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YOUNG MEASURES AND AN APPLICATION OF COMPENSATED COMPACTNESS TO ONE-DIMENSIONAL NONLINEAR ELASTODYNAMICS

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ABSTRACT. We study the existence problem for the equations of 1-dimensional nonlinear elastodynamics. We obtain the convergence of L^p ($p < \infty$) bounded approximating sequences generated by the method of vanishing viscosity and the Lax-Friedrichs scheme. The analysis uses Young measures, Lax entropies, and the method of compensated compactness.

1. INTRODUCTION

In this paper we consider the Cauchy problem for a system of one-dimensional nonlinear elasticity in Lagrangian coordinates which describes the balance of mass and linear momentum of the medium:

$$\partial_t u - \partial_x v = 0, \qquad \partial_t v - \partial_x \sigma(u) = 0, \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \tag{1.2}$$

where u is the strain, $\sigma(u)$ the stress, and v the velocity. Our assumptions about $\sigma(u)$ are as follows:

(A1) There exist constants $\sigma_0 > 0$ and $M > 0$ such that $\sigma(u) \in C^4(\mathbf{R})$, $|(\frac{d}{du})^k \sigma(u)| \le \sigma_0$, $\forall u \in \mathbf{R}$, $k = 2, 3, 4$, and $(\sigma'(u))^{-1/2}$ is concave for $u \ge M$, convex for $u \le -M$.

(A2) There is a constant $\delta_0 > 0$ such that $\sigma'(u) \ge \delta_0$, $\forall u \in \mathbf{R}$.

(A3) $u\sigma''(u) < 0$, $\forall u \in \mathbf{R} - \{0\}$.

(A2) guarantees that (1.1) is strictly hyperbolic, and admits two Riemann invariants

$$r(u, v) = v + \int_0^u \lambda(\tau)\, d\tau, \qquad s(u, v) = v - \int_0^u \lambda(\tau)\, d\tau, \tag{1.3}$$

where $\lambda(u) = (\sigma'(u))^{1/2}$ is one of the eigenvalues of ∇f, where $f = (v, \sigma(u))$.

We first consider the artificial viscosity approximation, that is, when (1.1) is approximated by its singular perturbation:

$$\partial_t u - \partial_x v = \varepsilon \partial_x^2 u, \qquad \partial_t v - \partial_x \sigma(u) = \varepsilon \partial_x^2 v, \tag{1.4}$$

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where $\varepsilon > 0$ is a perturbation parameter which measures the viscosity. We are concerned with the convergence of the viscosity solutions $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ generated by the Cauchy problem for (1.4) as ε tends to 0. We prove the following results.

Theorem 1.1. *Let* (A1)–(A3) *hold. Assume further that there exist real numbers* $\overline{u}$, $\overline{v}$, r^0, s^0 *such that* $u_0(x) - \overline{u} \in L^2(\mathbf{R})$, $v_0(x) - \overline{v} \in L^2(\mathbf{R})$, $r^0 > s^0$, *and*

$$r(u_0(x), v_0(x)) \geq r^0, \quad s(u_0(x), v_0(x)) \leq s^0 \quad \forall x \in \mathbf{R}.$$

Then there exist a subsequence $\{u^{\varepsilon_n}(x, t), v^{\varepsilon_n}(x, t)\}$ *of* $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ *and* $u(x, t)$, $v(x, t) \in L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ *such that*

$$\begin{aligned} &u^{\varepsilon_n}(x, t) \rightharpoonup u(x, t) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \\ &v^{\varepsilon_n}(x, t) \rightharpoonup v(x, t) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \\ &\sigma(u^{\varepsilon_n}(x, t)) \rightharpoonup \sigma(u(x, t)) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+) \end{aligned}$$

(*i.e., each sequence converges weakly in* $L^2(G \cap (\mathbf{R} \times \mathbf{R}^+))$ *for any bounded domain* $G \subset \mathbf{R}^2$). *Therefore,* $\{u(x, t), v(x, t)\}$ *is an admissible solution of the Cauchy problem* (1.1), (1.2).

We can also consider the approximation by finite difference schemes which are conservative in the sense of Lax-Wendroff (cf. Lax and Wendroff [1960]). For simplicity we are concerned with the convergence of the approximate solutions $\{u^l(x, t), v^l(x, t)\}$ generated by the Lax-Friedrichs scheme (cf. Lax [1954]) in the form:

$$\begin{aligned} u_{n+1,k} &= \tfrac{1}{2}(u_{n,k+1} + u_{n,k-1}) + \tfrac{1}{2}\kappa(v_{n,k+1} - v_{n,k-1}), \\ v_{n+1,k} &= \tfrac{1}{2}(v_{n,k+1} + v_{n,k-1}) + \tfrac{1}{2}\kappa(\sigma(u_{n,k+1}) - \sigma(u_{n,k-1})), \end{aligned} \tag{1.5}$$

where $\kappa = \frac{\Delta t}{\Delta x}$, $l = \Delta x$, and Δt and Δx are increments in the directions of t and x respectively. In §6 we shall give the details of the construction of $\{u^l(x, t), v^l(x, t)\}$. Our main result is similar to Theorem 1.1 and can be summarized as follows.

Theorem 1.2. *Let* (A1)–(A3) *hold. Assume further that there exist constants* $\overline{u}$, $\overline{v}$, r^0, s^0 *such that* $u_0(x) - \overline{u} \in L^2(\mathbf{R})$, $v_0(x) - \overline{v} \in L^2(\mathbf{R})$, $r^0 > s^0$, *and*

$$r(u_0(x), v_0(x)) \geq r^0, \quad s(u_0(x), v_0(x)) \leq s^0 \quad \forall x \in \mathbf{R}.$$

Then there exist a subsequence $\{u^{l_n}(x, t), v^{l_n}(x, t)\}$ *of* $\{u^l(x, t), v^l(x, t)\}$ *and* $u(x, t)$, $v(x, t) \in L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ *such that*

$$\begin{aligned} &u^{l_n}(x, t) \rightharpoonup u(x, t) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \\ &v^{l_n}(x, t) \rightharpoonup v(x, t) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \\ &\sigma(u^{l_n}(x, t)) \rightharpoonup \sigma(u(x, t)) && \text{in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+). \end{aligned}$$

Therefore, $\{u(x, t), v(x, t)\}$ *is an admissible solution of the Cauchy problem* (1.1), (1.2).

The hypotheses of both theorems imply that $u^\varepsilon(x, t)$ or $u^l(x, t)$ is uniformly bounded below by a positive constant. We remark, however, that similar results hold when $u^\varepsilon(x, t)$, $u^l(x, t)$ are uniformly bounded above by a negative constant.

Because of (A3), using the theory of invariant regions (cf. Chuech, Conley, and Smoller [1977], Hoff [1985]) we get

$$
\begin{aligned}
&r(u^\varepsilon(x, t), v^\varepsilon(x, t)) \geq r^0, \qquad r(u^l(x, t), u^l(x, t)) \geq r^0, \\
&s(u^\varepsilon(x, t), v^\varepsilon(x, t)) \leq s^0, \qquad s(u^l(x, t), u^l(x, t)) \leq s^0.
\end{aligned}
\tag{1.6}
$$

However, we do not know whether $\{u^\varepsilon, v^\varepsilon\}$ and $\{u^l, v^l\}$ are uniformly bounded even when the initial data $\{u_0, v_0\}$ is in $L^\infty(\mathbf{R})$. Dafermos [1987] proved that in the case of strain softening, that is, when $\sigma(u)$ satisfies (A3), the viscosity sequence $\{u^\varepsilon(\cdot, t), v^\varepsilon(\cdot, t)\}$ is uniformly bounded in the space $L^p(\mathbf{R})$, $0 < t < \infty$, $2 \leq p < \infty$, provided certain other conditions hold. Moreover, for the case of strain hardening, that is, $\sigma(u)$ is convex for u large and concave for u small, Dafermos [1987] proved that the viscosity sequence $\{u^\varepsilon(\cdot, t), v^\varepsilon(\cdot, t)\}$ is uniformly bounded in $L^\infty(\mathbf{R})$, $0 < t < \infty$, provided the initial data is in $L^\infty(\mathbf{R})$.

Our technique is to apply the method of compensated compactness. As we know, this method was established by Tartar [1979] and Murat [1978, 1981], motivated in part by the paper of Ball [1977] on nonlinear elasticity. This method has shown itself powerful in resolving some important problems in the theory of conservation laws. Tartar first succeeded in giving a new proof of convergence of the viscosity sequence for scalar conservation laws. Through an extremely novel use and generalization of Lax's [1971] entropy-entropy flux, DiPerna [1983a, 1983b] (see also Ding, Chen, and Luo [1985a, 1985b] and Chen [1986]) successfully proved existence of the Cauchy problem for the equations of isentropic gas dynamics in Eulerian coordinates.

We observe, however, that all the above papers require the local uniform boundedness in L^∞ of the approximate sequences of viscosity solutions, or the approximation constructed by a finite difference scheme. It is still an open problem to establish the convergence of more general approximate solution sequences of conservation laws. We remark that it seems very difficult to prove the local uniform boundedness of viscosity solutions of isentropic gas dynamics in Lagrangian coordinates.

We confront in the analysis the difficulty that the supports of the Young measures of an approximating sequence are no longer uniformly bounded, since the approximating sequence is not bounded in L^∞, so that consequently DiPerna's argument does not apply directly. In this paper we explore a technique which can deal with the problem of convergence for more general approximating sequences.

Based on condition (1.6), which results from the hypotheses on the initial data in Theorem 1.1 and Theorem 1.2, we are able to construct the required entropy-entropy flux pairs of Lax's type, via the method of Riemann functions from the standard theory of linear hyperbolic equations. A similar idea is used by Serre [1986], who obtains half-plane supported entropies by solving the Goursat problem for the related hyperbolic equations. However, his work is just concerned with uniformly bounded approximating sequences. Based on Tartar's commutation relation derived from Tartar and Murat's Div-Curl lemma, we prove that the Young measures are supported almost everywhere at at most four points.

We can then follow DiPerna's argument to deduce that the Young measure is indeed a Dirac mass.

One may use the Glimm scheme (cf. Glimm [1965], Liu [1977]) to solve the existence problem for (1.1). But we would then have to assume in particular that the initial data $u_0(\cdot)$, $v_0(\cdot)$ are of bounded variation.

The plan of this paper is as follows. In §2 we give an alternative proof of the representation of Young measures which enables us to establish a general framework for the application of compensated compactness. In §3 we are concerned with the viscosity solutions of the Cauchy problem for (1.4). In §4 we construct several families of entropy-entropy flux of Lax's type which, by applying Murat and Tartar's Div-Curl lemma, we use in §5 to prove that the resulting Young measures are indeed Dirac measures. Finally in §6 we consider the approximation by the Lax-Friedrichs finite difference scheme.

I had a chance to read part of the manuscript of a paper by J. W. Shearer [1989] which considers the same problem. We share many common methods such as vanishing viscosity, Lax entropies, compensated compactness and Young measures. Interestingly, we both divide the proof of reduction of Young measures into two steps. The first step is to prove that almost every Young measure is supported on at most four points. Then in the second step we prove that it is indeed a Dirac measure. However, the approach in the proof of each step is quite different. In the first step Shearer uses a class of half supported entropy-entropy flux pairs expressed through integral representations, while in the second step he uses another class of entropy-entropy flux pairs which are composed of complex functions. But in this paper we use the same class of entropy-entropy flux pairs of Lax's type in both steps.

2. Preliminaries

In this section we describe several fundamental results that we shall use. We first give for the reader's convenience a self-contained proof of a version of the representation theorem of Young measures that we use later, motivated in part by Tartar [1983], Slemrod [1985], and Ball [1988]. The Young measure was developed as a tool for analysing nonlinear partial differential equations by Tartar [1979]. For more details and comment we refer the reader to Berliocchi and Lasry [1973], Tartar [1979, 1983], Schonbek [1982], Balder [1984], Ball [1988], and Evans [1988].

We first specify some notation we shall use. $\mathbf{R}^N$ is N-dimensional real Euclidean space; $\overline{\mathbf{R}}^N = \mathbf{R}^N \cup \{\infty\}$; $\mathbf{R}^1 = \mathbf{R}$; $\mathbf{R}^+ = \{a \geq 0, a \in \mathbf{R}\}$; $C(\mathbf{R}^N)$ is the space of continuous functions, while $C_0(\mathbf{R}^N)$ is the space of continuous functions which tend to zero at infinity; $M(\mathbf{R}^N)$ is the dual space of $C_0(\mathbf{R}^N)$; and the symbol $\rightharpoonup$ means weak convergence in L^p with $1 \leq p < \infty$ (if $p = \infty$ we replace $\rightharpoonup$ by $\overset{*}{\rightharpoonup}$).

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^N$ be measurable. Suppose that $u^n(x): \Omega \to \overline{\mathbf{R}}^S$ is a sequence of measurable functions. Then there exist a subsequence $u^{n_k}(x)$ of $u^n(x)$ and a family of positive measures $\mu_x \in M(\mathbf{R}^S)$, depending measurably on $x \in \Omega$, such that for any $f \in C_0(\mathbf{R}^S)$*

$$f(u^{n_k}) \overset{*}{\rightharpoonup} \langle f(\lambda), \mu_x \rangle = \int_{\mathbf{R}^S} f(\lambda)\, d\mu_x \quad \text{in } L^\infty(\Omega). \tag{2.1}$$

Proof. Let $E = \{f^m\}$ be a dense set in $C_0(\mathbf{R}^S)$. Then $\{f^1(u^n)\}$ is bounded on Ω, and hence there exist a subsequence $\{u^{n_k^1}\}$ of $\{u^n\}$ and $\alpha(f^1)(x) \in L^\infty(\Omega)$ such that

$$f^1(u^{n_k^1}(x)) \overset{*}{\rightharpoonup} \alpha(f^1)(x).$$

Furthermore, $\{f^2(u^{n_k^1}(x))\}$ is also bounded on Ω, and hence there exist a subsequence $\{u^{n_k^2}\}$ of $\{u^{n_k^1}\}$ and $\alpha(f^2)(x) \in L^\infty(\Omega)$ such that

$$f^2(u^{n_k^2}(x)) \overset{*}{\rightharpoonup} \alpha(f^2)(x).$$

Proceeding in this way we obtain a series of subsequences $\{u^{n_k^m}\}$, $\alpha(f^m)$ such that

(i) $\{u^{n_k^1}\} \supset \{u^{n_k^2}\} \supset \{u^{n_k^3}\} \supset \cdots$, and

(ii) for each fixed m, $f^m(u^{n_k^m}) \overset{*}{\rightharpoonup} \alpha(f^m)$.

We let $\{u^{n_k}\} = \{u^{n_k^k}\}$, the diagonal sequence. Then from (ii) we get that for each fixed m,

$$f^m(u^{n_k}) \overset{*}{\rightharpoonup} \alpha(f^m). \tag{2.2}$$

For each $f^m \in E$, we define a bounded functional $I(f^m)$ on $L^1(\Omega)$ by

$$\langle I(f^m), \psi\rangle = \int_\Omega \psi\alpha(f^m)\,dx = \lim_{k\to\infty}\int_\Omega \psi f^m(u^{n_k})\,dx \quad \forall\psi \in L^1(\Omega).$$

Then for any given $f \in C_0(\mathbf{R}^S)$, suppose that $f = \lim_{l\to\infty} f^l$ in $C_0(\mathbf{R}^S)$, where $\{f^l\} \subset E$. We want to prove that the following limit exists, and hence we denote it by $I(f)$, namely,

$$\langle I(f), \psi\rangle = \lim_{k\to\infty}\int_\Omega \psi f(u^{n_k})\,dx \quad \forall\psi \in L^1(\Omega). \tag{2.3}$$

In fact, for any n_{k_1}, n_{k_2}, we notice that

$$\begin{aligned} &\left|\int_\Omega \psi[f(u^{n_{k_1}}) - f(u^{n_{k_2}})]\,dx\right| \\ &\quad\le \left|\int_\Omega \psi[f(u^{n_{k_1}}) - f^l(u^{n_{k_1}})]\,dx\right| + \left|\int_\Omega \psi[f(u^{n_{k_2}}) - f^l(u^{n_{k_2}})]\,dx\right| \\ &\qquad + \left|\int_\Omega \psi[f^l(u^{n_{k_1}}) - f^l(u^{n_{k_2}})]\,dx\right| \\ &\quad\le 2\|f - f^l\|_{C_0}\|\psi\|_1 + \left|\int_\Omega \psi[f^l(u^{n_{k_1}}) - f^l(u^{n_{k_2}})]\,dx\right|. \end{aligned} \tag{2.4}$$

We first choose l large enough such that the first term on the right-hand side of (2.4) is small, then by (2.2) the second term on the right-hand side of (2.4) can be small whenever n_{k_1} and n_{k_2} are large enough. Hence we prove that $\{\int_\Omega \psi f(u^{n_k})\,dx\}$ is a Cauchy sequence for any fixed $\psi \in L^\infty(\Omega)$, and so we have proved (2.3). Consequently, we obtain

$$|\langle I(f), \psi\rangle| \le \|f\|_{C_0}\|\psi\|_1 \quad \forall\psi \in L^1(\Omega). \tag{2.5}$$

We notice, by (2.5), that $I(f)$ is a bounded functional on $L^1(\Omega)$, and hence by the Riesz representation theorem there exists $\alpha(f)(x) \in L^\infty(\Omega)$ such that

$$\langle I(f), \psi\rangle = \int_\Omega \alpha(f)\psi\,dx \quad \forall\psi \in L^1(\Omega). \tag{2.6}$$

We also have

$$\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2) \quad \forall f_i \in C_0(\mathbf{R}^S),\ i = 1, 2,$$
$$\alpha(kf) = k\alpha(f) \quad \forall f \in C_0(\mathbf{R}^S),\ k \in \mathbf{R}.$$

At this moment we suppose, without loss of generality, that every point $x \in \Omega$ is a Lebesgue point of each function $\alpha(f)$. Then for any fixed $x_0 \in \Omega$, we set

$$\psi(x) = (\operatorname{meas} B_r(x_0))^{-1} \chi_{B_r(x_0)},$$

where $B_r(x_0)$ is the ball centred at x_0 with diameter r, and $\chi_{B_r(x_0)}$ is the characteristic function of $B_r(x_0)$. By (2.5) and (2.6) we get

$$\left| (\operatorname{meas} B_r(x_0))^{-1} \int_{B_r(x_0)} \alpha(f)\, dx \right| \le \|f\|_{C_0}.$$

We now pass to the limit as $r \to 0$ to obtain $|\alpha(f)(x_0)| \le \|f\|_{C_0}$. Combining this with the fact that $\alpha(f)$ is linear with respect to f we have that $\alpha(f)(x_0)$ is a bounded functional on $C_0(\mathbf{R}^S)$. Therefore applying the Riesz representation theorem we have that there is a $\mu_{x_0} \in M(\mathbf{R}^S)$ such that

$$\alpha(f)(x_0) = \langle f(\lambda), \mu_{x_0} \rangle = \int_{\mathbf{R}^S} f(\lambda)\, d\mu_{x_0}.$$

Since x_0 is arbitrary we get

$$\langle I(f), \psi \rangle = \int_\Omega \psi \langle f(\lambda), \mu_x \rangle\, dx \quad \forall \psi \in L^1(\Omega),$$

where $\mu_x \in M(\mathbf{R}^S)$ for a.e. $x \in \Omega$. So we have proved (2.1).

Finally, we notice that for any positive $f \in C_0(\mathbf{R}^S)$

$$\alpha(f)(x) = \langle f(\lambda), \mu_x \rangle \ge 0 \quad \text{a.e. } x \in \Omega,$$

which implies that μ_x is positive for almost all $x \in \Omega$. This completes the proof.

Remark 2.1. Using a dense set of $C_0(\mathbf{R}^S)$ in the proof of Theorem 2.1 is suggested in Tartar [1983] where the dense set of polynomials with rational coefficients is used. The idea in the proof of the following corollary is due to Ball [1988].

Corollary 2.2. *Suppose that $u^n(x)$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N; \mathbf{R}^S)$, where $1 \le p < \infty$. Then there exist a subsequence u^{n_k} of u^n and a family of positive measures $\mu_x \in M(\mathbf{R}^S)$, $x \in \mathbf{R}^N$, such that for any bounded set $A \subset \mathbf{R}^N$*

$$f(u^{n_k}) \rightharpoonup \langle f(\lambda), \mu_x \rangle \quad \text{in } L^1(A), \tag{2.7}$$

whenever $f \in C(\mathbf{R}^S)$ satisfies

$$\lim_{|\lambda| \to \infty} \frac{f(\lambda)}{|\lambda|^p} = 0. \tag{2.8}$$

Proof. Without loss of generality we assume that $f \ge 0$. Then we define $f^m \in C_0(\mathbf{R}^S)$ by $f^m = \theta^m f$, where $\theta^m \in C_0(\mathbf{R}^S)$ is defined by

$$\theta^m(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \le m, \\ 1 + m - |\lambda| & \text{for } m \le |\lambda| \le m + 1, \\ 0 & \text{for } |\lambda| \ge m + 1. \end{cases}$$

We claim that for each $\varphi \in L^\infty(A)$

$$\lim_{m\to\infty}\int_A \varphi f^m(u^n)\,dx = \int_A \varphi f(u^n)\,dx \tag{2.9}$$

uniformly in n. Indeed,

$$\begin{aligned}\left|\int_A \varphi[f^m(u^n) - f(u^n)]\,dx\right| &\le \|\varphi\|_{L^\infty(A)}\int_{\{x\in A\,;\,|u^n|\ge m\}} f(u^n)\,dx\\ &\le \|\varphi\|_{L^\infty(A)}\|u^n\|_{L^p(A)}\max_{|\lambda|\ge m}\left\{\frac{f(\lambda)}{|\lambda|^p}\right\},\end{aligned}$$

which tends to 0 uniformly in n as $m \to \infty$.

On the other hand, by Theorem 2.1 there exist a subsequence u^{n_k} of u^n and a family of positive measures $\mu_x \in M(\mathbf{R}^S)$ such that for each m

$$\lim_{n\to\infty}\int_A \varphi f^m(u^{n_k})\,dx = \int_A \varphi\langle f^m, \mu_x\rangle\,dx \quad \forall\varphi \in L^\infty(A). \tag{2.10}$$

Furthermore, from the monotone convergence theorem we get

$$\lim_{m\to\infty}\int_A \varphi\langle f^m, \mu_x\rangle\,dx = \int_A \varphi\langle f, \mu_x\rangle\,dx. \tag{2.11}$$

Combining (2.9), (2.10), and (2.11) we get (2.7) and complete the proof.

We now describe Murat and Tartar's Div-Curl lemma which is the prototype for the theory of compensated compactness (cf. Murat [1978], Tartar [1983], and Ding, Chen, and Luo [1985c]).

Div-Curl lemma. *Let $\Omega \subset \mathbf{R}^2$ be an open bounded set. Let $\{u_i^n(x)\}$ be a sequence in $L^2(\Omega)$ for each $i = 1, 2, 3, 4$. Suppose that $u_i^n \rightharpoonup u_i^0$ in $L^2(\Omega)$, $i = 1, 2, 3, 4$, and $\partial_{x_1}u_1^n + \partial_{x_2}u_2^n$ and $\partial_{x_1}u_3^n + \partial_{x_2}u_4^n$ are compact in $H^{-1}(\Omega)$. Then*

$$u_1^n u_4^n - u_2^n u_3^n \to u_1^0 u_4^0 - u_2^0 u_3^0 \quad \textit{in the sense of distributions.}$$

We finally describe an embedding theorem (see Ding, Chen, and Luo [1985a] and Evans [1988]) which is related to an earlier result of Murat (cf. Tartar [1979]).

Embedding Theorem. *Let $\Omega \subset \mathbf{R}^N$ be an open bounded set, and let $1 < q \le 2 < r < \infty$. Assume that $\{f_n\}$ is bounded in $W^{-1,r}(\Omega)$ and relatively compact in $W^{-1,q}(\Omega)$. Then $\{f_n\}$ is relatively compact in $H^{-1}(\Omega)$.*

3. Viscosity solutions

In this section we consider the Cauchy problem for the related parabolic system:

$$\partial_t u - \partial_x v = \varepsilon\partial_x^2 u, \qquad \partial_t v - \partial_x\sigma(u) = \varepsilon\partial_x^2 v, \tag{3.1}$$

with initial data

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x). \tag{3.2}$$

We assume that

(H) (A2), (A3) hold, and $\sigma(u) \in C^2(\mathbf{R})$.

Consequently, we have that

$$(3.3)\qquad \begin{aligned} &0 < \delta_0 \le \sigma'(u) \le \sigma'(0), \qquad \sigma''(0) = 0, \\ &|\sigma(u)| = \left| \int_0^u \sigma'(\tau)\, d\tau \right| \le \sigma'(0)|u|. \end{aligned}$$

A local existence result for (3.1), (3.2) can easily be obtained by applying the contraction mapping principle to an integral representation for a solution, following the standard theory of semilinear parabolic systems (cf. Ladyzhenskaya, Solonnikov, and Uraltseva [1968], Ding and Wang [1983], and Hoff and Smoller [1985]). Whenever we have a suitable a priori estimate, we can establish the global existence of a smooth solution of (3.1), (3.2). The following version of a result of Dafermos is just the a priori estimate we require.

Theorem 3.1 (Dafermos [1987]). *Let* (H) *hold. Assume further that there exist* $\overline{u}$, $\overline{v}$ *such that*

$$(3.4)\qquad u_0(x) - \overline{u} \in L^2(\mathbf{R}), \qquad v_0(x) - \overline{v} \in L^2(\mathbf{R}).$$

Suppose that $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ *is a smooth solution of* (3.1), (3.2) *defined in a strip* $\mathbf{R} \times (0, T]$ *with* $0 < T < \infty$, *and that* $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ *tends to* $\{\overline{u}, \overline{v}\}$ *as* $|x| \to \infty$, *for any* $t \in (0, T]$. *Then for each* $t \in [0, T]$

$$(3.5)\qquad \int_{\mathbf{R}} |u^\varepsilon(\cdot, t) - \overline{u}|^2\, dx \le c(t), \qquad \int_{\mathbf{R}} |v^\varepsilon(\cdot, t) - \overline{v}|^2\, dx \le c(t),$$

where $c(t)$ *is bounded on* $[0, T]$, *and* $c(t)$ *depends on* T, *but is independent of* ε.

Remark 3.1. Dafermos [1987, Proposition 3.1] proved precisely that

$$(3.6)\qquad \int_{\mathbf{R}} |u^\varepsilon(\cdot, t) - \overline{u}|^p\, dx \le c(t), \qquad \int_{\mathbf{R}} |v^\varepsilon(\cdot, t) - \overline{v}|^p\, dx \le c(t)$$

with $p \ge 2$, provided that

$$(3.7)\qquad \begin{aligned} &u_0(x) - \overline{u}, \qquad v_0(x) - \overline{v} \in L^p(\mathbf{R}) \cap L^2(\mathbf{R}), \\ &\int_{\mathbf{R}} |u|^{p-2} |a'(u)|\, du < \infty, \end{aligned}$$

where $a^2(u) = \sigma'(u)$. To deduce Theorem 3.1, we notice that if $p = 2$ the last condition in (3.7) is superfluous, due to (A3).

We can now state the global existence result, but we omit the proof (cf. Hoff and Smoller [1985], for example).

Theorem 3.2. *Let* (H) *and* (3.4) *hold. Then there is a solution*

$$\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$$

of the Cauchy problem (3.1), (3.2) *such that,* $u^\varepsilon, v^\varepsilon \in C^2(\mathbf{R} \times (0, \infty))$,

$$(3.8)\qquad \int_{\mathbf{R}} |u^\varepsilon(\cdot, t) - \overline{u}|^2\, dx \le c(t), \qquad \int_{\mathbf{R}} |v^\varepsilon(\cdot, t) - \overline{v}|^2\, dx \le c(t),$$

where $c(t)$ *is locally bounded in* $\mathbf{R}^+$ *and independent of* ε.

Corollary 3.3. *Under the assumptions of Theorem* 3.2, *we have*

(3.9) $\varepsilon^{1/2}\partial_x u^\varepsilon$ *and* $\varepsilon^{1/2}\partial_x v^\varepsilon$ *are uniformly bounded in* $L^2_{\mathrm{loc}}(\mathbf{R} \times (0, \infty))$.

Proof. Let $K \Subset \mathbf{R} \times (0, \infty)$, $\varphi \in C_0^\infty(\mathbf{R} \times (0, \infty))$, $\varphi|_K = 1$, $\varphi \geq 0$, and $G := \operatorname{supp}\{\varphi\}$. Since

$$
\begin{aligned}
&\partial_t \left[\frac{1}{2}(v^\varepsilon)^2 + \int_0^{u^\varepsilon} \sigma(\tau)\, d\tau\right] - \partial_x[v^\varepsilon \sigma(u^\varepsilon)] \\
&\quad = \varepsilon v^\varepsilon \partial_x^2 v^\varepsilon + \varepsilon \sigma(u^\varepsilon) \partial_x^2 u^\varepsilon \\
&\quad = -\varepsilon(\partial_x v^\varepsilon)^2 - \varepsilon \sigma'(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \varepsilon \partial_x^2 \left[\frac{1}{2}(v^\varepsilon)^2 + \int_0^{u^\varepsilon} \sigma(\tau)\, d\tau\right],
\end{aligned} \tag{3.10}
$$

we may multiply (3.10) by φ and integrate over $\mathbf{R}^2$ to get

$$
\begin{aligned}
&\varepsilon \int_0^\infty \int_{-\infty}^{+\infty} [(\partial_x v^\varepsilon)^2 + \sigma'(u^\varepsilon)(\partial_x u^\varepsilon)^2] \varphi \, dx\, dt \\
&\quad = \int_0^\infty \int_{-\infty}^{+\infty} \left[\frac{1}{2}(v^\varepsilon)^2 + \int_0^{u^\varepsilon} \sigma(\tau)\, d\tau\right] \partial_t \varphi \, dx\, dt \\
&\qquad - \int_0^\infty \int_{-\infty}^{+\infty} v^\varepsilon \sigma(u^\varepsilon) \partial_x \varphi \, dx\, dt \\
&\qquad + \varepsilon \int_0^\infty \int_{-\infty}^{+\infty} \left[\frac{1}{2}(v^\varepsilon)^2 + \int_0^{u^\varepsilon} \sigma(\tau)\, d\tau\right] \partial_x^2 \varphi \, dx\, dt \\
&\quad \leq C(\|u^\varepsilon\|_{L^2(G)} + \|v^\varepsilon\|_{L^2(G)}),
\end{aligned} \tag{3.11}
$$

where C depends on φ. Therefore, we get that $\varepsilon^{1/2}\partial_x u^\varepsilon$ and $\varepsilon^{1/2}\partial_x v^\varepsilon$ are uniformly bounded in $L^2(K)$, and hence we complete the proof.

Remark 3.2. From the proof above we observe that, to prove (3.9), it is sufficient that

$$
u^\varepsilon \text{ and } v^\varepsilon \text{ are uniformly bounded in } L^2_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \tag{3.12}
$$

which is weaker than (3.8).

By the theory of invariant regions (cf. Chueh, Conley, and Smoller [1977] and Smoller [1983]), the following result is obvious.

Theorem 3.4. *Suppose that* $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ *is a smooth solution of the Cauchy problem* (3.1), (3.2). *If there are* r^* *and* s^*, $r^* > s^*$, *such that*

$$
r(u_0(x), v_0(x)) \geq r^*, \quad s(u_0(x), v_0(x)) \leq s^* \quad \forall x \in \mathbf{R},
$$

then

$$
r(u^\varepsilon(x, t), v^\varepsilon(x, t)) \geq r^*, \quad s(u^\varepsilon(x, t), v^\varepsilon(x, t)) \leq s^* \quad \forall (x, t) \in \mathbf{R} \times (0, \infty).
$$

Consequently, u^ε is uniformly bounded below by a positive constant.

Similarly, if $r^* < s^*$ and

$$
r(u_0(x), v_0(x)) \leq r^*, \quad s(u_0(x), v_0(x)) \geq s^* \quad \forall x \in \mathbf{R},
$$

then

$$
r(u^\varepsilon(x, t), v^\varepsilon(x, t)) \leq r^*, \quad s(u^\varepsilon(x, t), v^\varepsilon(x, t)) \geq s^* \quad \forall (x, t) \in \mathbf{R} \times (0, \infty).
$$

Consequently, we also have that u^ε is uniformly bounded above by a negative constant.

Remark 3.3. From the initial data given in Theorem 1.1 we have that the supports of the viscosity sequence $\{u^\varepsilon(x, t), v^\varepsilon(x, t)\}$ are contained in $\{(r, s);\ r \geq r^0, s \leq s^0\}$.

4. Admissible solutions. Lax entropies

We first describe in standard fashion a definition of a weak solution for the Cauchy problem (1.1), (1.2).

Definition 4.1. A pair of functions $\{u(x, t), v(x, t)\}$, $u, v \in L^1_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$, is said to be a weak solution of the Cauchy problem (1.1), (1.2) if

$$\begin{aligned} &\iint_{t>0} (u\varphi_t - v\varphi_x)\,dx\,dt + \int_{-\infty}^{\infty} u_0(x)\varphi(x, 0)\,dx = 0, \\ &\iint_{t>0} (v\varphi_t - \sigma(u)\varphi_x)\,dx\,dt + \int_{-\infty}^{\infty} v_0(x)\varphi(x, 0)\,dx = 0, \end{aligned} \tag{4.1}$$

whenever $\varphi \in C_0^\infty(\mathbf{R}^2)$.

It is necessary to introduce an admissibility criterion which can pick out the physically relevant solutions. We recall the definition of generalized entropy formulated by Lax [1971]. A pair of smooth mappings (η, q), where $\eta(u, v)$, $q(u, v)$: $\mathbf{R}^2 \to \mathbf{R}$, is called an entropy-entropy flux pair if

$$\nabla q = \nabla\eta \cdot \nabla f, \qquad f = (-v, -\sigma(u))^{\mathrm{T}}, \tag{4.2}$$

for all $u, v \in \mathbf{R}$. In components, (4.2) reads

$$\partial_u q(u, v) = -\sigma'(u)\partial_v\eta(u, v), \qquad \partial_v q(u, v) = -\partial_u\eta(u, v), \tag{4.3}$$

from which it follows that η is a solution of

$$\partial_u^2\eta(u, v) = \sigma'(u)\partial_v^2\eta(u, v).$$

A typical example of an entropy-entropy flux pair is

$$\eta(u, v) = \frac{1}{2}v^2 + \int_0^u \sigma(\tau)\,d\tau, \qquad q(u, v) = -v\sigma(u). \tag{4.4}$$

Note that in this example η is convex.

For convenience we define some classes of entropy-entropy flux pairs as follows:

$$\begin{aligned} L := \{(\eta, q);\ &|\nabla^2\eta| \leq C,\ |\nabla\eta| \leq C(1 + |u|^\alpha + |v|^\alpha), \\ &|\eta| \leq C(1 + |u|^\alpha + |v|^\alpha), \\ &|q| \leq C(1 + |u|^\alpha + |v|^\alpha),\ 0 < \alpha < 1\}, \\ L_{\mathrm{con}} := \{(\eta, q);\ &\eta \text{ is convex, } |\eta| \leq C(1 + |u|^\beta + |v|^\beta), \\ &|q| \leq C(1 + |u|^\beta + |v|^\beta),\ 0 < \beta < 2\}. \end{aligned} \tag{4.5}$$

Here C, α, and β are constants depending on η and q.

We now can describe the definition of an admissible solution in the sense of Lax [1971] for the Cauchy problem (1.1), (1.2).

Definition 4.2. Suppose that $\{u(x,t), v(x,t)\}$ is a weak solution defined by Definition 4.1. Then $\{u(x,t), v(x,t)\}$ is said to be an admissible solution if, for any pair $(\eta, q) \in L_{\text{con}}$,

$$\iint_{t>0} [\eta(u,v)\partial_t\varphi + q(u,v)\partial_x\varphi]\,dx\,dt \ge 0, \tag{4.6}$$

whenever $\varphi \ge 0$, $\varphi \in C_0^\infty(\mathbf{R}\times(0,\infty))$.

Theorem 4.1. *Suppose that $\{u^\varepsilon(x,t), v^\varepsilon(x,t)\}$ is the sequence of viscosity solutions given in Theorem 3.2. Then we have that for each $(\eta, q) \in L$*

$$\begin{gathered}\partial_t\eta(u^\varepsilon(x,t), v^\varepsilon(x,t)) + \partial_x q(u^\varepsilon(x,t), v^\varepsilon(x,t)) \\ \textit{is relatively compact in } H^{-1}_{\text{loc}}(\mathbf{R}\times(0,\infty)).\end{gathered} \tag{4.7}$$

Proof. Given any bounded open set $\Omega \subset \mathbf{R}\times(0,\infty)$, $\overline{\Omega} \subset \mathbf{R}\times(0,\infty)$, we want to prove that

$$\partial_t\eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) \text{ is relatively compact in } H^{-1}(\Omega). \tag{4.8}$$

We first notice that $\partial_t\eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) = I_1^\varepsilon + I_2^\varepsilon$, with

$$\begin{aligned}I_1^\varepsilon &= \varepsilon\partial_x[\nabla\eta\cdot(u_x^\varepsilon, v_x^\varepsilon)],\\ I_2^\varepsilon &= -\varepsilon[\eta_{uu}(u_x^\varepsilon)^2 + 2\eta_{uv}u_x^\varepsilon v_x^\varepsilon + \eta_{vv}(v_x^\varepsilon)^2],\end{aligned}$$

where we use $\partial_x u$ or u_x as the derivative of u with respect to x, whichever is convenient. Since $|\nabla^2\eta| \le C$, by Corollary 3.3

$$\iint_\Omega |I_2^\varepsilon|\,dx\,dt \le C\iint_\Omega \varepsilon[(u_x^\varepsilon)^2 + (v_x^\varepsilon)^2]\,dx\,dt \le C,$$

where C is independent of ε. (For simplicity we may use the same C as various constants independent of ε.) Therefore, I_2^ε is bounded in $M(\Omega)$, the dual space of $C_0(\Omega)$, and hence, by the Schauder theorem (cf. Yosida [1968], Chapter 10]),

$$I_2^\varepsilon \text{ is relatively compact in } W^{-1,q_0}(\Omega),\ 1 < q_0 < 2. \tag{4.9}$$

Furthermore, because of the definition of L, we have that for each $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}\left|\iint_\Omega I_1^\varepsilon\varphi\,dx\,dt\right| &\le \varepsilon\iint_\Omega (|\eta_u u_x^\varepsilon| + |\eta_v v_x^\varepsilon|)|\varphi_x|\,dx\,dt\\ &\le \varepsilon^{1/2}C(\|\varepsilon^{1/2}u_x^\varepsilon\|_2 + \|\varepsilon^{1/2}v_x^\varepsilon\|_2)(1 + \|u^\varepsilon\|_2^\alpha + \|v^\varepsilon\|_2^\alpha)\|\varphi_x\|_{q_1}\\ &\to 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

where $\frac{1}{2}(1+\alpha) + 1/q_1 = 1$, $q_1 > 2$. This implies that $I_1^\varepsilon \to 0$ in $W^{-1,q_1'}$, and $1/q_1 + 1/q_1' = 1$. Combining the above with (4.9) we get

$$\partial_t\eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) \text{ is relatively compact in } W^{-1,q_1'}(\Omega). \tag{4.10}$$

On the other hand, for any $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}&\left|\iint_\Omega [\partial_t\eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon)]\varphi\,dx\,dt\right|\\ &\quad\le C(\|\eta\|_{2/\alpha} + \|q\|_{2/\alpha})\|\nabla\varphi\|_{2/(2-\alpha)}\\ &\quad\le C(1 + \|u^\varepsilon\|_2^\alpha + \|v^\varepsilon\|_2^\alpha)\|\nabla\varphi\|_{2/(2-\alpha)},\end{aligned}$$

which means that $\partial_t \eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon)$ is bounded in $W^{-1, 2/\alpha}(\Omega)$. Since $2/\alpha > 2$, combining the above with (4.10) we can apply the Embedding Theorem of §2 to get (4.8), which completes the proof.

We now discuss the construction of entropy-entropy flux pairs of Lax's type. We can regard the Riemann invariants (1.3) as a mapping I from the (u, v) plane to the (r, s) plane,

$$I: (u, v) \to (r, s),$$

which is a smooth, one-to-one mapping. We notice that (4.2) is equivalent to

$$\frac{\partial q}{\partial s} = \lambda \frac{\partial \eta}{\partial s}, \qquad \frac{\partial q}{\partial r} = -\lambda \frac{\partial \eta}{\partial r}, \tag{4.11}$$

where $\lambda = (\sigma'(u))^{1/2}$.

We first construct entropy-entropy flux pairs having the form

$$\begin{aligned} \eta_{\pm k}(r, s) &= e^{\pm ks}(A_0 + A_1(\pm k)^{-1}) + P_{\pm k}, \\ q_{\pm k}(r, s) &= e^{\pm ks}(B_0 + B_1(\pm k)^{-1}) + Q_{\pm k}, \end{aligned} \tag{4.12}$$

where $k = 2, 3, 4, \ldots$ and A_j, B_j, $P_{\pm k}$, and $Q_{\pm k}$ are smooth functions of r, s to be defined below.

Here we should mention that, by Remark 3.3, it is sufficient to construct entropy-entropy flux pairs $(\eta_{\pm k}, q_{\pm k})$ satisfying (4.11) in the region Σ_0,

$$\Sigma_0 = \{(r, s); r \geq r^0, s \leq s^0\} \subset \{(r, s); r > s\}, \tag{4.13}$$

and for this reason our discussion is focused on the region $\{(r, s); r > s\}$.

For convenience, we define

$$C_0^+(s) = \{h(s); h(s) \geq 0, h \in C_0^\infty(\mathbf{R}), \operatorname{supp}\{h\} \subset (-\infty, r^0)\}.$$

Given any fixed $h \in C_0^+(s)$, we suppose that $\operatorname{supp}\{h\} \subset [s^-, s^+]$, where $s^+ < r^0$. Then we define

$$A_0(r, s) = \lambda^{-1/2}(r, s)h(s), \qquad B_0(r, s) = \lambda^{1/2}(r, s)h(s), \tag{4.14}$$

and from (4.11) we see that A_1 and B_1 are defined by the recursion conditions

$$B_1 + \partial_s B_0 = \lambda(A_1 + \partial_s A_0), \qquad \partial_r B_1 = -\lambda \partial_r A_1, \tag{4.15}$$

or, equivalently,

$$\begin{aligned} &2\lambda \partial_r A_1 + (\partial_r \lambda) A_1 + \partial_r(\lambda \partial_s A_0 - \partial_s B_0) = 0, \\ &B_1 = \lambda A_1 + \lambda \partial_s A_0 - \partial_s B_0. \end{aligned} \tag{4.16}$$

Solving the ordinary differential equation with $A_1(s, s) = 0$, we get

$$\begin{aligned} A_1(r, s) = &-\frac{1}{2}\lambda^{-1/2}(r, s)[F_0(r, s) - F_0(s, s)] \\ &- \frac{1}{4}\lambda^{-1/2}(r, s) \int_s^r F_0(\tau, s) \partial_\tau \ln \lambda(\tau, s)\, d\tau, \\ B_1(r, s) = &\frac{1}{2}\lambda^{1/2}(r, s)[F_0(r, s) + F_0(s, s)] \\ &- \frac{1}{4}\lambda^{1/2}(r, s) \int_s^r F_0(\tau, s) \partial_\tau \ln \lambda(\tau, s)\, d\tau, \end{aligned} \tag{4.17}$$

where

$$F_0(r, s) = \lambda^{-1/2}(\lambda \partial_s A_0 - \partial_s B_0). \tag{4.18}$$

Having defined A_1 and B_1, we prove the following estimates.

Proposition 4.2. *Let* (A1)–(A3) *hold. On the half-plane* $\{(r, s); r \geq s\}$ *we have*

$$
\begin{aligned}
&|A_j(r, s)| \leq C_0 h(s), \qquad |B_j(r, s)| \leq C_0 h(s), \\
&|\partial_r^l \partial_s^m A_j(r, s)| \leq C, \qquad |\partial_r^l \partial_s^m B_j(r, s)| \leq C,
\end{aligned}
\tag{4.19}
$$

where $j = 0, 1$, $1 \leq l + m \leq 2$, $l, m = 0, 1, 2$, C *depends on* $h(\cdot)$ *and* $\sigma(\cdot)$, *and* C_0 *depends only on* $\sigma(\cdot)$. *Here we emphasize that* C_0 *is independent of* $h(\cdot)$.

Proof. Since from (1.3)

$$
\frac{\partial u}{\partial r} = -\frac{\partial u}{\partial s} = \frac{1}{2}\lambda^{-1} = \frac{1}{2}(\sigma')^{-1/2}, \qquad \frac{\partial v}{\partial r} = \frac{\partial v}{\partial s} = \frac{1}{2}, \tag{4.20}
$$

we see by (A1) and (3.3) that (4.19) holds for $A_0(r, s)$ and $B_0(r, s)$.

We now deal with $A_1(r, s)$. Since

$$
F_0(r, s) = -(\partial_s \ln \lambda) h(s) = \tfrac{1}{4}(\sigma')^{-3/2}\sigma'' h(s),
$$

we have that $F_0(s, s) = 0$, since $\sigma''(0) = 0$ and $r = s$ corresponds to $u = 0$. Furthermore, we notice from (A3) that

$$
\begin{aligned}
&\partial_r \ln \lambda \text{ is negative on } \{(r, s); r \geq s\}, \\
&\partial_s \ln \lambda \text{ is positive on } \{(r, s); r \leq s\}, \\
&\partial_s \ln \lambda = -\partial_r \ln \lambda.
\end{aligned}
\tag{4.21}
$$

Then we estimate

$$
|A_1(r, s)| \leq C_0 h(s) + C_0 h(s) \int_s^r -\partial_\tau \ln \lambda(\tau, s)\, d\tau \leq C_0 h(s),
$$

where C_0 is independent of $h(\cdot)$. A similar argument applies for $B_1(r, s)$.

Furthermore, we calculate that

$$
\begin{aligned}
\partial_s A_1(r, s) = {} & \frac{1}{4}\lambda^{-3/2}(r, s) F_0(r, s) \partial_s \lambda(r, s) - \frac{1}{2}\lambda^{-1/2}(r, s)\partial_s F_0(r, s) \\
& + \frac{1}{8}\lambda^{-3/2}(r, s)\partial_s \lambda(r, s) \int_s^r F_0(\tau, s)\partial_\tau \ln \lambda(\tau, s)\, d\tau \\
& - \frac{1}{4}\lambda^{-1/2}(r, s) \int_s^r \partial_s F_0(\tau, s)\partial_\tau \ln \lambda(\tau, s)\, d\tau \\
& - \frac{1}{4}\lambda^{-1/2}(r, s) \int_s^r F_0(\tau, s)\partial_s \partial_\tau \ln \lambda(\tau, s)\, d\tau.
\end{aligned}
$$

By (A1), (4.20), and (4.21),

$$
\begin{aligned}
|\partial_s A_1(r, s)| &\leq C + C\left|\int_s^r F_0(\tau, s)\partial_s \partial_\tau \ln \lambda(\tau, s)\, d\tau\right| \\
&= C + C\left|\int_s^r F_0(\tau, s)\partial_\tau^2 \ln \lambda(\tau, s)\, d\tau\right| \\
&\leq C + C\left|\int_s^r \partial_\tau F_0(\tau, s)\partial_\tau \ln \lambda(\tau, s)\, d\tau\right| \\
&\leq C\left(1 + \int_s^r -\partial_\tau \ln \lambda(\tau, s)\, d\tau\right) \leq C,
\end{aligned}
$$

where C depends on $h(\cdot)$ and $\sigma(\cdot)$. Using the same method we can prove the rest of (4.19), and hence we complete the proof.

We now define $P_{\pm k}$ and $Q_{\pm k}$. Our purpose is to obtain the following results.

Proposition 4.3. *There exist smooth functions $P_k(r, s)$ and $Q_k(r, s)$ defined on Σ_0 such that $(\eta_k(r, s), q_k(r, s))$ satisfies (4.11) in Σ_0. Moreover, we have*

$$\begin{aligned} & P_k(r, s) = 0, \quad Q_k(r, s) = 0 \quad \text{for } s \le s^-, \\ & |P_k(r, s)| \le C_0 k^{-1} \int_{s^-}^{s} e^{k\xi}[|h'(\xi)| + h(\xi)]\, d\xi, \qquad s \ge s^-, \\ & |Q_k(r, s)| \le C_0 k^{-1} \int_{s^-}^{s} e^{k\xi}[|h'(\xi)| + h(\xi)]\, d\xi, \qquad s \ge s^-, \end{aligned} \tag{4.22}$$

where C_0 is independent of $h(\cdot)$ and k, and

$$|\partial_r^l \partial_s^m P_k(r, s)| \le C(k, h), \qquad |\partial_r^l \partial_s^m Q_k(r, s)| \le C(k, h), \tag{4.23}$$

where $(r, s) \in \Sigma_0$, $1 \le l + m \le 2$, $l, m = 0, 1, 2$, and $C(k, h)$ is some constant depending on k and $h(\cdot)$.

Proposition 4.4. *There exist smooth functions $P_{-k}(r, s)$ and $Q_{-k}(r, s)$ defined on Σ_0 such that $(\eta_{-k}(r, s), q_{-k}(r, s))$ satisfies (4.11) in Σ_0. Moreover, we have*

$$\begin{aligned} & P_{-k}(r, s) = 0, \quad Q_{-k}(r, s) = 0 \quad \text{for } s \ge s^+, \\ & |P_{-k}(r, s)| \le C_0 k^{-1} \int_{s}^{s^+} e^{-k\xi}[|h'(\xi)| + h(\xi)]\, d\xi, \qquad s \le s^+, \\ & |Q_{-k}(r, s)| \le C_0 k^{-1} \int_{s}^{s^+} e^{-k\xi}[|h'(\xi)| + h(\xi)]\, d\xi, \qquad s \le s^+, \end{aligned} \tag{4.24}$$

where C_0 is independent of $h(\cdot)$ and k, and

$$|\partial_r^l \partial_s^m P_{-k}(r, s)| \le C(k, h), \qquad |\partial_r^l \partial_s^m Q_{-k}(r, s)| \le C(k, h), \tag{4.25}$$

where $(r, s) \in \Sigma_0$, $1 \le l + m \le 2$, $l, m = 0, 1, 2$, and $C(k, h)$ is some constant depending on k and $h(\cdot)$.

We notice from (4.11), (4.14), and (4.15) that $P_{\pm k}(r, s)$ and $Q_{\pm k}(r, s)$ should satisfy the linear hyperbolic equations:

$$\begin{aligned} & \partial_s Q_{\pm k} - \lambda \partial_s P_{\pm k} = (\pm k)^{-1} e^{\pm ks}(\lambda \partial_s A_1 - \partial_s B_1), \\ & \partial_r Q_{\pm k} + \lambda \partial_r P_{\pm k} = 0. \end{aligned} \tag{4.26}$$

Therefore, $P_k(r, s)$ is the solution of the linear hyperbolic equation

$$\partial_r \partial_s P_k + \tfrac{1}{2}(\partial_s \ln \lambda)\partial_r P_k + \tfrac{1}{2}(\partial_r \ln \lambda)\partial_s P_k = f_k, \tag{4.27}$$

where

$$f_k = -k^{-1} e^{ks} \frac{1}{2\lambda} \partial_r(\lambda \partial_s A_1 - \partial_s B_1).$$

By the standard theory (cf. Bitsadze [1964] and Sobolev [1964]), we can solve the initial value problem for (4.27).

We define the initial values along characteristics as follows:

$$\begin{aligned} & P_k(r, s) = 0 \quad \text{for } s = s^-, \\ & P_k(r, s) = 0 \quad \text{for } r = r^* := s^0 + \tfrac{1}{2}(r^0 - s^0) < r^0. \end{aligned} \tag{4.28}$$

By the Riemann representation we can express $P_k(r, s)$ in the form

$$P_k(r, s) = \int_{r^*}^{r} \int_{s^-}^{s} R_{r,s}(\alpha, \beta) f_k(\alpha, \beta)\, d\beta\, d\alpha \tag{4.29}$$

for $r \geq r^*$, $r^* \geq s \geq s^-$, where in (4.29) the Riemann function $R_{r,s}(\alpha, \beta)$ is the solution of the corresponding adjoint equation

$$\partial_\alpha \partial_\beta R_{r,s} - \tfrac{1}{2}\partial_\alpha[\partial_\beta \ln \lambda(\alpha, \beta) R_{r,s}] - \tfrac{1}{2}\partial_\beta[\partial_\alpha \ln \lambda(\alpha, \beta) R_{r,s}] = 0, \tag{4.30}$$

$r^* \leq \alpha \leq r$, $s^- \leq \beta \leq s$, with the following initial values along characteristics:

$$\begin{aligned} R_{r,s}(\alpha, s) &= [\lambda(r, s)]^{-1/2}[\lambda(\alpha, s)]^{1/2}, \\ R_{r,s}(r, \beta) &= [\lambda(r, s)]^{-1/2}[\lambda(r, \beta)]^{1/2}. \end{aligned} \tag{4.31}$$

Here we notice that $R_{r,s}(r, s) = 1$, and, by (3.3), that $R_{r,s}(\alpha, s)$ and $R_{r,s}(r, \beta)$ are uniformly bounded in r, s, α, and β.

Proposition 4.5. *We have*

$$\begin{gathered} R_{r,s}(\alpha, \beta), \partial_\alpha R_{r,s}(\alpha, \beta), \partial_\beta R_{r,s}(\alpha, \beta) \\ \text{are uniformly bounded in } r, s, \alpha, \beta, \end{gathered} \tag{4.32}$$

where $r^* \leq \alpha \leq r$ *and* $s^- \leq \beta \leq s \leq r^*$.

Proof. We first claim that there are r_M, s_M such that, if $(r, s) \in \{(r, s); r \geq r^*, s \leq r^*\}$,

$$\partial_r \ln \lambda(r, s) \text{ is monotone in } r, s, \text{ if } r \geq r_M, \text{ or } s \leq s_M. \tag{4.33}$$

The same results hold for $\partial_s \ln \lambda$, since $\partial_s \ln \lambda = -\partial_r \ln \lambda$.

We notice that

$$\begin{aligned} \partial_r^2 \ln \lambda(r, s) &= \partial_s^2 \ln \lambda(r, s) = -\partial_r \partial_s \ln \lambda(r, s) \\ &= -\frac{1}{4}(\lambda(u))^{-1}\frac{d^2}{du^2}\lambda^{-1}(u), \end{aligned}$$

where (r, s) corresponds to (u, v) under the mapping I. Therefore, by (A1), (4.33) holds if $u \geq M$. More precisely, (4.33) holds if we set

$$r_M = r^* + 2\int_0^M \lambda(\tau)\,d\tau, \qquad s_M = r^* - 2\int_0^M \lambda(\tau)\,d\tau,$$

where M is given by (A1).

Given any α_0, α_1, and β_1 with $r^* \leq \alpha_0 < \alpha_1 < r$, $s^- \leq \beta_1 < s$, and $\beta_1 < \alpha_1$, we take the integral of (4.30):

$$\int_{\beta_1}^{s}\int_{\alpha_1}^{r}\left\{\partial_\alpha\partial_\beta R_{r,s} - \frac{1}{2}\partial_\alpha[\partial_\beta \ln\lambda(\alpha, \beta)R_{r,s}] - \frac{1}{2}\partial_\beta[\partial_\alpha \ln\lambda(\alpha, \beta)R_{r,s}]\right\} d\alpha\, d\beta = 0.$$

By calculation we get

$$\begin{aligned} R_{r,s}(\alpha_1, \beta_1) &+ \frac{1}{2}\int_{\beta_1}^{s} \partial_\beta \ln\lambda(\alpha_1, \beta)R_{r,s}(\alpha_1, \beta)\,d\beta \\ &+ \frac{1}{2}\int_{\alpha_1}^{r} \partial_\alpha \ln\lambda(\alpha, \beta_1)R_{r,s}(\alpha, \beta_1)\,d\alpha = 1. \end{aligned}$$

Then

$$
\begin{aligned}
|R_{r,s}(\alpha_1, \beta_1)| \le{}& \frac{1}{2}\int_{\alpha_1}^{r} |\partial_\alpha \ln\lambda(\alpha, \beta_1)|\,|R_{r,s}(\alpha, \beta_1)|\, d\alpha \\
&+ 1 + \frac{1}{2}\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)| \max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta)|\, d\beta, \\
&\qquad \alpha_0 \le \alpha_1 \le r.
\end{aligned}
\tag{4.34}
$$

Regarding β_1 as a parameter in (4.34), we use Bellman's inequality[1] (cf. Bellman and Cooke [1963]) to get

$$
\begin{aligned}
|R_{r,s}(\alpha_1, \beta_1)| \le{}& e^{\frac{1}{2}\int_{\alpha_1}^{r} |\partial_\alpha \ln\lambda(\alpha, \beta_1)|\, d\alpha} \\
&\cdot \left[1 + \frac{1}{2}\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)| \max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta)|\, d\beta\right] \\
\le{}& C\left[1 + \frac{1}{2}\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)| \max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta)|\, d\beta\right],
\end{aligned}
\tag{4.35}
$$

where $\alpha_0 \le \alpha_1 \le r$, and where we have used (3.3) and (4.21).

(4.35) implies

$$
\begin{aligned}
&\max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta_1)| \\
&\quad \le C\left[1 + \frac{1}{2}\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)| \max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta)|\, d\beta\right].
\end{aligned}
$$

Again by Bellman's inequality we get

$$
\max_{\alpha_0\le\alpha_1\le r} |R_{r,s}(\alpha_1, \beta_1)| \le C e^{C\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)|\, d\beta}.
$$

On the other hand, from (A1) and (4.33),

$$
\begin{aligned}
&\int_{\beta_1}^{s} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)|\, d\beta \\
&\quad\le \int_{s_M}^{r^*} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)|\, d\beta + \int_{-\infty}^{s_M} \max_{\alpha_0\le\alpha_1\le r} |\partial_\beta \ln\lambda(\alpha_1, \beta)|\, d\beta \\
&\quad\le C|r^* - s_M| + \int_{-\infty}^{s_M} |\partial_\beta \ln\lambda(r, \beta)|\, d\beta + \int_{-\infty}^{s_M} |\partial_\beta \ln\lambda(\alpha_0, \beta)|\, d\beta \\
&\quad\le C,
\end{aligned}
$$

where C is independent of r, s, α_0, α_1, and β_1.

Combining the above we obtain

$$
|R_{r,s}(\alpha_1, \beta_1)| \le C, \qquad r^* \le \alpha_1 \le r,\ s^- \le \beta_1 \le s \le r^*,
$$

where C is independent of r, s, β_1, and α_1.

---

[1]Let $f(t)$, $a(t) \ge 0$ be continuous functions in $[a, b]$, let $A > 0$, and suppose that $f(t) \le A + \int_t^b a(\tau)f(\tau)\, d\tau$, $t \in [a, b]$. Then we have $f(t) \le A\exp(\int_t^b a(\tau)\, d\tau)$, $t \in [a, b]$.

Next, we integrate (4.30) with respect to β and obtain

$$\begin{aligned}\partial_\alpha R_{r,s}(\alpha, \beta_1) &+ \frac{1}{2}\int_{\beta_1}^{s}[\partial_\beta \ln\lambda(\alpha, \beta)]\partial_\alpha R_{r,s}(\alpha, \beta)\, d\beta \\ &+ \frac{1}{2}\int_{\beta_1}^{s}[\partial_\alpha\partial_\beta \ln\lambda(\alpha, \beta)]R_{r,s}(\alpha, \beta)\, d\beta \\ &= \partial_\alpha R_{r,s}(\alpha, s) - \frac{1}{2}[\partial_\alpha \ln\lambda(\alpha, s)R_{r,s}(\alpha, s) - \partial_\alpha \ln\lambda(\alpha, \beta_1)R_{r,s}(\alpha, \beta_1)].\end{aligned}$$

Hence

$$\begin{aligned}|\partial_\alpha R_{r,s}(\alpha, \beta_1)| \le \frac{1}{2}\int_{\beta_1}^{s}&[\partial_\beta \ln\lambda(\alpha, \beta)]|\partial_\alpha R_{r,s}(\alpha, \beta)|\, d\beta \\ &+ C\left(1 + \int_{\beta_1}^{s}|\partial_\alpha\partial_\beta \ln\lambda(\alpha, \beta)|\, d\beta\right).\end{aligned}$$

But, from (4.33) and (A1),

$$\begin{aligned}&\int_{\beta_1}^{s}|\partial_\alpha\partial_\beta \ln\lambda(\alpha, \beta)|\, d\beta \\ &\quad\le \int_{s_M}^{r^*}|\partial_\alpha\partial_\beta \ln\lambda(\alpha, \beta)|\, d\beta + \int_{-\infty}^{s_M}|\partial_\alpha\partial_\beta \ln\lambda(\alpha, \beta)|\, d\beta \le C,\end{aligned}$$

where C is independent of r, s, α, and β_1. Then, by the Bellman inequality, we get

$$|\partial_\alpha R_{r,s}(\alpha, \beta_1)| \le C, \qquad r^* \le \alpha \le r,\ s^- \le \beta_1 \le s \le r^*.$$

A similar argument applies for $\partial_\beta R_{r,s}(\alpha, \beta)$, and so we complete the proof.

Remark 4.1. From the proof above we see that in hypothesis (A1) the condition where $\lambda^{-1}(u)$ is concave for $u \ge M$ and convex for $u \le -M$ can be replaced by $\sigma'''(\cdot) \in L^1(\mathbf{R})$.

The existence of $Q_k(r, s)$ follows from that of $P_k(r, s)$. In fact, we get from (4.26) that

$$Q_k = \lambda P_k - \int_{s^-}^{s}(\partial_\beta\lambda)P_k\, d\beta + \frac{1}{k}\int_{s^-}^{s} e^{k\beta}(\lambda\partial_\beta A_1 - \partial_\beta B_1)\, d\beta. \tag{4.36}$$

Proof of Proposition 4.3. By (4.14) and (4.15) we calculate that

$$\begin{aligned}&\lambda^{-1/2}(\lambda\partial_s A_1 - \partial_s B_1) \\ &\quad= \lambda^{-1/2}[\lambda\partial_s A_1 - \partial_s(\lambda A_1 + \lambda\partial_s A_0 - \partial_s B_0)] \\ &\quad= (\partial_s \ln\lambda)h'(s) + (\partial_s^2 \ln\lambda)h(s) \\ &\qquad+ \frac{1}{4}(\partial_s \ln\lambda)\int_s^r F_0(\tau, s)\partial_\tau \ln\lambda(\tau, s)\, d\tau,\end{aligned} \tag{4.37}$$

where $F_0(\tau, s) = -[\partial_s \ln\lambda(\tau, s)]h(s)$, and we get

$$
\begin{aligned}
f_k(r, s) &= \frac{1}{2k}e^{ks}[-\lambda^{-1}\partial_r(\lambda\partial_s A_1 - \partial_s B_1)] \\
&= -\frac{1}{2k}e^{ks}\lambda^{-1}\partial_r(\lambda^{1/2}\partial_s \ln\lambda)h'(s) \\
&\quad - \frac{1}{2k}e^{ks}\left[\frac{1}{2}\lambda^{-1/2}(\partial_r \ln\lambda)(\partial_s^2 \ln\lambda) + \lambda^{-1/2}\partial_r\partial_s^2 \ln\lambda \right. \\
&\qquad\qquad \left. + \frac{1}{4}\lambda^{-1/2}(\partial_s \ln\lambda)^3\right] h(s) \\
&\quad - \frac{1}{2k}e^{ks}\left[\frac{1}{4}\lambda^{-1/2}(\partial_s^2 \ln\lambda) + \frac{1}{8}\lambda^{-1/2}(\partial_s \ln\lambda)^2\right] \\
&\qquad \times \int_s^r F_0(\tau, s)\partial_\tau \ln\lambda(\tau, s)\, d\tau.
\end{aligned} \tag{4.38}
$$

Then, as in the proof of Proposition 4.5,

$$
\begin{aligned}
|P_k(r, s)| &= \left|\int_{r^*}^r \int_{s-}^s R_{r,s}(\alpha, \beta) f_k(\alpha, \beta)\, d\beta\, d\alpha\right| \\
&\le C_0 \frac{1}{k}\int_{s-}^s e^{k\beta}[|h'(\beta)| + h(\beta)] \int_{r^*}^r [|\partial_\alpha^2 \ln\lambda| + |\partial_\alpha \ln\lambda|]\, d\alpha \\
&\quad + \frac{1}{2k}\left|\int_{s-}^s e^{k\beta}h(\beta)\, d\beta \int_{r^*}^r \lambda^{-1/2}(\partial_\alpha\partial_\beta^2 \ln\lambda) R_{r,s}(\alpha, \beta)\, d\alpha\right| \\
&\le C_0 \frac{1}{k}\int_{s-}^s e^{k\beta}[|h'(\beta)| + h(\beta)]\, d\beta \\
&\quad + \frac{1}{2k}\left|\int_{s-}^s e^{k\beta}h(\beta)\lambda^{-1/2}(r, \beta)\partial_\beta^2 \ln\lambda(r, \beta) R_{r,s}(r, \beta)\, d\beta\right| \\
&\quad + \frac{1}{2k}\left|\int_{s-}^s e^{k\beta}h(\beta)\lambda^{-1/2}(r^*, \beta)\partial_\beta^2 \ln\lambda(r^*, \beta) R_{r,s}(r^*, \beta)\, d\beta\right| \\
&\quad + \frac{1}{2k}\left|\int_{s-}^s e^{k\beta}h(\beta)\int_{r^*}^r \lambda^{-1/2}(\partial_\beta^2 \ln\lambda)(\partial_\alpha R_{r,s})\, d\alpha\, d\beta\right| \\
&\quad + \frac{1}{2k}\left|\int_{s-}^s e^{k\beta}h(\beta)\int_{r^*}^r \frac{1}{2}\lambda^{-1/2}(\partial_\alpha \ln\lambda)(\partial_\beta^2 \ln\lambda) R_{r,s}\, d\alpha\, d\beta\right| \\
&\le C_0 \frac{1}{k}\int_{s-}^s e^{k\beta}[|h'(\beta)| + h(\beta)]\, d\beta,
\end{aligned}
$$

where C_0 is independent of k and $h(\cdot)$.

By (4.36) we have a similar estimate for $Q_k(r, s)$, so we get (4.22).

Furthermore, integrating (4.27) with respect to s we obtain

$$
\begin{aligned}
\partial_r P_k(r, s) &+ \frac{1}{2}\int_{s-}^s [\partial_\xi \ln\lambda(r, \xi)]\partial_r P_k(r, \xi)\, d\xi \\
&+ \frac{1}{2}\int_{s-}^s [\partial_r \ln\lambda(r, \xi)]\partial_\xi P_k(r, \xi)\, d\xi \\
&= \int_{s-}^s f_k(r, \xi)\, d\xi.
\end{aligned}
$$

Applying integration by parts we get

$$\begin{aligned}\partial_r P_k(r, s) &+ \frac{1}{2}\int_{s^-}^{s} [\partial_\xi \ln\lambda(r, \xi)]\partial_r P_k(r, \xi)\, d\xi \\ &= \frac{1}{2}\int_{s^-}^{s} [\partial_\xi\partial_r \ln\lambda(r, \xi)] P_k(r, \xi)\, d\xi \\ &\quad + \int_{s^-}^{s} f_k(r, \xi)\, d\xi - \frac{1}{2}[\partial_r \ln\lambda(r, s)] P_k(r, s).\end{aligned}$$

Then it follows that

$$\begin{aligned}&\left|\partial_r P_k(r, s) + \frac{1}{2}\int_{s^-}^{s} \partial_\xi \ln\lambda(r, \xi)\partial_r P_k(r, \xi)\, d\xi\right| \\ &\quad \le Ck^{-1}e^{ks^+}\left(1 + \int_{s^-}^{s^+} |\partial_\xi\partial_r \ln\lambda(r, \xi)|\, d\xi\right) \le Ck^{-1}e^{ks^+},\end{aligned}$$

and, again by the Bellman inequality,

$$|\partial_r P_k(r, s)| \le Ck^{-1}e^{ks^+},$$

which is part of (4.23).

Similarly, we can prove the rest of (4.23), and we complete the proof.

From the proof of Proposition 4.3 we can obtain the following expressions.

Proposition 4.6. *For* $(r, s) \in \Sigma_0$, *we have*

$$\begin{aligned}P_k(r, s) &= \frac{1}{2k}\lambda^{-1/2}(r, s)h(s)e^{ks} \\ &\quad \cdot \int_{r^*}^{r} \left[-\partial_\alpha\partial_s \ln\lambda(\alpha, s) - \frac{1}{2}\partial_\alpha \ln\lambda(\alpha, s)\partial_s \ln\lambda(\alpha, s)\right] d\alpha \\ &\quad + P_k^0(r, s),\end{aligned} \tag{4.39}$$

with $|P_k^0| \le C_0 \int_{s^-}^{s} e^{k\beta}h(\beta)\, d\beta$, *and*

$$\begin{aligned}Q_k - \lambda P_k &= \frac{1}{k}e^{ks}\lambda^{-1/2}(\partial_s\lambda)h(s) - \int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h(\beta)\, d\beta \\ &\quad + \Delta_k^1(h) + \Delta_k^2(h),\end{aligned} \tag{4.40}$$

with

$$|\Delta_k^1(h)| \le \frac{C_0}{k}\int_{s^-}^{s} e^{k\beta}h(\beta)\, d\beta, \qquad |\Delta_k^2(h)| \le C_0\int_{s^-}^{s} d\beta \int_{s^-}^{\beta} e^{k\xi}h(\xi)\, d\xi,$$

where C_0 *is independent of* k *and* $h(\cdot)$.

Proof. By (4.38) we easily get

$$\begin{aligned}&\left|P_k - \int_{s^-}^{s}\int_{r^*}^{r} -\frac{1}{2k}e^{k\beta}\lambda^{-1}\partial_\alpha[\lambda^{1/2}\partial_\beta \ln\lambda]h'(\beta)R_{r,s}(\alpha, \beta)\, d\alpha\, d\beta\right| \\ &\quad \le \frac{C_0}{k}\int_{s^-}^{s} e^{k\beta}h(\beta)\, d\beta.\end{aligned}$$

On the other hand, by (4.31) we calculate

$$\begin{aligned}
&\int_{s^-}^{s}\int_{r^*}^{r} -\frac{1}{2k}e^{k\beta}\lambda^{-1}\partial_\alpha[\lambda^{1/2}\partial_\beta \ln\lambda]h'(\beta)R_{r,s}(\alpha,\beta)\,d\alpha\,d\beta\\
&\quad= -\frac{1}{2k}e^{ks}h(s)\int_{r^*}^{r}\lambda^{-1}(\alpha,s)\partial_\alpha[\lambda^{1/2}\partial_s\ln\lambda]R_{r,s}(\alpha,s)\,d\alpha\\
&\qquad+\frac{1}{2k}\int_{s^-}^{s}\int_{r^*}^{r} h(\beta)\partial_\beta\{e^{k\beta}\lambda^{-1}\partial_\alpha[\lambda^{1/2}\partial_\beta\ln\lambda]R_{r,s}(\alpha,\beta)\}\,d\alpha\,d\beta\\
&\quad= -\frac{1}{2k}\lambda^{-1/2}(r,s)e^{ks}h(s)\int_{r^*}^{r}\Big[\partial_\alpha\partial_s\ln\lambda(\alpha,s)\\
&\qquad\qquad+\frac{1}{2}\partial_\alpha\ln\lambda(\alpha,s)\partial_s\ln\lambda(\alpha,s)\Big]\,d\alpha\\
&\qquad+\frac{1}{2k}\int_{s^-}^{s}\int_{r^*}^{r} h(\beta)\partial_\beta\{e^{k\beta}\lambda^{-1}\partial_\alpha[\lambda^{1/2}\partial_\beta\ln\lambda]R_{r,s}(\alpha,\beta)\}\,d\alpha\,d\beta.
\end{aligned}$$

We see that the last term on the right-hand side of the above equality is bounded by $C_0\int_{s^-}^{s} e^{k\beta}h(\beta)\,d\beta$. Combining the above we get (4.39).

(4.40) follows from (4.15) and (4.36) since

$$\begin{aligned}
&\frac{1}{k}\int_{s^-}^{s} e^{k\beta}(\lambda\partial_\beta A_1-\partial_\beta B_1)\,d\beta\\
&\quad=\frac{1}{k}\int_{s^-}^{s} e^{k\beta}\{-(\partial_\beta\lambda)A_1+\partial_\beta[(\partial_\beta\lambda)A_0]\}\,d\beta\\
&\quad=\frac{1}{k}e^{ks}(\partial_s\lambda)A_0-\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)A_0\,d\beta-\frac{1}{k}\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)A_1\,d\beta.
\end{aligned}$$

Hence we complete the proof.

The discussion about $P_{-k}(r,s)$ and $Q_{-k}(r,s)$ is quite similar. Our purpose is to construct P_{-k} and Q_{-k} to be defined for $s\le s^+$, while they are identically zero for $s\ge s^+$. It is sufficient to propose the following initial values:

$$\begin{aligned}
P_{-k}(r,s)&=0 \quad \text{for } s=s^+,\ r\ge r^*,\\
P_{-k}(r,s)&=0 \quad \text{for } r=r^*,\ s\le s^+.
\end{aligned}$$

Then we proceed as before and prove the existence of P_{-k} and Q_{-k} defined on Σ_0 which admits Proposition 4.4. In particular, we have the following result similar to Proposition 4.6.

Proposition 4.7. *For $(r,s)\in\Sigma_0$, we have*

$$\begin{aligned}
P_{-k}(r,s)=&-\frac{1}{2k}\lambda^{-1/2}(r,s)h(s)e^{-ks}\\
&\cdot\int_{r^*}^{r}\Big[-\partial_\alpha\partial_s\ln\lambda(\alpha,s)-\frac{1}{2}\partial_\alpha\ln\lambda(\alpha,s)\partial_s\ln\lambda(\alpha,s)\Big]\,d\alpha\\
&+P^{-}_{-k}(r,s),
\end{aligned}\tag{4.41}$$

with $|P^0_{-k}|\le C_0\int_s^{s^+} e^{-k\beta}h(\beta)\,d\beta$, and

$$\begin{aligned}
Q_{-k}-\lambda P_{-k}=&-\frac{1}{k}e^{-ks}\lambda^{-1/2}(\partial_s\lambda)h(s)+\int_s^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h(\beta)\,d\beta\\
&+\Delta^1_{-k}(h)+\Delta^2_{-k}(h),
\end{aligned}\tag{4.42}$$

with

$$|\Delta^1_{-k}(h)| \le \frac{C_0}{k}\int_s^{s^+} e^{-k\beta}h(\beta)\,d\beta\,, \qquad |\Delta^2_{-k}(h)| \le C_0\int_s^{s^+} d\beta\int_\beta^{s^+} e^{-k\xi}h(\xi)\,d\xi\,,$$

where C_0 *is independent of* k *and* $h(\cdot)$.

From the procedure above we conclude that for each $h \in C_0^+(s)$ we can construct two pairs of entropy-entropy flux $(\eta_{\pm k}, q_{\pm k})$ in the form of (4.12) which satisfy (4.11) in Σ_0. For distinction and when it is convenient we sometimes may use one of the following notations:

$$\begin{gathered}\eta_{\pm k}(u, v) = \eta_{\pm k}(r, s) = \eta_{\pm k}(h(s)) = \eta_{\pm k}(r, s; h(s)) = \eta_{\pm k}(u, v; h(s)),\\ q_{\pm k}(u, v) = q_{\pm k}(r, s) = q_{\pm k}(h(s)) = q_{\pm k}(r, s; h(s)) = q_{\pm k}(u, v; h(s)),\\ A_i(r, s) = A_i(h(s)), \quad B_i(r, s) = B_i(h(s)), \qquad i = 0, 1,\\ P_{\pm k}(r, s) = P_{\pm k}(h(s)), \qquad Q_{\pm k}(r, s) = Q_{\pm k}(h(s)).\end{gathered}$$

We denote by E_s all those entropy-entropy flux constructed as above, namely,

$$E_s = \{(\eta_{\pm k}(h(s)), q_{\pm k}(h(s))); h(s) \in C_0^+(s)\}. \tag{4.43}$$

From Propositions 4.2–4.4 and (4.20), we can obtain the following results.

Theorem 4.8. *For each* $(\eta_{\pm k}(h), q_{\pm k}(h)) \in E_s$, *there exists* $C > 0$, *depending on* k, h, *and* σ, *such that*

$$\begin{gathered}|\partial_u^l\partial_v^m \eta_{\pm k}(u, v; h)| \le C, \qquad |\partial_u^l\partial_v^m q_{\pm k}(u, v; h)| \le C,\\ \forall (u, v) \in I^{-1}(\Sigma_0), \quad 0 \le l + m \le 2, \ l, m = 0, 1, 2,\end{gathered}$$

where I^{-1} *is the inverse mapping of* I.

Hence, as a consequence of Theorem 4.1, we have

Theorem 4.9. *For each* $(\eta_{\pm k}(h), q_{\pm k}(h)) \in E_s$,

$$\begin{gathered}\partial_t\eta_{\pm k}(u^\varepsilon(x, t), v^\varepsilon(x, t)) + \partial_x q_{\pm k}(u^\varepsilon(x, t), v^\varepsilon(x, t))\\ \textit{is relatively compact in } H^{-1}_{\mathrm{loc}}(\mathbf{R}\times(0, \infty)).\end{gathered} \tag{4.44}$$

We now begin the construction of another type of Lax's entropy, namely, we consider entropy-entropy flux pairs of the form

$$\begin{aligned}\overline{\eta}_{\pm k}(r, s) &= e^{\pm kr}(a_0 + a_1(\pm k)^{-1}) + W_{\pm k},\\ \overline{q}_{\pm k}(r, s) &= e^{\pm kr}(b_0 + b_1(\pm k)^{-1}) + Z_{\pm k},\end{aligned} \tag{4.45}$$

where $k = 2, 3, 4, \dots$, and a_j, b_j, $W_{\pm k}$, and $Z_{\pm k}$ are smooth functions of r and s defined by the following recursive compatibility conditions:

$$\begin{gathered}b_0 = -\lambda a_0, \qquad b_1 + \partial_r b_0 = -\lambda(a_1 + \partial_r a_0),\\ \partial_s b_j = \lambda\partial_s a_j, \qquad j = 0, 1,\end{gathered}$$

which are equivalent to

$$\begin{gathered}2\lambda\partial_s a_0 + (\partial_s\lambda)a_0 = 0, \qquad b_0 = -\lambda a_0,\\ 2\lambda\partial_s a_1 + (\partial_s\lambda)a_1 + \partial_s(\lambda\partial_r a_0 + \partial_r b_0) = 0,\\ b_1 = -\lambda a_1 - (\lambda\partial_r a_0 + \partial_r b_0).\end{gathered}$$

Given any $h \in C_0^+(r)$, where

$$C_0^+(r) = \{h(r)\,;\, h(r) \geq 0\,,\, h \in C_0^\infty(\mathbf{R})\,,\, \operatorname{supp}\{h\} \subset (s^0\,,\, +\infty)\}\,,$$

we suppose that $\operatorname{supp}\{h\} \subset [r^-\,,\, r^+]$, $r^- > s^0$. We define

$$a_0(r\,,\, s) = \lambda^{-1/2}(r\,,\, s)h(r)\,, \qquad b_0(r\,,\, s) = -\lambda^{1/2}(r\,,\, s)h(r).$$

Then a_1 and b_1 are given by

$$\begin{aligned} a_1(r\,,\, s) = & -\frac{1}{2}\lambda^{-1/2}(r\,,\, s)[G_0(r\,,\, s) - G_0(r\,,\, r)] \\ & - \frac{1}{4}\lambda^{-1/2}(r\,,\, s)\int_s^r G_0(r\,,\, \xi)\partial_\xi \ln\lambda(r\,,\, \xi)\, d\xi\,, \\ b_1(r\,,\, s) = & -\frac{1}{2}\lambda^{1/2}(r\,,\, s)[G_0(r\,,\, s) + G_0(r\,,\, r)] \\ & + \frac{1}{4}\lambda^{1/2}(r\,,\, s)\int_s^r G_0(r\,,\, \xi)\partial_\xi \ln\lambda(r\,,\, \xi)\, d\xi\,, \end{aligned}$$

where $G_0(r\,,\, s) = \lambda^{-1/2}(\lambda\partial_r a_0 + \partial_r b_0) = -(\partial_r \ln\lambda)h(r)$.

Moreover, $W_{\pm k}(r\,,\, s)$ and $Z_{\pm k}(r\,,\, s)$ can be defined by solving the linear hyperbolic equations

$$\begin{aligned} \partial_s Z_{\pm k} - \lambda\partial_s W_{\pm k} &= 0\,, \\ \partial_r Z_{\pm k} + \lambda\partial_r W_{\pm k} &= (\pm k)^{-1}e^{\pm kr}(\lambda\partial_r a_1 + \partial_r b_1). \end{aligned}$$

We now denote by E_r all these entropy-entropy flux pairs constructed for any $h \in C_0^+(r)$, that is,

$$E_r = \{(\overline{\eta}_{\pm k}(h(r))\,,\, \overline{q}_{\pm k}(h(r)))\,;\, h(r) \in C_0^+(r)\}.$$

We still have the results for E_r parallel to that for E_s, but we omit the details.

5. Reduction of Young measures

In this section we will use the notation

$$r^\varepsilon(x\,,\, t) = r(u^\varepsilon(x\,,\, t)\,,\, v^\varepsilon(x\,,\, t))\,, \qquad s^\varepsilon(x\,,\, t) = s(u^\varepsilon(x\,,\, t)\,,\, v^\varepsilon(x\,,\, t))\,,$$

where r and s are the Riemann invariants defined by (1.3). The first result in the following theorem can be derived from Corollary 2.2 and the Div-Curl lemma.

Theorem 5.1. *There exist a subsequence $\{r^{\varepsilon_n}\,,\, s^{\varepsilon_n}\}$ of $\{r^\varepsilon\,,\, s^\varepsilon\}$ and a family of positive measures $\mu_{x,t} \in M(\mathbf{R}^2)$ such that, for any entropy-entropy flux $(\eta^i_{\pm k}\,,\, q^i_{\pm k}) \in E_s \cup E_r$, $i = 1\,,\, 2$, we have*

$$\langle \eta^1_m q^2_l - q^1_m \eta^2_l\,,\, \mu_{x,t}\rangle = \langle \eta^1_m\,,\, \mu_{x,t}\rangle\langle q^2_l \mu_{x,t}\rangle - \langle q^1_m\,,\, \mu_{x,t}\rangle\langle \eta^2_l\,,\, \mu_{x,t}\rangle\,, \tag{5.1}$$

for $(x\,,\, t) \in \mathbf{R} \times (0\,,\, \infty)$ and $l\,,\, m = \pm 2, \pm 3, \pm 4, \ldots,$ and

$$\begin{aligned} \eta^i_{\pm k}(r^{\varepsilon_n}\,,\, s^{\varepsilon_n}) &\rightharpoonup \langle \eta^i_{\pm k}\,,\, \mu_{x,t}\rangle \quad \text{in } L^1_{\text{loc}}(\mathbf{R} \times (0\,,\, \infty))\,, \\ q^i_{\pm k}(r^{\varepsilon_n}\,,\, s^{\varepsilon_n}) &\rightharpoonup \langle q^i_{\pm k}\,,\, \mu_{x,t}\rangle \quad \text{in } L^1_{\text{loc}}(\mathbf{R} \times (0\,,\, \infty))\,, \end{aligned} \tag{5.2}$$

for $k = 1\,,\, 2\,,\, 3\,,\, \ldots$ and $i = 1\,,\, 2$.

Proof. By Theorem 3.2 we get that $\{r^\varepsilon, s^\varepsilon\}$ is bounded in $L^2_{\mathrm{loc}}(\mathbf{R}\times\mathbf{R}^+)$. Therefore, applying Corollary 2.2, we obtain that there exist a subsequence $\{r^{\varepsilon_n}, s^{\varepsilon_n}\}$ of $\{r^\varepsilon, s^\varepsilon\}$ and a family of positive measures $\mu_{x,t} \in M(\mathbf{R}^2)$ such that

$$(5.3) \qquad f(r^{\varepsilon_n}(x, t), s^{\varepsilon_n}(x, t)) \rightharpoonup \langle f(r, s), \mu_{x,t}\rangle \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}\times\mathbf{R}^+),$$

whenever $f \in C(\mathbf{R}^2)$ satisfies

$$\frac{f(r, s)}{\sqrt{r^2+s^2}} \to 0 \quad \text{as } r^2+s^2 \to \infty.$$

This implies (5.2). Furthermore, it follows from Theorem 4.8 and Theorem 4.13 that

$$\begin{aligned}&\eta^1_m(r^{\varepsilon_n}, s^{\varepsilon_n})q^2_l(r^{\varepsilon_n}, s^{\varepsilon_n}) - q^1_m(r^{\varepsilon_n}, s^{\varepsilon_n})\eta^2_l(r^{\varepsilon_n}, s^{\varepsilon_n})\\ &\qquad \rightharpoonup \langle \eta^1_m q^2_l - q^1_m\eta^2_l, \mu_{x,t}\rangle \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}\times\mathbf{R}^+),\ l, m = \pm2, \pm3, \pm4, \dots.\end{aligned}$$

On the other hand, by the Div-Curl lemma we get

$$\begin{aligned}&\eta^1_m(r^{\varepsilon_n}, s^{\varepsilon_n})q^2_l(r^{\varepsilon_n}, s^{\varepsilon_n}) - q^1_m(r^{\varepsilon_n}, s^{\varepsilon_n})\eta^2_l(r^{\varepsilon_n}, s^{\varepsilon_n})\\ &\qquad \to \langle \eta^1_m, \mu_{x,t}\rangle\langle q^2_l, \mu_{x,t}\rangle - \langle q^1_m, \mu_{x,t}\rangle\langle \eta^2_l, \mu_{x,t}\rangle,\end{aligned}$$

in the sense of distributions, $l, m = \pm2, \pm3, \pm4, \dots$. Combining the above we get (5.1) and complete the proof.

Remark 5.1. In the proof of Theorem 5.1 we need only (3.12), rather than (3.5).

Our purpose is to prove that each positive measure $\mu_{x,t}$ is indeed a Dirac measure (we will drop the index $\{x, t\}$ for simplicity). The ideas of the proof are as follows:

1. We prove that, in the s-axis direction, the following case, denoted by (S1), cannot happen. There exist s^1, s^2, s^3, l_0, $0 < l_0 < \frac{1}{2}(r^0 - s^0)$, $-\infty < s^1 < s^1 + 3l_0 < s^2 < s^2 + 3l_0 < s^3 \le s^0$, such that for any $0 < l \le l_0$

$$\begin{aligned}&\mu\{(r, s); s^1 - l < s < s^1 + l, -\infty < r < \infty\} \ne 0,\\ &\mu\{(r, s); s^2 - l < s < s^2 + l, -\infty < r < \infty\} \ne 0,\\ &\mu\{(r, s); s^3 - l < s < s^3 + l, -\infty < r < \infty\} \ne 0.\end{aligned}$$

Since the support of μ is a subset of Σ_0, we assume that

$$(5.4) \qquad \mu\{(r, s); s > s^3, -\infty < r < \infty\} = 0,$$

i.e., we take s^3 maximal.

2. Since (S1) is impossible, the support of μ must lie on two lines, that is, there exist s^- and s^+ such that

$$(5.5) \qquad \mathrm{supp}\{\mu\} \subset \{(r, s); s = s^-\} \cup \{(r, s); s = s^+\}.$$

Then we use DiPerna's argument to deduce that the support of μ lies only on one line, that is, there is an s^0 such that

$$(5.6) \qquad \mathrm{supp}\{\mu\} \subset \{(r, s); s = s^0\}.$$

3. We carry out a similar argument in the r-axis direction and prove that the support of μ lies only on one line, i.e., there is an r^0 such that

$$(5.7) \qquad \mathrm{supp}\{\mu\} \subset \{(r, s); r = r^0\}.$$

Combining (5.6) and (5.7) we get that μ is a Dirac measure with support at (r^0, s^0).

Theorem 5.2. (S1) *is impossible.*

Proof. If, conversely, (S1) holds, we would have a contradiction. To prove this, we argue in several steps. Our proof is motivated in part by the argument of Serre [1986] (cf. also Shearer [1989]).

Step 1. We choose $h_i \in C_0^+(s)$, $i = 1, 2, 3$, as follows:

$$\begin{aligned} &h_1 \le 1, \quad \operatorname{supp}\{h_1\} \subset [s^1 - l_0, s^1 + l_0], \\ &h_2 \le 1, \quad \operatorname{supp}\{h_2\} := [s^-, s^+] \subset [s^2 - l_0, s^2 + l_0], \\ &h_3 \le 1, \quad \operatorname{supp}\{h_3\} \subset [s^3 - l_0, s^3 + l_0], \\ &\qquad\qquad h_3(s) = 1 \text{ for } s \in [s^3 - \tfrac{1}{2}l_0, s^3 + \tfrac{1}{2}l_0]. \end{aligned}$$

We observe that h_i can be obtained by means of mollification.

By §4 we can construct the corresponding entropy-entropy flux pairs:

$$(\eta_{\pm k}(h_i), q_{\pm k}(h_i)) \in E_s, \qquad i = 1, 2, 3, \ k = 2, 3, 4, \dots.$$

We now give some basic results that we shall use later. First, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{\langle \eta_k(h_3), \mu\rangle}{\langle e^{ks}A_0(h_3), \mu\rangle} = 1, \\ (5.8)\qquad &(\delta_0)^{1/2} \le \lim_{k\to\infty} \frac{\langle q_k(h_3), \mu\rangle}{\langle e^{ks}A_0(h_3), \mu\rangle} \le (\sigma'(0))^{1/2}. \end{aligned}$$

In fact, we recall from Propositions 4.2 and 4.3 that

$$\begin{aligned} \eta_k(h_3) &= e^{ks}(\lambda^{-1/2} + O(k^{-1}))h_3 + P_k(h_3), \\ q_k(h_3) &= e^{ks}(\lambda^{1/2} + O(k^{-1}))h_3 + Q_k(h_3), \end{aligned}$$

with

$$\begin{aligned} |P_k(h_3)| &\le C_0 k^{-1} \int_{s^3 - l_0}^{s} e^{k\xi}[|h_3'(\xi)| + h_3(\xi)]\, d\xi, \\ |Q_k(h_3)| &\le C_0 k^{-1} \int_{s^3 - l_0}^{s} e^{k\xi}[|h_3'(\xi)| + h_3(\xi)]\, d\xi. \end{aligned}$$

We can choose h_3 such that $h_3'(s) \ge 0$ for $s^3 - l_0 \le s \le s^3$. We then have

$$|P_k(h_3)| \le Ck^{-1}e^{ks}h_3(s), \quad |Q_k(h_3)| \le Ck^{-1}e^{ks}h_3(s) \quad \text{if } s \le s^3 + \tfrac{1}{2}l_0,$$

where C is independent of k and h. Thus $(5.8)_1$ follows from the fact that μ is zero on the region $\{(r, s); s > s^3\}$. $(5.8)_2$ is also true since $(\delta_0)^{1/2} \le \lambda \le (\sigma'(0))^{1/2}$.

We also have

$$(5.9)\qquad \langle \eta_k(h_3)q_{-k}(h_1) - q_k(h_3)\eta_{-k}(h_1), \mu\rangle = 0 \quad \forall k.$$

This holds since the supports of $\eta_k(h_3)$ and $q_k(h_3)$ do not intersect the supports of $\eta_{-k}(h_1)$ and $q_{-k}(h_1)$, due to our construction in §4.

Step 2. (i) Suppose that there is a subsequence of $\{k\}$, also denoted by $\{k\}$, such that

$$(5.10)\qquad \langle \eta_{-k}(h_1), \mu\rangle \ne 0 \quad \forall k.$$

Then we have

$$\langle \eta_k(h_2)q_{-k}(h_2) - q_k(h_2)\eta_{-k}(h_2), \mu\rangle = 0 \quad \forall k. \tag{5.11}$$

Indeed, from (5.9) we have

$$\frac{\langle q_{-k}(h_1), \mu\rangle}{\langle \eta_{-k}(h_1), \mu\rangle} = \frac{\langle q_k(h_3), \mu\rangle}{\langle \eta_k(h_3), \mu\rangle}, \tag{5.12}$$

where we have used the fact that $\langle \eta_k(h_3), \mu\rangle \neq 0$ if k is large enough, due to $(5.8)_1$.

On the other hand, from

$$\begin{aligned} 0 &= \langle \eta_k(h_3)q_{-k}(h_2) - q_k(h_3)\eta_{-k}(h_2), \mu\rangle \\ &= \langle \eta_k(h_3), \mu\rangle\langle q_{-k}(h_2), \mu\rangle - \langle q_k(h_3), \mu\rangle\langle \eta_{-k}(h_2), \mu\rangle \end{aligned}$$

and

$$\begin{aligned} 0 &= \langle \eta_{-k}(h_1)q_k(h_2) - q_{-k}(h_1)\eta_k(h_2), \mu\rangle \\ &= \langle \eta_{-k}(h_1), \mu\rangle\langle q_k(h_2), \mu\rangle - \langle q_{-k}(h_1), \mu\rangle\langle \eta_k(h_2), \mu\rangle, \end{aligned}$$

we get

$$\langle q_{-k}(h_2), \mu\rangle = \frac{\langle q_k(h_3), \mu\rangle}{\langle \eta_k(h_3), \mu\rangle}\langle \eta_{-k}(h_2), \mu\rangle,$$

$$\langle q_k(h_2), \mu\rangle = \frac{\langle q_{-k}(h_1), \mu\rangle}{\langle \eta_{-k}(h_1), \mu\rangle}\langle \eta_k(h_2), \mu\rangle.$$

Substituting these into

$$\begin{aligned} &\langle \eta_k(h_2)q_{-k}(h_2) - q_k(h_2)\eta_{-k}(h_2), \mu\rangle \\ &\quad = \langle \eta_k(h_2), \mu\rangle\langle q_{-k}(h_2), \mu\rangle - \langle q_k(h_2), \mu\rangle\langle \eta_{-k}(h_2), \mu\rangle \end{aligned}$$

and using (5.12) we get (5.11).

(ii) Suppose that there is a subsequence of $\{k\}$, also denoted by $\{k\}$, such that

$$\langle q_{-k}(h_1), \mu\rangle \neq 0 \quad \forall k.$$

By the same argument we still have (5.11).

(iii) If

$$\langle \eta_{-k}(h_1), \mu\rangle = 0, \quad \langle q_{-k}(h_1), \mu\rangle = 0 \quad \forall k,$$

we then have

$$\langle \eta_k(h_1)q_{-k}(h_1) - q_k(h_1)\eta_{-k}(h_1), \mu\rangle = 0 \quad \forall k. \tag{5.13}$$

Step 3. If (5.11) holds, we would have

$$\begin{aligned} &\left\langle e^{-ks}h_2(s)\lambda^{-1/2}\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta, \mu\right\rangle \\ &\quad + \left\langle e^{ks}h_2(s)\lambda^{-1/2}\int_{s}^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta, \mu\right\rangle \\ &\quad \leq C\left[\frac{1}{k}(s^+ - s^-) + (s^+ - s^-)^2\right]\mu\{(r, s); s^- < s < s^+\}, \end{aligned} \tag{5.14}$$

where $C > 0$ is independent of k. (In the following we use the same C to denote various constants which are independent of k.)

In fact, we calculate that

$$
\begin{aligned}
&\eta_k(h_2)q_{-k}(h_2) - q_k(h_2)\eta_{-k}(h_2)\\
&\quad = [e^{ks}(A_0(h_2) + A_1(h_2)k^{-1}) + P_k(h_2)]\\
&\qquad \cdot [e^{-ks}(B_0(h_2) + B_1(h_2)(-k)^{-1}) + Q_{-k}(h_2)]\\
&\qquad - [e^{ks}(B_0(h_2) + B_1(h_2)k^{-1}) + Q_k(h_2)]\\
&\qquad \cdot [e^{-ks}(A_0(h_2) + A_1(h_2)(-k)^{-1}) + P_{-k}(h_2)]\\
&\quad = I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{2}{k}(A_1B_0 - A_0B_1),\\
I_2 &= e^{-ks}(B_0P_k - A_0Q_k) + e^{ks}(A_0Q_{-k} - B_0P_{-k}),\\
I_3 &= \frac{1}{k}e^{ks}(A_1Q_{-k} - B_1P_{-k}) + \frac{1}{k}e^{-ks}(A_1Q_k - B_1P_k),\\
I_4 &= P_kQ_{-k} - Q_kP_{-k}.
\end{aligned}
$$

First, by (4.16) and Propositions 4.6 and 4.7, we see that

$$
\begin{aligned}
I_1 &= \frac{2}{k}\lambda^{-1/2}h_2(s)(\lambda A_1 - B_1) = \frac{2}{k}h_2^2(s)\partial_s \ln\lambda,\\
I_2 &= e^{-ks}h_2(s)\lambda^{-1/2}(\lambda P_k - Q_k) + e^{ks}h_2(s)\lambda^{-1/2}(Q_{-k} - \lambda P_{-k})\\
&= -\frac{2}{k}h_2^2(s)\partial_s\ln\lambda + e^{-ks}h_2(s)\lambda^{-1/2}\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta\\
&\quad + e^{ks}h_2(s)\lambda^{-1/2}\int_s^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta + I_2^*,
\end{aligned}
$$

where

$$
\begin{aligned}
I_2^* = &-e^{-ks}h_2(s)\lambda^{-1/2}[\Delta_k^1(h_2) + \Delta_k^2(h_2)]\\
&+ e^{ks}h_2(s)\lambda^{-1/2}[\Delta_{-k}^1(h_2) + \Delta_{-k}^2(h_2)].
\end{aligned}
$$

Clearly, we have

$$
|I_2^*| \le \begin{cases} \frac{C}{k}(s^+ - s^-) + C(s^+ - s^-)^2, & s \in (s^-, s^+),\\ 0, & \text{otherwise.}\end{cases}
$$

We now deal with I_3. By (4.40) and (4.42), we get

$$
I_3 = I_3^1 + I_3^2,
$$

where

$$
\begin{aligned}
I_3^1 &= \frac{1}{k}e^{ks}(\lambda A_1 - B_1)P_{-k} + \frac{1}{k}e^{-ks}(\lambda A_1 - B_1)P_k,\\
I_3^2 &= \frac{1}{k}e^{ks}A_1\left[\int_s^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta + \Delta_{-k}^1 + \Delta_{-k}^2\right]\\
&\quad + \frac{1}{k}e^{-ks}A_1\left[-\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta + \Delta_k^1 + \Delta_k^2\right].
\end{aligned}
$$

We have from (4.39) and (4.41) that

$$|I_3^1| \le \begin{cases} \frac{C}{k}(s^+ - s^-), & s \in (s^-, s^+), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, from Proposition 4.2 we have

$$|I_3^2| \le \begin{cases} \frac{C}{k}(s^+ - s^-), & s \in (s^-, s^+), \\ 0, & \text{otherwise.} \end{cases}$$

Finally, by (4.40) and (4.42) we get

$$I_4 = I_4^1 + I_4^2,$$

where

$$\begin{aligned} I_4^1 &= -\frac{1}{k}\lambda^{-1/2}(\partial_s\lambda)h_2(s)[e^{-ks}P_k + e^{ks}P_{-k}], \\ I_4^2 &= P_k\left[\int_s^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta + \Delta_{-k}^1 + \Delta_{-k}^2\right] \\ &\quad - P_{-k}\left[-\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta + \Delta_k^1 + \Delta_k^2\right]. \end{aligned}$$

Clearly, by (4.39) and (4.41) we get

$$|I_4^1| \le \begin{cases} \frac{C}{k}(s^+ - s^-), & s \in (s^-, s^+), \\ 0, & \text{otherwise,} \end{cases}$$

$$|I_4^2| \le \begin{cases} \frac{C}{k}(s^+ - s^-) + C(s^+ - s^-)^2, & s \in (s^-, s^+), \\ 0, & \text{otherwise.} \end{cases}$$

Combining the above we get (5.14).

Step 4. We now prove that (5.14) leads to a contradiction.

Given any integer N, we write

$$\begin{aligned} [s^2 - l_0, s^2 + l_0] = &\bigcup_{n=1}^{N}\left[s^2 + \frac{n-1}{N}l_0, s^2 + \frac{n}{N}l_0\right] \\ &+ \bigcup_{n=1}^{N}\left[s^2 - \frac{n}{N}l_0, s^2 - \frac{n-1}{N}l_0\right]. \end{aligned}$$

We then choose h_2 such that

$$\operatorname{supp}\{h_2\} = [s^-, s^+] := \left[s^2 + \frac{n-2}{N}l_0, s^2 + \frac{n+1}{N}l_0\right],$$

$$h_2(s) = 1 \quad \text{if } s \in \left[s^2 + \frac{n-1}{N}l_0, s^2 + \frac{n}{N}l_0\right], \qquad n = 1, 2, \dots, N-1.$$

By the previous argument (5.14) holds for each h_2. Note that both terms on the right-hand side of (5.14) are positive. Given any $M > 0$, we estimate that

$$\begin{aligned} &\left\langle e^{-ks}h_2(s)\lambda^{-1/2}\int_{s^-}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta, \mu\right\rangle \\ &\ge \int_{\{s^2+(n-1)l_0/N \le s \le s^2+nl_0/N,\, |r|\le M\}} e^{-ks}\lambda^{-1/2}\int_{s^2+(n-1)l_0/N}^{s} e^{k\beta}(\partial_\beta\lambda)\lambda^{-1/2}\,d\beta\,d\mu \\ &\ge e^{-kl_0/N}c_0\int_{\{s^2+(n-1)l_0/N \le s \le s^2+nl_0/N,\, |r|\le M\}}\left(s - s^2 - \frac{n-1}{N}l_0\right)d\mu, \end{aligned}$$

where

$$c_0 = (\sigma'(0))^{-1/2} \max_{(r,s)\in\Sigma_0,\,|r|\le M,\,|s-s^2|\le l_0} |\partial_s\lambda|.$$

Note that $c_0 > 0$ because of (A3).

Similarly, we have

$$\left\langle e^{ks}h_2(s)\lambda^{-1/2}\int_s^{s^+} e^{-k\beta}(\partial_\beta\lambda)\lambda^{-1/2}h_2(\beta)\,d\beta\,,\,\mu\right\rangle$$
$$\ge e^{-kl_0/N}c_0\int_{\{s^2+(n-1)l_0/N\le s\le s^2+nl_0/N\,,\,|r|\le M\}}\left(s^2-s+\frac{n}{N}l_0\right)d\mu.$$

Combining the above we see that (5.14) implies

$$0\le \frac{l_0}{N}e^{-kl_0/N}c_0\mu\left\{(r\,,s)\,;\,s^2+\frac{n-1}{N}l_0\le s\le s^2+\frac{n}{N}l_0\,,\,|r|\le M\right\}$$
$$\le C\left[\frac{l_0}{kN}+\left(\frac{l_0}{N}\right)^2\right]\mu\left\{(r\,,s)\,;\,s^2+\frac{n-2}{N}l_0<s<s^2+\frac{n+1}{N}l_0\right\},$$
$$n=1\,,2\,,\dots\,,N-1.$$

We divide this by l_0/N and take the summation which deduces that

$$c_0e^{-kl_0/N}\mu\left\{(r\,,s)\,;\,s^2\le s\le s^2+l_0-\frac{l_0}{N}\,,\,|r|\le M\right\}$$
$$\le C\left(\frac{1}{k}+\frac{l_0}{N}\right)3\mu\left\{(r\,,s)\,;\,s^2-\frac{l_0}{N}\le s\le s^2+l_0\right\}.$$

We first let $N\to\infty$ and then let $k\to\infty$ which gives

$$\mu\{(r\,,s)\,;\,s^2\le s<s^2+l_0\,,\,|r|\le M\}=0.$$

With the same method we also get

$$\mu\{(r\,,s)\,;\,s^2-l_0<s\le s^2\,,\,|r|\le M\}=0.$$

The above contradicts the fact that $\mu\{(r\,,s)\,;\,|s-s^2|<l_0\}\ne 0$, since M is arbitrary.

If (5.13) holds, we also get a contradiction by a similar argument and hence we complete the proof.

The idea in the proof of the following result is due to DiPerna [1983a] (cf. Tartar [1983]).

Theorem 5.3. *The following case, denoted by* (S2), *is impossible. There are* $s^- < s^+$ *such that*

$$\operatorname{supp}\{\mu\}\subset\{(r\,,s)\,;\,s=s^+\}\cup\{(r\,,s)\,;\,s=s^-\},$$
$$\mu\{(r\,,s)\,;\,s=s^+\}\ne 0\,,\qquad \mu\{(r\,,s)\,;\,s=s^-\}\ne 0.$$

(Indeed, we can assume that μ has a support on four points.)

Proof. If, conversely, (S2) holds, we would have a contradiction. To prove it, we again argue in several steps.

Step 1. We choose $h_0 \in C_0^+(s)$ such that

$$\begin{aligned} &h_0(s) = 1 \quad \text{for } s \in [s^-, s^+], && \operatorname{supp}\{h_0\} = [s^- - \delta_0, s^+ + \delta_0], \\ &h_0'(s) > 0 \quad \text{in } (s^- - \delta_0, s^-), && h_0'(s) < 0 \quad \text{in } (s^+, s^+ + \delta_0), \end{aligned}$$

where $\delta_0 > 0$ is small.

By §4, we have

$$(\eta_{\pm k}(r, s; h_0), q_{\pm k}(r, s; h_0)) \in E_s, \qquad k = 1, 2, 3, \ldots.$$

We then define two sequences of measures ν_k, $\nu_{-k} \in M(\mathbf{R}^2)$ as follows:

$$\begin{aligned} \langle f, \nu_k \rangle &= \frac{\langle e^{ks} h_0(s) \lambda^{-1/2} f, \mu \rangle}{\langle e^{ks} h_0(s) \lambda^{-1/2}, \mu \rangle} \quad \forall f \in C_0(\mathbf{R}^2), \\ \langle f, \nu_{-k} \rangle &= \frac{\langle e^{-ks} h_0(s) \lambda^{-1/2} f, \mu \rangle}{\langle e^{-ks} h_0(s) \lambda^{-1/2}, \mu \rangle} \quad \forall f \in C_0(\mathbf{R}^2). \end{aligned}$$

We notice from (5.3) that, if $f \in C(\mathbf{R}^2)$, $\langle f, \nu_k \rangle$ and $\langle f, \nu_{-k} \rangle$ are well defined and bounded. Since $\|\nu_{\pm k}\| = 1$, there is a subsequence of $\{k\}$, also denoted by $\{k\}$, such that

$$\begin{aligned} \nu_k &\overset{*}{\rightharpoonup} \nu_+ \in M(\mathbf{R}^2) \quad \text{as } k \to \infty, \\ \nu_{-k} &\overset{*}{\rightharpoonup} \nu_- \in M(\mathbf{R}^2) \quad \text{as } k \to \infty. \end{aligned}$$

Clearly, $\nu_+, \nu_- \neq 0$, and

$$\operatorname{supp}\{\nu_+\} \subset \{(r, s); s = s^+\}, \qquad \operatorname{supp}\{\nu_-\} \subset \{(r, s); s = s^-\}.$$

Consequently, by Propositions 4.3 and 4.4 we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{\langle \eta_k(h_0), \mu \rangle}{\langle e^{ks} h_0(s) \lambda^{-1/2}, \mu \rangle} = 1, \\ &\lim_{k\to\infty} \frac{\langle q_k(h_0), \mu \rangle}{\langle e^{ks} h_0(s) \lambda^{-1/2}, \mu \rangle} = \langle \lambda, \nu_+ \rangle, \\ &\lim_{k\to\infty} \frac{\langle \eta_{-k}(h_0), \mu \rangle}{\langle e^{-ks} h_0(s) \lambda^{-1/2}, \mu \rangle} = 1, \\ &\lim_{k\to\infty} \frac{\langle q_{-k}(h_0), \mu \rangle}{\langle e^{-ks} h_0(s) \lambda^{-1/2}, \mu \rangle} = \langle \lambda, \nu_- \rangle. \end{aligned}$$

Step 2. We want to prove that

$$\langle q_m(h) - \lambda \eta_m(h), \nu_\pm \rangle = \langle q_m(h), \mu \rangle - \langle \lambda, \nu_\pm \rangle \langle \eta_m(h), \mu \rangle, \tag{5.15}$$

where $(\eta_m(h), q_m(h)) \in E_s$, $m = 1, 2, 3, \ldots$, $h \in C_0^+(s)$. Moreover, we have

$$\langle \lambda, \nu_+ \rangle = \langle \lambda, \nu_- \rangle. \tag{5.16}$$

Consequently, we get

$$\langle q_m(h) - \lambda \eta_m(h), \nu_+ \rangle = \langle q_m(h) - \lambda \eta_m(h), \nu_- \rangle. \tag{5.17}$$

In fact, since

$$\begin{aligned} &\langle \eta_{\pm k}(h_0) q_m(h) - q_{\pm k}(h_0) \eta_m(h), \mu \rangle \\ &\qquad = \langle \eta_{\pm k}(h_0), \mu \rangle \langle q_m(h), \mu \rangle - \langle q_{\pm k}(h_0), \mu \rangle \langle \eta_m(h), \mu \rangle, \end{aligned}$$

we divide the above by $\langle e^{ks}h_0(s)\lambda^{-1/2}, \mu\rangle$ and $\langle e^{-ks}h_0(s)\lambda^{-1/2}, \mu\rangle$ respectively and pass to the limit as $k \to \infty$, which gives (5.15).

To prove (5.16), we notice that

$$\begin{aligned}\langle \eta_k(h_0)q_{-k}(h_0) - q_k(h_0)\eta_{-k}(h_0), \mu\rangle \\ = \langle \eta_k(h_0), \mu\rangle\langle q_{-k}(h_0), \mu\rangle - \langle q_k(h_0), \mu\rangle\langle \eta_{-k}(h_0), \mu\rangle.\end{aligned} \tag{5.18}$$

Since

$$|\eta_k(h_0)q_{-k}(h_0) - q_k(h_0)\eta_{-k}(h_0)| = O(k^{-1}),$$

we divide (5.18) by $\langle e^{ks}h_0\lambda^{-1/2}, \mu\rangle\langle e^{-ks}h_0\lambda^{-1/2}, \mu\rangle$ and pass to the limit as $k \to \infty$, which gives (5.16).

Step 3. We claim that (5.17) leads to a contradiction. In fact, let $h(s^+) = 1$, $0 \le h(s) \le 1$, and $\operatorname{supp}\{h\} = [s^+-l, s^++l]$, where $s^+-s^- > l > 0$ is arbitrarily small. On the one hand, it follows from the construction of entropy-entropy flux pairs in §4 that

$$\langle q_m(h) - \lambda\eta_m(h), \nu_-\rangle = 0 \quad \forall m. \tag{5.19}$$

On the other hand, by (4.15) and Proposition 4.7 we calculate that

$$\begin{aligned} q_m(h) - \lambda\eta_m(h) &= \frac{1}{m}e^{ms}[B_1(h) - \lambda A_1(h)] + Q_m(h) - \lambda P_m(h) \\ &= -\int_{s^+-l}^{s} e^{m\beta}(\partial_\beta\lambda)\lambda^{-1/2}h(\beta)\,d\beta + \Delta_m^1(h) + \Delta_m^2(h).\end{aligned}$$

Therefore, from

$$\langle q_m(h) - \lambda\eta_m(h), \nu_+\rangle = 0 \quad \forall m$$

we get

$$\begin{aligned}&\left\langle \int_{s^+-l}^{s} e^{m\beta}(\partial_\beta\lambda)\lambda^{-1/2}h(\beta)\,d\beta, \nu_+\right\rangle \\ &\qquad \le C\left(\frac{1}{m} + l\right)\int_{s^+-l}^{s^+} e^{m\beta}h(\beta)\,d\beta\,\nu_+\{(r, s); s = s^+\},\end{aligned}$$

while for any $M > 0$

$$\begin{aligned}&\left\langle \int_{s^+-l}^{s} e^{m\beta}(\partial_\beta\lambda)\lambda^{-1/2}h(\beta)\,d\beta, \nu_+\right\rangle \\ &\qquad \ge c_*\int_{s^+-l}^{s^+} e^{m\beta}h(\beta)\,d\beta\,\nu_+\{(r, s); s = s^+, |r| \le M\},\end{aligned}$$

where

$$c_* = (\sigma'(0))^{-1/4}\max_{(r,s)\in\Sigma_0, |r|\le M, |s-s^+|\le s^+-s^-}|\partial_s\lambda|.$$

Note by (A3) that $c_* > 0$. We then get that $\nu_+\{(r, s); s = s^+, |r| \le M\} = 0$, which is impossible since M is arbitrary. Hence we complete the proof.

Having proved that the Young measures $\mu_{x,t}$ are Dirac measures, we can easily obtain the existence, i.e., Theorem 1.1. A brief argument is as follows.

We write $\mu_{x,t} = \delta_{r(x,t),s(x,t)} = \delta_{u(x,t),v(x,t)}$. Then

$$\begin{aligned} u^{\varepsilon_n}(x, t) &\rightharpoonup u(x, t) && \text{in } L^2_{\text{loc}}(\mathbf{R} \times (0, \infty)), \\ v^{\varepsilon_n}(x, t) &\rightharpoonup v(x, t) && \text{in } L^2_{\text{loc}}(\mathbf{R} \times (0, \infty)), \\ \sigma(u^{\varepsilon_n}(x, t)) &\rightharpoonup \sigma(u(x, t)) && \text{in } L^2_{\text{loc}}(\mathbf{R} \times (0, \infty)).\end{aligned}$$

Moreover, by the definition of L_{con} and Corollary 2.2, we have that for each $(\eta, q) \in L_{\text{con}}$

$$\eta(u^{\varepsilon_n}(x, t), v^{\varepsilon_n}(x, t)) \rightharpoonup \eta(u(x, t), v(x, t)) \quad \text{in } L^1_{\text{loc}}(\mathbf{R} \times (0, \infty)),$$
$$q(u^{\varepsilon_n}(x, t), v^{\varepsilon_n}(x, t)) \rightharpoonup q(u(x, t), v(x, t)) \quad \text{in } L^1_{\text{loc}}(\mathbf{R} \times (0, \infty)).$$

Combining the above we get that $\{u(x, t), v(x, t)\}$ is an admissible solution of the Cauchy problem (1.1), (1.2), as defined in Definition 4.2.

6. Lax-Friedrichs scheme

In this section we consider the convergence of the approximate solutions generated by the Lax-Friedrichs scheme. Let Δx and Δt be increments in the x and t axes respectively. We denote $x_k = k\Delta x$, $k = 0, \pm 1, \pm 2, \pm 3, \dots$, and $t_n = n\Delta t$, $n = 0, 1, 2, \dots$. The approximate values $\{u(x_k, t_n), v(x_k, t_n)\}$ are denoted by $\{u_{n,k}, v_{n,k}\}$. Then the Lax-Friedrichs scheme takes the form

$$\text{(6.1)} \qquad \begin{aligned} &\frac{u_{n+1,k} - \frac{1}{2}(u_{n,k+1} + u_{n,k-1})}{\Delta t} - \frac{v_{n,k+1} - v_{n,k-1}}{2\Delta x} = 0, \\ &\frac{v_{n+1,k} - \frac{1}{2}(v_{n,k+1} + v_{n,k-1})}{\Delta t} - \frac{\sigma(u_{n,k+1}) - \sigma(u_{n,k-1})}{2\Delta x} = 0. \end{aligned}$$

(6.1) can be rewritten in the form

$$\text{(6.2)} \qquad \begin{aligned} u_{n+1,k} &= \tfrac{1}{2}(u_{n,k+1} + u_{n,k-1}) + \tfrac{1}{2}\kappa(v_{n,k+1} - v_{n,k-1}), \\ v_{n+1,k} &= \tfrac{1}{2}(v_{n,k+1} + v_{n,k-1}) + \tfrac{1}{2}\kappa[\sigma(u_{n,k+1}) - \sigma(u_{n,k-1})], \end{aligned}$$

where $\kappa = \frac{\Delta t}{\Delta x}$ is the ratio of mesh lengths, which remains constant. We require that

$$\text{(6.3)} \qquad \max|\lambda(u_0(x))|^{-1} \le (\delta_0)^{-1/2} < \kappa.$$

We now describe the procedure of the construction of the approximating sequence $\{u^l(x, t), v^l(x, t)\}$, where we write $l = \Delta x$. We first define

$$\text{(6.4)} \qquad \begin{aligned} I_k^n := \{(x, t);\, &n\Delta t \le t < (n+1)\Delta t, \\ &(k-1)l < x < (k+1)l,\ n + k = \text{even}\}, \end{aligned}$$

and

$$\text{(6.5)} \qquad u^l_{0,k} = u^l_0(kl), \qquad v^l_{0,k} = v^l_0(kl),$$

where $u_0(x)$ and $v_0(x)$ are the initial data.

We first define $\{u^l(x, t), v^l(x, t)\}$ on each I_k^0, $k = 0, \pm 2, \pm 4, \dots$, as the solution of Riemann problem (1.1) with the initial data given by

$$u_0^l(x) = \begin{cases} u^l_{0,k+1}, & x > kl, \\ u^l_{0,k-1}, & x < kl, \end{cases} \qquad v_0^l(x) = \begin{cases} v^l_{0,k+1}, & x > kl, \\ v^l_{0,k-1}, & x < kl. \end{cases}$$

It is well known (cf. Di Perna [1983a], for example) that the values at intersecting points for the Lax-Friedrichs scheme can be expressed as the mean values of the corresponding Riemann solutions, namely,

$$u_{1,k} = \frac{1}{2l}\int_{(k-1)l}^{(k+1)l} u^l(x, \Delta t - 0)\,dx, \qquad v_{1,k} = \frac{1}{2l}\int_{(k-1)l}^{(k+1)l} v^l(x, \Delta t - 0)\,dx,$$

where

$$u^l(x\,,\,\Delta t-0)=\lim_{t\to\Delta t-0}u^l(x\,,\,t)\,,\qquad v^l(x\,,\,\Delta t-0)=\lim_{t\to\Delta t-0}v^l(x\,,\,t).$$

We can proceed in this way, since (6.3) holds for each step. Thus we can construct $\{u^l(x\,,\,t)\,,\,v^l(x\,,\,t)\}$ for all $x\in\mathbf{R}\,,\ t>0$, such that

$$\begin{aligned} u_{n,k}&=\frac{1}{2l}\int_{(k-1)l}^{(k+1)l}u^l(x\,,\,n\Delta t-0)\,dx\,,\\ v_{n,k}&=\frac{1}{2l}\int_{(k-1)l}^{(k+1)l}v^l(x\,,\,n\Delta t-0)\,dx\,,\end{aligned}\tag{6.6}$$

where $n=1\,,\,2\,,\,\dots\,,\quad k=0\,,\,\pm1\,,\,\pm2\,,\,\dots\,,\quad n+k=\text{odd}$.

From the properties of fundamental waves we know that $\Sigma_0=\{(r\,,\,s)\,;\,r\geq r^0\,,\ s\leq s^0\}$ is an invariant region of Riemann solutions. Combining this with the fact that $I^{-1}(\Sigma_0)$ is a convex set in the $(u\,,\,v)$ plane we get the following result.

Theorem 6.1. *Assume the hypotheses of Theorem* 1.2. *Then the approximate sequence* $\{u^l(x\,,\,t)\,,\,v^l(x\,,\,t)\}$ *satisfies*

$$(r^l(x\,,\,t)\,,\,s^l(x\,,\,t))\subset\Sigma_0\quad\forall(x\,,\,t)\in\mathbf{R}\times\mathbf{R}^+\,,$$

where

$$r^l(x\,,\,t)=r(u^l(x\,,\,t)\,,\,v^l(x\,,\,t))\,,\qquad s^l(x\,,\,t)=s(u^l(x\,,\,t)\,,\,v^l(x\,,\,t)).$$

The following basic results are parallel to Theorem 3.2 and Corollary 3.3. The technique we use is based on the argument of Ding, Chen, and Luo [1985a, Theorem 5] and DiPerna [1983a]. We first define some notation as follows:

$$\begin{gathered} w^l(x\,,\,t):=\{u^l(x\,,\,t)\,,\,v^l(x\,,\,t)\}\,,\qquad w_j^n:=\{u_{n,j}\,,\,v_{n,j}\}\,,\\ w_-^n:=\{u^l(x\,,\,n\Delta t-0)\,,\,v^l(x\,,\,n\Delta t-0)\}\,,\\ [f]:=f(w^l(x(t)+0\,,\,t))-f(w^l(x(t)-0\,,\,t))\,,\end{gathered}$$

where $S:=(t\,,\,x(t))$ denotes a shock wave in $w^l(x\,,\,t)$, so that $[f]$ describes the jump of f across S from the left side to the right side.

We will use the following specific entropy-entropy flux generated by (4.4):

$$\begin{gathered}\eta_*(u\,,\,v)=\frac12v^2+\int_0^u\sigma(\tau)\,d\tau-\frac12(\overline{v})^2-\int_0^{\overline{u}}\sigma(\tau)\,d\tau\\ -(\sigma(\overline{u})\,,\,\overline{v})\cdot(u-\overline{u}\,,\,v-\overline{v})\,,\end{gathered}$$

$$q_*(u\,,\,v)=-v\sigma(u)+\overline{v}\sigma(\overline{u})+(\sigma(\overline{u})\,,\,\overline{v})\cdot(v-\overline{v}\,,\,\sigma(u)-\sigma(\overline{u})).$$

We observe that

$$\eta_*(u\,,\,v)\leq\max\{1\,,\,\sigma'(0)\}(|u-\overline{u}|^2+|v-\overline{v}|^2)\,,$$

and hence that

$$\int_{\mathbf{R}}\eta_*(u_0(x)\,,\,v_0(x))\,dx<\infty.$$

Theorem 6.2. *Assume the hypotheses of Theorem* 1.2. *Then there is a* $C > 0$, *independent of* Δx *and* Δt, *such that for any* $T > 0$, *say* $T = M\Delta t$,

$$
\begin{aligned}
&\sum_{j,n} \int_{(j-1)l}^{(j+1)l} (w_-^n - w_j^n)^2 \, dx \le C, \qquad 1 \le n \le M-1, \\
&\int_0^T \sum \{\rho[\eta_*] - [q_*]\} \, dt \le C,
\end{aligned} \tag{6.7}
$$

where the summation in the last inequality is taken over all shock waves S *at fixed time* t, *and* $\rho = dx(t)/dt$ *is the speed of the discontinuity* (*which is constant in each block* I_k^n *since, within* I_k^n, $\{u^l(x, t), v^l(x, t)\}$ *is the exact Riemann solution*).

Proof. We calculate that for any $\varphi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$,

$$
\iint_{0 \le t \le T} (\eta_*(w^l)\varphi_t + q_*(w^l)\varphi_x) \, dx \, dt = M_*(\varphi) + L_*(\varphi) + \Sigma_*(\varphi), \tag{6.8}
$$

where

$$
M_*(\varphi) = \int \varphi(x, T)\eta_*(w^l(x, T)) \, dx - \int \varphi(x, 0)\eta_*(w^l(x, 0)) \, dx,
$$

$$
L_*(\varphi) = \sum_{j,n} \int_{(j-1)l}^{(j+1)l} [\eta_*(w_-^l) - \eta_*(w_j^n)]\varphi(x, n\Delta t) \, dx, \qquad 1 \le n \le M-1,
$$

$$
\Sigma_*(\varphi) = \int_0^T \sum \{\rho[\eta_*] - [q_*]\}\varphi(x(t), t) \, dt.
$$

We let $\varphi = 1$, and obtain

$$
\begin{aligned}
\int_0^T \sum \{\rho[\eta_*] - [q_*]\} \, dt &+ \sum_{j,n} \int_{(j-1)l}^{(j+1)l} [\eta_*(w_-^l) - \eta_*(w_j^n)] \, dx \\
&\le - \int \eta_*(w^l(x, T)) \, dx + \int \eta_*(w^l(x, 0)) \, dx \\
&\le \int \eta_*(w^l(x, 0)) \, dx \le C \int \eta_*(u_0(x), v_0(x)) \, dx.
\end{aligned} \tag{6.9}
$$

Furthermore, we notice (cf. Smoller [1983]) that $\rho[\eta_*] - [q_*] \ge 0$, since, in each block I_k^n, $\{u^l(x, t), v^l(x, t)\}$ is exactly the Riemann solution. On the other hand, since

$$
\xi \nabla^2 \eta_*(u, v) \xi^T = (\xi_1)^2 + \sigma'(u)(\xi_2)^2 \ge \min\{1, \delta_0\}|\xi|^2
$$

for any $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, we have, by the Taylor expansion and (6.6),

$$
\begin{aligned}
&\sum_{j,n} \int_{(j-1)l}^{(j+1)l} [\eta_*(w_-^l) - \eta_*(w_j^n)] \, dx \\
&\quad = \sum_{j,n} \int_{(j-1)l}^{(j+1)l} dx \int_0^1 (1-\theta)(w_-^n - w_j^n) \\
&\qquad \times \nabla^2 \eta_*(w_j^n + \theta(w_-^n - w_j^n))(w_-^n - w_j^n)^T \, d\theta \\
&\quad \ge \min\{1, \delta_0\} \sum_{j,n} \int_{(j-1)l}^{(j+1)l} |w_-^n - w_j^n|^2 \, dx.
\end{aligned}
$$

Therefore, (6.9) results in (6.7), and we complete the proof.

Corollary 6.3. *There exists* $C > 0$, *independent of* l, *such that for any* $T > 0$

$$\int_0^T \int_{\mathbf{R}} |u^l(x, t) - \overline{u}|^2 \, dx \, dt \leq CT, \quad \int_0^T \int_{\mathbf{R}} |v^l(x, t) - \overline{v}|^2 \, dx \, dt \leq CT. \tag{6.10}$$

Proof. We let $\varphi = t - T$, and then (6.8) becomes

$$\begin{aligned} \int_0^T \int_{\mathbf{R}} & \eta_*(u^l(x, t), v^l(x, t)) \, dx \, dt \\ & \leq T\left[\int \eta_*(w^l(x, 0)) \, dx + L_*(1) + \Sigma_*(1)\right] \leq 2T \int \eta_*(w^l(x, 0)) \, dx. \end{aligned}$$

This is (6.10), and we complete the proof.

The following result enables us to apply the argument of §5 to prove that the resulting Young measures are indeed Dirac measures, and hence to get Theorem 1.2.

Theorem 6.4. *Assume the hypotheses of Theorem* 1.2. *Then for each* $(\eta, q) \in E_s \cup E_r$ (*here we drop the index* $\{\pm k\}$ *for simplicity*)

$$\begin{gathered} \partial_t \eta(u^l(x, t), v^l(x, t)) + \partial_x q(u^l(x, t), v^l(x, t)) \\ \textit{is relatively compact in } H^{-1}_{\mathrm{loc}}(\mathbf{R} \times (0, \infty)). \end{gathered} \tag{6.11}$$

Proof. As in Theorem 6.2, we have that for any $\varphi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$

$$\begin{aligned} \iint_{0 \leq t \leq T} & (\eta(w^l)\varphi_t + q(w^l)\varphi_x) \, dx \, dt \\ & = M(\varphi) + L_1(\varphi) + L_2(\varphi) + \Sigma(\varphi), \end{aligned}$$

where

$$\begin{aligned} M(\varphi) &= \int \varphi(x, T)\eta(w^l(x, T)) \, dx - \int \varphi(x, 0)\eta(w^l(x, 0)) \, dx, \\ L_1(\varphi) &= \sum_{j, n} \varphi(jl, n\Delta t) \int_{(j-1)l}^{(j+1)l} [\eta(w^l_-) - \eta(w^n_j)] \, dx, \\ L_2(\varphi) &= \sum_{j, n} \int_{(j-1)l}^{(j+1)l} [\eta(w^l_-) - \eta(w^n_j)][\varphi(x, n\Delta t) - \varphi(jl, n\Delta t)] \, dx, \\ \Sigma(\varphi) &= \int_0^T \sum \{\rho[\eta] - [q]\}\varphi(x(t), t) \, dt. \end{aligned}$$

We notice that, since $|\nabla^2 \eta| \leq C$,

$$|\xi \nabla^2 \eta \xi^T| \leq 2C|\xi|^2 \leq \frac{2C}{\min\{1, \delta_0\}} \xi \nabla^2 \eta_* \xi^T \quad \forall \xi \in \mathbf{R}^2.$$

It follows from the argument of DiPerna [1983a] and Ding, Chen, and Luo [1985a] that

$$|M(\varphi) + L_1(\varphi) + \Sigma(\varphi)| \leq C\|\varphi\|_{C_0},$$

where C is independent of l. Hence we have

$$M + L_1 + \Sigma \text{ is compact in } W^{-1, q_0}_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+), \qquad 1 < q_0 < 2.$$

Furthermore, since $|\nabla\eta| \leq C$,

$$|L_2(\varphi)| \leq Cl^{\alpha_0-1/2}\|\varphi\|_{W_0^{1,p}} \to 0 \quad \text{as } l \to 0,$$

$\frac{1}{2} < \alpha_0 < 1$, $p > 2/(1-\alpha_0)$, which implies

$$\|L_2\|_{-1,q_1} \to 0 \quad \text{as } l \to 0, \qquad 1 < q_1 < \frac{2}{1+\alpha_0}.$$

Combining the above we get

$$M + L_1 + L_2 + \Sigma \text{ is compact in } W_{\mathrm{loc}}^{-1,q_1}(\mathbf{R}\times\mathbf{R}^+).$$

On the other hand, since $|\eta| \leq C$,

$$\begin{aligned}&\partial_t\eta(u^l(x,t), v^l(x,t)) + \partial_x q(u^l(x,t), v^l(x,t))\\ &\quad\text{is bounded in } W^{-1,r}(\mathbf{R}\times\mathbf{R}^+), \qquad 1 < r < \infty.\end{aligned}$$

Therefore, applying the Embedding Theorem of §2 we get (6.11) and complete the proof.

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