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SUPPORT ALGEBRAS OF σ-UNITAL C^*-ALGEBRAS AND THEIR QUASI-MULTIPLIERS

HUAXIN LIN

ABSTRACT. We study certain dense hereditary *-subalgebras of σ-unital C^*-algebras and their relations with the Pedersen ideals. The quasi-multipliers of the dense hereditary *-subalgebras are also studied.

1. INTRODUCTION

Let A be a C^*-algebra and $K(A)$ its Pedersen's ideal. When A is commutative, that is, $A = C_0(A)$, the algebra of all complex valued continuous functions which vanish at infinity on some locally compact Hausdorff space X, then $K(A) = C_{00}(X)$, the algebra of all complex valued continuous functions with compact support. In [15], we define a dense hereditary *-subalgebra A_{00} (we used the notation $C_{00}(A)$ there) of a σ-unital C^*-algebra which satisfies:

(i) For every a in (A_{00}), there is a b in (A_{00}) such that $[a] \leq b$, where $[a]$ is the range projection of a in A^{**}.
(ii) If A is nonunital, $A_{00} \neq A$.
(iii) When $A = C_0(X)$, $A_{00} = C_{00}(X)$.

Naturally, we may view A_{00} as a noncommutative analogue of $C_{00}(X)$. In fact the algebra A_{00} plays an important role in [15]. In this paper we shall study the relation between A_{00} and $K(A)$. We also study the quasi-multipliers of A_{00}. In the view of [11], where Lazer and Taylor studied the multipliers of $K(A)$ as a noncommutative analogue of (unbounded) continuous functions on locally compact Hausdorff space X, the quasi-multipliers of A_{00} is another noncommutative analogue of $C(X)$. The reason our attention is focused on the quasi-multipliers of A_{00} and not on the multipliers of A_{00} is that the set of multipliers of A_{00} may not contain A and is not closed under a natural topology.

We denote the quasi-multipliers of A_{00} by $QM(A_{00})$. In §2, we give some basic concepts and facts related to quasi-multipliers of A_{00}. In §3, we study the order structure $QM(A_{00})$. We also show that $QM(A_{00}) = LM(A_{00}) + RM(A_{00})$ (a similar equation for A has been studied in [16, 3, 13, 14]). In §4, we

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prove an extension theorem in the sense of Tietse. We also give a version of the Dauns-Hofmann theorem for $QM(A_{00})$. In §5, we study the dual and bidual spaces of $QM(A_{00})$. We find that $QM(A_{00})''$, the bidual of $QM(A_{00})$, is isomorphic to the quasi-multipliers of the support algebra of $M_0(A)$, the hereditary C^*-subalgebra of A^{**} generated by A. In §6, we study the problem when $A_{00} = K(A)$. Finally, in §7, we consider the uniqueness of A_{00} for certain C^*-algebras.

We shall be utilizing the following notations throughout this paper. Suppose that A is a C^*-algebra. Then $K(A)$ denotes the Pedersen's ideal (for a definition see [17 or 18, 5.6]), and $M(A)$, $LM(A)$, $RM(A)$, and $QM(A)$ denote the multipliers, left multipliers, right multipliers, and quasi-multipliers of A, respectively (see [18, 3.12]). For the element a in the C^*-algebra A, $[a]$ shall denote the range projection of a in the enveloping W^*-algebra A^{**}. Any other unexplained notation may be found in [18 or 4].

2. Preliminaries

2.1. Let A be a σ-unital C^*-algebra. Then A has a strictly positive element e. Let $f_n(t)$ be continuous functions satisfying

(i) $$0 \le f_n(t) \le 1;$$

(ii) $$f_n(t) = 0 \quad \text{if and only if} \quad 0 \le t \le 1/2n;$$

(iii) $$f_n(t) = 1 \quad \text{if } t \ge 1/n.$$

Define $e_n = f_n(e)$. Then $\{e_n\}$ forms an approximate identity for A. Moreover, $e_{n+1}e_n = e_n e_{n+1} = e_n$ for all n. Let χ_n be the characteristic function of the set $(1/2n, \|e\|)$. Then $p_n = \chi_n(e)$ is an open projection of A such that $[e_n] = p_n$ and $e_n \le p_n \le e_{n+1}$.

2.2. **Definition.** Let A and p_n be as in 2.1. Denote the hereditary C^*-subalgebra $p_n A^{**} p_n \cap A$ by A_n. We call $\bigcup_{n=1}^{\infty} A_n$ a support algebra of A and denote it by A_{00} (or $A_{00}(e)$, or $A_{00}(\{e_n\})$).

2.3. By [15, 1.1], A_{00} is a norm dense, hereditary *-subalgebra of A contained in $K(A)$. Since $e \notin A_{00}$, if A is not unital, then $A_{00} \ne A$. Moreover, for every $a \in (A_{00})_+$, there is an n such that $[a] \le e_n$. Thus, as in [15], we regard A_{00} as a noncommutative analogue of $C_{00}(X)$.

2.4. **Example.** Let X be a locally compact, σ-compact Hausdorff space and let $A = C_0(X)$. (σ-compact means $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is compact.) Then for any strictly positive element e, $A_{00}(e) = C_{00}(X)$.

2.5. **Example.** Let H be a separable Hilbert space and let $A = K$, the compact operators on H. Let $\{H_n\}$ be an increasing sequence of finite-dimensional subspaces of H such that $\bigcup_{n=1}^{\infty} H_n$ is dense in H. Denote by M_n the set of bounded linear operators on H_n. Then $\bigcup_n M_n$ is a support algebra for $A = K$. We shall see in §7 that, up to isomorphisms, $\bigcup_n M_n$ is the only support algebra for K.

2.6. **Lemma.** *Suppose that A is a C^*-algebra. Let $a, p \in A_+$ and $p \leq a \leq 1$. If p is a projection, the $ap = pa = p$.*

2.7. **Lemma.** *Suppose that $a_n \in A_+$, and p_n are open projections of A. If $\{a_n\}$ forms an approximate identity for A and $a_n \leq p_n \leq a_{n+1}$ for each n, then there is a support algebra A_{00} of A such that*

$$A_{00} = p_n A^{**} p_n \cap A.$$

2.8. By 2.7, we may define A_{00} by an approximate identity $\{e_n\}$ together with open projections $\{p_n\}$ satisfying:

$$e_n \leq p_n \leq e_{n+1} \quad \text{for all } n.$$

If $e_n \leq p_n \leq e_{n+1}$ for each n, then $e_{n+1}e_n = e_n e_{n+1} = e_n$. Conversely, if $e_{n+1}e_n = e_n e_{n+1} = e_n$, then $e_{n+1} \geq [e_n]$. Thus we will always assume that every support algebra A_{00} of A is defined by an approximate identity $\{e_n\}$ which satisfies $e_{n+1}e_n = e_n e_{n+1} = e_n$.

We now fix a σ-unital C^*-algebra A and a support algebra $A_{00} = A_{00}(\{e_n\})$.

2.9. **Definitions.** A linear map $\rho: A_{00} \to A_{00}$ is called a left, respectively right, multiplier if $\rho(ab) = \rho(a)b$, respectively $\rho(ab) = a\rho(b)$. A multiplier is a pair (ρ_1, ρ_2) consisting of a right multiplier ρ_1 and a left multiplier ρ_2 such that $\rho_1(a)b = a\rho_2(b)$ for all $a, b \in A_{00}$. A quasimultiplier is a bilinear map $\rho: A_{00} \times A_{00} \to A_{00}$ such that for each fixed $a \in A_{00}$ the map $\rho(a, \cdot)$ is a left multiplier and the map $\rho(\cdot, a)$ is a right multiplier. We denote by $M(A_{00})$, $LM(A_{00})$, $RM(A_{00})$, and $QM(A_{00})$ the sets of multipliers, left multipliers, right multipliers, and quasi-multipliers of A_{00}, respectively.

2.10. Suppose that $\rho \in QM(A_{00})$, and a and $b \in A_{00}$. Then we denote the element $\rho(a, b)$ by $a \cdot \rho \cdot b$. If $\rho \in LM(A_{00})$, we denote $\rho(a)$ by $\rho \cdot a$ and if $\rho \in RM(A_{00})$, we denote $\rho(a)$ by $a \cdot \rho$. If $z = (\rho_1, \rho_2) \in M(A_{00})$, we denote $\rho_1(a)$ by $a \cdot z$ and $\rho_2(a)$ by $z \cdot a$.

2.11. For $a, b \in A_{00}$, we have the following seminorms:

(i) $\quad z \to \|a \cdot z\| + \|z \cdot a\|, \quad z \in M(A_{00});$

(ii) $\quad z \to \|z \cdot a\|, \quad z \in LM(A_{00});$

(iii) $\quad z \to \|a \cdot z\|, \quad z \in RM(A_{00});$

(iv) $\quad z \to \|a \cdot z \cdot b\|, \quad z \in QM(A_{00}).$

We define (A_{00})-, L-A_{00}-, R-A_{00}-, and Q-A_{00}- topologies on $M(A_{00})$, $LM(A_{00})$, $RM(A_{00})$, and $QM(A_{00})$ to be those locally convex topologies generated by the seminorms (i), (ii), (iii), and (iv) (for all $a, b \in A_{00}$), respectively.

2.12. **Proposition.** *$QM(A_{00})$ is a locally convex complete topological vector space under the Q-A_{00}-topology.*

2.13. We define the following subsets of $QM(A_{00})$:

$QM_l(A_{00}) = \{\rho \in QM(A_{00})$: for each k, there exist $N(\rho, k)$ such that $\rho(e_n, e_k) = \rho(e_m, e_k)$ if $n, m > N(\rho, k)\}$,

$QM_r(A_{00}) = \{\rho \in QM(A_{00})$: for each k, there exists $N(\rho, k)$ such that $\rho(e_k, e_n) = \rho(e_k, e_m)$ if $n, m > N(\rho, k)\}$,

$QM_d(A_{00}) = QM_l(A_{00}) \cap QM_r(A_{00})$, and

$QM^b(A_{00})$ is the subset of those elements in $QM(A_{00})$ such that

$$\sup\{\|a \cdot \rho \cdot b\|: a, b \in A_{00}, \ \|a\| \le 1, \ \|b\| \le 1\} < \infty.$$

2.14. **Theorem.** *There are bijective correspondences between*

(i) $QM_l(A_{00})$ *and* $LM(A_{00})$;

(ii) $QM_r(A_{00})$ *and* $RM(A_{00})$;

(iii) $QM_d(A_{00})$ *and* $M(A_{00})$;

(iv) $QM^b(A_{00})$ *and* $QM(A)$.

2.15. We shall use notations $LM(A_{00})$, $RM(A_{00})$, $M(A_{00})$, and $QM(A)$ instead of $QM_l(A_{00})$, $QM_r(A_{00})$, $QM_d(A_{00})$, and $QM^b(A_{00})$. Thus

$$M(A_{00}) \subset LM(A_{00}) \subset QM(A_{00}),$$
$$LM(A_{00}) \cap RM(A_{00}) = M(A_{00}),$$

and

$$A_{00} \subset A \subset QM(A) \subset QM(A_{00}).$$

2.16. **Lemma.** *If A is not unital, then*

$$QM(A_{00}) \neq QM^b(A_{00}) \quad (= QM(A)).$$

Proof. We may assume that $e_n - e_{n-1} \neq 0$ for all n. Define

$$z = \sum_{n=1}^{\infty} n(e_n - e_{n-1}),$$

where the convergence is in $Q - A_{00}$-topology. Clearly $z \in QM(A_{00})$, but $z \notin QM^b(A_{00})$.

2.17. We notice that, in general, $A \not\subset M(A_{00})$ and $M(A_{00})$ is not complete under A_{00}-topology. These are the reasons why we choose $QM(A_{00})$ and not $M(A_{00})$ as our main subject.

2.18. **Proposition.** *A_{00} is L-A_{00}-dense (respectively, R-A_{00}-dense, Q-A_{00}-dense, and A_{00}-dense) in $LM(A_{00})$ (respectively in $RM(A_{00})$, $QM(A_{00})$, and $M(A_{00})$).*

2.19. We now define an operation "·" on some of the elements of $QM(A_{00})$. If $\rho \in QM(A_{00})$, $y \in LM(A_{00})$, and $z \in RM(A_{00})$, we denote by $\rho \cdot y$ the element $\rho(\cdot, y(\cdot))$ and $z \cdot \rho$ the element $\rho(z(\cdot), \cdot)$. It is easy to see that "·" is the "natural" extension of the multiplication on $M(A)$.

2.20. Let $\rho \in QM(A_{00})$. The involution ρ^* of ρ is a quasi-multiplier defined by $\rho^*: (a, b) \to [\rho(b^*, a^*)]^*$. It is easy to see that the involution is conjugate linear and Q-A_{00}-continuous. Moreover the involution is the extension of the original involution on $QM(A)$. Thus

$$LM(A_{00})^* = RM(A_{00}).$$

An element is called selfadjoint if $\rho = \rho^*$. We denote by $QM(A_{00})_{\text{s.a.}}$ the set of selfadjoint elements.

2.21. **Example.** Let X be a locally compact, σ-compact Hausdorff space, and let B be a unital C^*-algebra. Denote by A the C^*-algebra of all the continuous mappings from X into B vanishing at infinity. One of the support algebras (in fact, it is the only one) A_{00} is the set of all continuous mappings with compact supports. One can check that $QM(A_{00})$ is the set of all continuous mappings from X into B.

Throughout §§3–7, A will denote a σ-unital C^*-algebra, and A_{00} one of its support algebras. e, e_n, and A_n will be the same as in 2.1.

3. Decompositions

3.1. **Definition.** We say that an element $z \in QM(A_{00})$ is positive, denoted by $z \geq 0$, if $a^*za \geq 0$ for all $a \in A_{00}$. We let $QM(A_{00})_+$ denote the set of all positive elements in $QM(A_{00})$.

Suppose that y and $z \in QM(A_{00})$. We say that $z \geq y$ (or $y \leq z$), if $z - y \geq 0$.

3.2. **Corollary.** *The set $QM(A_{00})_+$ is a Q-A_{00}-closed real convex cone and $QM(A_{00})_+ \cap (-QM(A_{00})_+) = \{0\}$.*

3.3. **Proposition.** *Let $z \in QM(A_{00})$. Then*

(i) *If $-y \leq z \leq y$ for some $y \in QM(A)_+$, then $z \in QM(A)$.*
(ii) *If $-a \leq z \leq a$ for some $a \in A^+$, then $z \in A$.*
(iii) *If $z \in LM(A_{00})$ and there is an element $a \in A^+$ such that $z^*z \leq a$, then $z \in A$.*

Proof. (i) Since $y - z \geq 0$, $a^*(-y)a \leq a^*za \leq a^*ya$ for all $a \in A_{00}$. Therefore $a^*za \leq a^*ya$. It follows that $z \in QM^b(A_{00}) = QM(A)$.

(ii) By (i), $z \in QM(A)$. Then by [1, Proposition 4.5], $z \in A$.

(iii) For every $b \in A_{00}$, we have $b^*z^*zb \leq b^*ab$. Thus $\|zb\| \leq \|a^{1/2}b\|$. Hence $z \in QM(A) \cap LM(A_{00})$. It follows from [1, Proposition 4.5] that z is in A.

3.4. Let $LM(A_{00}, AA_{00})$ denote the set of those linear mappings ρ from A_{00} into AA_{00} satisfying $\rho(xy) = \rho(x)y$ for all $x, y \in A_{00}$. As in §2, we can view $LM(A_{00}, AA_{00})$ as a subset of $QM(A_{00})$. If $x \in LM(A_{00}, AA_{00})$, we define $x^* \cdot x(a, b) = (a \cdot x^*)(x \cdot b)$. Hence $x^* \cdot x \in QM(A_{00})_+$.

3.5. **Theorem.** *If* $z \in QM(A_{00})_+$, *then there is an* $x \in LM(A_{00}AA_{00})$ $(\subset QM(A_{00}))$ *such that* $x^* \cdot x = z$.

Proof. Let $\alpha_k = \|z|_{A_k \times A_k}\|$. Define $b_k = (1/\alpha_{k+1})(1/2)^k(e_k - e_{k-1})$ for $k = 1, 2, \ldots$ (where $e_0 = 0$), $a_k = \sum_{i=1}^k b_i$, and $b = \sum_{i=1}^\infty b_i$. Let $z_k = a_k z a_k$, $k = 1, 2, \ldots$. Then, if $k \geq m$

$$\begin{aligned} \|z_k - z_m\| &\leq \left\| \sum_{i=m+1}^{k} b_1 z a_k \right\| + \left\| \sum_{j=m+1}^{k} a_k z b_j \right\| \\ &= \left\| \sum_{i=m+1}^{k} \sum_{j=1}^{k} b_i z b_j \right\| + \left\| \sum_{j=m+1}^{k} \sum_{i=1}^{k} b_i z b_j \right\| \\ &\leq \sum_{i=m+1}^{k} \sum_{j=1}^{k} (1/2)^{i+j} + \sum_{j=m+1}^{k} \sum_{i=1}^{k} (1/2)^{i+j} \\ &\leq 1/(2)^{m-1}. \end{aligned}$$

Thus z_k converges to a positive element h in A in norm. It is easy to see that $e_k h e_k = e_k z_{k+1} e_k$ for every k. Take $u_n = h^{1/2}(b^2 + 1/n)^{-1}b$. Then, for every k,

$$\begin{aligned} \|u_n e_k\|^2 &= \|e_k b(b^2 + 1/n)^{-1} h (b^2 + 1/n)^{-1} b e_k\| \\ &= \|b(b^2 + 1/n)^{-1} e_k h e_k (b^2 + 1/n)^{-1} b e_k\| \\ &= \|b(b^2 + 1/n)^{-1} a_{k+1} e_k h e_k a_{k+1} (b^2 + 1/n)^{-1} b e_k\| \\ &\leq \alpha_k \|b(b^2 + 1/n)^{-1} b e_k a_{k+1}\|^2 \leq \alpha_k. \end{aligned}$$

So $\|u_n e_k\|$ is bounded for every k.

Put $d_{nm} = (1/n + b^2)^{-1} - (1/n + b^2)^{-1}$. Then, for each k,

$$\begin{aligned} \|u_n a_k - u_m a_k\|^2 &= \|h^{1/2} d_{nm} b a_k\|^2 \\ &= \|b d_{nm} a_k h a_k d_{nm} b\| \\ &\leq \alpha_{k+1} \|b d_{nm} a_k a_{k+1} a_k d_{nm} b\| \\ &= \alpha_{k+1} \|d_{nm} b a_k (a_{k+1})^{1/2}\|^2. \end{aligned}$$

From spectral theory we see that the sequence $\{(1/n + b^2)^{-1} b a_k (a_{k+1})^{1/2}\}$ is increasing to an element in A and by Dini's theorem it is uniformly convergent to it. Consequently

$$\|d_{nm} b a_k (a_{k+1})^{1/2}\| \to 0,$$

so that $\{u_n a_k\}$ is norm convergent to an element in A for each k. Since $\|u_n e_{k+1}\|$ is bounded and $\overline{a_k A} \supset A_k$, it follows that $\{u_n y\}$ is norm convergent for every $y \in A_k$. Thus we have an element $x \in LM(A_{00}, AA_{00})$ defined by

$$x(a) = \lim u_n a \quad \text{for every } a \in A_{00}.$$

It is easy to check that for every k,

$$a_{k+1}x^* \cdot a_{k+1} = a_{k+1}za_{k+1}.$$

Therefore $x^* \cdot x = z$.

3.6. The idea of the proof of 3.5 is taken from [3, 4.9; and 18, 1.44]. The element x in 3.5 is in $QM(A_{00})$ but not in $QM(A_{00})_+$. In general, x may not be taken from $LM(A_{00})$.

3.7. **Theorem.** $QM(A_{00}) = LM(A_{00}) + RM(A_{00})$.

Proof. Let $z \in QM(A_{00})$. Define

$$x = \sum_{k=1}^{\infty} e_k z(e_k - e_{k-1})$$

and

$$y = \sum_{k=1}^{\infty}(1 - e_k)z(e_k - e_{k-1}).$$

Both sums converge in Q-A_{00}-topology. It is easy to verify that $x \in LM(A_{00})$ and $y \in RM(A_{00})$. For every n,

$$\begin{aligned} e_n(x+y)e_n &= \left(\sum_{k=1}^{n-1} e_k z(e_k - e_{k-1} + e_n^2 z(e_n - e_{n-1})e_n + e_n z(e_{n+1} - e_n)e_n\right) \\ &\quad + \left(\sum_{k=1}^{n-1}(e_n - e_k)z(e_k - e_{k-1}) + (e_n - e_n^2)z(e_n - e_{n-1})e_n\right) \\ &= \left(\sum_{k=1}^{n-1} e_n z e_k - e_{k-1}\right) + e_n z(e_n - e_n) + e_n z(e_n^2 - e_{n-1}) \\ &= e_n z e_{n-1} + e_n z(e_n - e_{n-1}) = e_n z e_n. \end{aligned}$$

So $x + y = z$.

3.8. The problem when $QM(A) = LM(A) + RM(A)$ had been studied in [16, 3, 13, 14]. In general, $QM(A) \neq LM(A) + RM(A)$.

4. The Tietze theorem and Dauns-Hofmann theorem

This section is inspired by [11]. Our results are similar to the corresponding ones in [11].

4.1. Let B be a σ-unital C^*-algebra and let ϕ be a *-homomorphism from A onto B. Then $B_{00} = \phi(A_{00})$ is a support algebra of B and ϕ can be extended to a linear map $\tilde{\phi}$ from $LM(A_{00})$ into $LM(B_{00})$ as follows:

(i) $$\tilde{\phi}(z) \cdot \phi(a) = \phi(z \cdot a)$$

for $z \in LM(A_{00})$ and $a \in A_{00}$. We can further extend $\tilde{\phi}$ from $QM(A_{00})$ into $QM(B_0)$ by

(ii) $$\phi(a) \cdot \tilde{\phi}(z) \cdot \phi(b) = \phi(a \cdot z \cdot b)$$

for $z \in QM(A_{00})$ and $a, b \in A_{00}$. It can be verified that if $z \in QM(A_{00})$, $x \in LM(A_{00})$, $y \in RM(A_{00})$, and $a \in A_{00}$, then

(iii) $$\phi(a) \cdot \tilde{\phi}(y) = \phi(a \cdot y);$$

(iv) $$\tilde{\phi}(y \cdot z) = \tilde{\phi}(y) \cdot \tilde{\phi}(z);$$

(v) $$\tilde{\phi}(z \cdot x) = \tilde{\phi}(z) \cdot \tilde{\phi}(x);$$

(vi) $$\tilde{\phi}(z)^* = \tilde{\phi}(z^*) \quad \text{and} \quad \tilde{\phi}(z) \geq 0 \quad \text{if } z \in QM(A_{00})_+ .$$

4.2. **Proposition.** *The extension $\tilde{\phi}$ is continuous when $QM(A_{00})$ is considered with Q-A_{00}-topology and $QM(B_{00})$ with Q-B_{00}-topology.*

4.3. Next we shall show that the extension $\tilde{\phi}$ is surjective. In view of 2.20, the following theorem can be regarded as a noncommutative extension of Tietze's theorem. The same results for bounded multipliers $M(A)$ and bounded quasi-multipliers $QM(A)$ can be found in [9, 3]. A similar result for (unbounded) multipliers of $K(A)$ can be found in [11].

4.4. **Theorem.** *Let ϕ be a homomorphism from A onto B and $B_{00} = \phi(A_{00})$. Then*

(i) $$\tilde{\phi}(QM(A_{00})) = QM(B_{00});$$

(ii) $$\tilde{\phi}(LM(A_{00})) = LM(B_{00});$$

(iii) $$\tilde{\phi}(RM(A_{00})) = RM(B_{00});$$

(iv) $$\tilde{\phi}(M(A_{00})) = M(B_{00}).$$

Proof. (i) We shall show that $\tilde{\phi}$ is surjective. Let $\overline{z} \in QM(B_{00})$ and $\overline{z}_k = \overline{e}_k \overline{z} \overline{e}_k$, where $\overline{e}_k = \phi(e_k)$, $k = 1, 2, \ldots$. Suppose that $y_k \in A_{00}$ such that $\phi(y_k) = \overline{z}_k$. Let $z_1 = y_1$,

$$z_{k+1} = y_{k+1} - e_k y_{k+1} e_k + z_k, \qquad k = 1, 2, \ldots.$$

Then $z_{k+1} \in A_{00}$; moreover,

$$\phi(z_{k+1}) = \overline{z}_{k+1} - \overline{e}_k \overline{z}_{k+1} \overline{e}_k + \overline{z}_k = z_{k+1}.$$

If $k > m$, then

$$e_m(z_{k+1} - z_k)e_m = e_m y_{k+1} e_m - e_m e_k y_{k+1} e_k e_m + e_m z_k e_m - e_m z_k e_m.$$

Thus, if $k, k' > m$,

$$e_m(z_k - z_{k'})e_m = 0.$$

So $\{z_k\}$ is a Q-A_{00}-Cauchy sequence. Suppose that $z = \lim z_k$. Then, by the continuity of $\tilde{\phi}$ (4.2),

$$\tilde{\phi}(z) = \lim \phi(z_k) = \lim \overline{z}_k = \overline{z}.$$

Then $\tilde{\phi}$ is onto.

(ii) Let $\overline{x} \in LM(A_{00})$ and $\overline{x}_k = \overline{x}\overline{e}_k$, $k = 1, 2, \ldots$. Suppose that $a_k \in A_{00}$ such that $\phi(a_k) = \overline{x}_k$. Define $x_1 = a_1$ and $x_{k+1} = a_{k+1} - a_{k+1} \cdot e_k + x_k$,

$k = 1, 2 \dots$. Then $\phi(x_{k+1}) = \overline{x}_{k+1}$, $k = 1, 2, \dots$. As in (i), $\{x_{k+1}\}$ is an L-A_{00}-Cauchy sequence, hence a Q-A_{00}-Cauchy sequence. Let $x = \lim x_k$. Then $\tilde{\phi}(x) = x$. To show that $x \in LM(A_{00})$, take $a \in A_n$. Then

$$\begin{aligned} x_{k+1}a - x_k a &= x_{k+1}e_{n+1}a - x_k e_{n+1}a \\ &= (x_{k+1} - x_k)e_{n+1}a = 0 \end{aligned}$$

if $k > n+1$. So $x_k a = x_{k+2}a$ for every $k > n+1$. Thus $x \cdot a \in A_{00}$. We conclude that x is in $LM(A_{00})$.

We omit the proofs for (iii) and (iv).

4.5. Let $z \in QM(A_{00})$ and $a \in A_{00}$. Then $z \cdot a$, $a \cdot z \in QM(A_{00})$. In fact, $a \cdot z \in LM(A_{00})$, while $z \cdot a \in RM(A_{00})$. The center of $QM(A_{00})$ is the set $Z = \{z \in QM(A_{00}) : a \cdot z = z \cdot a \text{ for all } a \in A_{00}\}$.

4.6. **Proposition.** $Z \subset M(A_{00})$. *Moreover, Z is the center of $M(A_{00})$.*

Proof. Suppose that $z \in Z$. Then for every k, if $n, m > k$,

$$e_n z e_k = e_n e_k^{1/2} z e_k^{1/2} = e_k^{1/2} z e_k^{1/2} = e_m z e_k .$$

Thus $z \in QM_l(A_{00}) = LM(A_{00})$. Similarly, $z \in RM(A_{00})$, so $z \in M(A_{00})$.

Let $y \in M(A_{00})$. Then

$$z \cdot y \cdot a = (y \cdot a) \cdot z = y \cdot z \cdot a \quad \text{for every } a \in A_{00}.$$

Hence $z \cdot y = y \cdot z$. Z is in the center of $M(A_{00})$. The center of $M(A_{00})$ contained in Z is trivial.

4.7. **Lemma.** *Let $z \in Z$. Then for each $f \in P(A)$, the pure state space of A, $f(z) = \lim f(e_n z e_n)$ exists. Moreover, the function $f \to f(z)$ is a weak*-continuous function on $P(A)$.*

Proof. Let f be in $P(A)$, let π_f be the corresponding irreducible representation of A, and let H be the associated Hilbert space. Suppose that $z_n = z|_{A_n}$. Then z_n is in the center of $M(A_n)$. We may assume that $A_n \not\subset \ker \pi_f$. Then $(\pi_f|_{A_n}, \overline{\pi_f(A_n)H})$ is an irreducible representation of A_n. Let q_n be the projection corresponding to H_n, the closure of $\pi_f(A_n)H$. Then

$$\pi_f(z_n)|_{H_n} = \lambda_n q_n \quad \text{for some scalar } \lambda_n .$$

Since $\pi_f(z_{n+1})|_{H_n} = \pi_f(z_n)|_{H_n}$, $\lambda_{n+1} = \lambda_n$ for each n. Thus $\pi_f(z)$ is a scalar multiple of the identity. Moreover, $\pi_f(z) = f(z) \cdot \mathrm{id}_H$.

Next we shall show that $f \to f(z)$ is continuous. Let $f_0 \in P(A)$. There is k_0 such that $1 \geq f_0(e_{k_0}) > 1/2$. Let $V_0 = \{f \in P(A) : |f(e_{k_0}) - f_0(e_{k_0})| < 1/4\}$. Then for every $f \in V_0$, $f(e_{k_0}) > 1/4$.

Let π_f be the associated irreducible representation and H_f the associated Hilbert space. Then, since $\pi_f(z^* z)$ is a scalar, for every unit vector $\xi \in H_f$,

$$\langle \pi_f(z^* z)\xi, \xi \rangle = f(z^* z).$$

Suppose that $f(a) = \langle \pi_f(a)\xi_f, \xi_f\rangle$ for every $a \in A$. Then

$$\begin{aligned} f(z^*z) &= 1/f(e_{k_0})^2\langle \pi_f(z^*z)e_{k_0}\xi_f, e_{k_0}\xi_f\rangle \\ &\le 1/f(e_{k_0})^2\|e_{k_0}z^*ze_{k_0}\| \\ &\le 16\|e_{k_0}z^*ze_{k_0}\| \end{aligned}$$

for every $f \in V_0$.

Let $M = \max\{1, 16\|e_kz^*ze_k\|\}$. For $\varepsilon > 0$, choose $k \ge k_0$ such that $1 \ge f_0(e_k) > 1 - \varepsilon^2/8M$. Denote

$$V = V_0 \cap \{f \in P(A): |f(e_k) - f_0(e_k)| < \varepsilon^2/8M, |f(e_kz) - f_0(e_kz)| < \varepsilon/4\}.$$

So for every $f \in V$, $|f(z^*z)| < M$ and $|f(1-e_k)| < \varepsilon^2/4M$. Hence, if $f \in V$,

$$\begin{aligned} |f(z) - f_0(z)| &\le |f(z) - f(e_kz)| + |f(e_kz) - f_0(e_kz)| + |f_0(e_kz) - f_0(z)| \\ &< |f((1-e_k)z)| + \varepsilon/4 + |f_0((1-e_k)z)| \\ &\le f(1-e_k)^{1/2}f(z^*z)^{1/2} + f_0((1-e_k)^2)^{1/2}f_0(z^*z)^{1/2} + \varepsilon/4 \\ &\le f(1-e_k)^{1/2}M^{1/2} + f_0(1-e_k)^{1/2}M^{1/2} + \varepsilon/4 \\ &< \varepsilon/2 + \varepsilon/8 + \varepsilon/4 < \varepsilon. \end{aligned}$$

4.8. The idea of the proof of 4.7 was taken from [11, 5.41]. However, the proof of [11, 5.41] is not complete. (The number M there depends on the choice of a and a depends on ε, so M depends on ε.) Nevertheless, the proof could be easily completed. The same result as [11, 5.41] is not true for $QM(A_{00})$, as we shall see in 4.14.

4.9. In the proof of 4.7, we see that if π_{f_1} and π_{f_2} are equivalent, then $f_1(z) = f_2(z)$ for $z \in Z$. Thus every $z \in Z$ defines a continuous function z on $\widehat{A}$ by $\hat{z}(\pi_f) = f(z)$.

4.10. **Theorem.** *The mapping* $z \to \hat{z}$ *is a *-isomorphism of* Z *onto* $C(\widehat{A})$. *Moreover, the mapping is bicontinuous when* Z *is considered with the* A_{00}-*topology and* $C(\widehat{A})$ *with the compact open topology.*

Proof. Clearly, $z \to \hat{z}$ is a *-homomorphism. If $\hat{z}_1 = \hat{z}_2$ for $z_1, z_2 \in Z$, then $\pi(z_1) = \pi(z_2)$ for every $\pi \in \widehat{A}$. Thus $z_1 = z_2$. Hence the mapping is one-to-one.

Suppose that $f \in C(\widehat{A})$. For every k, by [11, 5.39], $\{\pi \in \widehat{A}: \pi(e_{k+1}) \ne 0\}$ is contained in a compact subset of $\widehat{A}$. Thus $\widehat{A}_k$ is contained in a compact subset of A. Thus $f|_{\widehat{A}_k}$ is bounded and by the Dauns-Hofmann theorem (we use the version [18, 4.4.6]), for every $a \in A_k$, there is $\rho(a) \in A_k \subset A_{00}$ such that

$$\pi(\rho(a)) = f(\pi)\pi(a) \quad \text{for } \pi \in \widehat{A}_k.$$

Hence, the above equality holds for all $\pi \in \widehat{A}$, and ρ defines a linear map from A_{00} into A_{00}. Let $a, b \in A_{00}$. We have

$$\pi(a\rho(b)) = f(\pi)\pi(a)\pi(b) = \pi(\rho(a)b)$$

for all $\pi \in \widehat{A}$. Thus $z = (\rho, \rho) \in M(A_{00}) \subset QM(A_{00})$ and, clearly, $z \in Z$. It is then easy to see that $\hat{z}(\pi) = f(\pi)$ for each $\pi \in \widehat{A}$. Thus the mapping is surjective.

The proof of the bicontinuity is essentially the same as the proof of [11, 5.44] with the obvious minor modifications.

4.11. **Corollary.** *Let $f \in C(\widehat{A})$. Then, for any $z \in QM(A_{00})$, there is $y \in QM(A_{00})$ such that $\pi(y) = f(\pi)\pi(z)$ for all $\pi \in \widehat{A}$.*

4.12. By [18, 4.417], we may replace $\widehat{A}$ by $\mathrm{Prim}(A)$ in 4.10 and 4.11.

4.13. We shall denote $FQM(A_{00}) = \{z \in QM(A_{00}) : f(z) = \lim f(e_n z e_n)$ exists for each $f \in P(A)\}$. Clearly, $FQM(A_{00})$ is a *-invariant linear space containing $QM(A)$.

4.14. **Theorem.** (i) *If $z \in FQM(A_{00})$, then $\tilde{\pi}(z) \in QM(\pi(A))$ for every $\pi \in \widehat{A}$.*

(ii) *If $C^b(\widehat{A}) \neq C(\widehat{A})$, then $FQM(A_{00}) \neq QM(A)$.*

(iii) *$FQM(A_{00}) = QM(A_{00})$ if and only if $\pi(A)$ is unital for each $\pi \in \widehat{A}$.*

Proof. (i) We may assume that $z = z^*$. Let $\pi \in \widehat{A}$, H be the associated Hilbert space, and ξ be a unit vector in H.

Since $\langle \pi(e_n z e_n)\xi, \xi\rangle$ converges, we may assume that there is a positive number M_ξ such that

$$|\langle \pi(e_n z e_n)\xi, \xi\rangle| \le M_\xi \quad \text{for all } n.$$

Hence

$$|\langle \pi(e_n z e_n)_+\xi, \xi\rangle| \le M_\xi \quad \text{for all } n.$$

So

$$\|(e_n z e_n)_+^{1/2}\xi\| \le M_\xi \quad \text{for all } n.$$

by the uniform boundedness theorem, $\{\|(e_n z e_m)_+^{1/2}\|\}$ is bounded. Hence $\{\|(e_m z e_n)_+\|\}$ is bounded. Similarly, $\{\|(e_n z e_n)_-\|\}$ is bounded, thus $\{\|(e_n z e_n)\|\}$ is bounded. This implies that $\tilde{\pi}(z) \in QM(\pi(A))$.

(ii) If $C^b(A) \neq C(A)$, then, by Theorem 4.10, there is $z \in Z \subset QM(A_{00})$ such that z is not bounded. Thus $z \notin QM(A)$. However $z \in FQM(A_{00})$.

(iii) Suppose that $\pi \in \widehat{A}$ and $\pi(A)$ has no unit. By taking a subsequence if necessary, we may assume that

$$\pi(e_{nm}) - \pi(e_{n-1}) \neq 0.$$

Thus there are $\xi_k \in H$ such that $\|\xi_k\| = 1$, and $\xi_k \perp \xi_j$ if $k \neq j$; and

$$\|(\pi(e_{2k+2}) - \pi(e_{2k}))^{1/2}\xi_k\| = a_k > 0$$

and

$$[\pi(e_{2k+2}) - \pi(e_{2k})]\xi_m = 0 \quad \text{if } m \neq k$$

for every k. Define

$$y = \sum_k (k+1)(2^{k+1}/a_k)(e_{2k+2} - e_{2k}).$$

Then it is easy to see that $y \in M(A_{00}) \subset QM(A_{00})$. Let $\xi = \sum_{k=1}^{\infty} (1/2)^{k/2}\xi_k$; then $\|\xi\| = 1$. So $f(\cdot) = \langle \cdot \xi, \xi \rangle$ is a pure state of A. But

$$f(e_{2k+2} y e_{2k+2}) \geq k.$$

So $y \in FQM(A_{00})$.

Conversely, if $\pi(A)$ is unital for each $\pi \in \widehat{A}$, then $\tilde{\pi}(QM(A_{00})) = QM(\pi(A))$. The conclusion is obvious.

5. Duals and biduals

In this section, we shall study $QM(A_{00})'$, the dual of $QM(A_{00})$ (the latter being considered with the Q-A_{00}-topology), and $QM(A_{00})''$, the bidual of $QM(A_{00})$.

5.1. **Theorem.** $QM(A_{00})' = \{f(a \cdot b) \colon a, b \in A_{00},\ f \in A^*, \textit{ and } \|f\| \leq 1\}$.

Proof. For $a, b \in A_{00}$, denote

$$U_{a,b} = \{z \in QM(A_{00}) \colon \|azb\| \leq 1\}.$$

Then $\{U_{a,b}\}$ forms a neighborhood base at 0. Let

$$U_{a,b}^0 = \{f \in QM(A_{00})' \colon |f(z)| < 1 \text{ if } z \in U_{a,b}\}.$$

Then

$$QM(A_{00})' = \bigcup \{U_{a,b}^0 \colon a, b \in A_{00}\}.$$

Suppose that $f \in U_{a,b}^0$; then $|f(z)| < 1$ for each $z \in U_{a,b}$, or, equivalently,

$$|f(z)| < \|azb\| \quad \text{for each } z \in QM(A_{00}).$$

Define a linear functional g on the normed linear space $\{azb \colon z \in QM(A_{00})\}$ of A by $g(azb) = f(z)$. Then g is well defined and $|g(azb)| < \|azb\|$. By the Hahn-Banach theorem, we can assume that g is in A^* and $\|g\| < 1$. Thus

$$U_{a,b}^0 \subset \{f(a \cdot b) \colon f \in A^*,\ \|f\| \leq 1\}.$$

This completes the proof.

5.2. Let $g \in A_n^*$ and $p_n = [e_n]$. For every $a \in A$, define $f(a) = g(p_n a p_n)$. Then $f \in A^*$ and $\|f\| = \|g\|$. Moreover,

$$\begin{aligned} f(e_{nm+1} a e_{n+1}) &= g(p_n e_{n+1} a e_{n+1} p_n) \\ &= g(p_n a p_n) = f(a) \quad \text{for every } a \in A. \end{aligned}$$

Define $\tilde{f}(z) = (e_{n+1}ze_{n+1})$; then $\tilde{f} \in QM(A_{00})'$. We denote by L_n the set

$$\{f\colon f(a) = g(p_n a p_n),\ g \in A_n^*,\ \text{for every } a \in A\}.$$

Then $L_n \subset QM(A_{00})'$. If $g \in QM(A_{00})'$, by Theorem 5.1, $g(\cdot) = f(a \cdot b)$ for some $a, b \in A_n$ and some n. Clearly $g(p_n \cdot p_n) = g$, so $g \in L_n$.

5.3. **Corollary.** $QM(A_{00})' = \bigcup_{n=1}^{\infty} L_n$.

5.4. By 5.2 we can identify L_n with A_n^*.

5.5. **Proposition.** *Let f be a positive Q-A_{00}-continuous functional on $QM(A_{00})$. Then there is a positive functional $g \in (A^*)_+$ and n such that*

$$f(z) = g(e_{n+1}ze_{n+1}) \quad \textit{for all } z \in QM(A_{00}).$$

Proof. It is an immediate consequence of 5.3.

5.6. **Proposition.** *$QM(A_{00})'$ is the linear span of its positive cone.*

Proof. Since L_n $(= A_n^*)$ is the linear span of its positive cone, by 5.3 $QM(A_{00})'$ is the linear span of its positive cone.

5.7. We shall denote by $M_0(A)$ the norm closure of $\bigcup_{n=1}^{\infty} A_n^{**}$ (cf. [15]). Then $\bigcup_{n=1}^{\infty} A_n^{**} = \bigcup_{n=1}^{\infty} p_n A^{**} p_n$ is a support algebra of $M_0(A)$, where $p_n = [e_n]$.

5.8. Let $QM(A_{00})''$ be the bidual of $QM(A_{00})$. The "strong" topology on $QM(A_{00})''$ is the locally convex topology generated by seminorms

$$\|F\|_{a,b} = \sup\{|F(f)|\colon f \in U_{a,b}^0\},$$

where $F \in QM(A_{00})''$, $a, b \in A_{00}$, and $U_{a,b}^0$ as in 5.1.

5.9. **Theorem.** *$QM(A_{00})''$ is isomorpic to $QM(\bigcup_{n=1}^{\infty} A_n^{**})$ as topological vector spaces, the former is considered with "strong" topology and the latter is considered with Q-$\bigcup_{n=1}^{\infty} A_n^{**}$-topology.*

Proof. Let L_n be the same as in 5.2. There is a natural isometry from L_n onto A_n^*. We may identify L_n with A_n^*.

Let $F \in QM(A_{00})''$. Define $F_n = F|_{L_n}$ $(= F|_{A_n^*})$. So there is $z_n(F) \in A^{**}$ such that

$$F_n(f) = z_n(F)(f) \quad \text{for all } f \in A^*.$$

We define a map Φ from $QM(A_{00})''$ into $QM(\bigcup_{n=1}^{\infty} A_n^{**})$ as follows:

$$\Phi\colon F \to \rho_F, \qquad \text{where } \rho_F(a, b) = az_n(F)b$$

for all $a, b \in A_n^{**}$, $n = 1, 2, \ldots$. Since $F_{n+1}|_{A_n^*} = F_n$, ρ_F is well defined and ρ_F is in $QM(\bigcup_{n=1}^{\infty} A_n^{**})$. Clearly Φ is a linear map.

If $\rho_F = 0$, then $F_n(f) = 0$ for all $f \in A_n^{**}$ and all n. So $F = 0$. Hence Φ is one-to-one.

Take $z \in QM(\bigcup_{n=1}^{\infty} A_n^{**})$. Then $p_n z p_n \in A_n^{**}$. For each $f \in A_n^*$ $(= L_n)$ define

$$F_z(f) = f(p_n z p_n) \quad \text{for } f \in A_n^* \ (= L_n).$$

Thus we define an element F_z in $QM(A_{00})''$. It is easy to see that $\Phi(F_z) = z$. Hence Φ is onto.

Now suppose that $F_\alpha, F \in QM(A_{00})''$ such that $F_\alpha \to F$ in the "strong" topology.

Let $U_n^0 = \{f \in QM(A_{00})' : |f(z)| < 1 \text{ if } \|e_{n+1} z e_{n+1}\| \le 1\}$. Then

$$\sup\{|F_\alpha(f) - F(f)| : f \in U_n^0\} \to 0.$$

If $f \in A_n^*$ $(= L_n)$ and $\|f\| \le 1$, then

$$|\tilde{f}(z)| = |f(p_n e_{n+1} z e_{n+1} p_n)|| \le \|p_n e_{n+1} z e_{n+1} p_n \le \|e_{n+1} z e_{n+1}\|.$$

Hence $f \in U_n^0$. Thus,

$$\begin{aligned} \|p_n(\rho_{F_\alpha} - \rho_F)p_n\| &= \sup\{|f(p_n e_n(z_n(F_\alpha) - z_n(F))p_n)| : f \in A_n^*, \ \|f\| \le 1\} \\ &= \sup\{|F_\alpha(f) - F(f)| : f \in L_n, \ \|f\| \le 1\} \\ &\le \sup\{|F_\alpha(f) - F(f)| : f \in U_n^0\} \to 0. \end{aligned}$$

Hence $\rho_{F_\alpha} \to \rho_F$ in $Q\text{-}\bigcup_{n=1}^\infty A_n^{**}$-topology.

Conversely, suppose that $\rho_{F_\alpha} \to \rho_F$ in $Q\text{-}\bigcup_{n=1}^\infty A_n^{**}$-topology. For each n, by 5.1,

$$U_n^0 \subset \{f(e_{n+1} \cdot e_{n+1}) : f \in A^*, \ \|f\| \le 1\}.$$

Thus

$$U_n^0 \subset \{f \in L_n : \|f\| < 1\}.$$

Hence

$$\begin{aligned} \|p_n(\rho_{F_\alpha} - \rho_F)p_n\| &= \sup\{|f(p_n(z_n(F_\alpha) - z_n(F))p_n)| : f \in L_n, \ \|f\| \le 1\} \\ &\ge \sup\{|f(F_\alpha) - f(F)| : f \in U_n^0\}. \end{aligned}$$

Thus $\|p_n(\rho_{F_\alpha} - \rho_F)p_n\| \to 0$ implies

$$\sup\{|f(F_\alpha) - f(F)| : f \in U_n^0\} \to 0.$$

So Φ is bicontinuous.

5.10. **Example.** Let K be the C^*-algebra of all compact operators on a separable Hilbert space. Let $A_{00} = \bigcup_{n=1}^\infty M_n$ be a support algebra of K, where each M_n is isomorphic to the $n \times n$ matrix algebra. Since $M_n^{**} = M_n$, $M_0(A) = A$. Hence $QM(\bigcup_{n=1}^\infty M_n^{**}) = QM(A_{00})$. By 5.9, $QM(A_{00})'' = QM(A_{00})$.

5.11. **Proposition.** *Every σ-unital dual C^*-algebra has reflexive quasi-multipliers.*

Proof. Let e be a strictly positive element of A. By [4, 4.7.20], every nonzero point of $\mathrm{Sp}(e)$ is isolated. So we may assume that e_n are projections. Consequently, $A_n = e_n A e_n$ and are unital dual C^*-algebras. Thus A_n are finite dimensional. This implies that $A_n^{**} = A_n$. Hence $M_0(A) = A$. By 5.9, $QM(A_{00})'' = QM(A_{00})$.

6. Pseudo-commutative C^*-algebras

In §3, we showed that $QM(A_{00}) = LM(A_{00}) + RM(A_{00})$. We now consider the problem when $QM(A_{00}) = M(A_{00})$. It turns out that the problem is equivalent to the problem when $K(A) = A_{00}$.

6.1. **Theorem.** *Let A be a σ-unital C^*-algebra and $A_{00}(\{e_n\})$ a support algebra of A. Then the following are equivalent:*

(i) $M(A_{00}) = QM(A_{00})$.

(ii) *For every n, there is an integer $N(n) < n$ such that $e_n a = e_n a e_{N(n)}$ for all $a \in A$.*

Proof. (i) $\Rightarrow$ (ii). Since $M(A_{00}) = QM(A_{00})$, $A \subset M(A_{00})$. So for every $a \in A$, $e_n a \in A_{00}$, that is, $e_n a \in A_k$ for some k. Thus $e_n a = e_n a e_{k+1}$. If (i) does not imply (ii), there are $a_k \in A$ such that

$$x_k = e_n a_k (e_{n_{k+1}} - e_{n_k}) \neq 0$$

for some subsequence $\{n_k\}$. We may assume that $\|x_k\| = 1$ for all k. Define $z = \sum_{k=1}^{\infty} (1/2)^k x_k$. Then $z \in A \subset QM(A_{00})$. But

$$e_{n=1} z = e_{n+1} \left(\sum_{k=1}^{\infty} (1/2)^k \right) = \sum_{k=1}^{\infty} (1/2)^k x_k = z \notin A_{00}.$$

Hence $z \notin M(A_{00})$, a contradiction.

(ii) $\Rightarrow$ (i) For fixed n,

$$(ae_n)^* = e_n a^* = e_n a^* e_{N(n)} \quad \text{for all } a \in A.$$

So $ae_n = e_{N(n)} a e_n$.

Suppose that $z \in QM(A_{00})$. For fixed k,

$$\begin{aligned} e_n z e_k &= e_{n+1} e_n z e_k e_{k+1} = e_{n+1} e_n e_{N(k+1)} z e_k \\ &= e_{N(k+1)} z e_k \quad \text{if } n > N(k+1). \end{aligned}$$

Thus $z \in QM_l(A_{00})$. Similarly, $z \in QM_r(A_{00})$, so $z \in M(A_{00})$.

6.2. **Definition.** A σ-unital C^*-algebra A (without unit) is called pseudo-commutative if A satisfies (i) or (ii) in 6.1.

6.3. **Proposition.** *Suppose that A is a pseudo-commutative C^*-algebra (without identity). Then the following are true:*

(i) *The Pedersen ideal $K(A)$ is a support algebra of A.*

(ii) $M(A) = QM(A)$.

(iii) *The spectrum $\hat{A}$ of A is not compact.*

(iv) *For every irreducible representation π of A, $\pi(A)$ has a unit.*

Proof. (i) By (ii) of 6.1, A_{00} is a dense ideal of A. Since $K(A) \subset A_{00}$, we conclude that $K(A) = A_{00}$.

(ii) Suppose that $z \in QM(A)$. Then $z \in M(A_{00})$. For every $a \in A$,

$$e_n a e_n z \in A_{00} \subset A.$$

Since z is bounded and $\|e_n a e_n - a\| \to 0$, we conclude that $az \in A$. Similarly $za \in A$. So $z \in M(A)$.

(iii) If $\widehat{A}$ is compact, by [11, 10.8], A is a PCS-algebra, that is, $M(A) = \Gamma(K(A))$. It follows from (i) that $\Gamma(K(A)) = M(A_{00})$. Hence $M(A) = M(A_{00}) = QM(A_{00})$. However, by Lemma 2.16, if A is not unital, $QM(A_{00}) \neq QM(A)$. A contradiction.

(iv) By [11, 10.4], $\pi(A)$ is a PCS-algebra, so, as in (iii), $QM(\pi(A)) = QM(\pi(A_{00}))$. By Lemma 2.16, it happens only when $\pi(A)$ has a unit.

The following lemma is taken from [11, 10.7] but in a slightly different setting. The terminology follows from [11].

6.4. **Lemma** (cf. [11, 10.7]). *Let A by a C^*-algebra and let $\{x_n\}$ be an orthogonal sequence in $(K(A))_+$ (that is, $x_n x_m = 0$, if $n \neq m$) such that the sequence of partial sum $\{\sum_{k=1}^{\infty} x_k\}$ is K-Cauchy. Let $a \in K(A)$, S be a subset of $\widehat{A}$, and let $\{\alpha_n\}$ be the sequence defined by*

$$\alpha_n = \sup\{\|\pi(a)\| \colon \pi \in S \text{ and } \|\pi(x_n)\| > \|x_n|_S\|/2\},$$

where $\|x_n|_S\| = \sup\{\|\pi(x_n)\| \colon \pi \in S\}$. If $\|x_n|_S\| \to \infty$, then $\alpha_n \to 0$.

Proof. The proof is the same as the proof of [11, 10.7]. We only need to change $\widehat{A}$ and $\|x_n\|$ into S and $\|x_n|_S\|$, respectively.

6.5. **Theorem.** *Suppose that A is a σ-unital C^*-algebra. Then A is pseudo-commutative if and only if one of its support algebras $A_{00} = K(A)$.*

Proof. Let $A_{00} = A_{00}(\{e_n\})$. For every n, denote

$$F_n = \{\pi \in \widehat{A} \colon \|\pi(e_n)\| \geq 1/n + 1\}.$$

We claim that there is a $b_n \in A_{00}$ such that

$$\pi(b_n) = 1 \quad \text{for each } \pi \in F_n.$$

If not, by taking a subsequence if necessary, we may assume that there are $\pi_k \in F_n$ such that

$$\pi_k(e_k - e_{k-1}) \neq 0.$$

Let $x_k = \beta_k(e_{2k} - e_{2k-1})$, where $\beta_k = k \cdot \max(1, 1/\|\pi_k(e_{2k} - e_{2k-1})\|)$, $k = 1, 2, \ldots$. Then $x_k x_m = 0$ if $n \neq m$ and $\sum_{k=1}^{\infty} x_k$ is A_{00}-Cauchy. By letting $a = e_n$, and $S = F_n$ in Lemma 6.4, we have $\|x_k|F_n\| \to \infty$ as $k \to \infty$, hence $\|\pi_k(e_n)\| \to 0$ as $k \to \infty$. This contradicts the fact $\|\pi(e_n)\| \geq 1/n + 1$ for all $\pi \in F_n$. So we complete the proof of the claim.

Now let $a_1 = b_1$. Then $a_1 \in A_{00}$, so $a_1 \in A_{N(1)}$ for some $N(1)$. Suppose that $a_1, a_2, \ldots, a_k$ have been chosen from A_{00}, and assume that $a_k \in A_{N(k)}$. Then

$$a_k e_{N(k+1)} = e_{N(k)+1} a_k = a_k.$$

So

$$\{\pi \in \widehat{A}: \pi(a_k) \neq 0\} \subset \{\pi \in \widehat{A}: \|\pi(e_{N(k)+1})\| \geq 1\} \subset F_{N(k)+1}.$$

We choose $a_{k+1} = b_{N(k)+1}$. Thus $\pi(a_{k+1}) = 1$ for all $\pi \in \{\pi \in \widehat{A}: \pi(a_k) \neq 0\}$. Hence $a_{k+1}a_k = a_k a_{k+1} = a_k$. For every $a \in A$,

$$\pi(a_k a) = \pi(a_k)\pi(a) = 0 \quad \text{if } \pi(a_k) = 0.$$

Thus

$$\pi(e_k a) = \pi(e_k)\pi(a)\pi(a_{k+1})$$

for all $\pi \in \widehat{A}$. We conclude that

$$a_k a = a_k a a_{k+1} \quad \text{for all } a \in A \text{ and } k.$$

Clearly $\{a_k\}$ forms an approximate identity for A. By 6.1 we conclude that A is pseudo-commutative.

The converse is (i) of 6.3.

6.6. **Theorem.** *Let A be a pseudo-commutative C^*-algebra. Then $K(A)$ is the only support algebra of A.*

Proof. By the proof of 6.5, there is an approximate identity $\{a_n\}$ satisfying $a_{k+1}a_k = a_k a_{k+1} = a_k$ for each k and $a_k a = a_k a a_{k+1}$ for every $a \in A$. Moreover, there are compact subsets F_n of A such that $F_n \subset F_{n+1}$, $\bigcup_{n=1}^{\infty} F_n = \widehat{A}$, and

$$\pi(a_n) = \begin{cases} 1 & \text{for all } \pi \in F_n, \\ 0 & \text{if } \pi \in \widehat{A} \setminus F_{n+1}. \end{cases}$$

Since $a_k a = a_k a a_{k+1}$ for every $a \in A$, $A_{00}(\{a_k\})$ is an ideal. So $A_{00}(\{a_n\}) = K(A)$.

Now suppose that $A_{00} = A_{00}(\{e_n\})$ is any support algebra of A. For every n, there is $k(n)$ such that

$$\|e_{k(n)}a_n - a_n\| < 1/2.$$

Hence

$$\|\pi(e_{k(n)}) - 1\| < 1/2 \quad \text{for all } \pi \in F_n.$$

Thus $\pi(A_{k(n)}) = \pi(A)$ for all $\pi \in F_n$. Since $\pi(a_{n-1}) = 0$ for $\pi \in \widehat{A} \setminus F_n$, we conclude that $e_{k(n)} \geq a_{n-1}$ for every n. Hence

$$A_{00} \supseteq A_{00}(\{a_n\}) = K(A).$$

This completes the proof.

6.7. **Definition.** An approximate identity $\{e_n\}$ of A is said to be central if $e_n a = a e_n$ for all $a \in A$ and all n.

6.8. **Theorem.** *Suppose that A is a σ-unital C^*-algebra such that* $\mathrm{Prim}(A)$ *is a Hausdorff space. Then A is pseudo-commutative if and only if A has a central approximate identity $\{e_n\}$ satisfying $e_{n+1}e_n = e_n e_{n+1} = e_n$ for all n.*

Proof. Suppose that A is pseudo-commutative. Let

$$T_n = \{\pi \in \mathrm{Prim}(A) \colon \|\pi(e_n)\| \geq 1/n\},$$
$$O_n = \{\pi \in \mathrm{Prim}(A) \colon \|\pi(e_n)\| > 1/n+1\},$$

and

$$F_n = \{\pi \in \mathrm{Prim}(A) \colon \|\pi(e_n)\| \geq 1/n+1\}.$$

by [18, 4.43 and 4.45], T_n and F_n are closed and compact and O_n is open. The element b_n in 6.5 satisfies $\pi(b_n) = 1$ for all $\pi \in F_n$. Since $\mathrm{Prim}(A)$ is a locally compact Hausdorff space, there is $f \in C(\mathrm{Prim}(A))$ such that $0 \leq f \leq 1$, $f|_{T_n} = 1$, and $f|_{(\mathrm{Prim}\,A)\setminus O_n} = 0$. By the Dauns-Hofmann theorem (cf. [6, Theorem 3]), there is $x_n \in A_+$ such that

$$\pi(x_n) = f(\pi)\pi(b_n) \quad \text{for all } \pi \in \mathrm{Prim}(A).$$

Notice that $T_n \subset O_n \subset F_n$; we have

$$\pi(x_n) = f(\pi) \quad \text{for all } \pi \in \mathrm{Prim}(A).$$

Hence x_n is in the center of A. Moreover, $\{x_n\}$ forms an approximate identity for A satisfying

$$x_{n+1}x_n = x_n x_{n+1} = x_n \text{ for all } n.$$

The converse follows from (ii) of 6.1.

6.9. **Proposition.** *Every homomorphic image of a pseudo-commutative C^*-algebra A is pseudo-commutative.*

Proof. Let ϕ be a homomorphism of A, $B = \phi(A)$, and $B_{00} = \phi(A_{00})$. Clearly, by (ii) of 6.1, for every n, $\phi(e_n)\phi(a) = \phi(e_n)\phi(a)\phi(e_{N(n)})$ for every $a \in A$. Thus B is also a pseudo-commutative C^*-algebra.

6.10. **Theorem.** *Suppose that A is a σ-unital C^*-algebra with continuous trace. Then A is pseudo-commutative if and only if A is a locally trivial continuous field of matrix algebras.*

Proof. Assume that A is a pseudo-commutative C^*-algebra. Since A has continuous trace, $\widehat{A}$ is a locally compact Hausdorff space. Fix $\pi \in A$. Let F be a compact (hence closed) neighborhood of π. Let $I = \{a \colon a \in A,\ \pi(a) = 0$ for $\pi \in F\}$, and ϕ be the canonical homomorphism from A onto A/I. So $\phi(A)^\wedge$ is compact. By the argument used in (iii) of 6.2 and 6.9, $\phi(A)$ has an identity. Thus, $\phi(A_n) = \phi(A)$ for some n. Let $a \in A_n$ such that $\pi(a_n) = 1$. Then $\pi(a_n) = 1$ for all $\pi \in F$. Since $A_n \subset K(A)$, $\mathrm{Tr}(\pi(a_n))$ is continuous. So $\mathrm{Tr}(\pi(a))$ is a constant in some neighborhood of . This implies that A is locally homogeneous of finite rank. By [7, Theorem 3.2], A is a locally trivial continuous field of matrix algebras.

Now we assume that A is a locally trivial continuous field of matrix algebras and $\{e_n\}$ is as usual. Denote

$$F_n = \{\pi \in \widehat{A} : \pi(e_n) \geq 1/2n\}.$$

Then F_n is compact. For each point $\pi \in F_n$, there is a neighborhood U_π such that A is trivial on $\overline{U}_\pi$, where $\overline{U}_\pi$ is the closure of U_π and we assume $\overline{U}_\pi$ is compact. Thus there is an $a_\pi \in A_{00}(\{e_n\})$ such that $\rho(a_\pi) = 1$ for all $\rho \in \overline{U}_\pi$. Since F_n is compact, we may assume that there are $\pi_1, \pi_2, \dots, \pi_k$, such that $\bigcup_{i=1}^k U_{\pi_i} \supset F_n$. There is m, such that

$$\|e_m a_{\pi_i} - a_{\pi_i}\| < 1/2 \quad \text{for } i = 1, 2, \dots, k.$$

So

$$\|\pi(e_m) - 1\| < 1/2 \quad \text{for all } \pi \in F_n.$$

Thus $\pi(A_m) = \pi(A)$ for each $\pi \in F_n$. Hence $\pi(e_{m+1}) = 1$ for each F_n. Now we can use the argument in 6.8 to construct a central approximate identity $\{a_n\}$ satisfying $a_{n+1}a_n = a_n a_{n+1} = e_n$. It follows then from 6.8 that A is pseudo-commutative.

6.11. **Examples.** Clearly every σ-unital commutative C^*-algebra is pseudo-commutative.

Let X be a locally compact and σ-compact Hausdorff space, and let B be a unital C^*-algebra. Let A be $C_0(X, B)$, the set of continuous mappings from X into B vanishing at infinity. It is easy to check that A has a central approximate identity $\{e_n\}$ such that $e_{n+1}e_n = e_n e_{n+1} = e_n$. So A is pseudo-commutative.

7. Singly supported C^*-algebras

7.1. We see from 6.7 that a pseudo-commutative C^*-algebra has a unique support algebra. It is evident that this may not be true for other C^*-algebras. But must every two support algebras of a given C^*-algebra be *-isomorphic?

7.2. **Definition.** We say that a σ-unital C^*-algebra is singly supported if every two support algebras are *-isomorphic.

7.3. **Corollary.** *Every pseudo-commutative C^*-algebra is singly supported.*

7.4. **Theorem.** *Let A be a C^*-algebra with approximate identities $\{e_n\}$ and $\{p_n\}$. Suppose that e_n and p_n are projections and*

$$A_{00} = \bigcup_{n=1}^{\infty} e_n A e_n, \qquad A'_{00} = \bigcup_{n=1}^{\infty} p_n A p_n.$$

Then there is a unitary $u \in M(A)$ (the multiplier algebra of A) such that $u^* A_{00} u = A'_{00}$.

Proof. We claim that there are subsequences $\{e_{n(k)}\}$ of $\{e_n\}$ and $\{p_{m(k)}\}$ of $\{p_n\}$, elements $\{f_k\}$, $\{f'_k\}$, $\{q_k\}$, $\{q'_k\}$, $\{v_k\}$, and $\{w_k\}$ in A, and unitary elements $\{u_k\}$ and $\{\overline{u}_k\}$ in $M(A)$ satisfying the following:

(i) f_k, f'_k, q_k, q'_k are projections in A, where f_k, $q'_k \in A_{00}$ and q_k, $f'_k \in A'_{00}$.

(ii) $f_if_j = 0$, $f_if_j = 0$, $q_iq_j = 0$, and $q_iq_j = 0$ if $i \neq j$.

(iii) $q'f_k = f_kq' = 0$ and $q_if' = f'q_i = 0$ for all i and k.

(iv) $e_1 = f_1$ and $\sum_{i=1}^k f_i + \sum_{i=1}^{k-1} q'_i = e_{n(k)}$.

(v) $p_{mk} = \sum_{i=1}^k q_i + \sum_{i=1}^k f'_i$.

(vi) $u_ke_{n(k)}u'_k = \sum_{i=1}^{k-1} q_i + \sum_i^k f'_i$ and $u^*_kp_{m(k)}u_k = \sum_{i=1}^k f_i + \sum_{i=1}^k q'_i$.

(vii) $v^*_kv_k = f_k$, $v_kv^*_k = f_k$, $w^*_kw_k = q'_k$, and $w_kw^*_k = q_k$.

We shall use induction.

Since A_{00} is dense in A, there is a selfadjoint element $a \in A'_{00}$ such that $\|a-e_1\| < 1/8$. We may assume that $a \in p_nAp_n$ for some $n(1)$. By [5, Lemma A.8.1], there is a projection $f'_1 \in p_{n(1)}Ap_{n(1)}$ such that

$$\|f'_1 - e_1\| < 1/4.$$

By [5, Lemmas A.8.1 and A.8.3], there is $v_1 \in A$ such that $\|v_1 - e_1\| < 1/2$, $v^*_1v_1 = e_1$, and $v_1v^*_1 = f'_1$, and there is a unitary element $u_1 \in M(A)$ such that $u_1e_1u^* = f'_1$ and $u^*_1f'_1u_1 = e_1$.

Let $q_1 = p_{n(1)} - f'_1$. Then $u^*_1q'_1u_1 \in (1-e_1)A(1-e_1)$ $(= (1-f_1)A(1-f_1))$. Since $(1-e_1)A_{00}(1-e_1)$ is dense in $(1-e_1)A(1-e_1)$, by the above argument there is a projection $q'_1 \in (1-e_1)A_{00}(1-e_1)$ such that

$$\|q'_1 - u^*_1q_1u_1\| < 1/4.$$

By [5, Lemmas A.8.1 and A.8.3], there is a $w'_1 \in (1-e_1)A(1-e_1)$ such that $(w'_1)^*(w'_1) = q'_1$, $w'_1w^*_1 = u^*_1q_1u_1$, and

$$\|w'_1 - q'_1\| < 1/2.$$

Moreover there is a unitary u' in $(1-e_1)M(A)(1-e_1)$ such that $(u')q'_1(u')^* = u^*q_1u_1$ and

$$(u')^*(u^*_1q'_1u_1)(u') = q'_1.$$

Let $w_1 = u_1w'_1$ and $\overline{u}_1 = (1-f'_1)u_1u' + f'_1u_1$. Then $w^*w_1 = q'$, $(w_1)(w_1)^* = q'_1$, and $\overline{u}_1$ is a unitary in $M(A)$ such that

$$\overline{u}^*_1p_{n(1)}\overline{u}_1 = e_1 + q'_1 = f_1 + q'_1.$$

Now we assume that we have chosen $e_{n(i)}$, $p_{m(i)}$, f_i, f'_i, q_i, q'_i, v_i, w_i, u_i, and $\overline{u}'_i$, $i = 1, 2, \dots, k$. Suppose that $q'_k \in e_{n(k+1)}Ae_{n(k+1)}$ and let

$$f_{k+1} = e_{n(k+1)} - \left(\sum_{i=1}^k f_i \sum_i^k q'_i\right).$$

Then $\overline{u}_k f_{k+1} \overline{u}_k^* \in (1-p_{n(k)})A(1-p_{n(k)})$. Since $(1-p_{n(k)})A_{00}(1-p_{n(k)})$ is dense in $(1-p_{n(k)})A(1-p_{n(k)})$, there is a projection $f'_{k+1} \in (1-p_{n(k)})A'_{00}(1-p_{n(k)})$ $(\subset A'_{00})$ such that

$$\|f'_{k+1} - \overline{u}_k f_{k+1} \overline{u}_k^*\| < 1/4.$$

By [5, Lemmas A.8.1 and A.8.3], there is $v'_{k+1} \in (1-p_{n(k)})A'_{00}(1-p_{n(k)})$ such that

$$(v'_{k+1})^*(v'_{k+1}) = f'_{k+1}, \qquad (v'_{k+1})(v'_{k+1})^* = \overline{u}_k f_{k+1} \overline{u}_k^*,$$

and a unitary $u'_1 \in (1-p_{n(k)})M(A)(1-p_{n(k)})$ such that

$$(u'_1) f_{k+1} (u'_1)^* = \overline{u}_k f_{k+1} \overline{u}_k^*$$

and

$$(u'_1)^* \overline{u}_k f_{k+1} \overline{u}_k^* (u'_1) = f'_{k+1}.$$

Define $v_{k+1} = v'_{k+1} \overline{u}_k$ and

$$u_{k+1} = (u'_1)^* \overline{u}_k \left(1 - \sum_{i=1}^{k} f_i - \sum_{i=1}^{k} q'_i\right) + \overline{u}_k \left(\sum_{i=1}^{k} f_i + \sum_{i=1}^{k} q'_i\right).$$

Then $v_{k+1}^* v_{k+1} = f_{k+1}$, $v_{k+1} v_{k+1}^* = f'_{k+1}$, and

$$u_{k+1} e_{n(k+1)} u_{k+1}^* = \sum_{i=1}^{k} q_i + \sum_{i}^{k+1} f'_i.$$

Let

$$\begin{aligned} q_{k+1} &= p_{m(k+1)} - \left(\sum_{i=1}^{k} q_i + \sum_{i=1}^{k+1} f'_i\right) \\ &= p_{m(k+1)} - u_{k+1} e_{n(k+1)} u_{k+1}^*. \end{aligned}$$

Then

$$u_{k+1}^* q_{k+1} u_{k+1} \in (1-e_{n(k+1)})A(1-e_{n(k+1)}).$$

Since $(1-e_{n(k+1)})A_{00}(1-e_{n(k+1)})$ is dense in $(1-e_{n(k+1)})A(1-e_{n(k+1)})$, there is a projection $q'_{k+1} \in (1-e_{n(k+1)1})A_{00}(1-e_{n(k+1)})$ $(\subset A_{00})$ such that

$$\|q'_{k+1} - u_{k+1}^* q_{k+1} u_{k+1}\| < 1/4.$$

By [5, Lemmas A.8.1 and A.8.3], there is a $w'_{k+1} \in (1-e_{n(k+1)})A(1-e_{n(k+1)})$ such that $(w'_{k+1})^*(w'_{k+1}) = q'_{k+1}$, $(w'_{k+1})(w'_{k+1})^* = u_{k+1}^* q_{k+1} u_{k+1}$, and

$$\|w'_{k+1} - q'_{k+1}\| < 1/2.$$

Moreover, there is a unitary u'_2 in $(1-e_{n(k+1)})M(A)(1-e_{n(k+1)})$ such that

$$(u'_2) q'_{k+1} (u'_2)^* = u_{k+1}^* q_{k+1} u_{k+1}$$

and

$$(u'_2)^* (u_{k+1}^* q_{k+1} u_{k+1})(u'_2) = q'_{k+1}.$$

Define $w_{k+1} = u_{k+1}w'_{k+1}$ and

$$\overline{u}_{k+1} = (1 - u_{k+1}e_{n(k+1)}u^*_{k+1})u_{k+1}u'_2 + u_{k+1}e_{n(k+1)}u^*_{k+1}.$$

Then $w^*_{k+1}w_{k+1} = q'_{k+1}$, $w_{k+1}w^*_{k+1} = q_{k+1}$, and

$$\overline{u}^*_{k+1}p_{m(k+1)}\overline{u}_{k+1} = \sum_{i=1}^{k+1} f'_{k+1} + \sum_{i=1}^{k+1} q'_i.$$

This completes the induction.

Now we define

$$u = \sum_{k=1}^{\infty} v_k + \sum_{k=1}^{\infty} w_k.$$

It is easily checked that u is a unitary in $M(A)$ and

$$u^*e_{n(k)}Ae_{n(k)}u = (f'_{n(k)} + p_{m(k-1)})A(f'_{n(k)} + p_{m(k-1)})$$

if $k \geq 2$. Thus

$$u^*A_{00}u = A'_{00}.$$

7.5. Let A be a C^*-algebra. We denote by $\mathrm{Aut}(A)$ the automorphism group of A. If u is a unitary in $M(A)$, we denote the automorphism $a \to u^*au$ by $\mathrm{aut}(u)$.

7.6. **Corollary.** *Let A be a C^*-algebra with an approximate identity $\{e_n\}$ consisting of projections. Define*

$$G = \{\rho \in \mathrm{Aut}(A) : \rho(A_{00}(\{e_n\})) = A_{00}(\{e_n\})\}.$$

Then for every $\phi \in \mathrm{Aut}(A)$ there are a unitary element $u \in M(A)$ and $\rho \in G$ such that $\phi = \mathrm{aut}(u) \circ \rho$.

Proof. Let $A'_{00} = \phi(A_{00}(\{e_n\}))$. It follows from 7.4 that there is a unitary $u \in M(A)$ such that

$$u(A'_{00})u^* = A_{00}.$$

Thus $\rho = \mathrm{aut}(u^*) \circ \phi \in G$. hence $\phi = \mathrm{aut}(u) \circ \rho$.

7.7. Recall that a C^*-algebra A is called scattered if every state of A is atomic, equivalently, if A has a composition series with elementary quotients (cf. [9, and 10]).

7.8. **Theorem.** *Every σ-unital scattered C^*-algebra is singly supported.*

Proof. It follows from [13, Lemma 5.1; 5, Lemma 9.4] that A has a support algebra $A_{00} = \bigcup_{n=1}^{\infty} e_nAe_n$, where the e_n are projections in A. Let a be any strictly positive element of A and $A'_{00} = A_{00}(a)$. By [12], $\mathrm{Sp}(a)$ is countable. Thus there are t_n, $0 < t_n < 1$, such that $t_n \searrow 0$ and $\chi_{(t_n, \|a\|]}(a)$ is in A. Let $p_n = \chi_{(t_n, \|a\|]}(a)$. Then

$$A'_{00} = \bigcup_{n=1}^{\infty} p_nAp_n.$$

By 7.6, A_{00} and A'_{00} are isomorphic.

7.9. Let A be a σ-unital C^*-algebra and e_n, p_n be as in 2.1. Let B^{**} be the enveloping Borel *-algebra of A. We denote the norm closure of $\bigcup_{n=1}^{\infty} p_n B^{**} p_n$ by $B_0(A)$. Clearly $B_0(A)$ is a σ-unital C^*-algebra. It follows from [15, Theorem 3.7] that $B_0(A)$ does not depend on the choices of $\{e_n\}$. We denote the norm closure of $\bigcup_{n=1}^{\infty} p_n A^{**} p_n$ by $M_0(A)$. Then $M_0(A)$ is a σ-unital C^*-algebra. By [15, Theorem 3.7], $M_0(A)$ is the hereditary C^*-subalgebra of A^{**} generated by A, hence it does not depend on the choices of $\{e_n\}$.

7.10. **Theorem.** *For every σ-unital C^*-algebra A, $B_0(A)$ and $M_0(A)$ are singly supported.*

Proof. Clearly, $\bigcup_{n=1}^{\infty} p_n B^{**} p_n$ is a support algebra of $B_0(A)$. Take any strictly positive element x of $B_0(A)$. By [15, Corollary 3.9], for every n, $\chi_{(1/n, \|x\|]}(x) \in B_0(A)$. Let $q_n = \chi_{(1/n, \|x\|]}(x)$. Then the support algebra associated with the strictly positive element x is $\bigcup_{n=1}^{\infty} q_n B^{**} q_n$. By 7.6, $B_0(A)$ is singly supported.

The proof for $M_0(A)$ is similar.

7.11. **Corollary.** *Let A be a σ-unital C^*-algebra, and let A_{00} and A'_{00} be two support algebras of A. Then $QM(A_{00})''$ is isomorphic to $QM(A'_{00})''$.*

Proof. By 7.10, $M_0(A)$ is singly supported. Therefore (up to isomorphism) there is only one quasi-multiplier space for supported algebras of $M_0(A)$. It follows from 5.9 that $QM(A_{00})''$ is isomorphic to $QM(A'_{00})''$.

7.12. The algebras in 7.8 and 7.10 have a rich structure of projections. Projectionless singly supported C^*-algebras can be found in pseudo-commutative C^*-algebras. The following is an example of a projectionless singly supported C^*-algebra which is not pseudo-commutative.

7.13. Let B be a separable nonelementary simple AF C^*-algebra with unique trace τ. Suppose that p is a nonzero projection of B. Then $pBp \cong B$ (see [2]). Let σ be a nonzero endomorphism of B, and A be the set of continuous functions from $[0, 1]$ into B such that $f(1) = \sigma(f(0))$. We assume that $\sigma(1) = p \neq 0$. By [2], A has no nonzero projections. A is nonunital but is a σ-unital C^*-algebra. Moreover, $\mathrm{Prim}(A)$ is homeomorphic to the unit circle. It follows from 6.3 that A is not pseudo-commutative.

Suppose that $\sigma(B) = pBp$ for some nonzero projection p in B. Let

$$e_n = \begin{cases} 1 & \text{if } 1/n < t \le 1; \\ p + n(n+1)(t - 1/n + 1)(1-p) & \text{if } 1/n + 1 \le t \le 1/n; \\ p & \text{if } 0 \le t < 1/n + 1. \end{cases}$$

Then $\{e_n\}$ forms an approximate identity for A, and

$$e_{n+1} e_n = e_n e_{n+1} = e_n \quad \text{for all } n.$$

Let $A = [e_n] A^{**} [e_n] \cap A$ and $A_{00} = \bigcup_{n=1}^{\infty} A_n$.

Suppose that $\{b_n\}$ is another approximate identity for A satisfying $b_{n+1}b_n = b_n b_{n+1} = b_n$ for all n. Define $A' = [b_n]A^{**}[b_n]$ A and $A'_{00} = \bigcup_{n=1}^{\infty} A'_n$. For each n, there is an $m(n)$ such that $\|b_m(t)e_n(t) - e_n(t)\| < 1/2$ for all $m \geq m(n)$ and $t \in [0, 1]$. Thus, if $m \geq m(n)$, $\|b_m(t) - 1\| < 1/2$ for all $t \in [1/n, 1]$ and $\|b_m(0) - p\| < 1/2$. So if $m \geq m(n)$, $b_m(t) = 1$ if $t \in [1/n, 1]$ and $b_m(0) = p$.

Without loss of generality we may assume that $b_n(t) = 1$ if $t \in [1/n, 1]$ and $b_n(0) = p$ for all n. For each n, there is a number $\alpha_n > 0$ such that $\|b_{n+1}(t) - p\| < 1/4$ and $\|b_n(t) - p\| < 1/4$ for $0 \leq t < \alpha_n$. Thus $\mathrm{Sp}(b_n(t)) \subset [0, 1/4] \cup [3/4, 1]$ and $\mathrm{Sp}(b_{n+1}(t)) \subset [0, 1/4] \cup [3/4, 1]$ for all $0 \leq t < \alpha_n$.

The characteristic function $\chi = \chi_{(1/4, 1]}$ is continuous on $\mathrm{Sp}(b_n(t))$ and $\mathrm{Sp}(b_{n+1}(t))$ for $0 \leq t < \alpha_n$, and thus $q_1 = \chi(b_n)$ and $q_2 = \chi(b_{n+1})$ are continuous on $[0, \alpha_n)$. Moreover.

$$\|q_1(t) - p\| < 1/2, \quad \|q_2(t) - p\| < 1/2 \quad \text{if } 0 \leq t < \alpha_n.$$

Clearly,

$$q_2(t) \geq [b_n(t)] \geq q_1(t).$$

Since $\tau(q_2(t)) = \tau(q_1(t))$ for $0 \leq t < \alpha_n$, we conclude that

$$q_2(t) = [b_n(t)] = q_1(t) \quad \text{for } 0 \leq t < \alpha_n.$$

Furthermore, since b_n is increasing,

$$[b_{n+k}(t)] = [b_n(t)] \quad \text{if } 0 \leq t < \min(\alpha_n, \alpha_{n+k}).$$

Let A_1 be the C^*-algebra $A|_{[0, (1/2)\alpha_1]}$. Since $[b_1(t)] = \chi_{(b_1(t))}$ for $t \in [0, (1/2)\alpha_1]$,

$$a_1 = [b_1(t)]|_{[0, (1.2)\alpha_1]} \in A_1.$$

Put $q(t) = p$ for all $t \in [0, (1/2)\alpha_1]$. Then $q(t) \in A_1$. By [5, Corollary A.8.3], there is a unitary $u_1 \in M(A_1)$ such that

$$u_1^* q u_1 = a_1 \quad \text{and} \quad u_1 a_1 u_1^* = q.$$

Define

$$u = \begin{cases} 1, & t = 0; \\ u_1(t), & 0 < t \leq (1/2)\alpha_1; \\ u_1(\alpha_1 - t), & (1/2)\alpha_1 < t \leq \alpha_1; \\ 1, & \alpha_1 < t \leq 1. \end{cases}$$

It is easy to verify that u is a unitary in $M(A)$. Moreover, $ub_n u^* \leq e_N$ and $ue_n u \leq b_N$, where $N > n$ and $1/N \leq (1.2)\alpha_n$.

We conclude that

$$u^* A_{00} u = A'_{00}.$$

So A is a singly supported C^*-algebra.

7.14. We denote $K_0 = \{a \in A_+ : \text{there is a } b \in (A_+)_1 \text{ such that } [a] \leq b\}$.

The following result may help to find a separable C^*-algebra which is not singly supported.

7.15. **Theorem.** *Let A be a separable C^*-algebra with an approximate identity consisting of projections. Suppose that A is singly supported. Then*

$$K_0^+ = \{a \in A_+ : a \leq p\,,\ p \text{ a projection in } A\}.$$

Proof. Suppose that a is a nonzero element in K_0^+ but no projection in A majorizes a. Let b be an element in $(A_+)_1$ such that $0 \leq [a] \leq b \leq 1$. Let B be the norm closure of $(1-b)A(1-b)$ and a' be a strictly positive element of B. We may assume that $0 \leq a' \leq 1$. Put $e = a' + b$. Then e is a strictly positive element of A. Since $a'[a] = [a]a' = 0$, it follows from Lemma 2.6 that $[a]e = e[a]$. By considering the abelian C^*-algebra generated by e, $[a]$, and 1, we obtain

$$p_n = \chi_{(1/n\,,e]}(e) \geq [a].$$

Thus $a \in \bigcup_{n=1}^{\infty} p_n A^{**} p_n \cap A$. We also notice that $A_{00} = \bigcup_{n=1}^{\infty} p_n A^{**} p_n \cap A$ is a support algebra of A.

Suppose that A'_{00} is a support algebra of A associated with an approximate identity $\{e_n\}$ consisting of projections. Since A is singly supported, there is an isometry ϕ such that $\phi(A_{00}) = A_{00}$. Thus we may assume that $\phi(a) \leq e_k$ for some k. Then $\phi^{-1}(e_k) \geq a$ and $\phi^{-1}(e_k)$ is a projection. A contradiction.

7.16. To conclude the paper, we state the following questions.

(1) Is $QM(A_{00})$ the linear span of its positive cone?

(2) Is every σ-unital C^*-algebra singly supported?

If the answer of (2) is negative one may consider (3):

(3) Let A be a σ-unital C^*-algebra. We denote by $s(A)$ the number of nonisomorphic support algebras of A. For every n, is there a σ-unital C^*-algebra A such that $s(A) = n$?

(4) Are the dual C^*-algebras the only C^*-algebras which have reflexive quasi-multipliers?

(5) Does every pseudo-commutative C^*-algebra have a central approximate identity?

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DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200062, CHINA