ASYMPTOTIC INTEGRATIONS OF NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The linear differential equation (1) $(r(t)x')' + (f(t) + q(t))x = 0$ is viewed as a perturbation of the equation (2) $(r(t)y')' + f(t)y = 0$, where $r > 0$, f and q are real-valued continuous functions. Suppose that (2) is nonoscillatory at infinity and y_1, y_2 are principal, nonprincipal solutions of (2), respectively. Adapted Riccati techniques are used to obtain an asymptotic integration for the principal solution x_1 of (1). Under some mild assumptions, we characterize that (1) has a principal solution x_1 satisfying $x_1 = y_1(1+o(1))$. Sufficient (sometimes necessary) conditions under which the nonprincipal solution x_2 of (1) behaves, in three different degrees, like y_2 as $t \to \infty$ are also established.

1. INTRODUCTION

Consider the second order differential equation

$$(1.1) \qquad (r(t)x')' + (f(t) + q(t))x = 0$$

as a perturbation of the equation

$$(1.2) \qquad (r(t)y')' + f(t)y = 0,$$

where r, f, q, x and y are continuous and real-valued functions on $[a, \infty)$, $r > 0$. Suppose that (1.2) is nonoscillatory at infinity. It is well known [3, p. 355] that (1.2) has a *principal* solution y_1 which is essentially unique (up to a constant factor) such that

$$(1.3) \qquad \int^{\infty} dt/(ry_1^2) = \infty$$

and for any solution y_2 linearly independent of y_1,

$$(1.4) \qquad y_1/y_2 \to 0, \qquad t \to \infty.$$

The solution y_2 is called a *nonprincipal* solution. Without loss of generality, we assume that $y_1(t) > 0$ for $t \geq a$. Set

$$(1.5) \qquad g(t) = \int_a^t ds/(ry_1^2);$$

then $y_1 g$ is a nonprincipal solution. We will assume throughout this paper that $y_2(t) = y_1(t)[g(t) - g(b)]$ for some $b \geq a$ and hence y_2 is eventually positive and satisfies

$$(1.6) \qquad\qquad r(y_1 y_2' - y_1' y_2) = 1.$$

We will be interested in finding conditions on q so that solutions of (1.1) will behave asymptotically like those of (1.2). Some remarks about known results on this seem to be appropriate here.

In what follows, L denotes the set of all continuous functions z on $[b, \infty)$ for some $b \geq a$ such that the improper integrals $\int_b^\infty z \, dt$ converge (at least conditionally), $z^*(t) = \sup_{s \geq t} |z(s)|$ whenever $z(t)$ is bounded for $t \geq b$ and

$$\overline{z}(t, b) = [g(t) - g(b)]^{-1} \int_b^t g' z \, ds,$$

the *weighted average* of z over $[b, t]$. We also write $\overline{z}(t) = \overline{z}(t, a)$ for short. The Landau notations "O" and "o" refer to behavior as $t \to \infty$.

Hartman and Wintner considered the equation (1.1) where q may be complex-valued and obtained

Theorem A [3, p. 379]. *If*

$$(1.7) \qquad\qquad y_1^2 q \in L$$

and

$$(1.8) \qquad\qquad \int^\infty g' Q^* dt < \infty,$$

where $Q(t) = \int_t^\infty y_1^2 q \, ds$, then (1.1) has solutions x_1 and x_2 such that $x_i = y_i(1 + o(1))$ and $x_i'/x_i - y_i'/y_i = o((ry_1 y_2)^{-1})$ for $i = 1, 2$.

Theorem A was improved by Trench:

Theorem B [6, Theorem 2]. *Suppose that*

$$(1.9) \qquad\qquad g y_1^2 q \in L$$

and $Q_1^* = O(\phi)$, where $Q_1(t) = \int_t^\infty g y_1^2 q \, ds$, $\phi(t) > 0$ and $\phi(t) \to 0$ monotonically as $t \to \infty$. If

$$(1.10) \qquad\qquad \int^\infty g' |Q| \phi \, dt < \infty,$$

and

$$(1.11) \qquad\qquad \limsup_{t \to \infty} (\phi(t))^{-1} \int_t^\infty g' |Q| \phi \, ds = A < 1/3,$$

then (1.1) has solutions x_1 and x_2 such that $x_1 = y_1(1 + O(\phi))$, $(x_1/y_1)' = O(\phi g'/g)$, $x_2 = y_2(1 + O(\overline{\phi}))$ and $(x_2/y_2)' = O(\overline{\phi} g'/g)$.

Note that $(1.9) \Rightarrow (1.7)$ but (1.7)–$(1.8) \Rightarrow (1.9)$–(1.11) (see [6]). Šimša [5] proved that $1/3$ can be replaced by 1 in (1.11) and obtained

Theorem C [5, Theorem 1]. *If* (1.9) *holds and*

$$(1.12) \qquad \int^{\infty} g' |Q_1| \phi/g \, dt < \infty,$$

then (1.1) *has a solution* x_1 *such that* $x_1 = y_1[1 + O(\phi + \int_t^{\infty} g' |Q_1| \phi/g \, ds)]$ *and* $(x_1/y_1)' = O(\phi g'/g)$.

The approach to proving Theorems B and C in [6,5] is mainly based on finding the principal solutions x_1 of (1.1) by a fixed point theorem. Instead a different method, the Riccati transformation, will be used here in the case where q is assumed to be real-valued; this method seems to be advantageous to establishing a rather accurate asymptotic integration for the principal solution x_1 of (1.1) which is asymptotic to y_1. For the nonprincipal solutions, more asymptotic properties, of interest in their own right, should be specified as follows:

$$(1.13) \qquad x_2 = y_2(1 + o(1)),$$

$$(1.14) \qquad x_2'/x_2 - y_2'/y_2 = \Delta = o((ry_1 y_2)^{-1}) \quad \text{and} \quad \Delta \in L,$$

$$(1.15) \qquad x_2 - y_2 = y_1(C + o(1)),$$

where C is a constant. Among them, (1.13) is obviously the weakest but (1.14) and (1.15) are independent of each other (Example 4.2). Motivations of investigating asymptotic properties of nonprincipal solutions in greater detail as above can also be found in [1, 2], where other kinds of differential equations were discussed.

In this paper, we will obtain sufficient (sometimes necessary) conditions on q under which (1.1) has a principal solution x_1 satisfying $x_1 = y_1(1 + o(1))$ and a nonprincipal solution x_2 satisfying each of the three asymptotic properties (1.13)–(1.15).

2. GENERALIZED RICCATI EQUATIONS AND NONOSCILLATION

Let x be a nonoscillatory solution of (1.1), $x(t) \neq 0$ for $t \geq b$ for some $b \geq a$. Let $u = rx'/x$. Then u satisfies

$$u' + u^2/r + f + q = 0, \qquad t \geq b.$$

Set $u = \eta + v$, where $\eta = ry_1'/y_1$. Then v satisfies

$$v' + 2\eta v/r + v^2/r + q = 0$$

or

$$(2.1) \qquad (y_1^2 v)' + y_1^2 v^2/r + y_1^2 q = 0, \qquad t \geq b.$$

Equation (2.1) is called the *generalized Riccati equation* for (1.1) with respect to (1.2). Where $f \equiv 0$ and $y_1 \equiv 1$, as in [3, pp. 331–332; 7, 8] (2.1) reduces to the ordinary Riccati equation. It is clear that (1.1) is nonoscillatory at infinity if and only if (2.1) has a solution $v(t)$ for $t \geq b$ for some $b \geq a$. Moreover, we can establish the following result involving a generalized Riccati integral equation for (1.1) with respect to (1.2).

Lemma 2.1. *If*

$$(2.2) \qquad \liminf_{t\to\infty}[g(t)]^{-1}\int_a^t g'(s)\left(\int_a^s y_1^2 q\,d\sigma\right)ds > -\infty,$$

then (1.1) *is nonoscillatory if and only if for some* $b \ge a$, *the limit*

$$(2.3) \qquad \lim_{t\to\infty}[g(t)-g(b)]^{-1}\int_b^t g'(s)\left(\int_b^s y_1^2 q\,d\sigma\right)ds = C_b$$

exists (finite) and there exists a continuous function $v(t)$ *for* $t \ge b$ *such that* $y_1^2 v^2/r \in L$ *and*

$$(2.4) \qquad y_1^2 v = Q(t) + \int_t^\infty y_1^2 v^2/r\,ds,$$

where

$$(2.5) \qquad Q(t) = C_b - \int_b^t y_1^2 q\,ds.$$

Note that $y_1^2 q \in L \Rightarrow (2.3) \Rightarrow (2.2)$, but in general the converses are not true. However, if (1.1) is nonoscillatory, then $(2.2) \Leftrightarrow (2.3)$. We also observe that if (2.3) holds for some $b_0 \ge a$ then it holds for all $b \ge a$. Letting $t \to \infty$ in the following identity

$$[g(t)-g(b)]^{-1}\int_b^t g'(s)\int_b^s y_1^2 q\,d\sigma\,ds$$

$$= [g(t)-g(b)]^{-1}\left\{\int_a^t g'(s)\int_a^s y_1^2 q\,d\sigma\,ds\right.$$

$$\left. -\int_a^b g'(s)\int_a^s y_1^2 q\,d\sigma\,ds - \int_b^t g'ds\int_a^b y_1^2 q\,d\sigma\right\}$$

gives $C_b = C_a - \int_a^b y_1^2 q\,ds$. Therefore, the function $Q(t)$ given by (2.5) is independent of the values of b. Thus in the sequel, by saying that $Q(t)$ exists (finite) we mean that (2.3) holds and $Q(t)$ is defined by (2.5) for $b \ge a$. In particular, if $y_1^2 q \in L$ then $Q(t) = \int_t^\infty y_1^2 q\,ds$.

We next introduce a generaltized Riccati equation of the second level. Suppose that (2.2) holds and (1.1) is nonoscillatory. Then $Q(t)$ exists (finite) and $y_1^2 v^2/r \in L$. Hence, if we let

$$(2.6) \qquad w(t) = \int_t^\infty y_1^2 v^2/r\,ds, \qquad t \ge b,$$

then w satisfies

$$(2.7) \qquad w' + 2g'Qw + g'(w^2 + Q^2) = 0, \qquad t \ge b.$$

Set $G(s,t) = \exp(2\int_t^s g'Q\,d\sigma)$ and

$$(2.8) \qquad P(t) = \int_t^\infty g'Q^2 G(s,t)ds$$

whenever the integrands make sense and the integrals converge. We now have

Lemma 2.2. *Let* (2.2) *hold. Then* (1.1) *is nonoscillatory if and only if* $Q(t)$ *and* $P(t)$ *exist (finite) and there exists a continuous function* $w(t)$, $t \geq b$ *for some* $b \geq a$, *such that*

$$(2.9) \qquad w(t) = P(t) + \int_t^\infty g'w^2 G(s, t)\,ds, \qquad t \geq b.$$

Remark. If (2.2) holds and x is a nonoscillatory solution of (1.1), then the function $w = y_1^2 v - Q$ must satisfy (2.9), where $v = rx'/x - \eta$.

The proofs of Lemmas 2.1 and 2.2 are omitted because they can be obtained either by employing the same arguments used in [7, 8] for ordinary Riccati equations or by letting $x = y_1 z$, $g(t) = s$ and then applying the known results on ordinary Riccati equations to the equation

$$d^2 z/ds^2 + ry_1^4 qz = 0.$$

The integral equation (2.9) has the advantage of involving the nonnegative functions P and w but no individual values of q. One can easily make use of (2.9) to estimate the function w and hence the nonoscillatory solution x of (1.1). This thought will partially be embodied in the next nonoscillation result (Theorem 2.4). Before giving this result, we introduce a lemma first.

Lemma 2.3. *Suppose that* $Q(t)$ *exists (finite) and* $g'Q \in L$. *Then*

$$(A_1) \qquad \psi(t) = \int_t^\infty gg'Q^2\,ds < \infty$$

if and only if $P(t)$ *exists (finite) and* $g'P \in L$.

Proof. Since $g'Q \in L$, there is an $M > 1$ such that

$$(2.10) \qquad M^{-1} \leq G(s, t) \leq M, \qquad t, s \geq a.$$

For any $T > t$, we have

$$\int_t^T g'P\,ds = \int_t^T g'(s) \int_s^\infty g'Q^2 G(\sigma, s)\,d\sigma\,ds$$

$$\leq M \int_t^T g'(s) \int_s^\infty g'Q^2\,d\sigma\,ds$$

$$= M \left(\int_t^T (g(s) - g(t))g'(s)Q^2(s)\,ds + (g(T) - g(t)) \int_T^\infty g'Q^2\,ds \right)$$

$$\leq M \int_t^\infty gg'Q^2\,ds,$$

and

$$\int_t^T g'P\,ds \geq M^{-1} \int_t^T g'(s) \int_s^\infty g'Q^2\,d\sigma\,ds$$

$$\geq M^{-1} \left(\int_t^T gg'Q^2\,ds - g(t) \int_t^\infty g'Q^2\,ds \right).$$

Then Lemma 2.3 follows immediately.

Theorem 2.4. *If $Q(t)$ exists (finite), $g'Q \in L$ and (A_1) holds, then (1.1) is nonoscillatory.*

Proof. Our proof proceeds on the basis of constructing a solution of (2.9).

By Lemma 2.3, (A_1) implies that P exists and $g'P \in L$. Let $b \geq a$ be so large that for all $t \geq b$,

$$(2.11) \qquad \int_t^\infty g'P ds < \frac{1}{4}M^{-3}.$$

Let X be the Banach space of all bounded, continuous functions on $[b, \infty)$ equipped with the norm given by $\|z\| = \sup_{t \geq b}\{|z(t)|\}$, $z \in X$. Let $S = \{z \in X : P(t) \leq z(t) \leq 2P(t), \ t \geq b\}$. Then S is a closed convex subset of X. Define an operator $T : S \to X$ by

$$(2.12) \qquad (Tz)(t) = P(t) + \int_t^\infty g'z^2 G(s, t)ds$$

for any $z \in S$. Since $g'Q \in L$, from (2.10), we have for $s \geq t$,

$$(2.13) \qquad \begin{aligned} P(s) &= \int_s^\infty g'Q^2 G(\sigma, s)d\sigma \leq M \int_s^\infty g'Q^2 d\sigma \\ &\leq M \int_t^\infty g'Q^2 d\sigma \leq M^2 \int_t^\infty g'Q^2 G(s, t)ds = M^2 P(t). \end{aligned}$$

From (2.12), (2.13) and (2.11),

$$P(t) \leq (Tz)(t) \leq P(t) + 4M^3 P(t) \int_t^\infty g'P ds \leq 2P(t), \qquad t \geq b.$$

Thus, $T : S \to S$. On the other hand, for every $t \geq b$, z_1, $z_2 \in S$, we have

$$\begin{aligned} |(Tz_1)(t) - (Tz_2)(t)| &= \left|\int_t^\infty g'(z_1 + z_2)(z_1 - z_2)G(s, t)ds\right| \\ &\leq 4M\|z_1 - z_2\| \int_t^\infty g'P ds \leq M^{-2}\|z_1 - z_2\|, \end{aligned}$$

which shows that T is a contraction. Hence, there exists a unique fixed point w of T in S, which is a solution of (2.9) on $[b, \infty)$. This completes the proof of Theorem 2.4.

It should particularly be noted at this point that under the hypotheses of Theorem 2.4, (2.9) has a unique solution w on $[b, \infty)$ satisfying

$$(2.14) \qquad P(t) \leq w(t) \leq 2P(t).$$

Solving $rx'/x = ry_1'/y_1 + (Q + w)/y_1^2$ for x we get a solution of (1.1)

$$(2.15) \qquad x_1(t) = y_1(t)\exp(-H(t) - W(t)),$$

where $W(t) = \int_t^\infty g'w ds$ and

$$(2.16) \qquad H(t) = \int_t^\infty g'Q ds.$$

Obviously, x_1 is a principal solution of (1.1). The asymptotic integration (2.15) of x_1 with w satisfying (2.14) will play a critical role in §3.

3. Main results

Under some mild assumptions, we first give a necessary and sufficient condition for (1.1) to have a principal solution $x_1 = y_1(1 + o(1))$ and a nonprincipal solution x_2 satisfying (1.13).

Theorem 3.1. *Let $Q(t)$ exist (finite) and $g'Q \in L$.*
   (i) *If (A_1) holds, then (1.1) has a principal solution x_1 satisfying*

$$(3.1) \qquad x_1 = y_1[1 - H + O(H^2 + \psi)],$$

$$(3.2) \qquad (x_1/y_1)' = g'Q[1 - H + O(H^2 + \psi)] + O(g'P)$$

and a nonprincipal solution x_2 satisfying

$$(3.3) \qquad x_2 = y_2[1 + O(\overline{H^*} + \overline{\psi})],$$

for some nonprincipal solution y_2 of (1.2).
   (ii) *Conversely, if there are solutions x of (1.1) and y of (1.2) such that $x/y \to 1$ as $t \to \infty$, then (A_1) holds.*

Proof. (i) If (A_1) holds, then by Theorem 2.4, (1.1) is nonoscillatory and (2.9) has a solution w satisfying (2.14) for $t \geq b$ for some $b \geq a$. Lemma 2.3 implies that $g'P \in L$ and, consequently, $g'w \in L$. Let x_1 be defined as in (2.15). It is then easy to verify that x_1 is a positive solution of (1.1) on $[b, \infty)$. From (2.10) and (2.14),

$$(3.4) \qquad \begin{aligned} W(t) &= \int_t^\infty g'w\,ds \leq 2\int_t^\infty g'P\,ds \leq 2M\int_t^\infty g'(s)\int_s^\infty g'Q^2\,d\sigma\,ds \\ &\leq 2M\int_t^\infty gg'Q^2\,ds = 2M\psi(t). \end{aligned}$$

Thus, (3.1) follows from (2.15) and (3.4). Since

$$(3.5) \qquad (x_1/y_1)' = (x_1/y_1)g'(Q + w),$$

(3.2) follows from (3.5), (3.1) and (2.14).

Since $\int^\infty dt/(rx_1^2) = \infty$, x_1 is a principal solution of (1.1). If we set $x_2 = x_1\int_b^t ds/(rx_1^2)$ and $y_2 = y_1[g(t) - g(b)]$, then x_2, y_2 are nonprincipal solutions of (1.1), (1.2), respectively. We note that $\overline{\psi}(t, b) = O(\overline{\psi}(t))$ and $\psi \leq \overline{\psi}$. From (2.15) and (3.1),

$$(3.6) \qquad \begin{aligned} x_2/y_2 &= (x_1/y_1)[g(t) - g(b)]^{-1}\int_b^t g'(y_1/x_1)^2\,ds \\ &= (x_1/y_1)[g(t) - g(b)]^{-1}\int_b^t g'[1 + 2H(1 + o(1)) + O(\psi)]\,ds \\ &= (x_1/y_1)[1 + O(\overline{H^*} + \overline{\psi})]. \end{aligned}$$

We then get (3.3) from (3.6) and (3.1).

(ii) Let x_1 and x_2 be principal and nonprincipal solutions of (1.1), respectively. Suppose that $x = Ax_1 + Bx_2$, $y = Cy_1 + Dy_2$ and $x/y \to 1$. Since neither x_1/y_2 nor x_2/y_1 can tend to a nonzero number, we may assume that either $B = D \neq 0$ or $B = D = 0$ and $A = C \neq 0$, and hence either $x_2/y_2 \to 1$ or $x_1/y_1 \to 1$. If $x_2/y_2 \to 1$, then

$$\lim_{t \to \infty} x_1/y_1 = \lim_{t \to \infty}(x_2/y_2) \int_t^\infty ds/(rx_2^2)\left[\int_t^\infty ds/(ry_2^2)\right]^{-1} = 1.$$

Thus, the assumption of (ii) is reduced to the case $x_1/y_1 \to 1$.

Assume that $x_1(t) > 0$, $t \geq b \geq a$. Let $w = y_1^2 v - Q$, where $v = rx_1'/x_1 - ry_1'/y_1$. Since (1.1) is nonoscillatory, by the remark after Lemma 2.2, w satisfies (2.9) and hence $0 \leq P \leq w$. Furthermore,

$$g'w = v/r - g'Q = x_1'/x_1 - y_1'/y_1 - g'Q = [\ln(x_1/y_1)]' - g'Q.$$

Since $g'Q \in L$ and $x_1/y_1 \to 1$, we have $g'w \in L$. As a result, $g'P \in L$. Thus (A_1) holds by the virtue of Lemma 2.3. The proof of Theorem 3.1 is complete.

Remark. In Theorem 3.1 (i), x_2 and y_2 have the same value zero at $t = b$. In case $y_2 = y_1[g(t) - g(c)]$ for some $c \neq b$, we still have

$$(3.7) \qquad\qquad x_2 = y_2[1 + O(g^{-1} + \overline{H^*} + \overline{\psi})].$$

Next, we give necessary and sufficient conditions for the nonprincipal solutions of (1.1) and (1.2) to have the property (1.14).

Theorem 3.2. *Suppose that $Q(t)$ exists (finite) and $g'Q \in L$.*
   (i) *If (A_1) holds, and*

$$(A_2) \qquad\qquad gQ \to 0 \text{ as } t \to \infty,$$

then (1.1) has a nonprincipal solution x_2 satisfying $x_2'/x_2 - y_2'/y_2 \in L$ and

$$(3.8) \qquad x_2'/x_2 - y_2'/y_2 = (ry_1y_2)^{-1}[(y_2/y_1)Q + O(\overline{H^*} + \overline{\psi})]$$

for some nonprincipal solution y_2 of (1.2).
   (ii) *Conversely, if (1.1) has a solution x such that $x'/x - y_2'/y_2 \in L$,*

$$(3.9) \qquad x'/x - y_2'/y_2 = o((ry_1y_2)^{-1}),$$

where y_2 is a nonprincipal solution of (1.2), then (A_1) and (A_2) hold.
Proof. (i) By Theorem 3.1, (1.1) has a solution x_1 satisfying (3.1) and (3.2) and a solution $x_2 = x_1 \int_b^t ds/(rx_1^2)$ satisfying (3.3) for $y_2 = y_1[g(t) - g(b)]$. Let $\Delta = x_2'/x_2 - y_2'/y_2$. Then $\Delta \in L$ because of (3.1). Moreover, from (3.5), (3.1) and (3.3) we have

$$(3.10) \qquad \begin{aligned} \Delta &= x_1'/x_1 + (rx_1x_2)^{-1} - y_1'/y_1 - (ry_1y_2)^{-1} \\ &= (x_1/y_1)'(x_1/y_1)^{-1} + [(y_1/x_1)(y_2/x_2) - 1](ry_1y_2)^{-1} \\ &= g'(Q + w) + (ry_1y_2)^{-1}O(\overline{H^*} + \overline{\psi}) \\ &= (ry_1y_2)^{-1}[(g(t) - g(b))(Q + w) + O(\overline{H^*} + \overline{\psi})]. \end{aligned}$$

Since

$$(3.11) \qquad gw \le 2gP \le 2Mg(t) \int_t^\infty g'Q^2 ds \le 2M \int_t^\infty gg'Q^2 ds = 2M\psi ,$$

we obtain (3.8) from (3.10).

(ii) Since $x'/x - y_2'/y_2 = [\ln(|x|/y_2)]' \in L$, integrating from b to ∞ gives that $|x(t)|/y_2(t)$ tends to a positive number as $t \to \infty$. Then (A_1) holds by Theorem 3.1(ii). Let $x = Ax_1 + Bx_2$, where x_1, x_2 are defined as in Theorem 3.1(i). From (3.9),

$$(3.12) \qquad \begin{aligned} x_2'/x_2 - y_2'/y_2 &= (x'/x - y_2'/y_2) + (x_2'/x_2 - x'/x) \\ &= o((ry_1y_2)^{-1}) + A[rx_2(Ax_1 + Bx_2)]^{-1}. \end{aligned}$$

If $B = 0$, then (3.1), (3.3) an (3.12) lead us to

$$(3.13) \qquad \begin{aligned} x_2'/x_2 - y_2'/y_2 &= o((ry_1y_2)^{-1}) + (rx_1x_2)^{-1} \\ &= (ry_1y_2)^{-1}[(y_1/x_1)(y_2/x_2) + o(1)] \\ &= (ry_1y_2)^{-1}[1 + o(1)]. \end{aligned}$$

Comparing (3.13) and (3.10) yields that $gQ \to 1$. Consequently,

$$\int^\infty gg'Q^2 dt = \int^\infty (gQ)^2 g'/g \, dt = \infty.$$

This contradicts (A_1), and hence $B \ne 0$. It then follows from (3.12) that

$$(3.14) \qquad x_2'/x_2 - y_2'/y_2 = (ry_1y_2)^{-1}[o(1) + O(g^{-1})].$$

A closer comparison between (3.14) and (3.10) gives that $gQ \to 0$, i.e. (A_2) holds. This completes the proof.

Remarks. (i) In the case where $r \equiv 1$ and $f \equiv 0$, Theorems 3.1 and 3.2 improve a result of Hartman and Wintner in [4] where a different method was used. The latter result was extended to a selfadjoint system of second order differential equations by the author in [2] using Riccati techniques.

(ii) The following lemma will show that our conditions used in Theorems 3.1 and 3.2 are weaker than the conditions (1.9), (1.10) used in Theorem B or (1.9), (1.12) used in Theorem C, provided that q is real-valued.

Lemma 3.3.

    (i) $gy_1^2q \in L \Leftrightarrow y_1^2q$, $g'Q \in L$ *and* (A_2);

    (ii) $g'QQ_1 \in L \Rightarrow (A_1)$;

    (iii) $g'Q_1^2/g \in L \Rightarrow (A_1)$.

Proof. Suppose that $gy_1^2q \in L$. As was indicated in [6], $gy_1^2q \in L \Rightarrow y_1^2q \in L$ and hence $Q(t) = \int_t^\infty y_1^2q \, ds$. Consequently,

$$gQ = -g \int_t^\infty Q_1'/g \, ds = Q_1 + g(t) \int_t^\infty Q_1 d((g(s))^{-1}) \to 0 \quad \text{as } t \to \infty.$$

Integration by parts produces

$$\int_t^T gy_1^2 q\,ds = g(t)Q(t) - g(T)Q(T) + \int_t^T g'Q\,ds.$$

Letting $T \to \infty$, we then have

(3.15) $$Q_1 = gQ + H,$$

that is, $g'Q \in L$. The sufficiency of the part (i) is clear. Therefore, the proof of (i) is complete.

It follows from (3.15) that

(3.16) $$\int_t^\infty g'QQ_1\,ds = \int_t^\infty gg'Q^2\,ds + \frac{1}{2}H^2,$$

(3.17) $$\int_t^\infty g'Q_1^2/g\,ds = \int_t^\infty gg'Q^2\,ds + H^2 + \int_t^\infty g'H^2/g\,ds.$$

We then get (ii) and (iii) in view of (3.16) and (3.17), respectively.

For (1.1) to have a solution x_2 satisfying the property (1.15), more assumptions are required.

(A_3) $$gg'Q \in L.$$

(A_4) $$\Psi(t) = \int_t^\infty g^2 g' Q^2\,ds < \infty.$$

The following result is similar to Lemma 3.3(i) and hence its proof is omitted.

Lemma 3.4. *If $Q(t)$ exists, then (A_3) holds if and only if $g'Q \in L$, $g'H \in L$ and $gH \to 0$ as $t \to \infty$, in which case*

(3.18) $$\int_t^\infty gg'Q\,ds = gH + R,$$

where $R(t) = \int_t^\infty g'H\,ds$.

It is evidently true that $(A_4) \Rightarrow (A_1)$. Meanwhile, we have more implications of (A_4).

Lemma 3.5. *If $Q(t)$ exists and (A_4) holds, then for any γ, $0 \le \gamma < \frac{1}{2}$, we have*

(3.19) $$\int^\infty g^\gamma g' |Q|dt < \infty.$$

Proof. By Schwarz's inequality, from (A_4) we have

$$\left(\int^\infty g^\gamma g'|Q|dt\right)^2 \le \int^\infty g^{2\gamma-2} g' dt \int^\infty g^2 g' Q^2 dt < \infty$$

for $\gamma < \frac{1}{2}$. This completes the proof.

Lemma 3.6. *If $Q(t)$ exists and $g'Q \in L$, then (A_4) holds if and only if $gg'P \in L$.*

The proof of Lemma 3.6 follows the steps of that of Lemma 2.3 and is therefore omitted.

Theorem 3.7. *Let* $Q(t)$ *exist (finite). If* (A_3) *and* (A_4) *hold, then* (1.1) *has a principal solution* x_1 *satisfying* (3.1) *and a nonprincipal solution* x_2 *such that*

$$(3.20) \qquad x_2 - y_2 = y_1[C - 2R - gH + O(H^* + \Psi)]$$

for some nonprincipal solution y_2 *of* (1.2) *and some constant* C.

Proof. Since $(A_4) \Rightarrow (A_1)$ and $(A_3) \Rightarrow g'Q \in L$ (Lemma 3.4), Theorem 3.1 provides a principal solution x_1 of (1.1) satisfying (2.15) and (3.1) for $t \geq b \geq a$. Set $x_2 = x_1 \int_b^t ds/(rx_1^2)$ and $y_2 = y_1[g(t) - g(b)]$. From (2.15),

$$x_2 = y_1 e^{-H-W} \int_b^t g' e^{2H+2W} ds$$

and hence

$$(3.21) \quad (x_2 - y_2)/y_1 = (e^{-H-W} - 1)\int_b^t g'ds + e^{-H-W}\int_b^t g'(e^{2H+2W} - 1)ds.$$

From (3.4) and $gW \leq 2Mg\psi \leq 2M\Psi$, we have

$$(3.22) \qquad (e^{-H-W} - 1)\int_b^t g'ds = -gH + O(gH^2 + \Psi) + O(|H|).$$

We claim that $g'(e^{2H+2W} - 1) \in L$. To this end, let $e^{2H} = 1 + 2H + 2Hh$. Then $h = O(H)$ and

$$(3.23) \qquad g'(e^{2H+2W} - 1) = 2g'H + 2g'Hh + g'O(\psi).$$

By Lemma 3.4, $(A_3) \Rightarrow g'H \in L$ and $gH \to 0$. Also, $(A_4) \Rightarrow g'\psi \in L$ and $\int_t^\infty g'\psi ds = O(\Psi(t))$. It then remains to show that $g'Hh \in L$. In fact, since $(Hh)' = -2g'QH(1 + h)$, $gH \to 0$ and $g'|Q| \in L$ (Lemma 3.5), it follows after some manipulation that

$$
(3.24) \quad
\begin{aligned}
\int_t^\infty g'Hh\,ds &= -gHh + 2\int_t^\infty gg'QH\,ds + 2\int_t^\infty g'QgHh\,ds \\
&= -gHh + gH^2 + \int_t^\infty g'H^2 ds + o(gH^2) = O(gH^{*2}).
\end{aligned}
$$

This asserts our claim. Now, if we let $C = \int_b^\infty g'(e^{2H+2W} - 1)ds$, then by (3.21)–(3.24) we are led to

$$
\begin{aligned}
(x_2 - y_2)/y_1 &= -gH + O(gH^2 + \Psi) + e^{-H-W}[C - 2R + O(gH^{*2} + \Psi)] \\
&\quad + O(|H|) \\
&= -gH + O(gH^2 + \Psi) + C - CH + O(H^2 + \Psi) \\
&\quad - 2R + O(gH^{*2}) + O(|H|) \\
&= C - gH - 2R + O(H^* + \Psi),
\end{aligned}
$$

which proves (3.20) and completes the proof.

In view of Theorem 3.7, if (A_3) and (A_4) hold, then for any solution x of (1.1) there is a unique solution y of (1.2) for which $x - y = o(y_1)$. In fact, if

$x = Ax_1 + bx_2$, then $y = (A + BC)y_1 + By_2$, where C is the constant in (3.20). Even so, in case y_1 is unbounded, x is still in the usual sense not asymptotic to y. In general, if y_1 is unbounded, stronger conditions than (A_3) and (A_4) are required so that for any given solution x of (1.1), (1.2) has a solution y such that $x - y = o(1)$. But we will not discuss this problem here. Another thing that should be mentioned is that we have not assumed (A_2) in Theorem 3.7 and therefore we are not able to ascertain the property (1.14) of the derivatives of nonprincipal solutions. However, this property can be obtained by assuming (A_2) additionally (see Example 4.2).

4. Examples

To indicate the scope of our results in §3, we give two examples in this section.

Example 4.1. Trench [6] and Šimša [5] have considered the equation

$$(4.1) \qquad x'' + K[t^{-\beta}(\ln t)^{-\alpha}\sin t]x = 0$$

as a perturbation of $y'' = 0$, where K, α, $\beta \neq 0$, $y_1 = 1$ and $y_2 = g = t$. Here we assume that K is a real constant. If $\beta = 1$, then Theorem B does not apply unless $\alpha \geq 1$ and Theorem C implies that (4.1) has a solution x_1 satisfying $x_1 = 1 + O((\ln t)^{-\alpha} + (\ln t)^{1-2\alpha})$, $x_1' = O(t^{-1}(\ln t)^{-\alpha})$, provided $\alpha > \frac{1}{2}$. For $\alpha > \frac{1}{2}$,

$$Q(t) = Kt^{-1}(\ln t)^{-\alpha}[\cos t + O(t^{-1})],$$
$$H(t) = -Kt^{-1}(\ln t)^{-\alpha}[\sin t + O(t^{-1})],$$
$$P(t) = O(t^{-1}(\ln t)^{-2\alpha}),$$
$$\psi(t) = O((\ln t)^{1-2\alpha}) = \overline{\psi}(t), \quad \overline{H^*} = o(\overline{\psi}),$$

and (A_1), (A_2) hold if and only if $\alpha > \frac{1}{2}$. Then Theorems 3.1 and 3.2 imply that if and only if $\alpha > \frac{1}{2}$, the equation (4.1) has two solutions x_1 and x_2 satisfying

$$x_1 = 1 + Kt^{-1}(\ln t)^{-\alpha}\sin t + O((\ln t)^{1-2\alpha}),$$
$$x_1' = Kt^{-1}(\ln t)^{-\alpha}\cos t + O(t^{-1}(\ln t)^{-2\alpha}),$$
$$x_2 = t[1 + O((\ln t)^{1-2\alpha})],$$
$$x_2' = 1 + K(\ln t)^{-\alpha}\cos t + O((\ln t)^{1-2\alpha}).$$

If $\beta = \frac{3}{2}$, $\alpha > \frac{1}{2}$, then (A_4) holds, $\Psi(t) = O((\ln t)^{1-2\alpha})$, $gH = o(\Psi)$ and $R = o(\Psi)$. Thus, Theorem 3.7 implies that

$$x_2 - t = C + O((\ln t)^{1-2\alpha}),$$

where C is a constant. Finally, if $\beta = 3$ and α is an arbitrary number, then $\Psi(t) = O(t^{-3}(\ln t)^{-2\alpha})$, $H^* = O(t^{-3}(\ln t)^{-\alpha})$ and so

$$x_2 - t = C + Kt^{-2}(\ln t)^{-\alpha}[\sin t + 2t^{-1}\cos t] + O(t^{-3}(\ln t)^{-\gamma}),$$

where $\gamma = \min\{\alpha, 1 - 2\alpha\}$.

Example 4.2. Let $\omega_n(t) = 4(-1)^n \sin(n\pi t)$ for $n \le t \le n+(1/n)$ and $\omega_n(t) = 0$ otherwise, $n = 1, 2, \dots$. Let $Q(t) = t^{-\beta}(\ln t)^{-\alpha} \sum_{n=1}^{\infty} \omega_n(t)$, $\beta > 0$ and $q = -Q'$. Consider the equation

$$(4.2) \qquad\qquad\qquad x'' + q(t)x = 0.$$

Observing that $\int_1^{\infty} \omega_n^2 dt = 6/n$, $tQ^2(t) = t^{1-2\beta}(\ln t)^{-2\alpha} \sum \omega_n^2(t)$, we can easily see that (A$_1$) holds if $\beta > \frac{1}{2}$, in which case Theorem 3.1 says that (4.2) has solutions $x_1 = 1 + o(1)$, $x_2 = t + o(t)$. But (A$_2$) does not hold unless $\beta > 1$ or $\beta = 1$, $\alpha > 0$. Thus, either of Theorems B and C does not apply for $\beta < 1$, because $tq \notin L$ by Lemma 3.3(i). If $\beta > 1$ or $\beta = 1$ and $\alpha > \frac{1}{2}$, then (A$_4$) and (A$_2$) hold and so $x_2 - t \to$ constant and $x_2' \to 1$ as $t \to \infty$.

However if we choose $Q(t) = t^{-\beta} \sum_{n=1}^{\infty} \omega_{n^2}(t)$ instead, then (A$_4$) holds for $\beta > \frac{1}{2}$ while (A$_2$) does not hold unless $\beta > 1$. In other words, for $\frac{1}{2} < \beta \le 1$, although every solution x of (4.2) is asymptotic to a certain straight line $A + Bt$ ($x(t) = A + Bt + o(1)$), the derivative of x diverges as $t \to \infty$. This example supports such a fact that the asymptotic property (1.14) does not have to do with the property (1.15) and therefore both of them should be classified.

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