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TERMS IN THE SELBERG TRACE FORMULA FOR $SL(3, \mathscr{Z})\backslash SL(3, \mathscr{R})/SO(3, \mathscr{R})$ ASSOCIATED TO EISENSTEIN SERIES COMING FROM A MINIMAL PARABOLIC SUBGROUP

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ABSTRACT. In this paper we compute the contribution to the trace formula for $SL(3, \mathscr{Z})$ of the integrals associated to inner products of Eisenstein series. We show these reduce to corresponding integrals for a lower rank trace formula plus a few residual terms.

The context of this paper is the Selberg trace formula for $SL(3, \mathscr{Z})$, a tool which we hope to use to study nonholomorphic cusp forms for $SL(3, \mathscr{Z})$. Work on the general trace formula is mostly due to Arthur [1, 2], but remains inexplicit compared to the kinds of expressions we get for $SL(2, \mathscr{Z})$, for example. A lot of the problem of making things explicit revolves around the computation of various truncated orbital integrals, described for $SL(3, \mathscr{Z})$ in [6, 7], and the computation of inner products of various sorts of Eisenstein series. For $SL(3, \mathscr{Z})$ there are fundamentally two types of Eisenstein series, those described in [8], and whose inner products are computed there also, and the ones described in this paper.

We want to compute the trace of an integral operator L_k on the discrete joint spectrum of invariant differential operators on $SL(3, \mathscr{Z})\backslash SL(3, \mathscr{R})/SO(3, \mathscr{R})$.

$$L_K(f) = \int_{\mathscr{F}} K(z, w) f(z)\, d\mu(z). \tag{1}$$

Here, $d_\mu(z)$ is the group invariant measure on $SL(3, \mathscr{R})/SO(3, \mathscr{R})$ and $\mathscr{F}$ is the fundamental region for the action of $SL(3, \mathscr{Z})$. The kernel K is given by

$$K(z, w) = \sum_{\gamma \in SL(3, \mathscr{Z})} k(z^{-1}\gamma w),$$

where k is $SO(3, \mathscr{R})$ bi-invariant and z, w are coset representatives of zK, wK in $SL(3, \mathscr{R})/SO(3, \mathscr{R})$, $(K = SO(3, \mathscr{R}))$.

Typically, L_k has some continuous spectrum and we must adjust by this to get a trace class operator. The continuous spectrum is spanned by a collection of Eisenstein series, to be described next.

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Let P_1 be a maximal parabolic subgroup of $SL(3, \mathscr{R})$:

$$P_1 = \left\{ z \in SL(3, R) \,\middle|\, z = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right) \right\},$$

and put coordinates

$$\begin{aligned} zK &= \begin{pmatrix} y^{1/6}u^{1/2} & y^{1/6}u^{-1/2}v & x_1 y^{-1/3} \\ 0 & y^{1/6}u^{-1/2} & x_2 y^{-1/3} \\ 0 & 0 & y^{-1/3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/6} & 0 & 0 \\ 0 & y^{1/6} & 0 \\ 0 & 0 & y^{-1/3} \end{pmatrix} \begin{pmatrix} u^{1/2} & u^{-1/2}v & 0 \\ 0 & u^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then we notice that

$$\begin{pmatrix} u^{1/2} & u^{-1/2}v \\ 0 & u^{-1/2} \end{pmatrix}$$

is a coset representative for $\tilde{z} \in SL(2, \mathscr{R})/SO(2, \mathscr{R})$ and we can think of $v + iu$ as a point in the Poincaré upper half plane. If we take an automorphic function $\psi(v+iu)$ for $SL(2, \mathscr{Z})$ acting on $SL(2, \mathscr{R})/SO(2, \mathscr{R})$ then $f(z) = \psi(v + iu)$ is invariant under $SL(3, \mathscr{Z}) \cap P_1 = \Gamma_{1,\infty}$. So we can construct an automorphic function for $\Gamma = SL(3, \mathscr{Z})$ by summing

$$\sum_{\gamma \in \Gamma_{1,\infty} \backslash \Gamma} f(\tilde{z}(\gamma z)) y(\gamma z)^s,$$

whenever it converges. If we want to find functions which are eigenfunctions of the invariant differential operators we choose f to be a Maass wave form on $SL(2, \mathscr{Z}) \backslash SL(2, \mathscr{R})/SO(2, \mathscr{R})$. If we choose f to be a cusp form φ we get the family of Eisenstein series whose inner products are computed in [8]. If we choose f to be an Eisenstein series for $SL(2, \mathscr{Z})$ we get another family of Eisenstein series for $SL(3, \mathscr{Z})$, and those are the ones we would like to consider here. We will write:

$$\psi(z) = \sum_{\gamma \in \Gamma_{1,\infty} \backslash \Gamma_\infty} E(v(\gamma z) + iu(\gamma z), t) y(\gamma z)^s = E(z, s, t). \tag{2}$$

These series were introduced in a slightly different form by Langlands [4], and have been studied extensively by Arthur, [1, 2]. The above definition gives a series which converges for $\operatorname{Re} s$ sufficiently large, which is due to Imai and Terras [5]. Venkov [9] is the source of basic information on Fourier expansions and other convergence properties used in this paper.

Another way to write $E(z, s, t)$ is the sum

$$E(z, s, t) = \sum_{\Gamma_{0,\infty} \backslash \Gamma} u(\gamma z)^t y(\gamma z)^s, \tag{3}$$

where $\Gamma_{0,\infty} = SL(3, \mathscr{Z}) \cap P_0$ and

$$P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in SL(3, \mathscr{R}) \right\}.$$

The two expressions give the same function whenever both converge and (3) converges in the region

$$3\,\mathrm{Re}\,s - \mathrm{Re}\,t > 2 \quad \text{and} \quad \mathrm{Re}\,t > 1,$$

with analytic continuation elsewhere and poles on the lines:

$$t = 1, \qquad 3s - t = 2, \qquad 3s + t = 3.$$

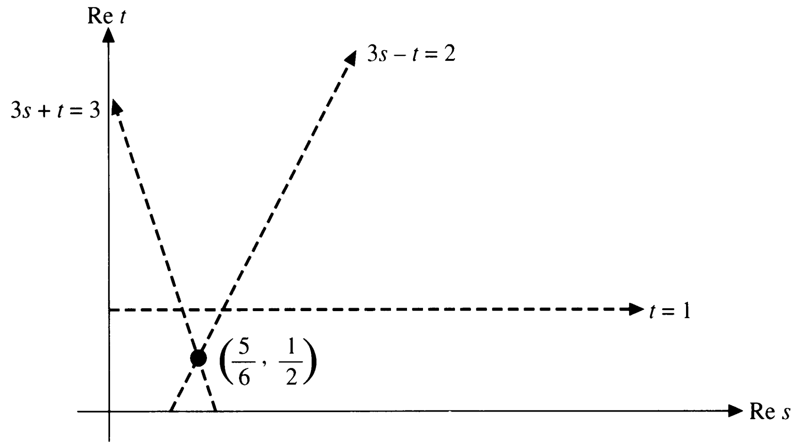

FIGURE 1

If we remain in the domain of convergence, Gelfand and Piatetskii-Shapiro, [3] show that the projection of L_k onto the complement of the cuspidal part of the spectrum can be obtained by an integral of K against these various Eisenstein series. As we move the series to $\mathrm{Re}\,s = \mathrm{Re}\,t = 1/2$, this property is lost.

All three lines are equivalent under the functional equations to $\mathrm{Re}\,t = 1$ (or 0). Here is the full set of functional equations for these series:

(a) $\quad c(1-t)E(z, s, t) = E(z, s, 1-t).$

(b) $\quad c\left(1 - \frac{3s-t}{2}\right) E(z, s, t) = E\left(z, \frac{1}{2}(1 - s + t), \frac{1}{2}(-1 + 3s + t)\right).$

(c)
$$c(1-t)c\left(1 - \frac{3s-t}{2}\right) c\left(1 - \frac{3s-t-1}{2}\right) E(z, s, t) = E\left(z, 1 - \frac{s}{2} - \frac{t}{2}, 1 - \frac{3s}{2} + \frac{t}{2}\right).$$

(d)
$$c\left(1 - \frac{3s-t}{2}\right) c\left(1 - \frac{3s-t-1}{2}\right) E(z, s, t) = E\left(z, \frac{1}{2}(1 - s + t), \frac{1}{2}(3 - 3s - t)\right).$$

$$\text{(e)}\qquad c(1-t)c\left(1-\frac{3s-t-1}{2}\right)E(z,s,t)=E\left(z,1-\frac{s}{2}-\frac{t}{2},\frac{3s-t}{2}\right).$$

Here the function c is given by

$$c(s)=\frac{\pi^{1/2}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Later in this paper we will use these equations to help us describe the automorphic forms which arise as poles of these Eisenstein series.

If we move an integral in s against one of these Eisenstein series across one of the lines we pick up a residue which is an integral against a lower rank Eisenstein series. All three lines are equivalent under the functional equations to $\operatorname{Re} t=1$ (or 0). Thus the poles all give rise to more or less the same Eisenstein series, up to multiplication by some function which we will call $\psi(s)$ and will describe later.

Let $h(s,t)$ be the Selberg transform of K. This transform is described in (7) and (8). Then the trace class operator associated to K is $L_{\widetilde{K}}$, where

$$\begin{aligned}\widetilde{K}=K-\sum_{\varphi}\int_{\operatorname{Re}s=1/2}&h(s,t_i)E(z,s,\varphi)\overline{E(w,s,\varphi)}\,ds\\ -\int_{\operatorname{Re}s=1/2}\int_{\operatorname{Re}t=1/2}&h(s,t)E(z,s,t)\overline{E(w,s,t)}\,ds\,dt\\ -\int_{\operatorname{Re}s=1/2}&h(s,0)E_r(z,s)\overline{E_r(z,s)}\psi(s)\,ds,\end{aligned}$$

where $E_r(z,s)$ is the residue at $t=0$ (or 1) of $E(z,s,t)$.

The purpose of this paper is to calculate the contributions of the terms in $T_r(L_{\widetilde{K}})$ coming from the integrals containing $E(z,s,t)$ and $E_r(z,s)$. We now state the main theorem in this paper.

Theorem. *The terms which contribute $O(\ln A)$ to the parabolic term of the trace formula and which come from $E(z,s,t)$ and $E_r(z,s)$ are equal to*

$$2\ln A\left[\frac{-1}{4\pi}\int_{\operatorname{Re}t=1/2}\int_{\tilde z\in\widetilde{\mathscr{F}}_B}\check{h}(t)E(\tilde z,t)\overline{E(\tilde z,t)}\,d\mu(\tilde z)\,dt\right.$$
$$\left.+\,c_E^2\operatorname{vol}\mathscr{F}_B\left(\frac{-1}{4\pi}\right)\int_{\operatorname{Re}s=1/2}h(s,0)\,ds\right].$$

The terms which are $O(1)$ are $(K_2+K_3)h(\frac{1}{2},\frac{1}{2})+K_4h(\frac{1}{3},1)+K_5h(\frac{2}{3},1)$ where the constants are independent of everything and are described in what follows. All other terms are $o(1)$ and have been suppressed completely. Here the function $\check{h}(t)$ is given by

$$\check{h}(t)=\int_{\operatorname{Re}s=c}h(s,t)\,ds.$$

In order to compute $T_r(L_{\widetilde{K}})$ we integrate $\widetilde{K}(z,z)$ over a truncated fundamental region $\mathscr{F}_{A,B}$. It is not known a priori whether $L_{\widetilde{K}}$ is trace class. For this we need the finiteness of the right-hand side of the trace formula, a project of which this paper is part. We will choose a copy of $\mathcal{F}$ with cusp at infinity and will truncate so that

$$\mathcal{F}_{A,B} = \{z \in \mathcal{F} | y(z) < A, u(z) < B\}.$$

$\mathcal{F}_{A,B}$ is a compact region. Integrating $\widetilde{K}(z, z)$ over $\mathcal{F}_{A,B}$ leads to the integral

$$\int_{\operatorname{Re} s=1/2} \int_{\operatorname{Re} t=1/2} \int_{\mathcal{F}_{A,B}} h(s, t) E(z, s, t) \overline{E(z, s, t)}\, d\mu(z)\, ds\, dt \tag{4}$$

and a similar one for $E_r(z, s)$. First we will compute (4) and later worry about the residue $E_r(z, s)$.

We compute (4) by replacing $\bar{s}$ and $\bar{t}$ by $1-s$, $1-t$, moving the line of integration the $\operatorname{Re} s$, $\operatorname{Re} t$ large, computing there and moving the lines of integration back to $\operatorname{Re} s = \operatorname{Re} t = \frac{1}{2}$. When we move the lines of integration we must now worry about poles of the function

$$g(s, t) = E(z, s, t)E(z, 1-s, 1-t). \tag{5}$$

Ignoring the poles of g for the moment, we will compute (4) where $\operatorname{Re} s$ and $\operatorname{Re} t$ are large. We can do an unwinding argument when $\operatorname{Re} s$ is large to replace

$$\int_{\operatorname{Re} s=c_1} \int_{\operatorname{Re} t=c_2} \int_{\mathcal{F}_{A,B}} h(s, t) E(z, s, t) E(z, 1-s, 1-t)\, d\mu(z)\, ds\, dt$$

by

$$\int_{\operatorname{Re} s=c_1} \int_{\operatorname{Re} t=c_2} h(s, t) \int_{\bigcup_{\gamma\in\Gamma_{1,\infty}\backslash\Gamma} \gamma^{-1}\mathcal{F}_{A,B}} y^s E(\tilde{z}, t) E(z, 1-s, 1-t)\, d\mu(z)\, ds\, dz,$$

and a messy estimate like that in [8] shows that the integral above can be replaced by

$$\int_{\operatorname{Re} s=c_1} \int_{\operatorname{Re} t=c_2} h(s, t) \int_{0<x_1, x_2<1} \int_{1/A<y<A} \times \int_{\widetilde{\mathcal{F}}_B} y^s E(\tilde{z}, t) E(z, 1-s, 1-t)\, d\mu(\tilde{z}) \frac{dx_1\, dx_2\, dy}{y^2}\, ds\, dt + o_A(1),$$

where $\widetilde{\mathcal{F}}_B$ is the fundamental region for $SL(2, \mathcal{Z})$ acting on $\tilde{z} = v + iu$, truncated at $u < B$, and $d\mu(\tilde{z})$ is the $SL(2, \mathcal{R})$ invariant measure for $\tilde{z}$.

Integrating over x and t we are left with the constant term in the Fourier expansion of $E(z, 1-s, 1-t)$ which (due to Venkov [9]) is

$$\begin{aligned} &y^{1-s} E(\tilde{z}, 1-t) + y^{1/2(1+s-t)} c\left(\frac{2-3s+t}{2}\right) E\left(\tilde{z}, \frac{3-3s-t}{2}\right) \\ &\quad + y^{(s+t)/2} c(1-t) c\left(\frac{3-3s-t}{2}\right) E\left(\tilde{z}, \frac{2-3s+t}{2}\right) \end{aligned} \tag{6}$$

where $\tilde{z} = v + iu$ and $E(\tilde{z}, *)$ is an Eisenstein series for $SL(2, \mathcal{Z})$.

The first term in the summand (6) gives

$$\int_{\operatorname{Re} s=c_1} \int_{\operatorname{Re} t=c_2} \int_{\tilde{z}\in\widetilde{\mathcal{F}}_B} \int_{1/A<y<A} y^s E(\tilde{z}, t) E(\tilde{z}, 1-t) y^{1-s} h(s, t) \frac{dy}{y^2} \, d\mu(\tilde{z}) \, ds \, dt.$$

We can take $\operatorname{Re} t = c_2 = 1/2$ because we only needed $\operatorname{Re} s$ large. Integrating out y gives (6) equal to

$$(7) \qquad 2 \ln A \int_{\operatorname{Re} t=1/2} \int_{\tilde{z}\in\widetilde{\mathcal{F}}_B} \check{h}(t) E(\tilde{z}, t) \overline{E(\tilde{z}, t)} \, d\mu(\tilde{z}) \, dt$$

where

$$\check{h}(t) = \int_{\operatorname{Re} s=c} h(s, t) \, ds.$$

Thus the term (7) becomes a term in the Selberg trace formula for $SL(2, \mathcal{Z})$ for some function whose Selberg transform in that context is $\check{h}(t)$. For a description of this function please see [8].

The other two terms in (6) behave differently. For example, the second gives

$$\int_{\operatorname{Re} s=c_1} \int_{\operatorname{Re} t=c_2} \int_{\tilde{z}_1\in\mathcal{F}_B} \int_{y_A<y<A} y^s E(z_1, t) y^{1/2(1+s-t)} \cdot c\left(\frac{2-3s+t}{2}\right) E\left(\tilde{z}, \frac{3-2s-t}{2}\right) \frac{dy}{y^2} \, d\mu(\tilde{z}) \, ds \, dt.$$

There are now no poles as we move $\operatorname{Re} s$ back to $1/2$ so that the above expression, after integrating out y is equal to

$$\int_{\operatorname{Re} t=1/2} \int_{s=1/2+i\alpha} \int_{\tilde{z}\in\mathcal{F}_B} \left[\frac{A^{i\alpha} - A^{-i\alpha}}{i\alpha}\right] E(\tilde{z}, t) \cdot c\left(\frac{2-3s+t}{2}\right) E\left(\tilde{z}, \frac{3-3s-t}{2}\right) h(s, t) \, d\alpha \, d\mu(\tilde{z}) \, dt$$

where $\alpha = \frac{3}{2} \operatorname{Im} s - \frac{1}{2} \operatorname{Im} t$. The expression in brackets is an approximate identity, so as A approaches ∞ we get the value of the integrand at $\alpha = 0$ or at $s = \frac{1}{2}$. So the above expression equals:

$$(8) \quad \int_{\operatorname{Re} t=1/2} \int_{\tilde{z}_1\in\widetilde{\mathcal{F}}_B} E(\tilde{z}, t) E\left(\tilde{z}, \frac{3/2-t}{2}\right) c\left(\frac{1/2+t}{2}\right) h\left(\frac{1}{2}, t\right) \, d\mu(\tilde{z}) \, dt.$$

Once again we would like to move the line of integration to $\operatorname{Re} t$ large. Again we pick up a residue at $t = 1$ which is given by

$$\int_{\tilde{z}=\widetilde{\mathcal{F}}_B} \operatorname{Res}_{t=1} E(\tilde{z}, t) \cdot E(\tilde{z}, \tfrac{1}{4}) c(\tfrac{3}{4}) h(\tfrac{1}{2}, \tfrac{1}{2}) \, d\mu(\tilde{z}).$$

Now $\operatorname{Res}_{t=1} E(\tilde{z}, t)$ is just some constant, and $E(\tilde{z}, \frac{1}{4})$ is equivalent via the functional equations for Eisenstein series for $SL(2, \mathcal{Z})$ to $E(z, \frac{3}{4})$ multiplied by a constant. So (8) has a residue of

$$c \int_{\tilde{z}\in\widetilde{\mathcal{F}}_B} E(\tilde{z}, \tfrac{3}{4}) \, d\mu(\tilde{z}).$$

But the Eisenstein series is L^1 at $\operatorname{Re} t = \frac{3}{4}$, so this integral just gives a constant as B approaches infinity. Call the constant K_2. So we can move the line of integration and unwind $E(\tilde{z}, t)$ to obtain from equation (8):

$$\int_{\operatorname{Re} t=c} \int_{1/B<u<B} \int_{0<v<1} u^t E\left(\tilde{z}, \frac{3/2-t}{2}\right) c\left(\frac{1/2+t}{2}\right) h\left(\frac{1}{2}, t\right) d\mu(\tilde{z})\, dt.$$

Integrating out v we get

$$\int_{\operatorname{Re} t=c} \int_{1/B<u<B} u^t \left[\tilde{c}\left(\frac{3/2-t}{2}\right) u^{(3/2-t)/2} + \tilde{c}\left(\frac{1/2+t}{2}\right) u^{(1/2+t)/2}\right] \times c\left(\frac{1/2+t}{2}\right) h\left(\frac{1}{2}, t\right) \frac{du}{u^2}\, dt.$$

Here $\tilde{c}$ is the function which appears in the constant term of the Fourier expansion of $E(\tilde{z}, t)$. That is

$$\tilde{c}(t) = \pi^{-t}\Gamma(t)\zeta(2t).$$

Moving the line of integration back to $\operatorname{Re} t = 1/2$ (notice there are no poles) we have

$$\int_{\operatorname{Re} t=1/2} \int_{1/2B<u<B} u^{-1+i\beta_1} c\left(\frac{1}{2}+i\beta_1\right) c\left(\frac{1}{2}-i\beta_1\right) + u^{-1-i\beta_1} c\left(\frac{1}{2}-i\beta_1\right) c\left(\frac{1}{2}-i\beta_1\right) du\, dt$$

where $\beta_1 = -\frac{1}{2}\operatorname{Im} t$. Again, integrating out u gives

$$\int_{\operatorname{Re} t=1/2} \left[\frac{B^{i\beta_1} - B^{-i\beta_1}}{i\beta_1}\right] \left[c\left(\frac{1}{2}+i\beta_1\right) c\left(\frac{1}{2}-i\beta_1\right) - c\left(\frac{1}{2}-i\beta_1\right)^2\right] dt,$$

which is an approximate identity as B approaches infinity, giving $C\left(\frac{1}{2}\right)^2 - c\left(\frac{1}{2}\right)^2 = 0$ as the final value.

Similarly the third term of (6) yields a constant (call it K_3) due to the residue of an Eisenstein series and similarly the value of the unwound integral is zero. (Note that $c(1/2) = 0$ so that exact cancellation isn't necessary.)

Up to now the total contribution of all terms discussed consists of:

(a) (8), which is a term in a lower rank trace formula times $2\ln A$,
(b) K_2 and K_3, constants which arise via residues of Eisenstein series for $SL(2, \mathscr{Z})$, yielding $(K_2 + K_3)h(\frac{1}{2}, \frac{1}{2})$,
(c) the residues of $g(s, t)$ in line (6), which we will describe next,
(d) the contribution of $E_r(z, s)$ which will come last in this paper.

We now examine the possible residue of

$$g(s, t) = E(z, s, t)E(z, 1-s, 1-t)$$

at $\operatorname{Re} t = 1/2$ as we move the line of integration for s from $\operatorname{Re} s = 1/2$ to $\operatorname{Re} s$ large. From Figure 1 we see that the only pole of the Eisenstein series along $\operatorname{Re} t = 1/2$ is at $\operatorname{Re} s = 5/6$. The constant term in the residue of $g(s, t)$ at a pole is just going to be the residue of the constant term, which in turn is equal to zero if the residue of the constant term of each of $E(z, s, t)$, and $E(z, 1-s, 1-t)$ is zero. For this we use the expression for the constant term due (in these coordinates) to Venkov which is given in (6) for $1-s$ and again below for s.

$$(9)\qquad \begin{aligned} & y^s E(\tilde{z}, t) + y^{1/2(1-s+t)} c\left(\frac{3s-t}{2}\right) E\left(\tilde{z}, \frac{3s+t-1}{2}\right) \\ & + y^{1-s/2-t/2} c(t) c\left(\frac{3s+t-1}{2}\right) E\left(\tilde{z}, \frac{3s-t}{2}\right). \end{aligned}$$

The poles of (9) along $\operatorname{Re} s = \frac{5}{6}$, $\operatorname{Re} t = \frac{1}{2}$ are at $(\frac{5}{6}, \frac{1}{2})$ and at $(\frac{5}{6} + i\alpha_1, \frac{5}{6} + i\alpha_2)$, where $3\alpha_1 - \alpha_2 = 0$, and at $(\frac{5}{6} + i\alpha, \frac{1}{2} + i\alpha_2)$, where $3\alpha_1 + \alpha_2 = 0$, because that's where the poles of the respective c's and $E(\tilde{z}, *)$ occur. Further, the function $E(z, 1-s, 1-t)$ has no pole at $(\frac{5}{6}, \frac{1}{2})$ and its constant term is $y^{1/6} E(\tilde{z}, \frac{1}{2})$ because $c(\frac{1}{2}) = c(0) = 0$.

Any pole of $E(z, s, t)$ other than those of (9) can be ignored because its residue has no constant term in its Fourier expansion and therefore the residue of the whole thing is cuspidal, hence orthogonal to $E(z, 1-s, 1-t)$ when integrated over $\mathscr{F}$. The pole of (9) along one of the two lines $3\alpha_1 + \alpha_2 = 1$ or $3\alpha_1 - \alpha_2 = 0$ gives an Eisenstein series in the remaining independent α as residue. This series in turn has a pole at $\alpha = 0$. If we think of α as a complex variable and shift the line of integration so that $E(z, 1-s, 1-t)$ is on $\operatorname{Re} s = \operatorname{Re} t = \frac{1}{2}$ we pick up the residue of $g(s, t)$ at $(\frac{5}{6}, \frac{1}{2})$. On the lines $\operatorname{Re} s = \operatorname{Re} t = \frac{1}{2}$ the Eisenstein series $E(z, 1-s, 1-t)$ is orthogonal to the residue of $E(z, s, t)$. Therefore as A and B approach infinity the integral over $\mathscr{F}_{A,B}$ of $g(s, t)$ along the moved contour is zero.

With this in mind we can safely claim that the only pole which can contribute anything to the trace formula is at $(\frac{5}{6}, \frac{1}{2})$ and that what it contributes comes from

$$\operatorname{Res}_{(5/6, 1/2)}(E(z, s, t)) E(z, \tfrac{1}{6}, \tfrac{1}{2}).$$

Lemma 1. *The constant term in the Fourier expansion of the residue of* $E(z, s, t)$ *at* $s = \frac{5}{6}$, $t = \frac{1}{2}$ *is zero.*

Proof. It follows from (9) that the only term having a residue is

$$y^{1/2(1-s+t)} c\left(\frac{3s-t}{2}\right) E\left(\tilde{z}, \frac{3s+t-1}{2}\right),$$

because the first term is analytic and the last has $c(t)$ in it, which goes to zero as $t \to 1/2$. In this expression both c and $E(\tilde{z}, *)$ have poles at $(\frac{5}{6}, \frac{1}{2})$. Using the formula for c from before:

$$c(s) = \frac{\pi^{1/2} \Gamma(s - 1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)},$$

and the functional equation for $E(\tilde{z}, s)$ which is

$$E(\tilde{z}, s) = \frac{\pi^{-1}\Gamma(1-s)\zeta(2(1-s))}{\Gamma(s)\zeta(2)} E(\tilde{z}, 1-s)$$

we obtain:

$$c\left(\frac{3s-t}{2}\right) E\left(\tilde{z}, \frac{3s+t-1}{2}\right) = \frac{\pi^{1/2}\Gamma\left(\frac{3s-t}{2} - 1/2\right)\zeta\left(2\left(\frac{3s-t}{2}\right)-1\right)}{\Gamma\left(\frac{3s-t}{2}\right)\zeta(3s-t)}$$
$$\cdot \frac{\pi^{-1}\Gamma\left(\frac{3-3s-t}{2}\right)\zeta\left(2\left(\frac{3-3s-t}{2}\right)\right)}{\Gamma\left(\frac{3s+t-1}{2}\right)\zeta(3s+t-1)} \cdot E\left(\tilde{z}, 1 - \frac{3s+t-1}{2}\right).$$

Plugging in $s = 5/6$, $t = 1/2$ we see that the pole is just the pole of

$$\zeta\left(2\left(\frac{3s-t}{2}\right)-1\right)\Gamma\left(\frac{3-3s-t}{2}\right)$$

at $(\frac{5}{6}, \frac{1}{2})$. Fixing $t = \frac{1}{2}$ this gives $\zeta(1-r)\Gamma(\frac{1}{2}r)$ where $r = \frac{5}{2} - 3s$ so that the above computation reduces to finding the residue of $\zeta(1-s)\Gamma(\frac{1}{2}s)$ at $s = 0$. We have

$$\zeta(1-s)\Gamma(\tfrac{1}{2}s) = \frac{\zeta(s)2\pi^{1-s}}{2\sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)} \cdot \frac{\Gamma(\frac{1}{2}s+1)}{\frac{1}{2}s},$$

which reduces the problem to finding the residue at $s = 0$ of

$$\frac{1}{s\sin\frac{s\pi}{2}}.$$

This is just equal to the constant term in the Laurent series at 0 of

$$\frac{1}{\sin\frac{s\pi}{2}}$$

which is zero. Therefore the constant term in the Fourier expansion of the residue of $E(z, s, t)$ at $s = \frac{5}{6}$, $t = \frac{1}{2}$ is zero.

We get as an immediate consequence of this lemma that

(1) $\operatorname{Res}_{(5/6, 1/2)} E(z, s, t) = 0$ because it must be orthogonal to the space of cusp forms, and

(2) $\operatorname{Res}_{(5/6, 1/2)} g(s, t) = 0$.

The last Eisenstein series to be considered in this paper is the residue of $E(z, s, t)$ as s and t move from the region of convergence to $\operatorname{Re} s = \operatorname{Re} t = \frac{1}{2}$. Looking at Figure 1 tells us that poles occur at $\operatorname{Re} t = 1$ and at $\operatorname{Re} s = \frac{5}{6}$, $\operatorname{Re} t = \frac{1}{2}$ if we follow a path from $\operatorname{Re} t = c_1$ to $\operatorname{Re} t = \frac{1}{2}$ (fixing s) and then from $\operatorname{Re} s = c_2$ to $\operatorname{Re} s = \frac{1}{2}$. When moving past $t = 1$ we pick up a residue $\operatorname{Res}_{t=1} E(z, s, t)$ which we will call $E_R(z, s)$. We can then move $E_R(z, s)$ to $\operatorname{Re} s = \frac{1}{2}$. As we do this we pass through two places where $E_R(z, s)$ has poles, namely at $\operatorname{Re} s = 1$ and $\operatorname{Re} s = \frac{2}{3}$. At these places $E_R(z, s)$ is equivalent via the functional equations to the residue of $E(z, s, t)$ at $\operatorname{Re} s = \frac{5}{6}$, $\operatorname{Re} t = \frac{1}{2}$. So from the L^2 point of view, $E_R(z, s)$ at $\operatorname{Re} s = \frac{1}{2}$ contributes the same subspace of $\mathscr{L}^2$ as the residue at $\operatorname{Re} s = \frac{5}{6}$, $\operatorname{Re} t = \frac{1}{2}$, up to residues of $E_R(z, s)$ at its own poles. We need consider first the integral

$$(10) \qquad \int_{\mathscr{F}_{A,B}} \int_{\operatorname{Re} s = \frac{1}{2}} h(s, 1) E_R(z, s) \overline{E_R(z, s)}\, ds\, d\mu(z),$$

and the residues of the $E_R(z, s)$ at $\operatorname{Re} s = 1$ and $\operatorname{Re} s = \frac{2}{3}$. Note that $h(s, 1) = h(s, 0)$ because t and $1 - t$ are equivalent under the functional equations and so t and $1 - t$ give the same eigenvalue of the pertinent differential operators.

Neglecting poles of $E_R(z, s)$ for the moment we will compute (10). As usual, replace s by $1 - s$, treat the resulting function

$$g_R = E_R(z, s) E_R(z, 1 - s) h(s, 1)$$

as a meromorphic function of s and move the line of integration to $\operatorname{Re} s$ large. The function $g_R(z, s)$ has three poles along the real axis for s because $E_R(z, s)$ has poles at $s = \frac{2}{3}$, $s = 1$, so $E_R(z, 1 - s)$ has poles at $\frac{1}{3}$ and 0. So we must check for residues at $\frac{1}{3}$, $\frac{2}{3}$ and 1. The constant term for $E_R(z, s)$ is easy to compute and is

$$y^s c_E + y^{1/2 - s/2} c_1 c\left(\frac{3s}{2}\right) E\left(\tilde{z}, \frac{3s - 1}{2}\right)$$

where $c_E =$ residue at $s = 1$ of $E(\tilde{z}, t)$ and $c_1 =$ residue at $s = 1$ of $c(t)$. Thus the constant term of $g_R(z, s)$ is

$$\left[y^s c_E + y^{1/2 - s/2} c_1 c\left(\frac{3s}{2}\right) E\left(\tilde{z}, \frac{3s - 1}{2}\right)\right] \times \left[y^{1-s} c_E + y^{s/2} c_1 c\left(\frac{3 - 3s}{2}\right) E\left(\tilde{z}, \frac{2 - 3s}{2}\right)\right] \times h(s, 1).$$

At $s = \frac{1}{3}$ the residue is

$$[y^{1/3} c_E] \times [y^{1/6} c_1^2 E(\tilde{z}, \tfrac{1}{2})] h(\tfrac{1}{3}, 1).$$

If we approximate $\mathscr{F}_{A,B}$ by the region $0 < x_1, x_2 < 1$, $c_1 < y < A$, $\tilde{z} \in \widetilde{\mathscr{F}_B}$ then the integral

$$\int_{y=c_1}^{A} \int_{0 < x_1, x_2 < 1} \int_{\tilde{z} \in \widetilde{\mathscr{F}}_B} g_R\, d\mu(z) = \int_{y=a}^{A} \int_{\tilde{z} \in \widetilde{\mathscr{F}}_B} y^{5/6} c_E c_1^2 E\left(\tilde{z}, \frac{1}{2}\right) \frac{dy}{y^2}\, d\mu(\tilde{z}) = 0$$

because $E(\tilde{z}, \frac{1}{2})$ is orthogonal to the constants. Therefore the residue at $s = \frac{1}{3}$ just contributes a constant, K_4, to the trace formula. The exact same argument works for $s = \frac{2}{3}$, to give K_5. At $s = 1$ the residue does something slightly different. Computing the constant term we have:

The constant term of $\operatorname{Res}_{s=1}(g_R(z, s))$ is $[c_1 c(\frac{3}{2}) c_E] \cdot [c_E \cdot h(1, 1)]$. Again, approximating $\mathscr{F}_{A,B}$ by one quarter of the strip $0 < x_1, x_2, v < 1$, $c_1 < y < A$, $c_2 < u < B$, we see that this term is finite as A, B tend to infinity.

Now we can proceed to unwind g_R for $\operatorname{Re} s$ large. For $\operatorname{Re} s$ large the Eisenstein series with constant term $y^s c_E + \cdots$ is just the one you get by summing

$$c_E \sum_{\gamma \in \Gamma_{1,\infty} \backslash \Gamma} y(\gamma z)^s \cdot 1,$$

which converges for $\operatorname{Re} s$ large enough. (See Imai and Terras [5].) So we can unwind $E_R(s, z)$ to obtain

$$\int_{\operatorname{Re} s=c} \int_{1/A<y<A} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} \int_{1<x_1, x_2<1} y^s c_E E_R(z, 1-s)\, d\mu(z)\, ds.$$

Again, we are suppressing a messy estimate which works the same way as for the Eisenstein series described in [8]. The above, after integrating out x_1 and x_2, becomes

$$\int_{\operatorname{Re} s=c} \int_{1/A<y<A} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} y^s c_E \left[y^{1-s} c_E + y^{s/2} c_1 c\left(\frac{3-3s}{2}\right) E\left(\tilde{z}, \frac{2-3s}{2}\right)\right] \times h(s, 1)\, d\mu(\tilde{z}) \frac{dy}{y^2}\, ds.$$

This expression is the sum of two terms, the first being:

$$\int_{\operatorname{Re} s=c} \int_{1/A<y<A} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} \frac{c_E^2}{y} h(s, 1)\, d\mu(z)\, dy\, ds$$

which yields

$$c_E^2 \operatorname{vol} \mathcal{F}_B \cdot 2 \ln A \int_{\operatorname{Re} s=c} h(s, 1)\, ds, \tag{11}$$

and we can move the line of integration to $\operatorname{Re} s = \frac{1}{2}$ and replace $h(s, 1)$ by $h(s, 0)$ if we want.

The second term is

$$\int_{\operatorname{Re} s=c} \int_{1/A<y<A} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} c_E y^{3s/2} c_1 c\left(\frac{3-3s}{2}\right) E\left(\tilde{z}, \frac{2-3s}{2}\right) h(s, 1)\, d\mu(\tilde{z}) \frac{dy}{y^2}\, ds.$$

The poles in this expression are at $s = \frac{1}{3}$ and $s = 0$. We can safely move $\operatorname{Re} s$ to $\frac{2}{3}$ thus giving the integral

$$\int_{2/3+i\alpha} \int_{1/A<y<A} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} c_E c_1 y^{-1+i\alpha} c\left(\frac{1-2i\alpha}{2}\right) E(\tilde{z}, -2i\alpha) \times h(\tfrac{2}{3} + i\tfrac{2}{3}\alpha, 1)\, d\mu(\tilde{z})\, dy\, ds.$$

Integrating out y we again obtain an approximate identity in α:

$$\int_{2/3+i\alpha} \int_{\tilde{z} \in \widetilde{\mathcal{F}}_B} \left[\frac{A^{i\alpha} - A^{-i\alpha}}{i\alpha}\right] c_E c_1 c\left(\frac{1-2i\alpha}{2}\right) E(\tilde{z}, -2i\alpha) \times h(\tfrac{2}{3} + i\alpha, 1)\, d\mu(\tilde{z})\, d\alpha,$$

and as A approaches infinity we get the value of the expression at $\alpha = 0$, which is

$$\int_{\tilde{z}\in\widetilde{\mathcal{F}}_B} c_E c_1 c(\tfrac{1}{2}) E(\tilde{z}, 0) h(\tfrac{2}{3}, 1)\, d\mu(\tilde{z})$$

and $c(1/2) = 0$.

Now, this calculation of E_R is sufficient to cover the other two poles also because the functional equations relate the three lines

(1) $s = \frac{1}{3} + i\alpha$, $t = 1$,
(2) $s = \frac{5}{6} + i\alpha$, $t = \frac{1}{2} + i\beta$, $3\alpha + \beta = 0$,
(3) $s = \frac{5}{6} + i\alpha$, $t = \frac{1}{2} + i\beta$, $3\alpha - \beta = 0$.

Functional equation (a) says

$$E_R\left(z, \frac{5}{6} + i\alpha, \frac{1}{2} - i3\alpha\right) = c\left(1 - \left(\frac{1}{2} + i3\alpha\right)\right) E_R\left(z, \frac{5}{6} + i\alpha, \frac{1}{2} + i3\alpha\right)$$

and functional equation (b) says

$$E_R\left(z, \frac{5}{6} + i\alpha, \frac{1}{2} + i3\alpha\right) = c\left(1 - \left(\frac{3(5/6 + i\alpha) - (1/2 + i3\alpha)}{2}\right)\right) \times E_R\left(z, \frac{1}{3} + \frac{1}{2}i(1 - 4\alpha), 1\right).$$

So the sum of both residues is

$$c(0)\left(1 + c\left(\frac{1}{2} - i3\alpha\right)\right) E_R\left(z, \frac{1}{3} + \frac{1}{2}i(1 - 4\alpha), 1\right).$$

But $c(0) = 0$, so the only contribution comes from the term already computed in (11). This concludes the proof of the theorem.

The value of our computation lies in its relation to the other terms of the trace formula, which have been computed in (5), (6) and (7). In these the various parabolic orbital integrals and the other family of Eisenstein series are computed and found to be $O(\ln A)$, just like the ones in this paper. It remains to compare the coefficients of $\ln A$ to see where and why terms cancel. This project is for a future paper, but we can say at this time that most cancellation occurs because of the presence of two sides of a lower rank trace formula in the coefficient of $\ln A$ in the final trace formula for $SL(3, \mathcal{Z})$.

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