for radii satisfying the Blaschke condition.

The author doesn't believe that the condition in Theorem 3 is optimal, but he has not been able to strengthen it or to find a condition on the radii (besides the failure of the Blaschke condition) implying that  $\sigma_{\{z_n\}}$  is not a Carleson measure a.s. However the event that  $\sigma_{\{z_n\}}$  is a Carleson measure is a tail event with respect to the  $\theta_n$ 's, so Kolmogorov's Zero-One Law produces the following proposition:

**Proposition 4.** For any sequence  $\{r_n\}$  satisfying the Blaschke condition  $\sum(1 - r_n) < \infty$ ,

$$P\{\omega : \sigma_{\{z_n(\omega)\}} \text{ is a Carleson measure}\} = 0 \text{ or } 1.$$

The proof of Proposition 4 will be given after the proof of Theorem 3.

The proof of Theorem 2 relies on the Borel-Cantelli Lemma and an interesting fact about the minimum distance between randomly chosen points. For the sake of completeness these tools will be stated as two lemmas below.

Given a sequence  $(A_k)$  of subsets of  $\Omega$ , define

$$\limsup A_k \equiv \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{\omega : \omega \in A_k \text{ for infinitely many } k\}$$

and

$$\liminf A_k \equiv \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \{\omega : \omega \in A_k \text{ for all but finitely many } k\}.$$

The next “lemma” is a fundamental tool throughout probability. For a proof see any measure-theory based probability textbook—for example, [Bil86] or [Chu74].

**The Borel-Cantelli Lemma.**

- (a) If  $\sum P(A_k) < \infty$  then  $P(\limsup A_k) = 0$ .
- (b) If the sets  $A_k$  are independent and  $\sum P(A_k) = \infty$  then  $P(\limsup A_k) = 1$ .

One can find the following probability lemma in Parzen’s book [Par60, Chapter VII, §7, pp. 304, 306].

**Lemma** (the probability of an uncrowded road). *Suppose  $n$  points are distributed randomly in an interval of length  $L$  (“distributed randomly” means “independent and uniformly distributed”). Then the probability that no two points are closer than  $d$  units apart is*

$$\left(1 - \frac{(n-1)d}{L}\right)^n \quad \text{for } (n-1)d \leq L.$$

The corresponding probability for a circle of circumference  $L$  is

$$\left(1 - \frac{nd}{L}\right)^{n-1} \quad \text{for } nd \leq L.$$

*Proof of Theorem 2.* To prove Theorem 2, first assume that  $\sum_{k=1}^{\infty} N_k^2 2^{-k} < \infty$ . If  $\theta$  and  $\psi$  are two numbers in the interval  $[0, 2\pi]$  then the notation “ $|\theta - \psi| \pmod{2\pi}$ ” refers to the length of the shorter arc on  $\partial D$  between the points  $e^{i\theta}$  and  $e^{i\psi}$ .

For each  $k = 1, 2, \dots$ , let

$$I_k = [1 - 2^{-(k-1)}, 1 - 2^{-k}]$$

and let  $\Omega_k$  be the set of all  $\omega$  in  $\Omega$  such that there exists two distinct indices  $i, j$ , with both  $r_i$  and  $r_j$  in  $I_k$  and with  $|\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-k} \pmod{2\pi}$ . Symbolically,

$$(2.1) \quad \Omega_k = \{ \omega \in \Omega : \exists (i, j), i \neq j, \text{ with } r_i, r_j \in I_k \text{ and } |\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-k} \pmod{2\pi} \}$$

$$(2.2) \quad = \bigcup_{\substack{i \neq j \\ r_i, r_j \in I_k}} \{ \omega : |\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-k} \pmod{2\pi} \}.$$

For each fixed  $t \in [0, 2\pi]$ ,

$$P(|\theta_i(\omega) - t| \leq \pi 2^{-k} \pmod{2\pi}) = 2^{-k}$$

since  $\theta_i$  is uniformly distributed. By independence the probability of each of the sets in the union (2.2) is also  $2^{-k}$ . So

$$P(\Omega_k) \leq \binom{N_k}{2} 2^{-k} \leq \frac{1}{2} N_k^2 2^{-k}.$$

By the Borel-Cantelli Lemma,

$$P\{\omega : \omega \text{ is in infinitely many } \Omega_k \text{'s}\} = 0.$$

Put into words, for almost every  $\omega$ , for all but finitely many  $k$  (depending on  $\omega$ ),  $|\theta_i - \theta_j| > \pi 2^{-k} \pmod{2\pi}$  for all  $i \neq j$ ,  $r_i, r_j \in I_k$ .

Given a point  $z$  in the disk with  $|z| \in I_k$ , let

$$R(z) \equiv \{re^{i\theta} : r \in I_k, |\theta - \arg z| \leq \pi 2^{-k}\}.$$

The sets  $R(z)$  are inner halves of Carleson rectangles. If  $\omega \notin \limsup \Omega_k$  then for all but finitely many  $j$ ,  $z_i(\omega) \notin R(z_j(\omega))$  for all  $i \neq j$ . So with probability 1 only finitely many of the "rectangles"  $R(z_j(\omega))$  contain a point of the sequence  $\{z_n\}$  besides  $z_j$  itself.

For  $k = 1, 2, \dots$ , let  $I'_k$  be the subinterval of  $(0, 1]$  whose endpoints are the midpoints of the intervals  $I_k$  and  $I_{k+1}$ . Then  $\#\{r_j : r_j \in I'_k\} \leq N_k + N_{k+1}$ . If  $\Omega'_k$  is the set defined by replacing  $I_k$  by  $I'_k$  in (2.1), then

$$\begin{aligned} P(\Omega'_k) &\leq \binom{N_k + N_{k+1}}{2} 2^{-k} \leq \frac{1}{2} (N_k + N_{k+1})^2 2^{-k} \\ &\leq (N_k^2 + N_{k+1}^2) 2^{-k} = N_k^2 2^{-k} + 2N_{k+1}^2 2^{-(k+1)}. \end{aligned}$$

In particular,  $\sum P(\Omega'_k) < \infty$ .

Given  $z \in D$  with  $|z| \in I'_k$ , let

$$R'(z) \equiv \{re^{i\theta} : r \in I'_k, |\theta - \arg z| \leq \pi 2^{-k}\}.$$

By repeating the argument above, one obtains that again with probability 1, only finitely many of the "rectangles"  $R'(z_j(\omega))$  contain more than one point of the sequence  $\{z_n\}$ .

Now let  $D_r(a)$  be the pseudo-hyperbolic disk centered at  $a$  with radius  $r$ :

$$D_r(a) = \{z \in D : \rho(z, a) < r\}.$$

*Claim.* There is a universal constant  $\delta > 0$  such that for every  $z$  in  $D$ , the pseudo-hyperbolic disk centered at  $z$  of radius  $\delta$  is entirely contained in one of the "rectangles"  $R(z)$  or  $R'(z)$ .

*Proof of Claim.* The pseudo-hyperbolic disk  $D_r(a)$  is a Euclidean disk with (Euclidean) radius

$$\gamma = r \frac{1 - |a|^2}{1 - |a|^2 r^2},$$

so

$$\text{the diameter of } D_r(a) < \left(\frac{2r}{1 - r^2}\right) (1 - |a|^2).$$

For  $k = 1, 2, \dots$ , partition the interval  $I_k$  into four consecutive subintervals of equal length and denote these subintervals by  $I_{k,j}$ ,  $j = 1, 2, 3, 4$ . Fix  $z$

in  $D$  and define

$$S(z) \equiv \begin{cases} R(z), & \text{if } |z| \in I_{k,2} \cup I_{k,3}; \\ R'(z), & \text{if } |z| \in I_{k,4}; \\ R'(z), & \text{if } |z| \in I_{k,1} \text{ and } k \geq 2; \\ R(z), & \text{if } |z| < 1/8. \end{cases}$$

In all cases one finds that

$$\text{dist}(z, \partial S(z)) \geq \frac{1}{4} 2^{-k} \geq \frac{1}{8} (1 - |z|).$$

Let  $\delta = \sqrt{1 + 16^2} - 16$ . Then  $2\delta/(1 - \delta^2) = 1/16$  and so

$$\text{diam } D_\delta(z) < \frac{1}{16} (1 - |z|^2) \leq \frac{1}{8} (1 - |z|) \leq \text{dist}(z, \partial S(z)).$$

Therefore  $D_\delta(z) \subset S(z)$  for all  $z \in D$  which proves the claim.

Thus if  $\omega \notin (\limsup \Omega_k) \cup (\limsup \Omega'_k)$ , then  $\rho(z_i(\omega), z_j(\omega)) \geq \delta$  for all but finitely many  $(i, j)$ ,  $i \neq j$ . For such  $\omega$ 's,

$$\inf_{\substack{i,j \\ i \neq j}} \rho(z_i(\omega), z_j(\omega)) \geq C(\omega) > 0$$

and so

$$\{ \omega : \{z_n(\omega)\} \text{ is weakly separated} \} \supset [(\limsup \Omega_k) \cup (\limsup \Omega'_k)]^C.$$

Thus

$$P(\{z_n\} \text{ is weakly separated}) \geq 1 - P(\limsup \Omega_k) - P(\limsup \Omega'_k) = 1.$$

Now assume that  $\sum_{k=1}^\infty N_k^2 2^{-k} = \infty$

A sequence  $\{z_n\} \subset D$  is *not* weakly separated if and only if for every  $l$ ,  $l = 0, 1, \dots$ , there is a pair of indices  $(i, j)$ ,  $i \neq j$ , such that  $\rho(z_i, z_j) \leq 2^{-l}$ . So

$$\{ \omega : \{z_n(\omega)\} \text{ is not weakly separated} \} = \bigcap_{l=0}^\infty A_l,$$

where

$$A_l = \{ \omega : \exists (i, j), i \neq j, \text{ with } \rho(z_i(\omega), z_j(\omega)) \leq 2^{-l} \}.$$

We will show that for each  $l$ ,  $P(A_l) = 1$ .

Fix  $l \geq 0$ . For each  $k = 1, 2, \dots$ , partition the interval  $I_k$  into  $2^l$  consecutive subintervals, each of length  $2^{-(k+l)}$ . Since there are  $N_k$  points of the sequence  $\{r_n\}$  in  $I_k$ , at least one of these subintervals must contain at least  $N_k/2^l$  points of  $\{r_n\}$ . Let  $J_k$  be such a subinterval and let  $M_k$  be the number of points of  $\{r_n\}$  that it contains, so that

$$\frac{N_k}{2^l} \leq M_k \leq N_k.$$

In particular  $\sum M_k 2^{-k} < \infty$  since the Blaschke condition  $\sum(1 - r_n) < \infty$  is equivalent to  $\sum N_k 2^{-k} < \infty$ .

Let

$$B_k = \{\omega : \exists (i, j), i \neq j, \text{ with } r_i, r_j \in J_k \\ \text{and } |\theta_i(\omega) - \theta_j(\omega)| < \pi 2^{-(k+l)} \pmod{2\pi}\}.$$

Apply the lemma on the probability of an uncrowded road with  $n = M_k$ ,  $d = \pi 2^{-(k+l)}$ ,  $L = 2\pi$ , and use the fact that  $M_k 2^{-k} \rightarrow 0$  to obtain

$$P(B_k^C) = (1 - M_k 2^{-(k+l+1)})^{M_k - 1}$$

for all  $k$  sufficiently large. So

$$P(B_k) = 1 - (1 - M_k 2^{-(k+l+1)})^{M_k - 1}$$

for  $k$  sufficiently large.

*Claim.*  $\sum_{k=1}^{\infty} P(B_k) = \infty$ .

*Proof of Claim.*

$$\begin{aligned} & \sum_{k=1}^{\infty} N_k^2 2^{-k} = \infty \\ \Rightarrow & \sum_{k=1}^{\infty} \left(\frac{N_k}{2^l}\right)^2 (2^{-k} 2^{-(l+1)}) = \infty \\ \Rightarrow & \sum_{k=1}^{\infty} M_k \cdot (M_k 2^{-(k+l+1)}) = \infty \quad (\text{since } M_k \geq N_k/2^l) \\ \Rightarrow & \sum_{k=1}^{\infty} M_k \log(1 - M_k 2^{-(k+l+1)}) = -\infty \quad (\text{since } \log(1 - x) \leq -x) \\ \Rightarrow & \log\left(\prod_{k=1}^{\infty} (1 - M_k 2^{-(k+l+1)})^{M_k}\right) = -\infty \\ \Rightarrow & \prod_{k=1}^{\infty} (1 - M_k 2^{-(k+l+1)})^{M_k} = 0 \\ \Rightarrow & \prod_{k=1}^{\infty} [(1 - M_k 2^{-(k+l+1)})^{M_k - 1} \cdot (1 - M_k 2^{-(k+l+1)})] = 0 \\ \Rightarrow & \prod_{k=1}^{\infty} [1 - M_k 2^{-(k+l+1)}]^{M_k - 1} = 0 \\ & \left(\prod (1 - M_k 2^{-(k+l+1)}) > 0 \text{ since } \sum M_k 2^{-k} \leq \sum N_k 2^{-k} < \infty\right) \\ \Rightarrow & \sum_{k=1}^{\infty} 1 - (1 - M_k 2^{-(k+l+1)})^{M_k - 1} = \infty \\ \Rightarrow & \sum_{k=1}^{\infty} P(B_k) = \infty. \end{aligned}$$

For each  $k = 1, 2, \dots$ , let  $\mathcal{A}_k$  be the  $\sigma$ -algebra generated by the random variables  $\{\theta_i : r_i \in J_k\}$ . Since  $\{\theta_n\}$  is an independent family of random variables, the  $\sigma$ -algebras  $\mathcal{A}_k$  are independent. But  $B_k \in \mathcal{A}_k$  for all  $k$ , so the sets  $(B_k)$  are independent. By the converse to the Borel-Cantelli Lemma,  $P(\limsup B_k) = 1$ .

If  $\omega \in \limsup B_k$  then there are infinitely many  $k$  for which there is a pair of distinct points  $z_i(\omega), z_j(\omega)$  in some Carleson "rectangle" with side lengths  $2^{-(k+l)}$  by  $\pi 2^{-(k+l)}$  with  $|z_i|, |z_j| \in I_k$ . For such a pair  $(z_i, z_j)$ , one has  $|z_i - z_j| \leq 5 \cdot 2^{-l} 2^{-k}$  and  $|1 - \overline{z_j} z_i| \geq 1 - |z_j| |z_i| \geq 2^{-k}$ . So  $\rho(z_i, z_j) \leq 5 \cdot 2^{-l}$ . Therefore

$$P\{\omega : \exists (i, j), i \neq j, \text{ with } \rho(z_i(\omega), z_j(\omega)) \leq 5 \cdot 2^{-l}\} = 1.$$

By letting  $l$  approach infinity, one obtains

$$P\{\omega : \{z_n(\omega)\} \text{ is not weakly separated}\} = 1.$$

This finishes the proof of Theorem 2.  $\square$

*Proof of Theorem 3.* A sequence  $\{z_n\}_{n=1}^\infty \subset D$  will be called a "Carleson sequence" if the measure  $\mu \equiv \sum_{n=1}^\infty (1 - |z_n|) \delta_{z_n}$  is a Carleson measure. The proof presented below uses the following two lemmas.

**Lemma 2.2.** *A sequence  $\{z_n\}$  is a Carleson sequence if and only if*

$$\sup_n \sum_{k=1}^\infty \frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \overline{z_n} z_k|^2} < \infty.$$

For a proof, see [Gar81, Chapter VI, Lemma 3.3, p. 239].

**Lemma 2.3.** *Let  $z_k = r_k e^{i\theta_k}$  and  $z_n = r_n e^{i\theta_n}$  with  $r_k$  and  $r_n \geq 1/2$ . Then  $|1 - \overline{z_n} z_k|^2 \geq |\theta_k - \theta_n|^2 / \pi^2$ , where  $|\theta_k - \theta_n|$  denotes the length of the shorter arc on  $\partial D$  between  $e^{i\theta_k}$  and  $e^{i\theta_n}$ .*

*Proof of Lemma 2.3.*

$$\begin{aligned} |1 - \overline{z_n} z_k|^2 &= 1 + r_k^2 r_n^2 - 2r_k r_n \cos(\theta_k - \theta_n) \\ &= (1 - r_k r_n)^2 + 2r_k r_n (1 - \cos(\theta_k - \theta_n)) \\ &= (1 - r_k r_n)^2 + 4r_k r_n \sin^2\left(\frac{\theta_k - \theta_n}{2}\right). \end{aligned}$$

If  $-\pi \leq t \leq \pi$  then  $|\sin(t/2)| \geq |t|/\pi$  (draw graphs). It follows that

$$|1 - \overline{z_n} z_k|^2 \geq 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{\pi^2} |\theta_k - \theta_n|^2.$$

This proves Lemma 2.3.  $\square$

We now prove Theorem 3. By Lemma 2.2 it is sufficient (and necessary) to show that

$$(2.3) \quad \sup_n \sum_{k=1}^{\infty} \frac{(1-r_k^2)(1-r_n^2)}{|1-\overline{z_n(\omega)}z_k(\omega)|^2} \leq M(\omega) < \infty \quad \text{a.s.},$$

where  $M(\omega)$  may depend on  $\omega$  but is independent of  $n$ .

Fix a positive integer  $n$ . Recall that

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{wz}|^2} = 1 - \rho(z, w)^2,$$

where  $\rho(z, w)$  is the pseudo-hyperbolic distance between  $z$  and  $w$ . Also  $\rho(z, w)$  is rotationally invariant—for any  $\theta$ ,  $\rho(z, w) = \rho(e^{i\theta}z, e^{i\theta}w)$ . So for all  $\omega \in \Omega$ ,

$$\rho(z_k(\omega), z_n(\omega)) = \rho(e^{-i\theta_n(\omega)}z_k(\omega), r_n).$$

Since the distribution of  $z_k(\omega)$  is rotationally invariant and since  $\theta_n$  and  $\{\theta_k\}_{k \neq n}$  are independent, it follows that for  $k \neq n$  the random variable  $e^{i\theta_n(\omega)}z_k(\omega)$  is again uniformly distributed on the circle  $|z| = r_k$ . Hence in equation (2.3) one may assume without loss of generality that  $\theta_n = 0$  or  $z_n = r_n$ .

Fix some number  $\alpha$  such that  $0 < \alpha < 1$ . (We will see later that the best choice for  $\alpha$  is  $\alpha = 1/3$ .) For  $k = 1, 2, \dots$ , let

$$\begin{aligned} \Omega_k &= \{ \omega \in \Omega : \exists j \neq n \text{ with } r_j \in I_k \text{ and } |\theta_j(\omega)| \leq \pi 2^{-\alpha k} \} \\ &= \bigcup_{\substack{j: r_j \in I_k \\ j \neq n}} \{ \omega : |\theta_j(\omega)| \leq \pi 2^{-\alpha k} \}. \end{aligned}$$

(This is not the same set as the set  $\Omega_k$  defined in the proof of Theorem 2.) Since  $\theta_j(\omega)$  is uniformly distributed on  $[0, 2\pi]$  then

$$P(|\theta_j| \leq \pi 2^{-\alpha k}) = 2^{-\alpha k}$$

and so

$$P(\Omega_k) \leq \sum_{\substack{j: r_j \in I_k \\ j \neq n}} P(|\theta_j| \leq \pi 2^{-\alpha k}) \leq N_k 2^{-\alpha k}.$$

Let  $A_k$  be the annulus  $\{z : |z| \in I_k\}$  and let  $S_k$  be the sector  $A_k \cap \{z : |\arg z| \leq \pi 2^{-\alpha k}\}$ . If one assumes that  $\sum_{k=1}^{\infty} N_k 2^{-\alpha k} < \infty$ , then

$$P(\limsup \Omega_k) = 0$$

by the Borel-Cantelli Lemma. But if  $\omega \notin \limsup \Omega_k$  then for all but finitely many  $k$  the sector  $S_k$  contains no points of the sequence  $\{z_j(\omega)\}$ . So for

all  $\omega \notin \limsup \Omega_k$ , there are only a finite number  $C(\omega)$  of  $z_j$ 's that lie in  $\bigcup_k S_k$ . Suppose  $z_j \in A_k \setminus S_k$ . Then by Lemma 2.3,

$$\frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n z_k|^2} \leq \frac{4 \cdot 2^{-k}(1 - r_n^2)}{2^{-2\alpha k}} \leq 4(1 - r_n^2)2^{-(1-2\alpha)k}.$$

Putting everything together one sees that if  $\omega \notin \limsup \Omega_k$  then

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{(1 - r_j^2)(1 - r_n^2)}{|1 - \bar{z}_n(\omega)z_j(\omega)|^2} &= \sum_{k=1}^{\infty} \sum_{j: r_j \in I_k} \left[ \quad \right] \leq \sum_{k=1}^{\infty} \left[ \sum_{z_j \in S_k} + \sum_{z_j \in A_k \setminus S_k} \right] \\ (2.4) \qquad \qquad \qquad &\leq \sum_{k=1}^{\infty} \left[ \sum_{z_j \in S_k} 1 + 4(1 - r_n^2) \sum_{z_j \in A_k \setminus S_k} 2^{-(1-2\alpha)k} \right] \\ &\leq C(\omega) + 4 \sum_{k=1}^{\infty} N_k 2^{-(1-2\alpha)k}. \end{aligned}$$

Thus if one assumes that both  $\sum N_k 2^{-\alpha k}$  and  $\sum N_k 2^{-(1-2\alpha)k}$  are finite, then with probability 1 (2.4) is smaller than some  $M(\omega) < \infty$  where  $M(\omega)$  is independent of  $n$ . These assumptions are equivalent to the hypothesis that  $\sum N_k 2^{-\phi(\alpha) \cdot k} < \infty$ , where  $\phi(\alpha) = \min(\alpha, 1 - 2\alpha)$ . The weakest possible hypothesis that works for this particular proof is then obtained by choosing  $\alpha$  to be the value that maximizes  $\phi(\alpha)$  over the interval  $(0, 1)$ . One can check that  $\phi(\alpha)$  attains its maximum value at  $\alpha = 1/3$  and that the corresponding hypothesis is that  $\sum_{k=1}^{\infty} N_k 2^{-(1/3)k} < \infty$ .

So if  $\sum_{k=1}^{\infty} N_k 2^{-(1/3)k} < \infty$  then  $(z_n(\omega))$  is Carleson a.s., and the proof of Theorem 3 is finished.  $\square$

*Proof of Proposition 4.* The proof of Proposition 4 relies on Kolmogorov's Zero-One Law. To state the law, we first define the notion of a "tail event". Given a sequence  $\mathcal{A}_k$  of  $\sigma$ -algebras of subsets of  $\Omega$ , the tail  $\sigma$ -algebra  $\mathcal{A}$  is

$$\mathcal{A} \equiv \bigcap_{n=1}^{\infty} \left[ \sigma \left( \bigcup_{k=n}^{\infty} \mathcal{A}_k \right) \right],$$

where  $\sigma(\bigcup \mathcal{A}_k)$  is the smallest  $\sigma$ -algebra containing  $\bigcup \mathcal{A}_k$ . (Recall that an intersection of  $\sigma$ -algebras is a  $\sigma$ -algebra but in general a union of  $\sigma$ -algebras is not.) A subset  $A$  of  $\Omega$  is a tail event with respect to the sequence  $\mathcal{A}_k$  if  $A$  is in the tail  $\sigma$ -algebra.

**Theorem** (Kolmogorov's zero-one law). *If  $\mathcal{A}_k$  is a sequence of independent  $\sigma$ -algebras then any tail event has probability 0 or 1.*

For a proof of this theorem see [Bil86] or [Chu74].

Now let  $\{r_n\}$  be a sequence of numbers in  $(0, 1)$  satisfying the Blaschke condition, let  $z_n(\omega) = r_n e^{i\theta_n(\omega)}$  where  $\theta_n(\omega)$  are independent random variables

uniformly distributed on  $[0, 2\pi]$ , and let

$$\sigma_{\{z_n\}} = \sum_n (1 - |z_n|) \delta_{z_n}.$$

By Lemma 2.2 above,  $\sigma_{\{z_n\}}$  is a Carleson measure almost surely if and only if

$$\sup_n S_n(\omega) < \infty,$$

where

$$S_n(\omega) = \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n z_k|^2}.$$

For each  $n$ ,

$$\frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n z_k|^2} \leq 4 \frac{(1 - r_k)(1 - r_n)}{(1 - r_n r_k)^2} \leq \frac{4}{1 - r_n} (1 - r_k),$$

so  $S_n(\omega) < \infty$  for every  $n$  and every  $\omega$  by the Blaschke condition.

Fix a positive integer  $N$ . Since

$$\frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n z_k|^2} = 1 - \rho(z_k, z_n)^2 < 1,$$

then for each  $n \geq N$

$$\begin{aligned} S_n(\omega) &= \sum_{k=1}^{N-1} \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n z_k|^2} + \sum_{k=N}^{\infty} \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n z_k|^2} \\ &\leq (N - 1) + \sum_{k=N}^{\infty} \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n z_k|^2}. \end{aligned}$$

So  $\sup_n S_n < \infty$  if and only if

$$\sup_{n \geq N} \sum_{k=N}^{\infty} \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n z_k|^2} < \infty,$$

or stated in terms of sets,

$$(2.5) \quad \left\{ \omega : \sigma_{\{z_n\}} \text{ is a Carleson measure} \right\} = \left\{ \omega : \sup_{n \geq N} \sum_{k=N}^{\infty} \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n z_k|^2} < \infty \right\}.$$

For each  $k$  and  $n$  the random variable

$$X_{k,n}(\omega) \equiv \frac{(1 - r_k^2)(1 - r_n^2)}{|1 - \bar{z}_n(\omega) z_k(\omega)|^2}$$

is measurable with respect to the  $\sigma$ -algebra  $\sigma(\mathcal{F}_{\theta_k} \cup \mathcal{F}_{\theta_n})$ . So for each fixed  $N$ , the random variable

$$X_N(\omega) \equiv \sup_{n \geq N} \sum_{k=N}^{\infty} X_{k,n}(\omega)$$

is measurable with respect to the  $\sigma$ -algebra

$$\mathcal{A}_N \equiv \sigma \left( \bigcup_{n=N}^{\infty} \mathcal{F}_{\theta_n} \right),$$

and so  $\{\omega : X_N(\omega) < \infty\} \in \mathcal{A}_N$  for all  $N$ .

Therefore by equation (2.5),  $\{\omega : \sigma_{\{z_n\}}$  is a Carleson measure  $\}$  lies in the tail  $\sigma$ -algebra  $\bigcap_N \mathcal{A}_N$  generated by the  $\sigma$ -algebras  $\mathcal{F}_{\theta_n}$ , and since these  $\sigma$ -algebras are independent,

$$P(\sigma_{\{z_n\}} \text{ is a Carleson measure}) = 0 \text{ or } 1$$

by Kolmogorov's 0-1 Law.  $\square$

### 3. NAFTALEVIČ'S INTERPOLATION THEOREM

In this section we will present a proof of the following theorem of A. G. Naftalevič. [Naf56, Lemma 2, pp. 13–14 and 17–19].

**Theorem 5** (Naftalevič). *Let  $\{r_n\}$ ,  $0 < r_n < 1$ , be a sequence of numbers satisfying the Blaschke condition  $\sum_{n=1}^{\infty} (1 - r_n) < \infty$ . Then there is an interpolating sequence of complex numbers  $(z_n)$  with  $|z_n| = r_n$  for every  $n$ .*

It is interesting to compare this result with Theorem 2 of §2. Suppose that the sequence  $\{r_n\}$  of radii satisfies the conditions  $\sum N_k 2^{-k} < \infty$  and  $\sum N_k^2 2^{-k} = \infty$ . Then one can choose (nonrandom) angles  $\theta_n$  so that the sequence  $z_n = r_n e^{i\theta_n}$  is interpolating, although if one chooses the angles  $\theta_n$  randomly in  $[0, 2\pi]$  then with probability one the sequence  $\{z_n\}$  is not even weakly separated.

The proof that is presented below is essentially the same as that contained in [Naf56]. It has been reorganized and additional details are provided. It is presented here because the journal containing [Naf56] is rather obscure and not readily available in the United States. Allen Shields gave the author a copy of the paper containing this theorem.

*Proof of Theorem 5.* Let  $\{r_n\}$  satisfy the Blaschke condition. Recall the notation used in §2:

$$I_k = [1 - 2^{-(k-1)}, 1 - 2^{-k}), \quad N_k = \#\{n : r_n \in I_k\}.$$

We first modify the sequence  $\{r_n\}$ :

**Lemma 3.1.** *Without loss of generality, one may prove Theorem 5 for a sequence of radii  $\{\alpha_n\}$  satisfying these three conditions:*

- (1)  $\{\alpha_n\}$  is nondecreasing,
- (2)  $\sum_{\alpha_n \in I_k} (1 - \alpha_n) > \sqrt{2^{-k}}$  for all  $k$  sufficiently large,
- (3)  $\sum_{n=1}^{\infty} (1 - \alpha_n) < 1/4$ .

*Proof of Lemma 3.1.* We first define a sequence  $\{\beta_n\}$  of radii satisfying conditions (1), (2), and the Blaschke condition  $\sum(1 - \beta_n) < \infty$ . Let  $n_0 = 0$ . For  $k = 1, 2, \dots$ , recursively define numbers  $n_k$  and  $\{\beta_n\}_{n_{k-1}+1}^{n_k} \subset I_k$  as follows.

If  $\sum_{r_n \in I_k} (1 - r_n) > \sqrt{2^{-k}}$ , then let  $n_k = n_{k-1} + N_k$  and let

$$\{\beta_n\}_{n_{k-1}+1}^{n_k} = \{r_n\}_{r_n \in I_k},$$

re-indexed if necessary so that  $\{\beta_n\}$  is nondecreasing in  $I_k$ . On the other hand, if  $\sum_{r_n \in I_k} (1 - r_n) \leq \sqrt{2^{-k}}$ , then choose  $m_k$  arbitrary numbers  $\gamma_j \in I_k$  such that

$$\sqrt{2^{-k}} < \sum_{r_n \in I_k} (1 - r_n) + \sum_{\gamma_j \in I_k} (1 - \gamma_j) \leq 2\sqrt{2^{-k}}.$$

This is possible since for all  $\gamma_j$  in  $I_k$ ,  $1 - \gamma_j < 2^{-k} \leq \sqrt{2^{-k}}$ . Let  $n_k = n_{k-1} + m_k + N_k$ , and let

$$\{\beta_n\}_{n_{k-1}+1}^{n_k} = \{r_n\}_{r_n \in I_k} \cup \{\gamma_j\},$$

indexed so that  $\{\beta_n\}$  is nondecreasing.

This procedure defines a sequence  $\{\beta_n\}_1^\infty$ . It is clear that  $\{\beta_n\}_1^\infty$  is nondecreasing and that  $\sum_{\beta_n \in I_k} (1 - \beta_n) > \sqrt{2^{-k}}$ . The  $\beta_n$ 's also satisfy the Blaschke condition since

$$\begin{aligned} \sum_{n=1}^\infty (1 - \beta_n) &= \sum_{k=0}^\infty \left[ \sum_{\beta_n \in I_k} (1 - \beta_n) \right] \\ &\leq \sum_{k=0}^\infty \left[ \sum_{r_n \in I_k} (1 - r_n) + 2\sqrt{2^{-k}} \right] \\ &= \sum_{n=1}^\infty (1 - r_n) + 2 \sum_{k=0}^\infty 2^{-k/2} \\ &< \infty. \end{aligned}$$

The sequence  $\{\beta_n\}$  was chosen so that the original sequence of radii  $\{r_n\}$  is a subsequence of  $\{\beta_n\}$ . Since subsequences of interpolating sequences are again interpolating, it suffices to prove Theorem 5 for the sequence  $\{\beta_n\}$ .

Now let

$$N = \min \left\{ n : \sum_{k=n}^\infty (1 - \beta_k) \leq 1/4 \right\}.$$

$N < \infty$  since the  $\beta_k$ 's satisfy the Blaschke condition. For  $n = 1, 2, \dots$ , let  $\alpha_n = \beta_{N+n}$ , so that  $\{\alpha_n\}$  is a tail end of  $\{\beta_n\}$ . Clearly  $\{\alpha_n\}$  satisfies the conditions of Lemma 3.1. Suppose that  $\{z_n\}$  is an interpolating sequence with  $|z_n| = \alpha_n$  for all  $n$ . Extend  $\{z_n\}$  to a larger sequence  $\{z'_n\}$  with  $|z'_n| = \beta_n$  for

all  $n$  by

$$z'_n = \begin{cases} \beta_n, & \text{if } 1 \leq n \leq N; \\ z_{n-N}, & \text{if } n > N. \end{cases}$$

*Claim.* If  $\{z_n\}$  is interpolating then so is  $\{z'_n\}$ .

*Proof of Claim.* Let

$$P_n = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \rho(z_k, z_n), \quad P'_n = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \rho(z'_k, z'_n),$$

where  $\rho(z, w)$  denotes the pseudo-hyperbolic metric (see §2). Since  $\{z_n\}$  is an interpolating sequence,  $\inf_n P_n \geq \delta > 0$  by Carleson's interpolation theorem (in §2). For each  $k \leq N$ ,  $\lim_{n \rightarrow \infty} \rho(z'_k, z_n) = 1$ , and so  $\inf_n \rho(z'_k, z_n) \geq \varepsilon_k > 0$ . Let  $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\}$ . Then if  $n > N$ ,

$$\begin{aligned} P'_n &= \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \rho(z'_k, z_{n-N}) \\ &= \left( \prod_{k=1}^N \rho(z'_k, z_{n-N}) \right) \left( \prod_{\substack{k=N+1 \\ k \neq n}}^{\infty} \rho(z_{k-N}, z_{n-N}) \right) \\ &\geq \varepsilon^N \delta. \end{aligned}$$

Also since  $\{z'_n\}$  satisfies the Blaschke condition,  $P'_n > 0$  for each  $n \leq N$ . Therefore

$$\inf_n P'_n = \min\{P'_1, P'_2, \dots, P'_N, \varepsilon^N \delta\} > 0$$

and so  $\{z'_n\}$  is an interpolating sequence by Carleson's interpolation theorem, finishing the proof of Lemma 3.1.  $\square$

To continue the proof of Theorem 5, assume that we're given a sequence  $\{\alpha_n\}$  of radii satisfying the conditions of Lemma 3.1. Let  $\theta_1 = 0$  and for  $n \geq 1$  let

$$\theta_n = 2\pi \sum_{k=1}^{n-1} (1 - \alpha_k).$$

Set  $z_n = \alpha_n e^{i\theta_n}$ . The proof that  $\{z_n\}$  is interpolating relies on the following estimate.

**Lemma 3.2.** *If  $k > n$  then*

$$|1 - \bar{z}_n z_k| > 4 \sum_{j=n}^{k-1} (1 - \alpha_j).$$

Similarly if  $k < n$  then

$$|1 - \overline{z_n} z_k| > 4 \sum_{j=k}^{n-1} (1 - \alpha_j).$$

*Proof of Lemma 3.2.* If  $k > n$  then

$$\arg(\overline{z_n} z_k) = 2\pi \sum_{j=n}^{k-1} (1 - \alpha_j) = \psi_1,$$

while if  $k < n$  then

$$\arg(\overline{z_n} z_k) = -2\pi \sum_{j=k}^{n-1} (1 - \alpha_j) = -\psi_2.$$

Both  $\psi_1$  and  $\psi_2$  are in the interval  $(0, \pi/2)$  since  $\sum(1 - \alpha_n) < 1/4$ . Let  $P$  be the point on the ray  $\{\arg z = \psi_1\}$  that is nearest to 1, and let  $|P1|$  denote the distance between the two points. Then the line segment  $\overline{0P}$  is orthogonal to  $\overline{P1}$  and the triangle  $0P1$  is a right triangle with a hypotenuse of length 1. So by elementary trigonometry  $|P1| = \sin \psi_1$ . If  $k > n$  then

$$|1 - \overline{z_n} z_k| \geq |P1| = \sin \psi_1 > \frac{2}{\pi} \psi_1 = 4 \sum_{j=n}^{k-1} (1 - \alpha_j).$$

A symmetric argument gives the result for  $k < n$ . This ends the proof of Lemma 3.2.  $\square$

We now prove that the sequence  $\{z_n\}$  defined above is an interpolating sequence. It is sufficient (and necessary) to show that both

(A) 
$$\inf_{\substack{k, n \\ k \neq n}} \rho(z_k, z_n) > 0$$

and

(B) 
$$\sup_n \sum_{k=1}^{\infty} \frac{(1 - \alpha_k^2)(1 - \alpha_n^2)}{|1 - \overline{z_n} z_k|^2} < \infty,$$

(see §2). We consider these two conditions separately.

**Condition (A).** First assume that  $k > n$  so that  $1 - \alpha_k < 1 - \alpha_n$ . Then

$$\begin{aligned} 1 - \rho(z_k, z_n)^2 &= \frac{(1 - \alpha_k^2)(1 - \alpha_n^2)}{|1 - \overline{z_n} z_k|^2} \\ &< \frac{(1 + \alpha_k)(1 - \alpha_k)(1 + \alpha_n)(1 - \alpha_n)}{\left[4 \sum_{j=n}^{k-1} (1 - \alpha_j)\right]^2} \\ &< \frac{4(1 - \alpha_k)(1 - \alpha_n)}{16(1 - \alpha_n)^2} \\ &< 1/4. \end{aligned}$$

So for all  $k > n$ ,  $\rho(z_k, z_n)^2 > 3/4$ . By switching the roles of  $k$  and  $n$ , the same inequality holds if  $k < n$ . Therefore

$$\inf_{\substack{k, n \\ k \neq n}} \rho(z_k, z_n) \geq \sqrt{3}/2$$

and Condition (A) holds.

**Condition (B).** Fix a number  $n$  and suppose that  $\alpha_n$  lies in the interval

$$I_k = [1 - 2^{-(k-1)}, 1 - 2^{-k}).$$

From Lemma 3.1 one may assume that  $n$  is sufficiently large so that

$$\sum_{\alpha_j \in I_{k-2}} (1 - \alpha_j) > \sqrt{2^{-(k-2)}}.$$

Let

$$L = \max\{j : \alpha_j < 1 - 2^{-(k-2)}\},$$

$$R = \min\{j : \alpha_j \geq 1 - 2^{-(k+1)}\}.$$

Split the sum

$$S_n \equiv \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{(1 - \alpha_j^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_j|^2}$$

into four parts

$$\begin{aligned} S_n &= \sum_{j=1}^L + \sum_{j=L+1}^{n-1} + \sum_{j=n+1}^{R-1} + \sum_{j=R}^{\infty} \\ &= \text{I} + \text{II} + \text{III} + \text{IV}, \end{aligned}$$

and estimate each part of the sum separately.

$\Sigma_I$ : If  $j \leq L$ , then by Lemma 3.2 and Lemma 3.1,

$$|1 - \bar{z}_n z_j| > 4 \sum_{i=j}^{n-1} (1 - \alpha_i) > 4 \sum_{\alpha_i \in I_{k-1}} (1 - \alpha_i) > 4\sqrt{2^{-(k-1)}}.$$

So

$$\begin{aligned} \frac{(1 - \alpha_j^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_j|^2} &< \frac{4(1 - \alpha_j)(1 - \alpha_n)}{16 \cdot 2^{-(k-1)}} \\ &< \frac{4(1 - \alpha_j)2^{-(k-1)}}{16 \cdot 2^{-(k-1)}} \\ &< \frac{1}{4}(1 - \alpha_j), \end{aligned}$$

and therefore

$$\sum_I < \frac{1}{4} \sum_{j=1}^L (1 - \alpha_j) < \frac{1}{4} \sum_{j=1}^{\infty} (1 - \alpha_j) < \frac{1}{16}.$$

$\sum_{IV}$ : If  $j \geq R$ , then

$$|1 - \bar{z}_n z_j| > 4 \sum_{i=n}^{j-1} (1 - \alpha_i) > 4 \sum_{\alpha_i \in I_{k+1}} (1 - \alpha_i) > 4\sqrt{2^{-(k+1)}}.$$

So

$$\begin{aligned} \frac{(1 - \alpha_j^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_j|^2} &< \frac{4(1 - \alpha_j)(1 - \alpha_n)}{16 \cdot 2^{-(k+1)}} \\ &< \frac{4(1 - \alpha_j)2^{-(k-1)}}{16 \cdot 2^{-(k+1)}} \\ &< 1 - \alpha_j, \end{aligned}$$

and therefore

$$\sum_{IV} < \sum_{j=R}^{\infty} (1 - \alpha_j) < \sum_{j=1}^{\infty} (1 - \alpha_j) < \frac{1}{4}.$$

$\sum_{II}$ : If  $L < j \leq n-1$  then  $(1 - \alpha_n) \leq (1 - \alpha_j) \leq 4(1 - \alpha_n)$ , so

$$\begin{aligned} \frac{(1 - \alpha_j^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_j|^2} &< \frac{4(1 - \alpha_j)(1 - \alpha_n)}{16 \left[ \sum_{i=j}^{n-1} (1 - \alpha_i) \right]^2} \\ &< \frac{4 \cdot 4 \cdot (1 - \alpha_n)^2}{16 (n-j)^2 (1 - \alpha_n)^2} \\ &= \frac{1}{(n-j)^2}. \end{aligned}$$

Therefore

$$\sum_{II} < \sum_{j=L+1}^{n-1} \frac{1}{(n-j)^2} < \frac{\pi^2}{6}.$$

$\sum_{III}$ : If  $n+1 \leq j \leq R-1$  then  $(1 - \alpha_n) \geq (1 - \alpha_j) \geq (1/4)(1 - \alpha_n)$ , so

$$\begin{aligned} \frac{(1 - \alpha_j^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_j|^2} &< \frac{4(1 - \alpha_j)(1 - \alpha_n)}{16 \left[ \sum_{i=n+1}^{j-1} (1 - \alpha_i) \right]^2} \\ &< \frac{4(1 - \alpha_n)^2}{16 \left[ (1/4)(j-n)(1 - \alpha_n) \right]^2} \\ &= 4 \frac{1}{(j-n)^2}. \end{aligned}$$

Therefore

$$\sum_{III} < 4 \sum_{j=n+1}^{R-1} \frac{1}{(j-n)^2} < \frac{4\pi^2}{6}.$$

Finally when  $k = n$ , then

$$\frac{(1 - \alpha_k^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_k|^2} = 1.$$

Putting everything together one sees that

$$\sum_{k=1}^{\infty} \frac{(1 - \alpha_k^2)(1 - \alpha_n^2)}{|1 - \bar{z}_n z_k|^2} < 1 + \frac{5}{16} + \frac{5\pi^2}{6}$$

for all  $n$  and so Condition (B) holds. Therefore the sequence  $\{z_n\}$  is an interpolating sequence and the proof of Theorem 5 is finished.  $\square$

### REFERENCES

- [And84] J. M. Anderson, *Bloch functions: the basic theory*, Operators and Function Theory, (Proceedings of the NATO Advanced Study Institute on Operators and Function Theory, Lancaster, U.K., July 16–26, 1984), ed. by S. C. Power, NATO ASI Series C: Math. Phys. Sci. 153, Reidel, 1985, pp. 1–17. MR 87h:30072.
- [ACP74] J. M. Anderson, J. G. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37. MR 50 #13536.
- [AFS] J. M. Anderson, J. L. Fernández and A. L. Shields, *Inner functions and cyclic vectors in the little Bloch space*, preprint.
- [Bil86] P. Billingsley, *Probability and measure*, 2nd ed., Wiley, 1986. MR87f:60001.
- [Bish90] C. J. Bishop, *Bounded functions in  $B_0$* , Pacific J. Math. **142** (1990), 209–225.
- [Car58] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930. MR 22 #8129.
- [Car62] ———, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547–559. MR 25 #5186.
- [Chu74] K. L. Chung, *A course in probability theory*, 2nd ed., Academic Press, 1974. MR 49 #11579.
- [Dur70] P. L. Duren, *Theory of  $H^p$  spaces*, Academic Press, 1970. MR 42 #3552.
- [Gar81] J. B. Garnett, *Bounded analytic functions*, Academic Press, 1981. MR 83g:30037.
- [Hof67] K. Hoffman, *Bounded analytic functions and Gleason parts*, Ann. of Math. **86** (1967), 74–111. MR 35 #5945.
- [Hung] G. J. Hungerford, *Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch space*, preprint from a doctoral thesis, California Institute of Technology, 1988.
- [Kah69] J. P. Kahane, *Trois notes sur les ensembles parfaits linéaires*, Enseignement Math. **15** (1969), 185–192. MR 39 #7040.
- [Naf56] A. G. Naftalevič, *On the interpolation of functions of bounded characteristic*, [Russian], Vilnius Valst. Univ. Mokslu Darbai., Mat. Fiz. Chem. Mokslu **5** (1956), 5–27. MR 22 #11141.
- [Par60] E. Parzen, *Modern probability theory and its applications*, Wiley, 1960. MR 22 #B3021.
- [Sar78] D. Sarason, *Function theory on the unit circle*, notes for lectures given at a conference at Virginia Polytechnic Institute and State University, Blacksburg Va., June 19–23, 1978. MR 80d:30035.
- [Sar84] ———, *Blaschke products in  $B_0$* , Linear and Complex Analysis Problem Book, ed. by Havin, Hruščëv, and Nikol'skii. Lecture Notes in Math., vol. 1043, Springer-Verlag, 1984, pp. 337–338. MR 85k:46001.
- [SSH61] H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532. MR 24 #A3280.

- [Ste88] K. Stephenson, *Construction of an inner function in the little Bloch space*, *Trans. Amer. Math. Soc.* **308** (1988), 713–720.
- [Zyg59] A. Zygmund, *Trigonometric series*, Vol. I, 2nd ed., Cambridge Univ. Press, 1959. MR **21** #6498.

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