A MODIFIED SCHUR ALGORITHM
AND AN EXTENDED HAMBURGER MOMENT PROBLEM

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ABSTRACT. An algorithm for a Pick-Nevanlinna problem where the interpolation points coalesce into a finite set of points on the real line is introduced, its connection with certain multipoint Padé approximation problems is discussed, and the results are used to obtain the solutions of an extended Hamburger moment problem.

1. INTRODUCTION

The Pick-Nevanlinna problem is the following: Let $\{z_\alpha : \alpha \in A\}$ be a set of distinct points in the open upper half-plane $H^0_+ = \{z : \operatorname{Im} z > 0\}$, and let $\{w_\alpha : \alpha \in A\}$ be a set of points in $\mathbf{C}$. Find a function $F(z)$ which is analytic in H^0_+ such that $\operatorname{Im} F(z) \geq 0$ for $z \in H^0_+$ and $F(z) = w_\alpha$ for all $\alpha \in A$. (A function $F(z)$ which is analytic for $z \in H^0_+$ with $\operatorname{Im} F(z) \geq 0$ is called a Nevanlinna function.) The problem was solved by Pick [27, 28] in the case that A is finite, by Nevanlinna [15, 16] in the case that A is countable, and by Krein and Rekhtman [14] in the general case.

A variant of the problem for finite or countable A arises when all points z_α coalesce to a single point a, and given values w_α at the points are replaced by the Taylor coefficients at a. A problem closely related to this is Carathéodory's coefficient problem (see [2, 3, 33]): Given a finite sequence $\{\gamma_0, \ldots, \gamma_m\}$ or an infinite sequence $\{\gamma_n : n \in \mathbf{N}\}$, find a function $F(z)$ which is analytic in the open unit disc $D^0 = \{z : |z| < 1\}$ such that $\operatorname{Re} F(z) \geq 0$ for $z \in D^0$ and $F(z) = \sum_{k=0}^{m} \gamma_k z^k + \sum_{k=m+1}^{\infty} \delta_k^{(m)} z^k$, or $F(z) = \sum_{k=0}^{\infty} \gamma_k z^k$. (For historical remarks on this problem, see [12].)

By introducing the linear fractional transformation $F \to \frac{1-F}{1+F}$ we can reformulate the requirement $\operatorname{Re} F(z) \geq 0$ to read $|F(z)| \leq 1$. (A function $F(z)$ which is analytic for $z \in D^0$ is called a Carathéodory function if $\operatorname{Re} F(z) \geq 0$ for $z \in D^0$, or a Schur function if $|F(z)| \leq 1$ for $z \in D^0$.) Schur [31] invented an algorithm called the Schur algorithm to deal with this problem. The technique was adapted by Nevanlinna to deal with the Pick-Nevanlinna problem.

For modern applications of the Schur algorithm and variants, see the collection [5]. For an operator theoretic approach, see, e.g., [29, 30].

A modification of the Pick-Nevanlinna problem with coinciding points $z_n = a$, $n \in \mathbb{N}$, arises when the point a is moved to the boundary of the region of definition of $F(z)$, i.e., to the real axis. The Taylor series expansion then has to be replaced by the asymptotic series expansions $\sum_{k=0}^{\infty} \gamma_k (z - a)^k$ at a, valid in arbitrary angular regions $R_{a,\delta} = \{z : \delta < \operatorname{Arg}(z - a) < \pi - \delta\}$, $\delta > 0$. If moreover the point a is moved to the point at infinity, then we obtain the following modified Pick-Nevanlinna problem: Given a sequence of numbers $\{\gamma_n : n \in \mathbb{N}\}$, find a Nevanlinna function $F(z)$ with the asymptotic expansion $F(z) \approx \sum_{k=0}^{\infty} \gamma_k z^{-k}$ at ∞, valid in every angular region $R_\delta = \{z : \delta < \operatorname{Arg} z < \pi - \delta\}$, $\delta > 0$. This problem is equivalent to the classical Hamburger moment problem (see, e.g., [1, 6, 32]).

If the points z_n, $n \in \mathbb{N}$, coalesce into the two points 0 and ∞ according to the rule $z_{2m} = 0$, $m = 1, 2, \ldots$, $z_{2m+1} = \infty$, $m = 0, 1, 2, \ldots$, then the corresponding modification of the Pick-Nevanlinna problem can be formulated as follows: Let $\{\gamma_n : n = 0, \pm 1, \pm 2, \ldots\}$ be a bi-infinite sequence of numbers. Find a Nevanlinna function $F(z)$ with the asymptotic expansions $F(z) \approx \sum_{k=0}^{\infty} \gamma_k z^{-k}$ at ∞, $F(z) \approx \sum_{k=1}^{\infty} \gamma_{-k} z^k$ at 0, in every angular region $R_\delta = \{z : \delta < \operatorname{Arg} z < \pi - \delta\}$. This problem is equivalent to the strong Hamburger moment problem. For definition and general treatment of this problem, see, e.g., [10, 11, 25]. The connection between the strong moment problem and the asymptotic expansions problem was treated in [10] for the case that the moment problem is nonsingular, and in [20–23] for the general case.

Again consider the sequence $\{z_n : n \in \mathbb{N}\}$, and now let z_n, $n = 1, 2, \ldots$, coalesce into p points $a_1, \ldots, a_p$ on the real axis as follows: $z_{pq+1} = a_1$, $z_{pq+2} = a_2, \ldots, z_{pq+p} = a_p$, $q = 0, 1, 2, \ldots$. A corresponding Pick-Nevanlinna problem can be formulated as follows: Let $\{\gamma_j^{(i)} : j = 0, 1, 2, \ldots\}$, $i = 1, \ldots, p$, be p sequences of given numbers. Find a Nevanlinna function $F(z)$ which has the asymptotic expansions $F(z) \approx \sum_{j=0}^{\infty} \gamma_j^{(i)} (z - a_i)^j$ at a_i in the angular regions $R_{i,\delta} = \{z : \delta < \operatorname{Arg}(z - a_i) < \pi - \delta\}$, $\delta > 0$, $i = 1, \ldots, p$. This problem will later be shown to be related to the extended Hamburger moment problem.

The use of the Schur algorithm and its variants has two steps. The first step is to successively construct rational functions $f_n(z)$ solving the truncated (finite) interpolation problem. (In the classical Hamburger moment case, e.g., this consists of finding for each n a rational function with power series expansion

$$f_n(z) = \sum_{k=0}^{2n} \gamma_k z^{-k} + \sum_{k=2n+1}^{\infty} \delta_k^{(n)} z^{-k} .$$

This is the Padé approximation problem for the series $\sum_{k=0}^{\infty} \gamma_k z^{-k}$, or for the solutions of the original problem, if they exist.) The second step is to obtain from the sequence $\{f_n(z)\}$ one or more of the original functions $F(z)$ which solve the nontruncated problem, if they exist.

In this paper we treat the above-mentioned version of the Pick-Nevanlinna problem related to the extended Hamburger moment problem. We describe an algorithm which generalizes the above-mentioned adaptions of the Schur algorithm. The algorithm produces certain (weak) multipoint Padé approximants $f_n(z) = A_n(z)/B_n(z)$ to the given power series. For reasons that will become clear later, we include a condition $F(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1})$ at ∞, $z \in R_\delta = \{z : \delta < \text{Arg } z < \pi - \delta\}$, $\delta > 0$. (For convenience we shall also call an expansion of this form on asymptotic expansion.) We shall in this paper call this problem the *modified Pick-Nevanlinna problem*, and the algorithm the *modified Schur algorithm* (associated with the series $\gamma_{-1}^{(\infty)} z^{-1}$, $\sum_{j=1}^{\infty} \gamma_j^{(i)}(z - a_i)^j$, $i = 1, \ldots, p$).

(For treatments of multipoint Padé approximants to the series $\gamma_{-1}^{(\infty)} z^{-1}$, $\sum_{j=1}^{\infty} \gamma_j^{(i)}(z - a_i)^j$, see also [7, 8, 19, 24, 34].)

The rational functions $A_n(z)/B_n(z)$ are Nevanlinna functions. We show that $\{A_n(z)/B_n(z)\}$, together with a related sequence $\{C_n(z)/D_n(z)\}$, give rise to solutions of the modified Pick-Nevanlinna problem (when solutions exist).

For each $z \in H_+^0$ there exists a sequence $\{\Delta_n(z)\}$ of nested disks such that $f_n(z, \tau) \in \partial \Delta_n(z)$ for every $\tau \in \mathbf{R}$, where

$$f_n(z, \tau) = \frac{A_n(z) + \tau C_n(z)}{B_n(z) + \tau D_n(z)}.$$

Let $\Delta(z)$ denote the intersection $\bigcap_{n=1}^{\infty} \Delta_n(z)$ (which may be a closed disk or a single point). Then every Nevanlinna function $F(z)$ which satisfies $F(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1})$ and $F(z) \in \Delta(z)$ for all z, is a solution of the problem, and every solution has this property. Every function $f_n(z, \tau)$, $\tau \in \mathbf{R}$, is a Nevanlinna function, and there exist sequences $\{f_{n_k}(z, \tau_{n_k})\}$ converging to Nevanlinna functions $F(z)$ which satisfy $F(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1})$ and $F(z) \in \Delta(z)$ for all z. Thus there exist Nevanlinna functions with the required property. These results answer in the affirmative the question of solvability of the interpolation problem (under the required conditions on the given series), and give a characterization of all the solutions. The solution is unique in the situation that $\Delta(z)$ reduces to a single point for all z (the limit point case).

The extended Hamburger moment problem associated with the points $a_1, \ldots, a_p$ and sequences of real numbers c_0, $\{c_j^{(i)} : j = 1, 2, \ldots\}$, $i = 1, \ldots, p$, is the following. Find distribution functions $\psi(t)$ (i.e.: bounded, nondecreasing functions) such that

$$\int_{-\infty}^{\infty} d\psi(t) = c_0, \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t - a_i)^j} = c_j^{(i)}, \quad j = 1, 2, \ldots, i = 1, \ldots, p.$$

This problem was introduced and treated in [17, 18] in the context of the theory of positive linear functionals and of orthogonal rational functions. (In [17, 18] we considered the case where the distribution functions $\psi(t)$ are required to have infinitely many points of increase.) In this paper we discuss the problem

on the basis of our results on the modified Pick-Nevanlinna problem. We point out that a sufficient condition for the existence of a solution $\psi(t)$ is that in the power series $-c_0 z^{-1}$, $\sum_{j=0}^{\infty} c_{j+1}^{(i)}(z-a_i)^j$, those conditions are satisfied that lead to solutions of the modified Pick-Nevanlinna problem. The solutions are exactly those distribution functions $\psi(t)$ whose Stieltjes transform $\hat{\psi}(z) = \int_{-\infty}^{\infty} d\psi(t)/(t-z)$ (which are Nevanlinna functions) satisfies $\hat{\psi}(z) = -c_0 z^{-1} + o(z^{-1})$ and $\hat{\psi}(z) \in \Delta(z)$ for all z.

2. The modified Schur algorithm

Let $F(z)$ be a Nevanlinna function. We recall that this means that $F(z)$ is analytic for $\operatorname{Im} z > 0$, and that $\operatorname{Im} F(z) \geq 0$ for $\operatorname{Im} z > 0$. For brevity we shall write N-function for Nevanlinna function.

Let $a_1, \ldots, a_p$ be given distinct points on the real axis, $p \geq 3$. For convenience we assume that they are ordered by size: $a_1 < a_2 < \cdots < a_p$. We assume that $F(z)$ has the asymptotic expansions (with real coefficients)

$$(2.1a) \qquad F(z) \approx \sum_{j=0}^{\infty} \gamma_j^{(i)}(z-a_i)^j, \qquad i = 1, \ldots, p,$$

$$(2.1b) \qquad F(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1}).$$

(By asymptotic expansions we shall always mean expansions in the angular regions R_δ, $R_{i,\delta}$, $i = 1, \ldots, p$, defined in the introduction. For the basic definitions and results on asymptotic expansions, see, e.g., [9].)

We note that every integer $n \in \mathbf{N}$ can be written in a unique way in the form

$$(2.2) \qquad n = p \cdot q_n + r_n,$$

where $q_n, r_n \in \mathbf{N}$, $1 \leq r_n \leq p$. When there is no danger of confusion, we write q for q_n and r for r_n.

By a_{r-1} we mean a_p when $r = 1$, by a_{r+1} we mean a_1 when $r = p$, and so on. Statements relating to a_{r-1} when $r = 1$ ¡ to a_{r+1} when $r = p$, etc. will be understood from the context.

We shall make repeated use of the following important result.

Lemma 2.1. *Let $H(z)$ be an N-function. Then the following statements are true:*

The limit $d_\infty = \lim_{z \to \infty, \operatorname{Re} z = 0}(-1/zH(z))$ exists and is nonnegative, and $\operatorname{Im}(-1/H(z) - d_\infty z) \geq 0$ when $\operatorname{Im} z > 0$.

The limits $d_i = \lim_{z \to a_i, \operatorname{Re}(z-a_i)=0}(-(z-a_i)/H(z))$ exist and are nonpositive for $i = 1, \ldots, p$, and $\operatorname{Im}(-1/H(z) - d_i/(z-a_i)) \geq 0$ when $\operatorname{Im} z > 0$.

Proof. It follows from Julia-Carathéodory-Landau-Valiron's theorem (see, e.g., [4, p. 236]) that when $G(z)$ is an N-function, the limits

$$d = \lim_{z \to \infty, \operatorname{Re} z = 0} z^{-1} G(z) \quad \text{and} \quad d' = \lim_{z \to 0, \operatorname{Re} z = 0} z G(z)$$

exist and are nonnegative, respectively nonpositive, and $\mathrm{Im}[G(z) - dz] \geq 0$, $\mathrm{Im}[G(z) - d'z^{-1}] \geq 0$ when $\mathrm{Im}\, z \geq 0$. For any real number a, the function $z - a \to -1/H(z)$ is an N-function. The lemma follows by application of the above-mentioned consequence of Julia-Carathéodory-Landau-Valiron's theorem to the functions $z \to -1/H(z)$, $z - a_i \to 1/H(z)$. $\square$

We shall describe an algorithm which successively determines rational functions $f_n(z)$ with Taylor series expansions that progressively coincide for more and more terms with the given expansions (2.1) of the N-function $F(z)$. We shall assume that $F(z)$ itself is not a rational function.

Since $F(z) - \gamma_0^{(i)}$ is an N-function, it follows from (2.1a) that $\gamma_1^{(i)} > 0$ for $i = 1, \ldots, p$. This is because $\gamma_1^{(i)}(z - a_i)$ is the dominating term in the asymptotic expansion. Similarly it follows from (2.1b) that $\gamma_{-1}^{(\infty)} < 0$.

We define linear fractional transformations $s_1(z, w)$, $S_1(z, w)$ by

$$(2.3) \qquad s_1(z, w) = \frac{-(1 - \delta_1) - \alpha_1^{-1}\delta_1 w}{-\alpha_1 + (1 - \delta_1)\beta_1 z + w}, \qquad S_1(z, w) = s_1(z, w),$$

where

$$(2.4a) \qquad \delta_1 = \begin{cases} 1 & \text{if } \gamma_0^{(1)} = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$(2.4b) \qquad \beta_1 = -1/\gamma_{-1}^{(\infty)},$$

$$(2.4c) \qquad \alpha_1 = \begin{cases} \beta_1 a_1 + 1/\gamma_0^{(1)} & \text{if } \gamma_0^{(1)} \neq 0, \\ \text{an arbitrary positive constant} & \text{if } \gamma_0^{(1)} = 0. \end{cases}$$

We define a new function $F_1(z)$ by

$$(2.5) \qquad F_1(z) = \alpha_1 - (1 - \delta_1)\beta_1 z - \frac{1}{\delta_1\alpha_1^{-1} + F(z)}.$$

Then

$$(2.6) \qquad F(z) = s_1(z, F_1(z)) = S_1(z, F_1(z)).$$

By (2.1), (2.4), and (2.5) we obtain expansions of the following forms:

$$(2.7a) \quad F_1(z) = \gamma_0^{(\infty, 1)} + \gamma_{-1}^{(\infty, 1)}z^{-1} + o(z^{-1}),$$

$$(2.7b) \quad F_1(z) \approx \gamma_{-1}^{(i, 1)}(z - a_i)^{-1} + \gamma_0^{(i, 1)} + \gamma_1^{(i, 1)}(z - a_i) + \cdots \quad \text{for } i \neq 1,$$

$$(2.7c) \quad F_1(z) \approx \gamma_1^{(1, 1)}(z - a_1) + \cdots.$$

Here $\gamma_0^{(\infty, 1)}$, $\gamma_{-1}^{(i, 1)}$, etc. are new real constants. In the same way new constants will be introduced later.

From (2.1b) and (2.4b) it follows that

$$(2.8) \qquad d_\infty = \lim_{\substack{z \to \infty \\ \mathrm{Re}\, z = 0}} \left(\frac{-1}{zF(z)}\right) = -\frac{1}{\gamma_{-1}^{(\infty)}} = \beta_1.$$

Since $F(z)$ is an N-function, it then follows by (2.5) and Lemma 2.1 that

$$(2.9) \qquad \operatorname{Im} F_1(z) = \operatorname{Im}\left(-\frac{1}{\delta_1\alpha_1^{-1} + F(z)} - (1 - \delta_1)\beta_1 z\right) \geq 0$$

when $\operatorname{Im}(z) > 0$. (Note that $-1/(\alpha_1^{-1} + F(z))$ is an N-function.) Thus $F_1(z)$ is an N-function. Therefore, as in the argument preceding the introduction of the transformations $s_1(z, w)$ and $S_1(z, w)$, we conclude that $\gamma_{-1}^{(\infty, 1)} < 0$, $\gamma_1^{(1, 1)} > 0$, and $\gamma_{-1}^{(i, 1)} < 0$ or $\gamma_{-1}^{(i, 1)} = 0$, $\gamma_1^{(i, 1)} > 0$ for $i \neq 1$.

Now let $n \in \mathbf{N}$, $n \geq 2$. We assume that N-functions $F_1(z), \ldots, F_{n-1}(z)$ and linear fractional transformations $s_1(z, w), \ldots, s_{n-1}(z, w), S_1(z, w), \ldots, S_{n-1}(z, w)$ have been defined such that

$$(2.10) \qquad S_t(z, w) = S_{t-1}(z, s_t(z, w)), \qquad F(z) = S_t(z, F_t(z))$$

for $t = 1, \ldots, n - 1$. As before, we write r for r_n and q for q_n. (For $r = 1, 2$ the following argument must be modified slightly.) We further assume that expansions for $F_{n-1}(z)$ of the following forms are valid:

$$(2.11a) \qquad F_{n-1}(z) = \gamma_1^{(\infty, n-1)} z + \gamma_0^{(\infty, n-1)} + \gamma_{-1}^{(\infty, n-1)} + o(z^{-1}),$$

$$(2.11b) \qquad F_{n-1}(z) \approx \gamma_{-1}^{(i, n-1)}(z - a_i)^{-1} + \cdots \quad \text{for } i \neq r-1, r-2,$$

$$(2.11c) \qquad F_{n-1}(z) \approx \gamma_0^{(r-2, n-1)} + \gamma_1^{(r-2, n-1)}(z - a_{r-2}) + \cdots,$$

$$(2.11d) \qquad F_{n-1}(z) \approx \gamma_1^{(r-1, n-1)}(z - a_{r-1}) + \cdots.$$

We note that all the assumptions are satisfied for $n = 2$.

Since $F_{n-1}(z)$ is assumed to be an N-function, it follows that $\gamma_1^{(\infty, n-1)} > 0$ or $\gamma_1^{(\infty, n-1)} = 0$, $\gamma_{-1}^{(\infty, n-1)} < 0$, and $\gamma_1^{(r-2, n-1)} > 0$, $\gamma_1^{(r-1, n-1)} > 0$, and $\gamma_{-1}^{(i, n-1)} < 0$ or $\gamma_{-1}^{(i, n-1)} = 0$, $\gamma_1^{(i, n-1)} > 0$ for $i \neq r-2, r-1$.

We define linear fractional transformations $s_n(z, w)$, $S_n(z, w)$ by

$$(2.12) \qquad \begin{aligned} s_n(z, w) &= \frac{(\delta_n - 1) + \alpha_n^{-1}\delta_n w}{\alpha_n + (\delta_n - 1)\beta_n/(z - a_{r-1}) + w}, \\ S_n(z, w) &= S_{n-1}(z, s_n(z, w)), \end{aligned}$$

where

$$(2.13a) \qquad \delta_n = \begin{cases} 1 & \text{if both } \gamma_{-1}^{(r, n-1)} \text{ and } \gamma_0^{(r, n-1)} = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$(2.13b) \qquad \beta_n = 1/\gamma_1^{(r-1, n-1)},$$

$(2.13c)$

$$\alpha_n = \begin{cases} \beta_n/(a_r - a_{r-1}) & \text{if } \gamma_{-1}^{(r, n-1)} \neq 0, \\ \beta_n/(a_r - a_{r-1}) - 1/\gamma_0^{(r, n-1)} & \text{if } \gamma_{-1}^{(r, n-1)} = 0, \ \gamma_0^{(r, n-1)} \neq 0, \\ \text{an arbitrary positive constant} & \text{if } \gamma_{-1}^{(r, n-1)} = \gamma_0^{(r, n-1)} = 0. \end{cases}$$

We define a new function $F_n(z)$ by

$$(2.14) \qquad F_n(z) = -\alpha_n + \frac{(1 - \delta_n)\beta_n}{z - a_{r-1}} - \frac{1}{-\alpha_n^{-1}\delta_n + F_{n-1}(z)} \, .$$

Then

$$(2.15) \qquad F_{n-1}(z) = s_n(z, F_n(z)), \qquad F(z) = S_n(z, F_n(z)) \, .$$

By using (2.11), (2.13), and (2.14), we obtain expansions of the following forms:

$$(2.16a) \qquad F_n(z) = \gamma_1^{(\infty, n)} z + \gamma_0^{(\infty, n)} + \gamma_{-1}^{(\infty, n)} z^{-1} + o(z^{-1}),$$

$$(2.16b) \qquad F_n(z) \approx \gamma_{-i}^{(i, n)}(z - a_i)^{-1} + \cdots \quad \text{for } i \neq r, \, r - 1,$$

$$(2.16c) \qquad F_n(z) \approx \gamma_0^{(r-1, n)} + \gamma_1^{(r-1, n)}(z - a_{r-1}) + \cdots,$$

$$(2.16d) \qquad F_n(z) \approx \gamma_1^{(r, n)}(z - a_r) + \cdots.$$

From (2.11d) and (2.13b) it follows that

$$(2.17) \qquad d_{r-1}^{(n-1)} = \lim_{\substack{z \to a_{r-1} \\ \mathrm{Re}(z - a_{r-1}) = 0}} \left(\frac{-(z - a_{r-1})}{-\alpha_n^{-1}\delta_n + F_{n-1}(z)} \right) = \frac{-1}{\gamma_1^{(r-1, n-1)}} = -\beta_n \, .$$

Since $F_{n-1}(z)$ is an N-function, it then follows by (2.14) and Lemma 2.1 that

$$(2.18) \qquad \mathrm{Im}\, F_n(z) = \mathrm{Im} \left(-\frac{1}{\alpha_n^{-1}\delta_n + F_{n-1}(z)} + \frac{(1 - \delta_n)\beta_n}{z - a_{r-1}} \right) \geq 0$$

when $\mathrm{Im}\, z \geq 0$. (Note that $-1/(\alpha_n^{-1}\delta_n + F_{n-1}(z))$ is an N-function.) Thus $F_n(z)$ is an N-function.

We may sum up the results of the foregoing discussion as follows:

Theorem 2.2. *Let* $F(z)$ *be an N-function with expansions (2.1). Then the algorithm described above is well defined and determines inductively linear fractional transformations* $s_1(z, w), \ldots, s_n(z, w), \ldots, S_1(z, w), \ldots, S_n(z, w), \ldots,$ *and N-functions* $F_1(z), \ldots, F_n(z), \ldots$. *These are connected by the relations* $F_{n-1}(z) = s_n(z, F_n(z))$, $F(z) = S_n(z, F_n(z))$. *For each n, either* $\gamma_{-1}^{(i, n)} < 0$ *or* $\gamma_{-1}^{(i, n)} = 0$, $\gamma_1^{(i, n)} > 0$ *for* $i \neq r - 1, \, r - 2$, $\gamma_1^{(r-1, n)} > 0$, $\gamma_1^{(r, n)} > 0$, *and either* $\gamma_1^{(\infty, n)} > 0$ *or* $\gamma_1^{(\infty, n)} = 0$, $\gamma_{-1}^{(\infty, n)} < 0$.

Now let

$$(2.19a) \qquad \sum_{j=0}^{\infty} \gamma_j^{(i)}(z - a_i)^j, \qquad i = 1, \ldots, p,$$

$$(2.19b) \qquad \gamma_{-1}^{(\infty)} z^{-1}$$

be given formal power series (with real coefficients). We shall denote this collection by $\phi(z)$. It is readily verified under suitable conditions the algorithm

we have constructed (with the function $F(z)$ replaced by the series (2.19)) produces linear fractional transformations $s_n(z, w)$, $S_n(z, w)$ of the form (2.3), (2.12), and collections $\phi_n(z)$ of formal power series such that

(2.20) $\phi_0(z) = \phi(z)$, $\phi_{n-1}(z) = s_n(z, \phi_n(z))$ for $n = 1, 2, \ldots$.

We note that then also $\phi(z) = S_n(z, \phi_n(z))$. The collections $\phi_n(z)$ are of the form

(2.21a) $$\sum_{j=-1}^{\infty} \gamma_j^{(i,n)}(z - a_i)^j \quad \text{for } i \neq r, r - 1,$$

(2.21b) $$\sum_{j=0}^{\infty} \gamma_j^{(r-1,n)}(z - a_{r-1})^j,$$

(2.21c) $$\sum_{j=1}^{\infty} \gamma_j^{(r,n)}(z - a_r)^j,$$

(2.21d) $$\sum_{j=0}^{2} \gamma_j^{(\infty,n)} z^{1-j} + o(z^{-1}).$$

We shall call the collection $\phi(z)$ *positive* if the following conditions hold:

(P_0) $\gamma_1^{(i)} > 0$ for $i = 1, \ldots, p$, $\gamma_{-1}^{(\infty)} < 0$,

and for $n = 1, 2, \ldots$,

(P_n) $\begin{cases} \gamma_{-1}^{(i,n)} < 0 \text{ or } \gamma_{-1}^{(i,n)} = 0, \gamma_1^{(i,n)} > 0, \ i = 1, \ldots, p, \\ \qquad\qquad\qquad \text{and} \\ \gamma_1^{(\infty,n)} > 0 \text{ or } \gamma_1^{(\infty,n)} = 0, \gamma_{-1}^{(\infty,n)} < 0. \end{cases}$

We observe that if $(P_0), \ldots, (P_{n-1})$ are satisfied, then $s_n(z, w)$, $S_n(z, w)$ can be defined.

It follows from the expressions (2.4b), (2.13b) that for a positive collection ϕ, the following inequalities hold:

(2.22) $\beta_n > 0$ for $n = 1, 2, \ldots$.

We shall call the index n *singular* if $\delta_n = 1$ and *regular* if $\delta_n = 0$. Thus by (2.4a), 1 is singular iff $\gamma_0^{(1)} = 0$, and by (2.13a), n is singular iff $\gamma_{-1}^{(r,n-1)} = \gamma_0^{(r,n-1)} = 0$ for $n = 2, 3, \ldots$.

We have shown above that if the collection $\phi(z)$ consists of asymptotic expansions for a Nevanlinna function $F(z)$, then $(P_0), \ldots, (P_n), \ldots$ are automatically satisfied.

We shall in this paper call the algorithm described here the *modified Schur algorithm*.

3. Recurrence relations

Let $\phi(z)$ be a positive collection of formal power series, with associated transformations $s_n(z, w)$, $S_n(z, w)$.

Since the $S_n(z, w)$ are linear fractional transformations, there exist functions $A_n(z), B_n(z), C_n(z), D_n(z), n = 1, 2, \ldots,$ such that

$$(3.1) \qquad S_n(z, w) = \frac{A_n(z) + wC_n(z)}{B_n(z) + wD_n(z)} .$$

For each n, A_n, B_n, C_n, D_n are defined up to a common factor $\varphi(z)$.

We call the functions

$$(3.2) \qquad f_n(z) = S_n(z, 0) = \frac{A_n(z)}{B_n(z)}$$

the approximants for the algorithm. We shall also make use of the functions

$$(3.3) \qquad g_n(z) = S_n(z, \infty) = \frac{C_n(z)}{D_n(z)} .$$

Since

$$(3.4) \qquad S_1(z, w) = \frac{-(1 - \delta_1) - \alpha_1^{-1}\delta_1 w}{-\alpha_1 + (1 - \delta_1)\beta_1 z + w} ,$$

we may set

$$(3.5) \quad A_1 = -(1 - \delta_1), \quad B_1 = -\alpha_1 + (1 - \delta_1)\beta_1 z, \quad C_1 = -\alpha_1^{-1}\delta_1, \quad D_1 = 1 .$$

From the relation $S_n(z, w) = S_{n-1}(z, s_n(z, w))$, (2.12), and (3.1) we obtain for $n = 2, 3, \ldots$

$$(3.6)$$
$$\frac{A_n + wC_n}{B_n + wD_n}$$
$$= \frac{[(-\alpha_n + \frac{(\delta_n - 1)\beta_n}{z - a_{r-1}})A_{n-1} + (\delta_n - 1)C_{n-1}] + [A_{n-1} + \alpha_n^{-1}\delta_n C_{n-1}]w}{[(-\alpha_n + \frac{(\delta_n - 1)}{z - a_{r-1}})B_{n-1} + (\delta_n - 1)D_{n-1}] + [B_{n-1} + \alpha_n^{-1}\delta_n D_{n-1}]w} .$$

Thus A_n, B_n, C_n, D_n satisfy the following recurrence relations for $n = 2, 3, \ldots$:

$$(3.7a) \qquad A_n = \left(\alpha_n + \frac{(\delta_n - 1)\beta_n}{z - a_{r-1}}\right) A_{n-1} + (\delta_n - 1)C_{n-1} ,$$

$$(3.7b) \qquad C_n = A_{n-1} + \delta_n \alpha_n^{-1} C_{n-1} ,$$

$$(3.7c) \qquad B_n = \left(\alpha_n + \frac{(\delta_n - 1)\beta_n}{z - a_{r-1}}\right) B_{n-1} + (\delta_n - 1)D_{n-1} ,$$

$$(3.7d) \qquad D_n = B_{n-1} + \delta_n \alpha_n^{-1} D_{n-1} .$$

When n is a regular index, these relations can be written as

$$(3.8a) \qquad A_n = \left(\alpha_n - \frac{\beta_n}{z - a_{r-1}}\right) A_{n-1} - C_{n-1} ,$$

$$(3.8b) \qquad C_n = A_{n-1} ,$$

$$(3.8c) \qquad B_n = \left(\alpha_n - \frac{\beta_n}{z - a_{r-1}}\right) B_{n-1} - D_{n-1} ,$$

$$(3.8d) \qquad D_n = B_{n-1}$$

while when n is a singular index, the relations may be written as

$$(3.9a) \qquad A_n = \alpha_n A_{n-1}, \qquad C_n = A_{n-1} + \alpha_n^{-1} C_{n-1},$$

$$(3.9b) \qquad B_n = \alpha_n B_{n-1}, \qquad D_n = B_{n-1} + \alpha_n^{-1} D_{n-1}.$$

We note that when all n are regular, the algorithm is a continued fraction algorithm. It can be shown that the continued fractions obtained in this way are equivalent to MP-fractions, treated in [7, 8, 24]. (For basic concepts of continued fractions, including the concept of equivalence, see [13].)

For further use we introduce polynomials $N_n(z)$ defined as follows:

$$(3.10) \qquad \begin{aligned} &N_0(z) \equiv 1, \\ &N_n(z) = (z-a_1)^{q+1} \cdots (z-a_r)^{q+1}(z-a_{r+1})^q \cdots (z-a_p)^q \end{aligned}$$

$$\text{for } n = 1, 2, \ldots .$$

Proposition 3.1. *The function A_n, B_n, C_n, D_n can be written as*

$$A_n(z) = \frac{U_n(z)}{N_{n-1}(z)}, \quad B_n(z) = \frac{V_n(z)}{N_{n-1}(z)}, \qquad n = 1, 2, \ldots,$$

$$C_n(z) = \frac{X_n(z)}{N_{n-2}(z)}, \quad D_n(z) = \frac{Y_n(z)}{N_{n-2}(z)}, \qquad n = 2, 3, \ldots,$$

where U_n, V_n, X_n, Y_n are polynomials, $\deg U_n \leq n-1$, $\deg V_n \leq n$, $\deg X_n \leq n-2$, and $\deg Y_n \leq n-1$ (except $\deg X_1 = 0$).

Proof. It follows by (3.4) that the result holds for $n = 1$, resp. $n = 2$, and by use of (3.6)–(3.7) the result is obtained for general n by induction. □

For further use it will also be natural to introduce the functions P_n, Q_n by

$$(3.12) \qquad P_n(z) = \frac{A_n(z)}{z - a_r} = \frac{U_n(z)}{N_n(z)}, \qquad Q_n(z) = \frac{B_n(z)}{z - a_r} = \frac{V_n(z)}{N_n(z)}.$$

We note for further reference that the recurrence relations (3.7) can be written in terms of U_n, V_n, X_n, Y_n as follows (for $n = 2, 3, \ldots$):

$$(3.13a) \qquad \begin{aligned} U_n = &[\alpha_n(z-a_{r-1}) + (\delta_n - 1)\beta_n]U_{n-1} \\ &+ (\delta_n - 1)(z-a_{r-2})(z-a_{r-1})X_{n-1}, \end{aligned}$$

$$(3.13b) \qquad X_n = U_{n-1} + \delta_n \alpha_n^{-1}(z-a_{r-2})X_{n-1},$$

$$(3.13c) \qquad \begin{aligned} V_n = &[\alpha_n(z-a_{r-1}) + (\delta_n - 1)\beta_n]V_{n-1} \\ &+ (\delta_n - 1)(z-a_{r-2})(z-a_{r-1})Y_{n-1}, \end{aligned}$$

$$(3.13d) \qquad Y_n = V_{n-1} + \delta_n \alpha_n^{-1}(z-a_{r-2})Y_{n-1}.$$

Proposition 3.2. *Assume that $V_n(a_r) \neq 0$ for all n. Then for every n, V_n and Y_n do not both have a zero at a_i for any fixed $i \in \{1, \ldots, p\}$. For every n, either $\deg V_n = n$ or $\deg Y_n = n-1$.*

Proof. Let $V_n(a_i) = Y_n(a_i) = 0$. It follows by (3.13) that $V_{n+1}(a_i) = Y_{n+1}(a_i) = 0$. By continuing this argument a finite number of steps, we reach the equality $Y_{pq+i}(a_i) = 0$ for some q. This contradicts the assumption $V_{p+i}(a_i) \neq 0$.

A similar argument shows that if $\deg V_n < n$, $\deg Y_n < n - 1$, then we must have $Y_1 \equiv 0$, which contradicts (3.4). $\square$

We present some basic formulas that will be needed later.

Proposition 3.3. *The following formulas hold:*

$$(3.14a) \qquad (i) \quad A_n D_n - B_n C_n = -1,$$

$$(3.14b) \qquad (ii) \quad A_n B_{n-1} - A_{n-1} B_n = (\delta_n - 1),$$

$$(3.14c) \qquad (iii) \quad C_n D_{n-1} - C_{n-1} D_n = -1,$$

$$(3.14d) \qquad (iv) \quad A_n D_{n-1} - B_n C_{n-1} = -\left(\alpha_n + \frac{(\delta_n - 1)\beta_n}{z - a_{r-1}} \right),$$

$$(3.14e) \qquad (v) \quad A_{n-1} D_n - B_{n-1} C_n = -\delta_n \alpha_n^{-1}.$$

Proof. By using (3.7) we obtain $A_n D_n - B_n C_n = A_{n-1} D_{n-1} - B_{n-1} C_{n-1} = \cdots = A_1 D_1 - B_1 C_1 = -1$. Again by using (3.7) and the result of (i) we get $A_n B_{n-1} - A_{n-1} B_n = (\delta_n - 1)[A_{n-1} D_{n-1} - B_{n-1} C_{n-1}] = 1 - \delta_n$. The formulas (iii), (iv), and (v) are proved analogously. $\square$

We next deduce some formulas analogous to the classical Christoffel-Darboux formula and related formulas. (See also [18] for similar formulas, proved within the framework of orthogonal rational functions.)

Proposition 3.4. *The following formulas are valid for $n = 1, 2, \ldots$:*

$$(3.15a) \qquad \begin{aligned} &B_n(z) D_n(\zeta) - B_n(\zeta) D_n(z) \\ &\quad = (z - \zeta) \sum_{m=1}^{n} (1 - \delta_m) \beta_m Q_{m-1}(z) Q_{m-1}(\zeta), \end{aligned}$$

$$(3.15b) \qquad \begin{aligned} &A_n(z) C_n(\zeta) - A_n(\zeta) C_n(z) \\ &\quad = (z - \zeta) \sum_{m=1}^{n} (1 - \delta_m) \beta_m P_{m-1}(z) P_{m-1}(\zeta), \end{aligned}$$

$$(3.15c) \qquad \begin{aligned} &B_n(z) C_n(\zeta) - A_n(\zeta) D_n(z) \\ &\quad = 1 + (z - \zeta) \sum_{m=1}^{n} (1 - \delta_m) \beta_n P_{m-1}(\zeta) Q_{m-1}(z). \end{aligned}$$

Proof. By multiplying the difference equation (3.7a) with argument z by the difference equation (3.7b) with argument ζ, and then subtracting the same equation with z and ζ interchanged, we get for $m = 2, 3, \ldots$

$$(3.16) \qquad \begin{aligned} B_m(z) D_m(\zeta) - B_m(\zeta) D_m(z) &= (z - \zeta)(1 - \delta_m) \beta_m \frac{B_{m-1}(z) B_{m-1}(\zeta)}{(z - a_{r-1})(\zeta - a_{r-1})} \\ &\quad + B_{m-1}(z) D_{m-1}(\zeta) - B_{m-1}(\zeta) D_{m-1}(z). \end{aligned}$$

Furthermore for $m = 1$ we get from (3.4)

$$(3.17) \qquad B_1(z) D_1(\zeta) - B_1(\zeta) D_1(z) = \beta_1(z - \zeta).$$

By adding these equations for $m = 1, 2, \ldots, n$, taking into account (3.12), we obtain (3.15a).

The proofs of (3.15b) and (3.15c) are similar. □

Proposition 3.5. *The following formula is valid for* $n = 1, 2, \ldots$:

(3.18)

$$B_n(z) - B_n(\zeta) = (z - \zeta) \sum_{m=1}^{n} (1 - \delta_m) \beta_m [B_n(\zeta) P_{m-1}(\zeta) - A_n(\zeta) Q_{m-1}(\zeta)] Q_{m-1}(z).$$

Proof. Subtraction of (3.15a) multiplied by $A_n(\zeta)$ from (3.15c) multiplied by $B_n(\zeta)$, with (3.14a) being taken into account, gives (3.18). □

Proposition 3.6. *The following formula is valid for* $n = 1, 2, \ldots$:

(3.19) $$V_n'(z) Y_n(z) = V_n(z) Y_n'(z) = N_{n-1}(z)^2 \sum_{m=1}^{n} (1 - \delta_m) \beta_m Q_{m-1}(z)^2.$$

Proof. Formula (3.15a) may be written as

(3.20)
$$V_n(z) Y_n(\zeta) - V_n(\zeta) Y_n(z)$$
$$= (z - \zeta) N_{n-1}(z) N_{n-1}(\zeta) \sum_{m=1}^{m} (1 - \delta_m) \beta_m Q_{m-1}(z) Q_{m-1}(\zeta).$$

The left side of this equation may be written as

$$[V_n(z) V_n(\zeta)] Y_n(\zeta) - [Y_n(z) - Y_n(\zeta)] V_n(\zeta).$$

By dividing (3.20) by $(z - \zeta)$ and letting $\zeta \to z$, we obtain (3.19). □

Corollary 3.7. *Assume that* $V_n(a_{r_n}) \neq 0$ *for all* n. *Then for each* a_i *and each* n, $V_n(z)$ *has a zero of multiplicity at most one at* a_i. *For each* n, $\deg V_n \geq n - 1$.
Proof. It follows from (3.13c) and the assumption $V_m(a_{r_m}) \neq 0$ that for at least one $m \in \{n - p + 1, \ldots, n\}$, $\delta_m = 0$. Let $m = pq + i$. For this m, the right side of equation (3.19) is different from zero for $z = a_i$. Consequently by (3.19), not both $V_n(a_i)$ and $V_n(a_i)$ can be zero.

The result on the degree can be proved analogously. □

We shall see in §4 that the condition $V_n(a_{r_n}) \neq 0$ for all n is always satisfied when the transformations $s_n(z, w)$, $S_n(z, w)$ and thereby A_n, B_n, C_n, D_n are obtained from a positive collection of power series.

4. Multipoint Padé approximants

In this section we show that the functions $f_n(z) = S_n(z, 0) = A_n(z)/B_n(z)$, or their Taylor series expansions, have certain interpolation properties with respect to the series expansions (2.19) (in particular, to (2.1), when the series originate as expansions of a Nevanlinna function). More precisely, we show that $f_n(z)$ is a multipoint Padé approximant for the given series. (For a detailed discussion of multipoint Padé approximants in this setting, we refer the reader to [19].)

We first give an interpolation result for $f_n(z)$ at a_r by a direct argument from the construction of the algorithm in §2. (As before, we write r for r_n and q for q_n.)

Proposition 4.1. *For $f_n(z)$ the following expansion is valid at a_r:*

$$(4.1) \qquad f_n(z) = \sum_{j=0}^{2q} \gamma_j^{(r)} (z - a_r)^j + O((z - a_r)^{2q+1}).$$

Proof. We define $H_{n-1}(z) = s_n(z, 0)$, $H_{n-2}(z) = s_{n-1}(z, H_{n-1}(z))$, and inductively $H_{n-m}(z) = s_{n-m+1}(z, H_{n-m+1}(z))$ for $m \le n$. Then $H_0(z) = f_n(z)$. For $H_{n-1}(z)$ we have the expression

$$H_{n-1}(z) = \frac{(\delta_n - 1)}{\alpha_n + \beta_n(\delta_n - 1)/(z - a_{r-1})},$$

while for $F_{n-1}(z)$ we have the expression

$$(4.2) \qquad F_{n-1}(z) \approx \frac{(\delta_n - 1) + \alpha_n^{-1}\delta_n[\gamma_1^{(r,n)}(z - a_r) + \cdots]}{(\delta_n - 1)\beta_n/(z - a_{r-1}) + \gamma_1^{(r,n)}(z - a_r) + \cdots}.$$

It follows that if $\gamma_{-1}^{(r,n-1)} = 0$, then

$$(4.3) \qquad H_{n-1}(z) = \gamma_0^{(r,n-1)} + O((z - a_r)).$$

On the other hand, if $\gamma_{-1}^{(r,n-1)} \neq 0$, then

$$(4.4) \qquad H_{n-1}(z) = \frac{\delta_{-1}^{(r,n-1)}}{(z - a_r)} + O(1).$$

Similarly by repeated comparison of $s_{n-m}(z, H_{n-m}(z))$ and $s_{n-m}(z, F_{n-m}(z))$ we find that

$$(4.5) \qquad H_{n-p+2}(z) = \gamma_0^{(r,n-p+2)} + O((z - a_r))$$

if $\gamma_{-1}^{(r,n-p+2)} = 0$, while

$$(4.6) \qquad H_{n-p+2}(z) = \frac{\delta_{-1}^{(r,n-p+2)}}{(z - a_r)} + O(1)$$

if $\gamma_{-1}^{(r,n-p+2)} \neq 0$.

Now we know that $\gamma_{-1}^{(r,n-p+2)} = 0$ (cf. (2.11c) with n replaced by $n - p + 1$). By comparison of $s_{n-p+2}(z, F_{n-p+2}(z))$ and $s_{n-p+2}(z, H_{n-p+2}(z))$ we see that the constant term in the expansion of $H_{n-p+1}(z)$ coincides with that for $F_{n-p+1}(z)$ both if $\gamma_{-1}^{(r,n-p+2)} = 0$ and if $\gamma_{-1}^{(r,n-p+2)} \neq 0$. Thus, in any case

$$(4.7) \qquad H_{n-p+1}(z) = \gamma_0^{(r,n-p+1)} + O((z - a_r)).$$

By comparison of $s_{n-p+1}(z, F_{n-p+1}(z))$ and $s_{n-p+1}(z, H_{n-p+1}(z))$, using (2.13b) with n replaced by $n - p + 1$, we then find

$$(4.8) \qquad H_{n-p}(z) = \gamma_1^{(r,n-p)}(z - a_r) + \gamma_2^{(r,n-p)}(z - a_r)^2 + O((z - a_r)^3).$$

By comparison of $s_{n-p}(z, F_{n-p}(z))$ and $s_{n-p}(z, H_{n-p}(z))$, using (2.13c) with n replaced by $n-p$, we find

(4.9)
$$H_{n-p-1}(z) = \gamma_0^{(r, n-p-1)} + \gamma_1^{(r, n-p-1)}(z - a_r)$$
$$+ \gamma_2^{(r, n-p+1)}((z - a_r)^2) + O((z - a_r)^3)$$

if $\gamma_{-1}^{(r, n-p-1)} = 0$, and

(4.10)
$$H_{n-p-1}(z) = \gamma_{-1}^{(r, n-p-1)} \frac{1}{z - a_r} + \gamma_0^{(r, n-p-1)} + O((z - a_r))$$

if $\gamma_{-1}^{(r, n-1)} \neq 0$.

By repeating these arguments until we reach $H_0(z) = f_n(z)$, noting that $\gamma_{-1}^{(r)} = 0$, we obtain the expansion

(4.11)
$$f_n(z) = \gamma_0^{(r)} + \gamma_1^{(r)}(z - a_r) + \cdots + \gamma_{2q}^{(r)}(z - a_r)^{2q} + O((z - a_r)^{2q+1}).$$

Since the functions $f_n(z)$ are rational, the asymptotic series for $f_n(z)$ are Taylor series. □

Proposition 4.2. Let $\phi(z)$ be a positive collection. Then for all n, $V_n(a_r) \neq 0$.

Proof. It follows directly from (2.4c) and (3.5) that $V_1(a_1) \neq 0$. Let $n > 1$. We recall that $f_n(z) = S_n(z, 0)$, while $\phi(z) = S_n(z, \phi_n(z))$. (For brevity we write $\phi(z)$ for the appropriate series.) It follows that

(4.12)
$$\frac{U_n(z)}{V_n(z)} - \phi(z) = \frac{A_n}{B_n} - \frac{A_n + \phi_n C_n}{B_n + \phi_n D_n} = \frac{U_n}{V_n} - \frac{U_n + (z - a_{r-1})X_n\phi_n}{V_n + (z - a_{r-1})Y_n\phi_n}$$
$$= \frac{(z - a_{r-1})\phi_n[U_n Y_n - X_n V_n]}{V_n[U_n + (z - a_{r-1})\phi_n Y_n]}.$$

By (4.1) the left side of (4.12) has expansion of the form $f_n(z) - \phi(z) \approx \delta_{2q+1}^{(r)}(z - a_r)^{2q+1} + \cdots$. The numerator of the right side of (4.12) has expansion of the form $-\phi_n(z)N_{n-1}(z)^2 \approx \gamma_1^{(r, n)}(z - a_r)^{2q+1} + \cdots$. For (4.12) to hold it is thus necessary that $V_n(a_r) \neq 0$. □

We note that in exactly the same way as (4.12) we get

(4.13)
$$\frac{X_n(z)}{Y_n(z)} - \phi(z) = \frac{[V_n X_n - U_n Y_n]}{Y_n[V_n + (z - a_{r-1})\phi_n Y_n]}.$$

We now state the interpolation theorem for the approximants $f_n(z)$ with respect to $F(z)$, or to the series expansions (2.19).

Theorem 4.3. The approximant $f_n(z) = U_n(z)/V_n(z)$ has the following interpolation properties:

$$(4.14) \quad \frac{U_n(z)}{V_n(z)} = \sum_{j=0}^{2q+1} \gamma_j^{(i)} (z - a_i)^j + O((z - a_i)^{2q+2}) \quad \text{if } i < r, \; V_n(a_i) \neq 0,$$

$$(4.15) \quad \frac{U_n(z)}{V_n(z)} = \sum_{j=0}^{2q} \gamma_j^{(r)} (z - a_r)^j + O((z - a_r)^{2q+1}),$$

$$(4.16) \quad \frac{U_n(z)}{V_n(z)} = \sum_{j=0}^{2q-1} \gamma_j^{(i)} (z - a_i)^j + O((z - a_i)^{2q}) \quad \text{if } i > r, \; V_n(a_i) \neq 0,$$

$$(4.17) \quad \frac{U_n(z)}{V_n(z)} = -\gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1}) \quad \text{if } \deg V_n = n.$$

In general, the approximant has the following (weak) multipoint Padé approximation property:

$$(4.18) \quad U_n(z) - V_n(z) \sum_{j=0}^{2q+1} \gamma_j^{(i)} (z - a_i)^j = O((z - a_i)^{2q+2}) \quad \text{if } i < r,$$

$$(4.19) \quad U_n(z) - V_n(z) \sum_{j=0}^{2q} \gamma_j^{(r)} (z - a_r)^j = O((z - a_r)^{2q+1}),$$

$$(4.20) \quad U_n(z) - V_n(z) \sum_{j=0}^{2q-1} \gamma_j^{(i)} (z - a_i)^j = O((z - a_i)^{2q}) \quad \text{if } i > r,$$

$$(4.21) \quad U_n(z) - V_n(z)(-\gamma_{-1}^{(\infty)} z^{-1}) = o(z^{-1}).$$

Proof. Formula (4.15) follows immediately from Proposition 4.1.

Let, e.g., $i < r$. For each of the numbers $m \in \{pq + i + 1, \ldots, n - 1\}$ let Z_m and W_m denote X_m and Y_m, or U_m and V_m, such that $W_m(a_i) \neq 0$ (cf. Proposition 3.2). According to Proposition 3.3 and (3.11) we may write for $m \in \{pq + i + 2, \ldots, n - 1\}$:

$$(4.22) \quad \frac{Z_m}{W_m} - \frac{Z_{m-1}}{W_{m-1}} = \frac{Z_m W_{m-1} - Z_{m-1} W_m}{W_m W_{m-1}} = \frac{c(z) N_{m-1}(z) N_{m-2}(z)}{W_m(z) W_{m-1}(z)},$$

where $c(z)$ is a constant or a simple rational function without a pole at a_i. Thus,

$$(4.23) \quad \frac{Z_m}{W_m} - \frac{Z_{m-1}}{W_{m-1}} = O((z - a_i)^{2q+2}) \quad \text{for } m = pq + i + 2, \ldots, n - 1.$$

Similarly we conclude that

$$(4.24) \quad \frac{U_n}{V_n} - \frac{Z_{n-1}}{W_{n-1}} = \frac{c(z) N_{n-1}(z) N_{n-2}(z)}{V_n(z) W_{n-1}(z)},$$

where $c(z)$ is as above. From this we conclude that we may write

$$(4.25) \quad \frac{U_n(z)}{V_n(z)} - \frac{Z_{n-1}(z)}{W_{n-1}(z)} = \frac{d(z - a_i)^{2a+2} + \cdots}{V_n(z)}.$$

From (4.12)–(4.13) together with (3.11) and Proposition 3.3 we obtain

$$(4.26) \quad \frac{Z_{pq+i+1}(z)}{W_{pq+i+1}(z)} - \phi(z) = \frac{\omega(z)N_{pq+i}(z)^2}{W_{pq+i+1}(z)[V_{pq+i+1}(z) + (z - a_i)Y_{pq+i+1}(z)]},$$

where $\omega(z)$ is regular at a_i. Thus,

$$(4.27) \quad \frac{Z_{pq+i+1}(z)}{W_{pq+i+1}(z)} - \phi(z) = O((z - a_i)^{2q+2}).$$

We may write
(4.28)
$$f_n(z) - \phi(z) = \left(\frac{U_n(z)}{V_n(z)} - \frac{Z_{n-1}(z)}{W_{n-1}(z)} \right) + \left(\frac{Z_{n-1}(z)}{W_{n-1}(z)} - \frac{Z_{n-2}(z)}{W_{n-2}(z)} \right)$$
$$+ \cdots + \left(\frac{Z_{pq+i+2}(z)}{W_{pq+i+2}(z)} - \frac{Z_{pq+i+1}(z)}{W_{pq+i+1}(z)} \right) + \left(\frac{Z_{pq+i+1}(z)}{W_{pq+i+1}(z)} - \phi(z) \right).$$

It follows by (4.23), (4.25), and (4.27) that we may write

$$(4.29) \quad f_n(z) - \phi(z) = \frac{e(z - a_i)^{2q+2}}{V_n(z)} + O((z - a_i)^{2q+2}).$$

Thus if $V_n(a_i) \neq 0$ we get $f_n(z) - \phi(z) = O((z - a_i)^{2q+2})$, and in all cases $U_n(z) - V_n(z) \cdot \phi(z) = O((z - a_i)^{2q+2})$. This implies (4.14) and (4.18).

Similarly we prove the formula

$$(4.30) \quad f_n(z) - \phi(z) = \frac{e(z - a_i)^{2q+2} + \cdots}{V_n(z)} + O((z - a_i)^{2q})$$

in the case that $i > r$. Thus if $V_n(a_i) \neq 0$ we get $f_n(z) - \phi(z) = O((z - a_i)^{2q})$, and in all cases $U_n(z) - V_n(z) \cdot \phi(z) = O((z - a_i)^{2q})$. This implies (4.16) and (4.20).

Finally a completely analogous argument shows that $f_n(z) - \phi(z) = o(z^{-1})$ if $\deg V_n = n$ and, in general, $U_n(z) - V_n(z) \cdot \phi(z) = o(z^{n-1})$. (Here it has to be used that $f_1(z) - \phi(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1})$, which follows immediately from the construction of $s_1(z, w)$.) This implies (4.17) and (4.21). This completes the proof. $\square$

Remark 4.4. It easily follows that if $V_n(a_i) = 0$, and hence $Y_n(a_i) \neq 0$, then (4.14)–(4.16) hold with $U_n(z)/V_n(z)$ replaced by $X_n(z)/Y_n(z)$ and r by $r - 1$. Similarly (4.17) holds with $U_n(z)/V_n(z)$ replaced by $X_n(z)/Y_n(z)$ if $\deg V_n < n$, and hence $\deg Y_n = n - 1$.

Remark 4.5. From Theorem 4.3 and the recurrence relations (3.13) it can be seen that among the pq first indices there are at least q regular ones (cf. the proof of Corollary 3.7).

We define the generalized approximants for the algorithm by

$$(4.31) \qquad f_n(z, \tau) = \frac{A_n(z) + \tau C_n(z)}{B_n(z) + \tau D_n(z)}, \qquad \tau \in \mathbf{R}.$$

We may clearly also write

$$(4.32) \qquad f_n(z, \tau) = \frac{U_n(z) + \tau(z - a_{r-1})X_n(z)}{V_n(z) + \tau(z - a_{r-1})Y_n(z)}.$$

Proposition 4.6. *For every* n *and every* $\tau \in \mathbf{R}$ *the following formulas hold (except possibly for one value for* τ *in each case):*

$$(4.33) \qquad f_n(z, \tau) = \sum_{j=0}^{2q+1} \gamma_j^{(i)}(z - a_i)^j + O((z - a_i)^{2q+2}) \quad \text{if } i < r - 1,$$

$$(4.34) \qquad f_n(z, \tau) = \sum_{j=0}^{2q-1} \gamma_j^{(i)}(z - a_i)^j + O((z - a_i)^{2q}) \quad \text{if } i \geq r - 1,$$

$$(4.35) \qquad f_n(z, \tau) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1}) \quad \text{if } \deg V_n = n.$$

Proof. We may write

$$(4.36) \qquad \begin{aligned} f_n(z, \tau) - \frac{U_n(z)}{V_n(z)} &= \frac{A_n(z) + \tau C_n(z)}{B_n(z) + \tau D_n(z)} - \frac{A_n(z)}{B_n(z)} \\ &= \frac{\tau N_{n-1}(z)^2}{V_n(z)[V_n(z) + \tau(z - a_{r-1})Y_n(z)]}, \end{aligned}$$

$$(4.37) \qquad \begin{aligned} f_n(z, \tau) - \frac{X_n(z)}{Y_n(z)} &= \frac{A_n(z) + \tau C_n(z)}{B_n(z) + \tau D_n(z)} - \frac{C_n(z)}{D_n(z)} \\ &= \frac{N_{n-1}(z)N_{n-2}(z)}{Y_n(z)[U_n(z) + \tau(z - a_{r-1})Y_n(z)]}. \end{aligned}$$

We note that since $V_n(z)$ and $Y_n(z)$ are not both zero for any a_i, then $[V_n(z) + \tau(z - a_{r-1})Y_n(z)]$ has no zero except possibly for one value of τ, when $i \neq r-1$. When $i = r-1$, $[V_n(z) + \tau(z - a_{r-1})Y_n(z)]$ may, except for one value of τ, have a zero of multiplicity at most one (cf. Corollary 3.7). This together with (4.36) and Theorem 4.3 or (4.37) and Remark 4.4 gives (4.33)–(4.34). The proof of (4.35) is analogous. $\square$

5. Mapping properties

In this section we discuss mapping properties of the linear fractional transformations $s_n(z, w)$, $S_n(z, w)$ (see also [8]). Let $z \in H_+^0$. Each of the linear fractional transformations $w \to s_n(z, w)$ maps the closed upper half-plane H_+ onto a disk $\delta_n(z)$ (possibly a half-plane), and each of the linear fractional transformations $w \to S_n(z, w)$ maps H_+ onto a disk (possibly a half-plane) $\Delta_n(z)$.

Propostiion 5.1. *The following inclusions hold for all* $z \in H_+^0$:

(A) $\Delta_1(z) \subset H_+$.

(B) $\Delta_n(z) \subset \Delta_{n-1}(z)$ *for* $n = 2, 3, \ldots$.

Proof. Since $\beta_n > 0$ for $n = 1, 2, \ldots$, it is readily verified that $s_n(z, w) \in H_+$ for all n and all $w \in H_+$. This means that $\delta_n(z) \subset H_+$. It follows that $\Delta_1(z) = \delta_1(z) \subset H_+$ and $\Delta_n(z) = S_n(z, H_+) = S_{n-1}(z, \delta_n(z)) \subset S_{n-1}(z, H_+) = \Delta_{n-1}(z)$. $\square$

We recall that the mapping $w \to S_n(z, w)$ is described by the formula

$$(5.1) \qquad S_n(z, w) = \frac{A_n(z) + wC_n(z)}{B_n(z) + wD_n(z)}.$$

$S_n(z, \tau)$ traces the boundary $\partial\Delta_n(z)$ when τ traces the real axis (plus ∞). For convenience we shall in the following write h_{m-1} for $\sqrt{(1 - \delta_m)\beta_m}$. (Recall that $\beta_m > 0$, $\delta_m = 0$ or $\delta_m = 1$.)

Proposition 5.2. *For every* $n = 1, 2, 3, 4, \ldots$, *the radius* $\rho_n(z)$ *of the disk* $\Delta_n(z)$ *is given by the formula*

$$(5.2) \qquad \frac{1}{\rho_n(z)} = |z - \bar{z}| \cdot \sum_{m=1}^{n} h_{m-1}^2 |Q_{m-1}(z)|^2.$$

Proof. By standard results on linear fractional transformations it follows (using formula (5.1)) that $\rho_n(z)$ can be expressed as

$$(5.3) \qquad \rho_n(z) = \left| \frac{A_n(z)D_n(z) - C_n(z)B_n(z)}{B_n(z)D_n(\bar{z}) - C_n(z)B_n(\bar{z})} \right|.$$

(We recall that all the coefficients in $A_n(z), B_n(z), C_n(z), D_n(z)$ are real.) From (3.14a) we get

$$|A_n(z)D_n(z) - C_n(z)B_n(z)| = 1.$$

From (3.15a) we get (setting $\zeta = \bar{z}$)

$$(5.4) \qquad |B_n(z)D_n(\bar{z}) - D_n(z)B_n(\bar{z})| = |z - \bar{z}| \cdot \sum_{m=1}^{n} h_{m-1}^2 |Q_{m-1}(z)|^2.$$

Formula (5.2) now immediately follows. $\square$

We write $\Delta(z)$ for the intersection $\bigcap_{n=1}^{\infty} \Delta_n(z)$. Then (by Proposition 5.2) $\Delta(z)$ is a closed disk or reduces to a single point.

Proposition 5.3. *The following statements are equivalent:*

(A) $\Delta(z)$ *reduces to a single point.*

(B) $\sum_{m=1}^{\infty} h_{m-1}^2 |B_{m-1}(z)|^2 = \infty$.

(C) $\sum_{m=1}^{\infty} h_{m-1}^2 |A_{m-1}(z)|^2 = \infty$.

Proof. The radius $\rho(z)$ of $\Delta(z)$ is obviously given by $\rho(z) = \lim_{n \to \infty} \rho_n(z)$, hence

$$(5.5) \qquad \frac{1}{\rho(z)} = |z - \bar{z}| \sum_{m=1}^{\infty} h_{m-1}^2 |Q_{m-1}(z)|^2.$$

Clearly $\sum_{m=1}^{\infty} h_{m-1}^2 |Q_{m-1}(z)|^2 = \infty$ iff $\sum_{m=1}^{\infty} h_{m-1}^2 |B_{m-1}(z)|^2 = \infty$, since $\min(|z - a_i|, \ldots, |z - a_p|) > 0$ for a given $z \in H_+^0$. Thus (A) and (B) are equivalent.

The transformation $w \to \frac{1}{S_n(z, w)}$ gives rise to a sequence $\{\Pi_n(z)\}$ of nested disks, where

$$\Pi_n(z) = \{\zeta = 1/S_n(z, w) : w \in H_+^0\}.$$

The intersection $\Pi(z)$ is a disk when $\Delta(z)$ is a disk and reduces to a single point when $\Delta(z)$ reduces to a single point. The transformation $w \to 1/S_n(z, w)$ can be written as $w \to (B_n(z) + wD_n(z))/(A_n(z) + wC_n(z))$. The radius $\pi_n(z)$ of $\Pi_n(z)$ is given by

$$(5.6) \qquad \pi_n(z) = \left| \frac{B_n(z)C_n(z) - A_n(z)D_n(z)}{A_{(z)}C_n(\bar{z}) - C_n(z)A_n(\bar{z})} \right|.$$

Just as we obtained (5.2) from (3.14a), (3.15a), and (5.3), we obtain from (3.14a), (3.15b) and (5.6) the formula

$$(5.7) \qquad \frac{1}{\pi_n(z)} = |z - \bar{z}| \cdot \sum_{m=1}^{n} h_{m-1}^2 |P_{m-1}(z)|^2.$$

Clearly $\sum_{m=1}^{\infty} h_{m-1}^2 |P_{m-1}(z)|^2 = \infty$ iff $\sum_{m=1}^{\infty} h_{m-1}^2 |A_{m-1}(z)|^2 = \infty$. Thus $\Pi(z)$, and hence $\Delta(z)$, reduces to a single point iff $\sum_{m=1}^{\infty} h_{m-1}^2 |A_{m-1}(z)|^2 = \infty$, i.e., (A) and (C) are equivalent. $\square$

Theorem 5.4. *If $\Delta(z)$ reduces to a single point for some $z \in H_+^0$, then $\Delta(z)$ reduces to a single point for every $z \in H_+^0$.*

Proof. The argument we give builds on a lemma of Perron [26], and is similar to the one given in [10] for APT-fractions.

It follows from [26] that when b_n is given recursively by

$$(5.8) \qquad b_n = \sum_{i=0}^{n-1} a_{n,i} b_i + c_n, \qquad n = 1, 2, 3, \ldots,$$

then

$$(5.9) \qquad \ln\left(1 + \sum_{i=1}^{n} |b_i|^2\right) \le \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} |a_{ij}|^2 + \sum_{i=1}^{n-1} |c_i|^2.$$

Set

$$(5.10) \qquad b_n = h_n B_n(z), \qquad c_n = h_n B_n(\zeta),$$

$$(5.11) \qquad a_{n,i} = h_i h_n [B_n(\zeta) P_i(\zeta) - A_n(\zeta) Q_i(\zeta)] \frac{(\zeta - z)}{(z - a_{r_i})}.$$

Then it follows from Proposition 3.5 that (5.8) is satisfied for $n = 1, 2, \ldots$. Hence from (5.9) we get

(5.12)
$$\ln\left(1 + \sum_{i=1}^{n-1} |B_i(z)|^2\right)$$
$$\leq \sum_{i=1}^{n-1} |B_i(\zeta)|^2 + |\zeta - z|^2 \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} h_i^2 \frac{|[B_i(\zeta)A_j(\zeta) - A_i(\zeta)B_j(\zeta)]|^2}{|z - a_{n_j}|^2 |\zeta - a_{n_j}|^2}.$$

Set $d = \min\{|z - a_1|, \ldots, |z - a_p|, |\zeta - a_1|, \ldots, |\zeta - a_p|\}$. By elementary inequalities we get

(5.13)
$$\sum_{i=1}^{n-1} \sum_{j=0}^{i-1} \frac{h_j^2 h_j^2}{|\zeta - a_{r_j}|^2 |\zeta - a_{r_j}|^2} |B_i(\zeta)A_j(\zeta) - A_i(\zeta)B_j(\zeta)|^2$$
$$\leq \frac{2}{d^4}\left(\sum_{i=1}^{n-1} h_i^2 |A_i(\zeta)|^2\right) \cdot \left(\sum_{i=1}^{n-1} h_i^2 |B_i(\zeta)|^2\right).$$

Substitution of (5.13) in (5.12) leads to

(5.14)
$$\ln\left(1 + \sum_{i=1}^{n-1} h_i^2 |B_i(z)|^2\right)$$
$$\leq \sum_{i=1}^{n-1} h_i^2 |B_i(\zeta)|^2 + \frac{2|\zeta - z|^2}{d^4}\left(\sum_{i=1}^{n-1} h_i^2 |A_i(\zeta)|^2\right) \cdot \left(\sum_{i=1}^{n-1} h_i^2 |B_i(\zeta)|^2\right).$$

Now assume that $\Delta(\zeta)$ is a disk for some ζ. Let $z \in H_+^0$ be arbitrary. Then by Proposition 5.3, the right-hand side of (5.14) has a finite upper bound M independent of n, and consequently $\sum_{i=1}^{\infty} h_i^2 |B_i(z)|^2 < \infty$. Then by Proposition 5.3, $\Delta(z)$ is a disk. $\square$

In view of Theorem 5.4 we may use the terms limit point situation (when $\Delta(z)$ reduces to a single point) and limit circle situation (when $\Delta(z)$ is a disk) without reference to a specified point z.

For further use we shall also give a formula for the center $\sigma_n(z)$ of the disk $\Delta_n(z)$.

Proposition 5.5. *For every $n = 1, 2, \ldots$, the center $\sigma_n(z)$ of the disk $\Delta_n(z)$ is given by the formula*

(5.15)
$$\sigma_n(z) = \frac{1 + (\bar{z} - z)\sum_{m=1}^{n} h_{m-1}^2 P_{m-1}(z)Q_{m-1}(\bar{z})}{(\bar{z} - z)\sum_{m=1}^{n} h_{m-1}^2 |Q_{m-1}(z)|^2}.$$

Proof. By standard results on linear fractional transformations (using (5.1)) it follows that

(5.16)
$$\sigma_n(z) = \frac{C_n(z)B_n(\bar{z}) - A_n(z)D_n(\bar{z})}{B_n(z)D_n(\bar{z}) - D_n(z)B_n(\bar{z})}.$$

(Recall that all the coefficients in $A_n(z)$, $B_n(z)$, $C_n(z)$, $D_n(z)$ are real.) From (3.15a) we conclude by setting $\zeta = \bar{z}$ that the denominator of (5.16) equals $(\bar{z} - z) \sum_{m=1}^{n} h_{m-1}^2 |Q_{m-1}(z)|^2$. Similarly we conclude from (3.15c) that the numerator of (5.16) equals $1 + (\bar{z} + z) \sum_{m=1}^{n} h_{m-1}^2 P_{m-1} Q_{m-1}(\bar{z})$. □

6. Asymptotic expansions and Nevanlinna functions

In this section we discuss the solution of the modified Pick-Nevanlinna problem.

Theorem 6.1. *Let $F(z)$ be a Nevanlinna function with the asymptotic expansions (2.1a)–(2.1b). Let $\Delta(z)$ be the disk (or point) associated with $F(z)$ through the modified Schur algorithm, as described in §§2 and 5. Then $F(z) \in \Delta(z)$ for every $z \in H_+^0$.*

Proof. For every $z \in H_+^0$ we have $F(z) = S_n(z, F_n(z))$ (cf. (2.15)), where $F_n(z)$ are the tail functions defined by the algorithm. The functions $F_n(z)$ are Nevanlinna functions. Therefore $F_n(z) \in H_+$ and so $S_n(z, F_n(z)) \in \Delta_n(z)$ for all $z \in H_+^0$. Thus $F(z) \in \Delta_n(z)$ for all n, i.e., $F(z) \in \Delta(z)$. □

Theorem 6.2. *Let a positive collection of power series (2.19) be given. Let $F(z)$ be a function which is analytic in H_+^0, and such that $F(z) \in \Delta(z)$ for all $z \in H_+^0$, where $\Delta(z)$ are the disks or points associated with the series through the modified Schur algorithm as described in §§2 and 5. Then $F(z)$ is a Nevanlinna function, and $F(z)$ has the asymptotic expansions (2.19a).*

Proof. That $F(z)$ is a Nevanlinna function follows from the fact that $\Delta(z) \subset H_+$ for all $z \in H_+^0$.

Let $\delta > 0$ be given, and let $r \in \{1, \ldots, p\}$ be fixed. Then for every $z \in R_{r,\delta}$ we have $|\operatorname{Im}(z - a_r)/(z - a_r)| > \delta$, hence $|(z - \bar{z})/(z - a_r)| > \delta$. Let N be an arbitrary natural number, and let n be a natural number such that $r_n = r$ and $2q - 2 \geq N$, where $q = q_n$. We see that when $\{\sum_{m=1}^{n+1} h_{m-1}^2 Q_{m-1}(z)^2\}^{-1}$ is expanded in power series in terms of $(z - a_r)$, the leading term will be of the form $k(z - a_r)^{2q}$ (recall that $V_n(a_r) \neq 0$). Thus there exists a constant c_r such that for $z \in R_{r,\delta}$ and $z - a_r$ sufficiently small, we have

$$(6.1) \qquad \rho_{n+1}(z) \leq c_r |z - a_r|^{2q-1}.$$

(Recall that $|(z - a_r)/(z - \bar{z})| \leq \delta$.) We have $f_{n+1}(z, \tau) \in \Delta_{n+1}(z)$, while $F(z) \in \Delta(z) \subset \Delta_{n+1}(z)$. It follows that

$$(6.2) \qquad |F(z) - f_{n+1}(z, \tau)| \leq 2c_r |z - a_r|^{2q-1}.$$

It follows from (4.16) and Remark 4.4 (with n replaced by $n + 1$) that

$$(6.3) \quad f_{n+1}(z, \tau) = \sum_{j=0}^{2q+1} \gamma_j^{(r)} (z - a_r)^j + O((z - a_r)^{2q+2}) \quad \text{for } \tau = 0 \text{ or for } \tau = \infty.$$

The function $f_{n+1}(z, \tau)$ is a rational function, and it has therefore a convergent Taylor series in a neighborhood of a_r. (Note that $f_n(z, 0)$ is analytic at a_r if $V_{n+1}(a_r) \neq 0$, and $f_n(z, \infty)$ is analytic at a_r if $Y_{n+1}(a_r) \neq 0$.) Consequently there exists a constant d_r such that

$$(6.4) \quad \left| f_{n+1}(t, \tau) - \sum_{j=0}^{2q+1} \gamma_j^{(r)}(z - a_r)^j \right| \leq d_r |z - a_r|^{2q+2} \quad \text{for } \tau = 0 \text{ or } \tau = \infty ,$$

when $z \in R_{r,\delta}$, $z - a_r$ sufficiently small. Combining (6.2) and (6.4) we see that there exists a constant k_r such that

$$(6.5) \quad \left| F(z) - \sum_{j=0}^{2q+1} \gamma_j^{(r)}(z - a_r)^j \right| \leq k_r |z - a_r|^{2q-1}$$

when $z \in R_{r,\delta}$, $z - a_r$ sufficiently small. Since $N \leq 2q - 2$, it follows that there exists a constant K_r such that

$$(6.6) \quad \left| F(z) - \sum_{j=0}^{N} \gamma_j^{(r)}(z - a_r)^j \right| \leq K_r |z - a_r|^{N+1}$$

when $z \in R_{r,\delta}$, $z - a_r$ sufficiently small. This means that

$$(6.7) \qquad\qquad F(z) \approx \sum_{j=0}^{\infty} \gamma_j^{(r)}(z - a_r)^j . \quad \square$$

Theorem 6.2 together with Theorem 6.1 characterize the solutions of the modified Pick-Nevanlinna problem for a positive collection (2.19) of power series. ($F(z)$ is a solution iff (2.1b) holds and $F(z) \in \Delta(z)$ for all z.) However, Theorem 6.2 does not in itself insure that there exist any solutions. We now turn to this problem.

Theorem 6.3. *Let a positive collection of power series (2.19) be given. Let* z_0 *be an arbitrary point in* H_+^0, *and let* ω_0 *be an arbitrary point on* $\partial\Delta(z_0)$. *Then for each* n *there exists a* $\tau_n \in \mathbf{R}$ *such that a subsequence of* $\{f_n(z, \tau_n)\}$ *converges on* H_+^0 *(uniformly on compact subsets) to a function* $F(z)$ *which has the asymptotic expansions (2.19) and which satisfies* $F(z_0) = \omega_0$.

Proof. Let $\delta > 0$, $\eta > 0$ be given, and set $S_{\delta,\eta} = \{z \in R_\delta : \operatorname{Im} z > \eta\}$. For $z \in R_\delta$ we have $|z - \bar{z}| = |2 \operatorname{Im} z| \geq 2|z| \sin \delta$. Let m be the smallest regular index. In view of (5.2) we have $\rho_m(z) = 1/h_{m-1}^2 |z - \bar{z}|$, hence there exists a constant A such that $|\rho_1(z)| \leq A/|z|$ for $z \in R_\delta$. Similarly, in view of (5.2) we have $\sigma_m(z) = 1/h_{m-1}^2 |z - \bar{z}|$, hence also $|\sigma_1(z)| \leq A/|z|$ for $z \in R_\delta$. Since $f_n(z, \tau) \in \partial\Delta_n(z)$, we then have

$$(6.8) \qquad\qquad |z f_n(z, \tau)| \leq 2A$$

for every generalized approximant $f_n(z, \tau)$, $z \in R_\delta$.

For each n we can choose a $\tau_n \in \mathbf{R}$ such that $f_n(z_0, \tau_n) = S_n(z_0, \tau_n) = \omega_n$, where $\omega_n \to \omega_0$ as $n \to \infty$. It is easily seen that every compact subset of H_+^0 is contained in a set R_δ. It follows by using standard results on compactness of families of analytic functions (see, e.g., [9, 13]) that $\{f_n(z, \tau_n)\}$ contains a subsequence $\{f_{n_k}(z, \tau_{n_k})\}$ such that $\deg V_{n_k} = n_k$ or $\deg Y_{n_k} = n_k - 1$, which converges on H_+^0 to a function $F(z)$ which is analytic in H_+^0, and such that $\{zf_{n_k}(z, \tau_{n_k})\}$ converges uniformly to $zF(z)$ in every $S_{\delta, \eta}$, $\delta, \eta > 0$. Also, $\{f_{n_k}(z, \tau_{n_k})\}$ converges uniformly to $F(z)$ on compact subsets of H_+^0. Since $f_n(z_0, \tau_n) = S_n(z_0, \tau_n) = \omega_n$, we have $F(z_0) = \lim_{n \to \infty} \omega_n = \omega_0$.

Let $\varepsilon > 0$, $\delta > 0$, $\eta > 0$ be given. We choose N so large that

$$(6.9) \qquad |zf_{n_k}(z, \tau_{n_k}) - zF(z)| < \varepsilon/2$$

for $z \in S_{\delta, \eta}$, $n_k \geq N$. (This is possible by the uniform convergence of $zf_{n_k}(z, \tau_{n_k})$ to $zF(z)$.) We fix one such n_k and choose $e \geq \eta$ so large that

$$(6.10) \qquad |zf_{n_k}(z, \tau_{n_k}) - \gamma_{-1}^{(\infty)}| < \varepsilon/2$$

for $z \in S_{\delta, e}$. (This is possible by (4.17) and Remark 4.4.) We therefore have

$$(6.11) \qquad |zF(z) - \gamma_{-1}^{(\infty)}| < \varepsilon$$

for $z \in S_{\delta, e}$. This shows that

$$(6.12) \qquad F(z) = \gamma_{-1}^{(\infty)} z^{-1} + o(z^{-1}).$$

For each $z \in H_+^0$ and each n, $f_n(z, \tau_n) \in \Delta_n(z)$. Consequently $F(z) \in \Delta(z)$, and by Theorem 6.2, $F(z)$ has the expansions (2.19a). $\square$

Theorem 6.4. *Let a positive collection of power series (2.19) be given. Then the corresponding modified Pick-Nevanlinna problem has solutions. An analytic function which satisfies (2.1b) is a solution iff $F(z) \in \Delta(z)$ for all $z \in H_+^0$. The problem has a unique solution iff the limit point situation obtains.*

Proof. It follows from Theorem 6.3 that the problem always has solutions. In the limit point situation there is exactly one analytic function $F(z)$ satisfying $F(z) \in \Delta(z)$ for all $z \in H_+^0$, hence by Theorem 6.1 there is exactly one analytic function with the expansions (2.19). For the limit circle situation there is for an arbitrary z_0 an infinity of analytic functions having the asymptotic expansions (2.19). $\square$

Remark 6.5. We observe the following. Let $F_0(z)$ be a given Nevanlinna function with the asymptotic expansions (2.19). Then the collection of asymptotic series is positive. It follows that the convergent subsequences of generalized approximants used in the proof of Theorem 6.3 converge to N-functions $F(z)$

with the same asymptotic expansions as $F_0(z)$. (However, no such subsequence need convergence to $F_0(z)$.)

7. The extended Hamburger moment problem

The extended Hamburger moment problem (associated with the set $\{a_1, \ldots, a_p\}$) is the following: Let $\{c_j^{(i)} : j = 1, 2, \ldots\}$, $i = 1, \ldots, p$, be sequences of real numbers and c_0 a real number. Find distribution functions $\psi(t)$ (i.e., bounded, nondecreasing functions) which have the property

(7.1)
$$\int_{-\infty}^{\infty} d\psi(t) = c_0, \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t - a_i)^j} = c_j^{(i)}, \quad i = 1, \ldots, p, \ j = 1, 2, \ldots.$$

(For convenience we do not require here that $\psi(t)$ has infinitely many points of increase.) This problem was treated in [17, 18] within the framework of a theory of positive linear functionals and orthogonal rational functions. In this paper we show (without making use of the material in [17, 18]) how the moment problem fits into the framework of the theory of asymptotic expansions of Nevanlinna functions.

We associate with the given sequences the series

(7.2a)
$$\sum_{j=0}^{\infty} c_{j+1}^{(i)} (z - a_i)^j, \qquad i = 1, \ldots, p,$$

(7.2b)
$$-c_0 z^{-1}.$$

That is; we set

(7.3) $\gamma_j^{(i)} = c_{j+1}^{(i)}$ for $i = 1, \ldots, p, \ j = 1, 2, \ldots,$ $\gamma_{-1}^{(\infty)} = -c_0,$

and consider the series (2.19) with these coefficients. Asymptotic expansions are as usual in the angular regions $R_{i,\delta}$, $i = 1, \ldots, p$, and R_δ, $\delta > 0$.

For each distribution function $\psi(t)$ we define its Stieltjes transform

(7.4)
$$\hat{\psi}(z) = \int_{-\infty}^{\infty} \frac{d\psi(t)}{t - z}.$$

We note that $\hat{\psi}(z)$ is a Nevanlinna function.

Theorem 7.1. *The following result holds for given* $\{c_j^{(i)}\}$, c_0: *If the distribution function* $\psi(t)$ *is a solution of the moment problem* (7.1), *then the Stieltjes transform* $\hat{\psi}(z)$ *has the asymptotic expansions*

(7.5a)
$$\hat{\psi}(z) \approx \sum_{j=0}^{\infty} c_{j+1}^{(i)} (z - a_i)^j \quad \text{for } i = 1, \ldots, p,$$

(7.5b) $\hat{\psi}(z) = -c_0 z^{-1} + o(z^{-1}).$

Proof. Set $z = x + iy$. We then have

(7.6)
$$z\hat{\psi}(z) = \int_{-\infty}^{\infty} \frac{[x(t - x) - y^2] + ity}{(t - x)^2 + y^2} \, d\psi(t).$$

We note that $(x(t-x) - y^2)/((t-x)^2 + y^2)$ is bounded for $z \in R_\delta$, and tends to -1 as z tends to ∞ in R_δ. Consequently,

$$(7.7) \qquad z\hat{\psi}(z) \to -\int_{-\infty}^{\infty} d\psi(t) = -c_0.$$

This means that

$$(7.8) \qquad \hat{\psi}(z) - (-c_0/z) = o(z^{-1}),$$

which is (7.5b).

Next fix an $i \in \{i, \ldots, p\}$. By expanding $1/(t-z)$ in a finite geometric series in terms of powers of $(z-a_i)/(t-a_i)$ we get

$$\frac{1}{t-z} = \sum_{j=0}^{n} \frac{(z-a_i)^j}{(t-a_i)^{j+1}} + \frac{(z-a_i)^{n+1}}{(t-a_i)^{n+1}(t-z)},$$

and hence

$$\hat{\psi}(z) = \sum_{j=0}^{n}(z-a_i)^j \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t-a_i)^{j+1}} + (z-a_i)^{n+1}\int_{-\infty}^{\infty} \frac{d\psi(t)}{(t-a_i)^{n+1}(t-z)}.$$

Thus we have

$$(7.9) \qquad \hat{\psi}(z) = \sum_{j=0}^{n} c_{j+1}^{(i)}(z-a_i)^j + h_n^{(i)}(z)\cdot(z-a_i)^{n+1},$$

where

$$(7.10) \qquad h_n^{(i)}(z) = \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t-a_i)^{n+1}(t-z)}.$$

It can easily be seen that $|t-z| \geq (t-a_i)\sin\delta$, hence

$$|h_n^{(i)}(z)| \leq \frac{1}{\sin\delta}\int_{-\infty}^{\infty} \frac{d\psi(t)}{|t-a_i|^{n+2}}$$

when $z \in R_{i,\delta}$. Note that

$$\int_{-\infty}^{\infty} d\psi(t)/|t-a_i|^{n+2} < \infty$$

since $\int_{-\infty}^{\infty} d\psi(t)/(t-a_i)^{n+2}$ is assumed to exist. Thus there exists a constant $s_{i,n}$ such that

$$(7.11) \qquad \left| \hat{\psi}(z) - \sum_{j=0}^{n} c_{j+1}^{(i)}(z-a_i)^j \right| \leq s_{i,n}|z-a_i|^{n+1}$$

when $z \in R_{i,\delta}$.

This shows that $\hat{\psi}(z)$ has the asymptotic expansions (7.5b). □

Theorem 7.2. *The following result holds for given* $\{c_j^{(i)}\}$, c_0: *If the distribution function* $\psi(t)$ *is such that its Stieltjes transform* $\hat{\psi}(z)$ *has the asymptotic expansions* (7.5a)–(7.5b), *then* $\psi(t)$ *is a solution of the moment problem* (7.1).

Proof. Again set $z = x + iy$. We then have

$$(7.12) \qquad \operatorname{Re}(iy\hat{\psi}(iy)) = \int_{-\infty}^{\infty} \frac{-y^2}{t^2 + y^2} \, d\psi(t).$$

It follows from (7.5b) that $iy\hat{\psi}(iy) \to -c_0$, and since c_0 is real, this implies

$$(7.13) \qquad \operatorname{Re}(iy\hat{\psi}(iy)) \to -c_0 \quad \text{as } y \to \infty.$$

The family $y^2/(t^2 + y^2)$ tends to 1 monotonically as y tends to ∞, and therefore

$$(7.14) \qquad \int_{-\infty}^{\infty} \frac{y^2}{t^2 + y^2} \, d\psi(t) \to \int_{-\infty}^{\infty} d\psi(t) \quad \text{as } y \to \infty.$$

From (7.12), (7.13), and (7.14) we conclude that

$$(7.15) \qquad \int_{-\infty}^{\infty} d\psi(t) = c_0.$$

Let k be a given index. We have

$$(7.16) \qquad \operatorname{Re}(\hat{\psi}(a_k + iy)) = \int_{-\infty}^{\infty} \frac{(t - a_k) \, d\psi(t)}{(t - a_k)^2 + y^2}.$$

It follows from (7.5a) that

$$(7.17) \qquad \hat{\psi}(a_k + iy) \to c_1^{(k)} \quad \text{as } y \to 0.$$

Since $c_1^{(k)}$ is real, this implies

$$(7.18) \qquad \operatorname{Re}(\hat{\psi}(a_k + iy)) \to c_1^{(k)} \quad \text{as } y \to 0.$$

By considering the cases $t < a_k$ and $t > a_k$ separately, we see that

$$\frac{t - a_k}{(t - a_k)^2 + y^2}$$

tends to $1/(t - a_k)$ monotonically when y tends to zero, and therefore

$$(7.19) \qquad \int_{-\infty}^{\infty} \frac{(t - a_k) \, d\psi(t)}{(t - a_k)^2 + y^2} \to c_1^{(k)} \quad \text{as } y \to 0.$$

It follows from (7.16), (7.18), and (7.15) that

$$(7.20) \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{t - a_k} = c_1^{(k)}.$$

Now assume that for some n it is known that

$$(7.21) \qquad \int_{-\infty}^{\infty} \frac{d\psi(t)}{(t - a_k)^{j+1}} = c_{j+1}^{(k)}, \qquad j = 0, \dots, n.$$

It follows from (7.5a) that

(7.22)
$$\frac{1}{(z-a_k)^{n+1}}\left(\hat{\psi}(z) - \sum_{j=0}^{n} c_{j+1}^{(k)}(z-a_k)^j\right) \to c_{n+2}^{(k)}$$

when $z \to a_k$ in $R_{k,\delta}$. This means that

$$\frac{1}{(z-a_k)^{n+1}} \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \sum_{j=0}^{n}\frac{(z-a_k)^j}{(t-a_k)^{j+1}}\right) d\psi(t)$$

tends to $c_{n+2}^{(k)}$ when z tends to a_k in $R_{k,\delta}$. By summing the finite geometric series under the integral sign we can write this as

(7.23)
$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{(t-z)(t-a_k)^{n+1}} \to c_{n+2}^{(k)}.$$

Since $c_{n+2}^{(k)}$ is real, the real part of the left-hand side of (7.23) tends to $c_{n+2}^{(k)}$, hence

(7.24)
$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{[(t-a_k)^2+y^2](t-a_k)^n} \to c_{n+2}^{(k)} \quad \text{as } y \to 0.$$

Again $1/[(t-a_k)^2+y^2](t-a_k)^n$ tends to $1/(t-a_k)^{n+2}$ monotonically for $t < a_k$ and $t > a_k$ when y tends to zero, and therefore

(7.25)
$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{[(t-a_k)^2+y^2](t-a_k)^n} \to \int_{-\infty}^{\infty}\frac{d\psi(t)}{(t-a_k)^{n+2}}.$$

Thus by (7.24) and (7.25) we have

(7.26)
$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{(t-a_k)^{n+2}} = c_{n+2}^{(k)}.$$

It follows by induction that all the moments of $\psi(t)$ exist, and that $\psi(t)$ is a solution of the moment problem. $\square$

Theorem 7.3. *Let a positive collection of power series (2.19) be given. Then the Stieltjes transforms of the solutions of the moment problem (7.1) are exactly those analytic functions $F(z)$ in H_+^0 for which (7.5b) is satisfied and for which $F(z) \in \Delta(z)$ for all $z \in H_+^0$. Equivalently the Stieltjes transforms are those analytic functions $F(z)$ in H_+^0 which have the asymptotic expansions (7.5). Such functions exist. The moment problem has a unique solution iff the limit point situation obtains.*

Proof. Let $F(z)$ be a Nevanlinna function with the series (7.2) as asymptotic expansions. From the expansion $F(z) = -c_0 z^{-1} + o(z^{-1})$ it follows in particular that for y real,

(7.27)
$$\sup_{y \geq 1} |iy F(iy)| < \infty.$$

Then there exists a distribution function $\psi(t)$ such that $F(z) = \hat{\psi}(z)$ (see, e.g., [1, pp. 92–94]). Thus all Nevanlinna functions with the series (7.2) as asymptotic expansions are Stieltjes transforms of distribution functions. The theorem now follows by combining Theorems 6.1–6.4 and Theorems 7.1–7.2. □

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