TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 326, Number 2, August 1991

VANISHING OF $H_w^2(M, K(H))$ FOR CERTAIN FINITE VON NEUMANN ALGEBRAS

FLORIN RĂDULESCU

ABSTRAGT. We prove the vanishing of the second Hochschild cohomology group $H_w^2(M, K(H))$, whenever $M \subset B(H)$ is a finite countably decomposable von Neumann algebra not containing a non Γ-factor or a factor without Cartan subalgebra as a direct summand. Here H is a Hilbert space, and $K(H)$ the compact operators.

1. INTRODUCTION

The cohomology of operator algebras introduced by B. E. Johnson, R. V. Kadison, and J. R. Ringrose in a series of three papers is a useful tool for obtaining new invariants for operator algebras or to prove stability results by the vanishing of their cohomology groups (see [14]).

If X is a von Neumann algebra and a Banach bimodule over M, and if n is a positive integer, then the nth cohomology group of M is denoted by $H_c^n(M, X)$. If X is also a normal dual bimodule, then $H_w^n(M, X)$ is the nth weakly continuous cohomology group and it is proved in [14] that $H_c^n(M, X)$ is isomorphic to $H_w^n(M, X)$ and vanishes whenever M is approximately finite dimensional. Actually, the results of Alain Connes show that the vanishing of $H_c^1(M, X)$ for every normal dual Banach bimodule over M is equivalent to the injectivity of M (see [2]).

If $M \subseteq B(H)$ (the space of all linear bounded operators on a Hilbert space H), then $B(H)$ itself is a normal dual Banach bimodule over M; the most interesting examples of dual normal bimodules are $B(H)$ and M.

By the work of E. Christensen (see [3]) it is known that $H_c^1(M, B(H)) = 0$ in most cases (see also [5] for results concerning the higher cohomology groups). It was very well known that $H_c^k(M, M)$ for nonnegative M vanishes for $k = 1$, but for $k = 2$ nothing was known (excepting an example of B. E. Johnson [9]) until recently. It was proved by E. Christensen and A. Sinclair [6] that it vanishes if M has property Γ.

When X is no longer a normal dual bimodule, the proof of vanishing of $H_c^n(M, X)$ can be difficult even for injective M (see [9]).

Received by the editors July 10, 1989. Presented to the International Conference on Operator Algebras and Ergodic Theory, Craiova, Romania, September '89.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L40, 46M20.

An interesting example of nondual bimodule over $M \subseteq B(H)$ is $K(H)$ (the compact linear operators on H). Computation of the cohomology groups with coefficients into $K(H)$ may be interesting for its connections with the Brown-Douglas-Fillmore theory of extensions of C^*-algebras.

Johnson and Parrott proved in the early 70's [10] that $H_c^1(M, K(H))$ vanishes whenever M is abelian. They deduced that the same holds true if M does not contain type II_1 factors without Cartan subalgebras as direct summands. Sorin Popa proved the striking results that $H_c^1(M, K(H)) = 0$ for arbitrary type II_1 factors M, so that this group vanishes for any von Neumann algebra M [12] (see also [13] for a related result).

Note also that in the case of the first cohomology no mention of topology is necessary since the cochains are this time derivations and, by [14], any derivation from M into $B(H)$ is automatically ultraweakly continuous.

In particular, $H_w^1(M, K(H))$ is isomorphic to $H_c^1(M, K(H))$ and hence also null.

In this paper we are concerned with the vanishing of the second cohomology group $H_w^2(M, K(H))$ for finite von Neumann algebras of countable type not containing certain type II_1 factors as direct summands. (The subscript w means that we consider only cochains which are ultraweakly continuous with respect to the restriction of the $\sigma(B(H), B(H)_*)$-topology on $K(H)$.)

Our main result (Theorem 13) is $H_w^2(M, K(H))$ vanishes whenever $M \subseteq B(H)$ is a finite von Neumann algebra of countable type not containing type II_1 factors without Cartan subalgebras, or type II_1 factors without property Γ as direct summands.

(In fact, instead of property Γ we use a more general property whose definition is given at the end of §2.) This partially answers a question raised by S. Popa.

As a technical tool we prove (Proposition 3) that any ultraweakly continuous linear map defined on finite von Neumann algebras of countable type with values into $K(H)$ is also continuous from the unit ball of M with the strong operator topology into $K(H)$ with the norm topology on $K(H)$ (and an analoguous result for bilinear mappings). This fact was known only for derivations by [12], but the arguments there, combined with the fact (Corollary 2) that any linear map $\xi : l^\infty \mapsto K(H)$ which is $\sigma(l^\infty, l^1) - \sigma(K(H), B(H))$ continuous is also continuous with respect to the norm topology on $K(H)$, yield this more general result. As a by-product, which is also used in the proof of Theorem 13, we show that if $M \subseteq B(H)$ has a Cartan subalgebra A, then M has certain properties which are very close to the vanishing of $H_w^2(M, K(H))$. In fact for every cochain η (i.e., η is a separately ultraweakly continuous bilinear map from $M \times M$ into $K(H)$ with $\Delta\eta = 0$) we prove that there exists an ultraweakly continuous linear map ξ such that $\Delta\xi = \eta$ (where Δ is the Hochschild coboundary operator) and such that $\xi(u)$ belongs to $K(H)$ for every u in the normalizer $N_M(A)$ of A. From this we easily infer vanishing of $H_w^2(M, K(H))$ in the case of diffuse center.

If we could prove that $\hat{\zeta}$ (the composition of ζ with the canonical surjection onto the Calkin algebra $Q(H) = B(H)/K(H)$) is continuous from the unit ball of M with the strong operator topology into $Q(H)$ with the norm topology, then it would follow that $H_2^2(M, K(H)) = 0$. We are able to prove only a weaker continuity property for $\hat{\zeta}$ but which is fortunately sufficient when combined with the arguments from [12, Proof of Proposition] to prove the vanishing of the cohomology group when M is a type II_1 factor with property Γ.

Note also that ξ is obtained by perturbing with a suitable element in $Q(H)$ the linear map that is obtained from η by the averaging techniques of [14].

Finally the analysis of the discrete abelian case (which is also based on the continuity result) gives a procedure for reduction in the case of discrete center. In fact we prove (Corollary 8) that $H_w^2(M, K(H)) = 0$ whenever M is finite of countable type and there are nonzero projections p_n in $Z(M)$ with sum 1 and such that $H_w^2(M_{p_n}, K(p_n H)) = 0$ (with a certain control of the norm) for every n.

2. Definitions

Let $M \subseteq B(H)$ be a von Neumann algebra, $P(M)$ the set of projections in M, $U(M)$ the unitary group of M, $Z(M)$ the center of M, and M_* the predual of M. By $\pi_H \colon B(H) \mapsto Q(H) = B(H)/K(H)$ we denote the usual projection of $B(H)$ into the Calkin algebra. If $A \subseteq M$ is a von Neumann subalgebra, then the normalizer of A in M is the multiplicative group $N_M(A) = \{u \in U(M) | u^* A u = A\}$ and the groupoid normalizer is the set

$$GN_M(A) = \{ue | u \in N_M(A),\ e \in P(A)\}.$$

If $S \subseteq M$ is a subset of M, then S'' is the von Neumann subalgebra of M generated by S. For a normal form φ on M and x in M we denote $\|x\|_\varphi = \varphi(x^* x)^{1/2}$. A von Neumann subalgebra B of M is diffuse if it has no minimal projections and discrete if it is generated by its minimal projections. Recall that a maximal abelian diffuse subalgebra A of M is a Cartan subalgebra if $(N_M(A))'' = M$.

Moreover M is of countable type if each orthogonal family of nonzero projections in M is at most countable.

Let $X = B(H)$ or $K(H)$ with the canonical structure of Banach bimodules over M, and let n be a positive integer. Let $C_w^n(M, X)$ be the space of n-linear maps from M into X which are separately $\sigma(M, M_*)$–$\sigma(B(H), B(H)_*)$ continuous. (When $X = K(H)$ we take the restriction of the $\sigma(B(H), B(H)_*)$-topology on X.) By an obvious application of the Banach-Steinhaus principle we obtain that any element η in $C_w^n(M, X)$ is also globally norm-bounded so that the quantity

$$\|\eta\| = \sup\{\|\eta(x_1, \dots, x_n)\| \mid x_i \in M,\ \|x_i\| \leq 1,\ i = 1, 2, \dots, n\}$$

is finite.

Let $\Delta\colon C_w^n(M, X) \mapsto C_w^{n+1}(M, X)$ be the usual Hochschild coboundary operator defined by

$$\begin{aligned}(\Delta\eta)&(x_1, \dots, x_{n+1})\\ &= x_1\eta(x_2, \dots, x_{n+1}) + \sum_{k=1}^{n}(-1)^k\eta(x_1, \dots, x_kx_{k+1}, \dots, x_{n+1})\\ &\quad + (-1)^{n+1}\eta(x_1, \dots, x_n)x_{n+1}\end{aligned}$$

for η in $C_w^n(M, X)$ and $x_1, \dots, x_{n+1}$ in M.

Recall that the corresponding space of cochains is

$$Z_w^n(M, X) = \{\eta \in C_w^n(M, X) | \Delta\eta = 0\}$$

and for $n \geq 1$ the space of coboundaries is $B_w^n(M, X) = \{\Delta\xi | \xi$ is an element in $C_w^{n-1}(M, X)\}$. The Hochschild cohomology groups $H_w^n(M, X)$ are then defined as the vector space quotient $Z_w^n(M, X)/B_w^n(M, X)$. Note that $C_w^0(M, X)$ is simply identified with X, and $Z_w^1(M, X)$ is the space of derivations from M into X (i.e., the space of linear mappings $\delta\colon M \to X$ that satisfy $\delta(xy) = x\delta(y) + \delta(x)y$ for x, y in M; by [14] any derivation from M into $B(H)$ is automatically $\sigma(M, M_*) - \sigma(B(H), H(H)_*)$ continuous). Also, the vanishing of $H_w^1(M, X)$ means that any derivation δ of M into X is inner, i.e., there is T in X such that $\delta(x) = \operatorname{Ad} T(x) = [T, x] = Tx - xT$ for x in M.

The vanishing of $H_w^2(M, X)$ means that for every bilinear map η which is separately $\sigma(M, M_*) - \sigma(B(H), B(H)_*)$ continuous and $\Delta\eta = 0$ there is a $\sigma(M, M_*) - \sigma(B(H), B(H)_*)$ continuous linear map $\xi\colon M \mapsto X$ such that η measures the obstruction of ξ to be a derivation, i.e., $\eta(x, y) = x\xi(y) - \xi(xy) + \xi(x)y$ for all x, y in M.

Let e be a nonzero projection in M, M_e the reduced von Neumann algebra, and $\rho_e\colon M_e \to Q(eH) = B(eH)/K(eH)$ the composition of the canonical inclusion of M_e into $B(eH)$ with the canonical map into the Calkin algebra $Q(eH)$. A linear mapping $\xi\colon M_e \mapsto Q(eH)$ is a ρ_e derivation if $\xi(xy) = \rho_e(x)\xi(y) + \xi(x)\rho_e(y)$ for all x, y in M_e.

For any C^*-algebras A, B and for a completely bounded linear map $\xi\colon A \mapsto B$ we denote $\|\xi\|_{\mathrm{cb}} = \sup\{\|\xi\|_n | n \in \mathbb{N}\}$, where $\xi_n\colon A \otimes M_n(\mathbb{C}) \mapsto B \otimes M_n(\mathbb{C})$ is the canonical extension of ξ.

Our main technical definition is the following.

Definition. A type II_1 factor $M \subseteq B(H)$ with faithful normalized trace τ will be said to have property S with respect to H if there is a strictly positive number C such that for any nonzero projection e in M with $\tau(e) \in \mathbb{Q}$ and for any ρ_e derivation $\xi\colon M_e \mapsto Q(eH)$ we have

$$\|\xi\|_{\mathrm{cb}} \leq C\|\xi\|.$$

Note that by the work of Christensen [4], factors with property Γ always have property S (independent on the Hilbert space H) and that in this case $C = 119$.

3. Proof of the continuity result

Recall that l^1 is the Banach space of absolutely summable scalar sequences $\{(\lambda_n)_{n\in\mathbb{N}} \mid \|(\lambda_n)\|_1 = \sum|\lambda_n| < \infty, \lambda_n \in \mathbb{C}\}$, and that its dual is canonically identified with the Banach space l^∞ of all bounded scalar sequences with the uniform norm. Moreover by a well-known result of Schur (see [7]), every weakly convergent sequence in l^1 is also norm convergent. Using a vector-valued version of this result we prove that any continuous linear mapping from l^∞ with the weak* topology into the space of compact operators $K(H)$ on an arbitrary Hilbert space H with the $\sigma(K(H), B(H)_*)$-topology is also continuous with respect to the uniform norm on $K(H)$.

Finally, using this and the arguments from the Appendix of [12] we prove that weakly continuous linear mappings from a finite von Neumann algebra M of countable type into $K(H)$ automatically have the special continuity properties we mentioned in the Introduction.

Lemma 1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be Banach spaces, and let $\mathscr{F}_n \subseteq X_n^*$ be linear subspaces such that $\|x\| = \sup\{|\varphi(x)| \mid \varphi \in \mathscr{F}_n, \|\varphi\| \le 1\}$ for every x in X_n and every $n \in \mathbb{N}$. Suppose further that $F_n: l^\infty \mapsto X_n$ are linear and weak*-$\sigma(X_n, \mathscr{F}_n)$ continuous and that $\|F_n(a)\| \to 0$ for every a in l^∞. Then $\|F_n\| \mapsto 0$.*

Proof. Suppose on the contrary that there exist $\{a_n\}_{n\in\mathbb{N}}$ in l^∞ and $c > 0$ such that $\|F_n(a_n)\| > 2c$, $\|a_n\| \le 1$.

By the property of $\mathscr{F}_n$ it follows that there exists φ_n in φ_n such that

$$|\varphi_n(F_n(a_n))| \ge c, \qquad \|\varphi_n\| \le 1. \tag{1}$$

Then $\psi_n = \varphi_n \circ F_n$ are weak* continuous functionals on l^∞ and hence may be identified with an element in l^1 for each n in $\mathbb{N}$. Moreover for every a in l^∞ we have

$$|\psi_n(a)| \le \|F_n(a)\| \mapsto 0$$

so that $\{\psi_n\}_n$ is weakly convergent to zero, viewed as a sequence in l^1. But then by the Schur lemma, $\|\psi_n\|_1 \mapsto 0$ which obviously contradicts (1).

The next corollary is the basic ingredient for the continuity result, but will also be useful in the proof of the discrete abelian case.

Corollary 2. *Let*

$$\xi: l^\infty \mapsto K(H)$$

be a linear mapping which is weak-$\sigma(K(H), B(H)_*)$ continuous. Then for any sequence $\{p_n\}_n \subseteq B(H)$ of projections increasing to 1, we have*

$$\sup\{\|\xi(a)(1-p_n)\| \mid a \in l^\infty, \|a\| \le 1\} \to 0.$$

In particular, ξ is also continuous from l^∞ with respect to the weak topology into $K(H)$ with the norm topology.*

Proof. We apply the preceding lemma with $X_n = B(H)(1-p_n)$ and $\mathscr{F}_n = \{\varphi|_{X_n} | \varphi \in \sigma(B(H), B(H)_*)\}$. Lemma 1.9 from [15] shows that $\mathscr{F}_n$ has the required properties.

Moreover it is clear that $F_n\colon l^\infty \mapsto X_n$ defined by $F_n(a) = \xi(a)(1-p_n)$ for any a in l^∞ is weak*-$\sigma(X_n, \mathscr{F}_n)$ continuous and $\|F_n(a)\| = \|\xi(a)(1-p_n)\| \mapsto 0$ since $\xi(a)$ is compact. But then $\|F_n\| \mapsto 0$ and this is exactly the first part of the statement. To prove the second let $\{a_i\}_{i\in I}$ be any net in l^∞ weak* convergent to 0, and let $\varepsilon > 0$ be arbitrary. Let $M = \sup\{\|a_i\| \mid i \in I\}$. By what we have just proved there exist two finite-dimensional projections p, q in $B(H)$ such that $\|\xi(a) - p\xi(a)q\| \le \varepsilon/2$ whenever a is any element in l^∞ of norm smaller than M. Since the linear mapping defined on l^∞ by $a \mapsto p\xi(a)q$ has finite dimensional range and hence is weak*-norm continuous it follows that there exists i_ε such that $\|p\xi(a_i)q\| \le \varepsilon/2$ for $i \ge i_\varepsilon$. The rest of the argument is standard.

We come now to the continuity result previously announced.

Proposition 3. *Let M be a finite von Neumann algebra with finite faithful normal trace τ, and $\eta\colon M\times M \mapsto K(H)$ a bilinear map which is separately $\sigma(M, M_*)$–$\sigma(K(H), B(H)_*)$ continuous. Then for every $\varepsilon > 0$ there is a δ such that $\|\eta(x, y)\| < \varepsilon$ whenever $x, y \in M$, $\|x\|, \|y\| \le 1$, $\|x\|_\tau, \|y\|_\tau \le \varepsilon$.*

In particular, any $\sigma(M, M_*) - \sigma(K(H), B(H)_*)$ continuous linear map ξ: $M \mapsto K(H)$ is also continuous from the unit ball of M with the strong operator topology into $K(H)$ with the norm topology.

Proof. It is sufficient to show that $\|\eta(x_n, y_n)\| \mapsto 0$ whenever $\{x_n\}, \{y_n\}$ are sequences in the unit ball of M with $\|x_n\|_\tau \mapsto 0$, $\|y_n\|_\tau \mapsto 0$. By the arguments in [12, Appendix] we may reduce ourselves to the case when each of the sequences $\{x_n\}, \{y_n\}$ consists of orthogonal projections. Let $A = \{x_n\}''$, $B = \{y_n\}''$ so that A, B are isomorphic as von Neumann algebras (and hence as Banach spaces) with l^∞. Let $F_n\colon A \mapsto K(H)$ be defined by $F_n(a) = \eta(a, y_n)$, $a \in A$. By the preceding corollary we have that $\|F_n(a)\| \to 0$ and therefore $\|F_n\| \to 0$ by Lemma 1. Consequently $\|\eta(x_n, y_n)\| \mapsto 0$ and this completes the proof.

As a corollary we obtain that whenever M is a finite von Neumann algebra and $\{p_n\}$ is a partition of the unity in the center of M, then we may reduce the analysis of the vanishing of $H^2_w(M, K(H))$ for certain cochains to the analysis of each term $H^2_w(M_{p_n}, K(p_nH))$.

Corollary 4. *Let M be a finite von Neumann algebra of countable type, and $\{p_n\}$ an orthogonal sequence of nonzero projections in $Z(M)$ and η in $Z^2_w(M, K(H))$ such that $\eta(x, y) = 0$ whenever x or y belong to the von Neumann algebra $Z = \{p_n\}''$. For each n, the restriction of η to M_{p_n} defines an* *element η_n in $Z_w^2(M_{p_n}, K(p_nH))$. Moreover if there is a number $C > 0$ and ξ_n in $C_w^1(M_{p_n}, K(p_nH))$ such that $\Delta\xi_n = \eta_n$, $\|\xi_n\| \le C\|\eta_n\|$ for every n in $\mathbb{N}$, then there is ξ in $C_w^1(M, K(H))$ such that $\Delta\xi = \eta$, $\|\xi\| \le C\|\eta\|$.*

Proof. Since $\eta(x, y) = 0$ whenever x or y belong to Z and $\Delta\eta = 0$ it follows that $p_n\eta(x, y) = \eta(p_nx, y) = \eta(x, p_ny) = \eta(x, y)p_n$ for every x, y in M, $n \in \mathbb{N}$. This shows that η_n is well defined as an element in $Z_w^2(M_{p_n}, K(p_nH))$. For x in M we let $\xi(x)$ be the Hilbert space direct sum (after n) of the compact operators $\xi_n(p_nx)$. Clearly $\|\xi\| \le C\|\eta\|$, $\xi \in C_w^1(M, B(H))$, and $\Delta\xi = \eta$. To end the proof we have only to prove that ξ takes its values into the compact operators. To do this it is sufficient to prove that $\lim_{n\to\infty} \|\xi_n(xp_n)\| = 0$ for every x in M. Since $\|\xi_n\| \le C\|\eta_n\|$, it is also sufficient to prove that $\|\eta_n\| \to 0$.

Let τ be a normal finite faithful trace on M. For every $\varepsilon > 0$ let δ be given by the preceding proposition. Let $n_0 \in \mathbb{N}$ such that $\tau(p_n) < \varepsilon$ whenever $n \ge n_0$. Then for $n \ge n_0$ and for every x_n, y_n in M_{p_n} with $\|x\|, \|y_n\| \le 1$, we have $\|x_n\|_\tau$, $\|y_n\|_\tau \le \varepsilon$. Hence $\|\eta(x_n, y_n)\| \le \varepsilon$ and therefore $\|\eta_n\| \le \varepsilon$ whenever $n \ge n_0$. This ends the proof.

4. The technical results

If $M \subseteq B(H)$ is a finite von Neumann algebra of countable type, $A \subseteq M$ is an injective von Neumann subalgebra, and η is an element in $Z_w^2(M, K(H))$, then the averaging technique from [14] combined with the results from [8] allows us to find a ξ in $C_w^1(M, B(H))$ such that $(\Delta\xi - \eta)(x, y) = 0$ whenever x or y are in A.

If A is discrete and abelian, then the statements in the preceding section will show that we may further assume that ξ takes its values into $K(H)$. This will complete the reduction procedure begun in Corollary 4 and will show in particular that $H_w^2(A, K(H)) = 0$ (for discrete abelian A).

When A is abelian and diffuse we shall show that we may still assume that $\xi(x) \in K(H)$ for x in $A' \cap M$ or for x in $N_M(A)$. Again this will imply that $H_w^2(A, K(H)) = 0$ and, using the fact that there is no nonzero compact operator which commutes with a diffuse abelian algebra, it will follow that $(\Delta\xi - \eta)(x, y) = 0$ if x or y belong to the von Neumann subalgebra of M generated by $N_M(A)$.

Lemma 5. *Let M be a finite von Neumann algebra and $A \subseteq M$ an injective von Neumann subalgebra. Let m be a mean on $l^\infty(U(A))$ such that*

$$\int_{U(A)} V(xu^*, u)\,dm(u) = \int_{U(A)} V(u^*, ux)\,dm(u),$$

for any x in A and for any separately weakly continuous bilinear form V on A

[8]. *Let* η *be any element in* $Z_w^2(M, K(H))$. *Then for* x *in* M *the formulas*

$$\xi_0(x) = \int_{U(A)} u^*\eta(u, x)\, dm(u), \tag{3}$$

$$\xi_1(x) = \int_{U(A)} (-\eta(xu^*, u) + \xi_0(xu^*)u + xu^*\xi_0(u))\, dm(u) \tag{4}$$

define elements in $C_w^1(M, B(H))$, *and* $(\eta - \Delta\xi_1)(x, y) = 0$ *whenever* x *or* y *are in* A *and* $\|\xi_1\| \le 3\|\eta\|$.

Proof. Since $\eta \in Z_w^2(M, B(H))$ we have

$$u^*\eta(u, x) = \eta(1, x) - \eta(u^*, ux) + \eta(u^*, u)x, \qquad u \in U(A),\ x \in M.$$

To show that ξ_0 defines an element in $C_w^1(M, B(H))$ it is therefore sufficient to prove that the mapping $x \mapsto \int_{U(A)} \eta(u^*, ux)\, dm(u)$ defines an element in $C_w^1(M, B(H))$—this is essentially done in [8, proof of Lemma 2.1].

Let $\eta_1 = \eta - \Delta\xi_0$ and

$$\xi' = \int -\eta_1(xu^*, u)\, dm(u).$$

Similarly, $\xi' \in C_w^1(M, B(H))$. The computations made in [14] show that $(\eta - \Delta(\xi_0 + \xi'))(x, y) = 0$ whenever x or y is in A. But

$$\begin{aligned} \xi'(x) &= -\int \eta(xu^*, u)\, dm(u) + \int \Delta\xi_0(xu^*, u)\, dm(u) \\ &= -\int \eta(xu^*, u)\, dm(u) + \int \xi_0(xu^*)u\, dm(u) \\ &\quad + \int xu^*\xi_0(u)\, dm(u) - \xi_0(x), \end{aligned}$$

whence $\xi_0 + \xi' = \xi_1$ and hence $\xi_1 \in C_w^1(M, B(H))$, $(\eta - \Delta\xi_1)(x, y) = 0$ for x or y in A. The last inequality in the statement is obvious since $\|\xi_0\| \le \|\eta\|$.

If $M \subseteq B(H)$ is finite dimensional, then M itself is injective and the mean invariant coincides with Haar measure on the compact group $U(M)$. Hence if $\eta \in Z_w^2(M, K(H))$, then the integrals in the definition of ξ_0 (which coincides this time with ξ_1) are norm-convergent and therefore ξ_0 belongs to $C_w^1(M, K(H))$. Consequently, we obtain

Corollary 6. *Let* $M \subseteq B(H)$ *be a finite-dimensional von Neumann algebra. Then for any* η *in* $Z_w^2(M, K(H))$ *there is* ξ *in* $C_w^1(M, K(H))$ *such that* $\Delta\xi = \eta$ *and* $\|\xi\| \le \|\eta\|$.

We come now to the analysis of the discrete case.

As we said before an obvious corollary of the next proposition will be that $H_w^2(A, K(H)) = 0$ for discrete abelian A of countable type.

Proposition 7. *Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type and $A \subseteq M$ a discrete abelian von Neumann subalgebra. For any η in $Z^2_w(M, K(H))$ there exists ξ_1 in $C^1_w(M, K(H))$ such that $(\eta - \Delta\xi_1)(x, y) = 0$ whenever x or y is in A and $\|\xi_1\| \leq 3\|\eta\|$.*

Proof. Fix η in $Z^2_w(M, K(H))$. Let ξ_0, ξ_1 be defined by (3), (4) and ξ' as in the proof of Lemma 5. We prove first that ξ_0 belongs to $C^1_w(M, K(H))$. Suppose on the contrary that $\xi_0(x)$ is noncompact for some x in M. Then there exist a sequence of finite dimensional projections $p_n \in B(H)$ with $p_n \uparrow 1$, and a number $c > 0$ such that $\|\xi_0(x)(1 - p_n)\| > c$ for every n in $\mathbb{N}$. Since $\xi_0(x)$ is a weak limit of convex combinations of elements from the set $\{u^*\eta(u, x) | u \in U(A)\}$, it follows by the weak inferior semicontinuity of the uniform norm on $B(H)$ that there exists u_n in $U(A)$ such that

$$\|\eta(u_n, x)(1 - p_n)\| = \|u_n^*\eta(u_n, x)(1 - p_n)\| > c, \quad \text{for every } n \in \mathbb{N}.$$

Since x is fixed this contradicts Corollary 2. Hence ξ_0 takes values into $K(H)$ and consequently $\eta_1 = \eta - \Delta\xi_0 \in Z^2_w(M, K(H))$. Moreover (by [14]), $\eta_1(a, y) = 0$ for a in A and y in M and hence for any x in M and u in $U(A)$:

$$0 = (\Delta\eta_1)(x, u^*, u) = x\eta_1^*(u^*, u) - \eta_1(xu^*, u) + \eta_1(x, 1) - \eta_1(x, u^*)u,$$

whence

$$\eta_1(xu^*, u) = \eta_1(x, 1) - \eta_1(x, u^*)u.$$

From the proof of Lemma 5 we know that $\xi_1 = \xi_0 + \xi'$, where

$$\xi'(x) = -\int_{U(A)} \eta(xu^*, u)\, dm(u) = -\eta_1(x, 1) + \int_{U(A)} \eta_1(x, u^*)u\, dm(u)$$

for x in M. Since $\eta_1(x, 1) \in K(H)$ for x in M, we prove similarly as for ξ_0 that ξ takes values into $K(H)$. This ends the proof.

By the previous proposition and using Corollary 4, we obviously obtain the argument for the reduction process in the case of discrete center:

Corollary 8. *Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type, and $\{p_n\} \subseteq Z(M)$ a sequence of orthogonal nonzero projections with $\sum p_n = 1$. If there is a positive C such that for every n and for every η_n in $Z^2_w(M_{p_n}, K(p_nH))$ there exists ξ_n in $C^1_w(M_{p_n}, K(p_nH))$ with $\Delta\xi_n = \eta_n$ and $\|\xi_n\| \leq C\|\eta_n\|$, then for every η in $Z^2_w(M, K(H))$ there exists ξ in $C^1_w(M, K(H))$ with $\Delta\xi = \eta$, $\|\xi\| \leq (4C + 3)\|\eta\|$. (In particular, under this hypothesis $H^2_w(M, K(H)) = 0$.)*

Proof. Let $Z = \{p_n\}''$ and let η be any element in $Z^2_w(M, K(H))$. By Proposition 7 we may find ξ_1 in $C^1_w(M, K(H))$ with $\|\xi_1\| \leq 3\|\eta\|$ and $\eta'(x, y) = (\eta - \Delta\xi_1)(x, y) = 0$ whenever x or y are in Z. Note that $\|\eta'\| \leq 4\|\eta\|$. The hypotheses of Corollary 4 are thus fulfilled for η' and therefore there is ξ_2 in

$C_w^1(M, K(H))$ such that $\eta' = \Delta\xi_2$, $\|\xi_2\| \leq C\|\eta'\| \leq 4C\|\eta\|$. Hence $\eta - \Delta\xi_1 = \Delta\xi_2$, $\eta = \Delta(\xi_1 + \xi_2)$, and $\xi_1 + \xi_2 \in Z_w^2(M, K(H))$, $\|\xi_1 + \xi_2\| \leq (4C+3)\|\eta\|$.

We turn now to the case when the injective von Neumann subalgebra A of M is diffuse. We first make a remark which shows that in the case of Cartan diffuse subalgebras the ξ_1 defined by Lemma 5 has in fact the property that $\Delta\xi_1 = \eta$ on M.

Remark 9. *Let $M \subseteq B(H)$ be an arbitrary von Neumann subalgebra, $B \subseteq M$ a diffuse (i.e., without minimal projections) von Neumann subalgebra. If η belongs to $Z_w^2(M, K(H))$ and ξ is in $C_w^1(M, B(H))$ such that $(\eta - \Delta\xi)(x, y) = 0$ whenever x or y is in B and $\xi(u)$ is compact for u in $N_M(B)$, then $(\eta - \Delta\xi)(x, y) = 0$ also when x and y belong to the von Neumann subalgebra of M generated by $N_M(B)$.*

Proof. Denote by N the von Neumann subalgebra of M generated by $N_M(B)$ and $\eta_1 = \eta - \Delta\xi$. Since $\eta_1(x, y) = 0$ for x or y in C and $\Delta\eta_1 = 0$ we have

$$b\eta_1(x, y) = \eta_1(bx, y), \quad \eta_1(xb, y) = \eta_1(x, by), \quad \eta_1(x, yb) = \eta_1(x, y)_b$$

for all x, y in M and b in B.

Therefore for any u, v in $N_M(B)$ and b in B we obtain

$$\begin{aligned} bu^*\eta_1(u, v)v^* &= u^*(ubu^*)\eta_1(u, v)v^* = u^*\eta_1((ubu^*)u, v)v^* \\ &= u^*\eta_1(ub, v)v^* = u^*\eta_1(u, bv)v^* = u^*\eta_1(u, v(v^*bv))v^* \\ &= u^*\eta_1(u, v)(v^*bv)v^* = u^*\eta_1(u, v)v^*b. \end{aligned}$$

Hence $u^*\eta_1(u, v)v^*$ belongs to the commutant of B in $B(H)$ and by hypothesis it is a compact operator.

Since B is diffuse, it follows that $u^*\eta_1(u, v)v^* = 0$ and consequently $\eta_1(u, v) = 0$ for every u, v in $N_M(B)$. By the separately weak continuity of η_1 it follows that $\eta_1(x, y) = 0$ if x or y are in N. But $\eta_1 = \eta - \Delta\xi$ and this completes the proof.

We are now able to prove our main technical result.

Proposition 10. *Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type, and $A \subseteq M$ a diffuse abelian subalgebra. Then for any η in $Z_w^2(M, K(H))$ there is a ξ in $C_w^1(M, B(H))$ such that $(\eta - \Delta\xi)(x, y) = 0$ whenever x or y are in A, or where x and y belong to the von Neumann subalgebra of M generated by $N_M(A)$. Moreover $\|\xi\| \leq 10\|\eta\|$ and $\xi(x)$ belongs to $K(H)$ if x is any element in $A' \cap M$ or in $N_M(A)$.*

Finally if $\hat{\xi}$ is the composition of ξ with the canonical surjection from $B(H)$ onto $B(H)/K(H)$ and τ is any normal finite faithful trace on M, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\hat{\xi}(exf)\| \leq \varepsilon$ whenever $x \in M$, $e, f \in P(A)$, and $\|e\|_\tau, \|f\|_\tau \leq \delta$, $\|x\| \leq 1$.

Proof. The proof of the proposition is divided into six steps. For any fixed η in $Z_w^2(M, K(H))$ we let ξ_0, ξ_1 be defined by (3) and (4). We prove first that ξ_0, ξ_1 have similar continuity properties (to those required for $\hat{\xi}$) and we construct an element $\widehat{T}$ in $Q(H)$ such that if T in $B(H)$ is any lift of $\widehat{T}$, then $\xi = \xi_1 + \operatorname{Ad} T$ has the stated properties.

Step I. For any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|e\xi_1(xf)\| \leq \varepsilon/2$ whenever $x \in M$, $e, f \in P(A)$, and $\|x\| \leq 1$, $\tau(e) < \delta$, $\tau(f) < \delta$.

Indeed, by Proposition 3, there is a $\delta > 0$ such that $\|\eta(x, y)\| \leq \varepsilon/6$ for every x, y in the unit ball of M with $\|x\|_\tau, \|y\|_\tau \leq \delta$. Then by the invariance of m

$$e\xi_0(xf) = \int_{U(A)} u^* \eta(ue, xf)\, dm(u),$$

whence

$$\|e\xi_0(xf)\| \leq \int_{U(A)} u^* \eta(ue, xf)\, dm(u) \leq \varepsilon/6.$$

Similarly,

$$\|e\xi_1(xf)\| \leq \int_{U(A)} \|e\eta(xfu^*, uf)\|\, dm(u) + \int_{U(A)} \|e\xi_0(xfu^*)\|\, dm(u) + \int_{U(A)} \|exfu^*\xi_0(uf)\|\, dm(u) \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

Step II: *Construction of a canonical element* $\widehat{T}$ *in* $Q(H)$. Let $\{e_i^n | i = 1, 2, \dots, k_n,\ n \in \mathbb{N}\}$ be a family of projections in A with the following properties:

(a) $\sum_{i=1}^{k_n} e_i^n = 1$ (e_i^n are orthogonal for any fixed $n \in \mathbb{N}$);

(b) for each i, n there is a set A_i^n such that $e_i^n = \sum_{j \in A_i^n} e_j^{n+1}$;

(c) $\{e_i^n\}'' = A$;

(d) $\sup\{\|e_i^n\|_\tau | i = 1, 2, \dots, k_n\} \leq \delta_n$,

where δ_n is the number given by Step I corresponding to $\varepsilon_n = 3\|\eta\|/2^{n+1}$. Let $\hat{\xi}_1 = \pi_H \circ \xi_1$ and observe that the restriction of $\hat{\xi}_1$ to A is a derivation into $Q(H)$ (where we identify the elements from M with their image into $Q(H)$).

Let $\widehat{T}_n \in Q(H)$ be defined by $\widehat{T}_n = \sum_i e_i^n \hat{\xi}_1(e_i^n)$, $n \in \mathbb{N}$. $\{\widehat{T}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $Q(H)$ since

$$\begin{aligned}
\|\widehat{T}_{n+1} - \widehat{T}_n\| &= \left\| \sum_j e_j^{n+1} \hat{\xi}_1(e_j^{n+1}) - \sum_i e_i^n \hat{\xi}_1(e_i^n) \right\| \\
&= \left\| \sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} (\hat{\xi}_1(e_j^{n+1}) - e_j^{n+1} \hat{\xi}_1(e_i^n)) \right) \right\| \\
&= \left\| \sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} (\hat{\xi}_1(e_i^n e_j^{n+1}) - e_j^{n+1} \hat{\xi}_1(e_i^n)) \right) \right\| \\
&= \left\| \sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} \hat{\xi}_1(e_j^{n+1}) \right) e_i^n \right\|
\end{aligned}$$

(continued)

(continued)

$$\begin{aligned}
&= \sup_i \left\| e_i^n \left(\sum_{j\in A_i^n} e_j^{n+1} \hat{\xi}_1(e_j^{n+1}) \right) e_i^n \right\| \\
&= \sup_i \left\| e_i^n \int_{\mathbb{T}^{a_i^n}} \left(\sum_{j\in A_i^n} z_j e_j^{n+1} \right)^* \hat{\xi}_1 \left(\sum_{k\in A_i^n} z_k e_k^{n+1} \right) d\lambda_i^n(z) e_i^n \right\| \\
&\le \sup_i \int_{\mathbb{T}^{a_i^n}} \left\| \left(\sum_{j\in A_i^n} z_j e_j^{n+1} \right)^* e_i^n \xi_1 \left(\sum_{k\in A_i^n} z_k e_k^{n+1} \right) e_i^n \right\| d\lambda_i^n(z) \\
&\le 3\|\eta\|/2^{n+1}.
\end{aligned}$$

Here λ_i^n is the normalized Lebesgue measure on the cartesian product of a_i^n copies of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, a_i^n is the cardinality of A_i^n. For the last inequality we used property (d) and Step I. Since obviously $\|\widehat{T}_1\| \le \|\xi_1\| \le 3\|\eta\|$, it follows that the sequence $\{\widehat{T}_n\}$ converges to an element $\widehat{T}$ in $Q(H)$ which has the property that $\|\widehat{T}\| \le 6\|\eta\|$.

Let T be any element in $Q(H)$ such that $\pi_H(\widehat{T}) = T$ and $\|T\| \le 7\|\eta\|$. We define $\xi = \xi_1 + \mathrm{Ad}\, T$, $\hat{\xi} = \pi_H \circ \xi$ and we obviously have $\|\xi\| \le 10\|\eta\|$ and $(\Delta\xi - \eta)(x, y) = 0$ for x or y in A. We will show now that ξ is the element we are looking for.

Step III:

$$\hat{\xi}_1(x) + [\widehat{T}_n, x] = \sum_{i,v} e_v^n \hat{\xi}_1(e_v^n x e_i^n) e_i^n.$$

Indeed,

$$\begin{aligned}
\hat{\xi}_1(x) + [\widehat{T}_n, x] &= \hat{\xi}_1(x) + \widehat{T}_n x - x\widehat{T}_n \\
&= \hat{\xi}(x) - \sum_i x e_i^n \hat{\xi}_1(e_i^n) + \sum_r e_r^n \hat{\xi}_1(e_r^n) x \\
&= \hat{\xi}_1(x) - \sum_i \hat{\xi}_1(x e_i^n) + \sum_i \hat{\xi}(x e_i^n) e_i^n + \sum_r e_r^n \hat{\xi}_1(e_r^n) x \\
&= \sum_{i,r} e_r^n (e_r^n \hat{\xi}_1(x e_i^n) + \hat{\xi}_1(e_r^n) x e_i^n) e_i^n \\
&= \sum_{i,r} e_r^n \hat{\xi}_1(e_r^n x e_i^n) e_i^n.
\end{aligned}$$

Step IV. If x belongs to $A' \cap M$, then $\hat{\xi}(x) = 0$ (or, what is the same, $\xi(x)$ belongs to $K(H)$).

This follows from

$$\begin{aligned}\|\hat{\xi}(x)\| &= \lim_{n\to\infty} \|\hat{\xi}_1(x) + [\widehat{T}_n, x]\| \\ &= \lim_{n\to\infty} \left\| \sum_{i,r} \hat{\xi}_1 e_r^n (e_r^n x e_i^n) e_i^n \right\| \\ &= \lim_{n\to\infty} \left\| \sum_i e_i^n \hat{\xi}_1 (e_i^n x e_i^n) e_i^n \right\| \\ &= \lim_n \sup_i \|e_i^n \widehat{\xi}_1 (e_i^n x e_i^n) e_i^n\| = 0,\end{aligned}$$

where the last equality follows from Step I and assumption (d).

Step V. For every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\hat{\xi}(exf)\| \le \varepsilon$ when x belongs to the unit ball of M and e, f are projections in $P(A)$ with $\tau(e), \tau(f) < \delta$.

To prove this let $\varepsilon > 0$ be arbitrary, let $\delta > 0$ be the corresponding number given by Step I, and assume x, e, f are fixed with the properties in the statement of Step V.

Let n_0 be a natural number such that $\|\widehat{T}_n - \widehat{T}\| \le \varepsilon/4$ for every $n \ge n_0$ and, therefore, $\|[\widehat{T}_n, x] - [\widehat{T}, x]\| \le \varepsilon/2$ for any x in M with $\|x\| \le 1$.

Letting n big enough, we may assume that there exist two sets of indices A, B with cardinalities a, b and such that

$$e \le \sum_{i\in A} e_i^n, \quad f \le \sum_{j\in B} e_j^n, \quad \left\| \sum_{i\in A} e_i^n \right\|_\tau^2 < \delta, \quad \left\| \sum_{j\in B} f_j^n \right\|_\tau^2 < \delta.$$

By Step III we obtain

$$\begin{aligned}&\|\hat{\xi}_1(exf) + [\widehat{T}_n, exf]\| \\ &\quad = \left\| \sum_{i\in A} \sum_{j\in B} e_i^n \hat{\xi}_1 (e_i^n x e_j^n) e_j^n \right\| \\ &\quad \le \left\| \int_{\mathbb{T}^a} \int_{\mathbb{T}^b} \left(\sum_{i\in A} z_i e_i^n \right)^* \hat{\xi}_1 \left(\left(\sum_{i\in A} z_i e_i^n \right) x \left(\sum_{j\in B} w_j e_j^n \right) \right) \right. \\ &\qquad\qquad \left. \cdot \left(\sum_{j\in B} w_j e_j^n \right)^* d\lambda_A(z)\, d\lambda_B(w) \right\| \\ &\quad \le \int_{\mathbb{T}^a} \int_{\mathbb{T}^b} \left\| e\hat{\xi}_1 \left(\left(\sum_{i\in A} z_i e_i^n \right) x \left(\sum_{j\in B} w_j e_j^n \right) f \right) \right\| d\lambda_A(z)\, d\lambda_B(w) \le \frac{\varepsilon}{2},\end{aligned}$$

where λ_A, λ_B are the normalized Lebesgue measures on the cartesian products $\mathbb{T}^a$ and $\mathbb{T}^b$, and where for the last inequality we used Step I.

Therefore $\|\hat{\xi}_1(exf) + [\widehat{T}, x]\| \le \varepsilon/2 + \|[\widehat{T} - \widehat{T}_n, exf]\| \le \varepsilon$.

Step VI: $\hat{\xi}(u) = 0$ *(or, what is the same, $\xi(u) \in K(H)$ if u belongs to $N_M(A)$)*. Indeed, since $(\Delta\xi - \eta)(x, y) = 0$ whenever x or y are in A, the restriction of ξ to A takes values into $K(H)$ and since η belongs to $Z^2_w(M, K(H))$ it follows that $a\hat{\xi}(x) = \hat{\xi}(ax)$, $\hat{\xi}(xa) = \hat{\xi}(x)a$ for x in M and a in A.

Consequently, for any u in $N_M(A)$, $u^*\hat{\xi}(u)$ commutes in $Q(H)$ with A (since $au^*\hat{\xi}(u) = u(uau^*)\hat{\xi}(u) = u^*\hat{\xi}((uau^*)u) = u^*\hat{\xi}(ua) = u^*\hat{\xi}(u)a$ for a in A).

We deduce therefore that for any natural n

$$\|u^*\hat{\xi}(u)\| = \left\| \sum_i e_i^n (u^*\hat{\xi}(u)) e_i^n \right\| = \sup_i \|e_i^n (u^*\hat{\xi}(u)) e_i^n\|$$

and the last term tends to zero when n tends to infinity. Hence $u^*\hat{\xi}(u) = 0$ and $\hat{\xi}(u) = 0$ for every u in $N_M(A)$.

The proof of the proposition is now accomplished by Remark 9.

As an obvious corollary we obtain that $H^2_w(M, K(H)) = 0$ whenever M has diffuse center. Precisely, we have

Corollary 11. *Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type with diffuse center $Z(M)$. Then for every η in $Z^2_w(M, K(H))$ there is a ξ in $C^1_w(M, K(H))$ such that $\Delta\xi = \eta$ and $\|\xi\| \le 10\|\eta\|$.*

Proof. Applying the preceding proposition with $A = Z(M)$ it follows that there is ξ in $C^1_w(M, B(H))$ such that $\|\xi\| \le 10\|\eta\|$, $(\Delta\xi - \eta)(x, y) = 0$ if x and y are in $N_M(A)''$, and $\xi(u)$ belongs to $K(H)$ if u is in $N_M(A)$.

Since M is this time the linear span of $N_M(A)$, it follows that ξ has compact values (by linearity) and $\Delta\xi = \eta$.

5. Proof of the main result

Using Proposition 10 and the arguments from [4, proof of Theorem 3.5] we prove that $H^2_w(M, K(H)) = 0$ for a type II_1 factor $M \subset B(H)$ with property S with respect to H and with a Cartan subalgebra A (see §2 for the definition of property S).

The general result will then follow easy from this and from Corollaries 6 and 11.

Proposition 12. *Let $M \subseteq B(H)$ be a type II_1 factor with property S with respect to H and with a Cartan subalgebra A. Then for every η in $Z^2_w(M, K(H))$ there exists ξ in $C^1_w(M, K(H))$ with $\Delta\xi = \eta$ and $\|\xi\| \le 10\|\eta\|$.*

Proof. Let ξ be given by Proposition 10, relative to the abelian diffuse von Neumann subalgebra A. (We have only to show that ξ has compact values or, what is the same, that $\hat{\xi}$ is identically zero, where $\hat{\xi}$ is the composition of ξ with the projection from $B(H)$ onto the Calkin algebra.)

Since $N_M(A))'' = M$ it follows that $\Delta\xi = \eta$ and hence that $\hat{\xi}: M \mapsto Q(H)$ is a derivation (since η has compact values). Moreover $\hat{\xi}$ vanishes on $GN_M(A)$. It remains only to prove that $\hat{\xi}$ vanishes everywhere.

Suppose on the contrary that this is not the case. Similar to the arguments in [4] it follows that there is an orthogonal sequence of nonzero projections $\{f_n\}_n \subseteq A$ and x_n in $f_n M f_n$ such that $\|x_n\| \le 1$ and $\|\hat{\xi}(x_n)\| > 1$. But this obviously contradicts the continuity properties of $\hat{\xi}$ stated in Proposition 10.

For the sake of completeness we repeat here the construction of the x_n's and f_n's. Let $\{e_{ij}^n\}$, $i, j = 1, 2, \dots, 2^{k_n}$, be a matrix unit for each n in $\mathbb{N}$ with the following properties [11]:

(α) e_{ii}^n belongs to A, $\sum_i e_{ii}^n = 1$ for each n;
(β) e_{ij}^n belongs to $N_M(A)$ for each n, i, j;
(γ) e_{rs}^p is the sum of some e_{ij}^n for every pn.

By (β) and Proposition 10, we have that $\hat{\xi}(e_{ij}^n)$ is zero for all n, i, j. Let $f_n = e_{s_n s_n}^n$ be any sequence of nonzero orthogonal projections in A. Moreover M is canonically identified with $M_{2^{k}n}(\mathbb{C}) \otimes M_{f_n}$ (by means of e_{ij}^n); with this identification $\hat{\xi}$ becomes $I_{M_{2^{k}n}(\mathbb{C})} \otimes \hat{\xi}^n$, where $\hat{\xi}^n$ is the restriction of $\hat{\xi}$ to $M_{f_n} = f_n M f_n$. Since $\hat{\xi} \ne 0$, we may suppose that $\|\hat{\xi}\| > C$, where C is the constant arising from the property S of M. Since (by assumption S) $\hat{\xi}^n$ is completely bounded with $\|(\hat{\xi}^n)\|_{\text{cb}} \le C\|\hat{\xi}^n\|$, it follows that there are x_n in $f_n M f_n$ with $\|x_n\| \le 1$ and $\|\hat{\xi}(f_n x_n f_n)\| > 1$. This completes the proof.

We are now able to prove our main result.

Theorem 13. *Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type and suppose that there is no nonzero central projection p in $Z(M)$ such that M_p is a type* II_1 *factor without property S with respect to pH or a type* II_1 *factor without Cartan subalgebra.*

Then $H_w^2(M, K(H)) = 0$ and moreover for every η in $Z_w^2(M, K(H))$ there exists ξ in $C_w^1(M, K(H))$ such that $\Delta\xi = \eta$, $\|\xi\| \le 43\|\eta\|$.

Proof. There exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq Z(M)$ of nonzero orthogonal projections with sum 1 such that Mp_1 has diffuse center and $\{Mp_n\}_{n\ge 2}$ is a factor. By hypothesis it follows that Mp_n for $n \ge 2$ is a type II_1 factor with property S relative to $p_n H$ and with a Cartan subalgebra, or it follows that Mp_n is a finite-dimensional factor.

By Corollaries 6 and 11 and by Proposition 12 we deduce that for every n in $\mathbb{N}$ and for every η_n in $Z_w^2(M_{p_n}, K(p_n H))$ there exists ξ_n in $C_w^1(M_{p_n}, K(p_n H))$ such that $\Delta\xi_n = \eta_n$ and $\|\xi_n\| \le 10\|\eta\|$. Hence by Corollary 8 we obtain that for every η in $Z_w^2(M, K(H))$ there exists ξ in $C_w^1(M, K(H))$ such that $\Delta\xi = \eta$ and $\|\xi\| \le 43\|\eta\|$.

Corollary 14. *If M is a type II_1 factor with property Γ and with Cartan subalgebra, or if M is finite, of countable type with diffuse center, then* $H_w^2(M, K(H)) = 0$.

REFERENCES

1. L. Brown, R. G. Douglas, and P. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. (2) **105** (1977), 265–324.
2. A. Connes, *Classification des facteurs*, Operator Algebras and Applications, Proc. Sympos. Pure Math., vol. 38, part 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 43–110.
3. E. Christensen, *Derivations and their relations to perturbations of operator algebras*, in Pure Math., vol. 38, part 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 261–274.
4. ——, *Similarities of II_1 factors with property Γ*, J. Operator Theory **15** (1980), 281–288.
5. E. Christensen, E. G. Effros, and A. Sinclair, *Completely bounded multilinear maps and C^*-algebraic cohomology*, preprint, 1987.
6. E. Christensen and A. Sinclair, *On the Hochschild cohomology for von Neumann algebras*, preprint, 1988.
7. J. Conway, *A course of functional analysis*, Springer-Verlag, New York, 1985.
8. U. Haagerup, *All nuclear C^*-algebras are amenable*, Invent. Math. **74** (1983), 305–319.
9. B. E. Johnson, *Low dimensional cohomology of operator algebras*, Proc. Sympos. Pure Math., vol. 38, part 2, Amer. Math. Soc., Providence, R. I., pp. 253–260.
10. B. E. Johnson and S. K. Parrott, *Operators commuting modulo the set of compact operators with a von Neumann algebra*, J. Funct. Anal. **11** (1972), 39–61.
11. S. Popa, *Notes on Cartan subalgebras in type II_1 factors*, Math. Scand. **57** (1985), 171–188.
12. ——, *The commutant modulo the set of compact operators of a von Neumann algebra*, J. Funct. Anal. **71** (1987), 393–408.
13. S. Popa and F. Rădulescu, *Derivations into the compact ideal space of a semifinite von Neumann algebra*, **57** (1988).
14. J. R. Ringrose, *Cohomology theory for operator algebras*, Proc. Sympos. Pure Math., vol. 38, part 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 229–253.
15. S. Strătilă and L. Zsido, *Lectures on von Neumann algebras*, Editura Tehnică, Bucureşti and Abacus Press, London, 1979.

DEPARTMENT OF MATHEMATICS, THE NATIONAL INSTITUTE FOR SCIENTIFIC AND TECHNICAL CREATION, BD. PĂCII 220, 79622 BUCHAREST, ROMANIA

Current address: Department of Mathematics, University of California, 405 Hilgard Avenue, Los Angeles, California 90024