SUPERHARMONIC FUNCTIONS ON FOLIATIONS

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ABSTRACT. We use techniques from geometric analysis to prove that any posi-
tive, leafwise superharmonic, measurable function on a Riemannian measurable
foliation with transverse invariant measure, finite total volume and complete
leaves is, in fact, constant on a.e. leaf.

0. INTRODUCTION

In [G], L. Garnett considers foliations of compact manifolds in which each
leaf is given a Riemannian structure. She proves:

Theorem [G, Theorem 1(b), p. 286]. *Any bounded Borel function* h *which is
harmonic on each leaf must be constant on almost all leaves, relative to any finite
harmonic measure.* □

We will only consider here the case of a transverse invariant measure and
finite total volume where the "finite harmonic measure" of this theorem reduces
to the transversal measure.

The theorem is remarkable because it could be the case that every leaf is
isometric to hyperbolic space. Thus, by the axiom of choice, one could simply
choose a bounded harmonic function on each leaf. The ergodic nature of the
theorem is that these choices cannot be made in a Borel way.

Her proof uses probability theory and is quite elegant. Here we take a geo-
metric analysis approach and prove

Theorem 5.1. *Let* $(M, \mathscr{F}, \mathscr{R})$ *be a finite volume oriented measurable Rieman-
nian foliation such that a.e. leaf is complete. Let* $h: M \to (0, \infty)$ *be a measur-
able function such that, for a.e. leaf* $L \subseteq M$, *the function* $h|L$ *is superharmonic
on the Riemannian manifold* L. *Then* $h|L$ *is constant, for a.e. leaf* $L \subseteq M$.

It is possible to remove the orientability assumption in the theorem by intro-
ducing the orientable double cover. A positive superharmonic function will lift
to a positive superharmonic function in the double cover, where the Theorem
5.1 applies. Therefore the lift is constant on a.e. leaf and it follows that the
same holds for the original function.

The proof of Theorem 5.1 is best understood as an evolution of theorems.

First is the very simple

Proposition 0.1. *A closed Riemannian manifold carries no nonconstant positive superharmonic functions.*

Proof. If h is positive superharmonic, then $u := \ln h$ satisfies $\nabla u + |\Delta u|^2 \le 0$. By Stokes' theorem,

$$\int \Delta u = \int_\partial \frac{\partial u}{\partial \vec{n}} = 0,$$

since $\partial = \varnothing$. However $\Delta u \le -|\nabla u|^2$, so $\int |\nabla u|^2 \le 0$, so $\nabla u \equiv 0$, so u is constant, so $h = e^u$ is also constant. $\square$

The next step is

Proposition 0.2. *A complete finite volume Riemannian manifold carries no nonconstant positive superharmonic functions.*

The proof of this proposition is somewhat more complicated because Stokes' theorem only applies to compactly supported forms. Thus it is necessary to introduce a gently sloping cutoff function with compact support. In the case of a single leaf, these functions are easily obtained as radial functions (depending only on the distance to some chosen point). Modulo finding such cutoff functions, the proof of Proposition 0.2 is essentially the same as the proof of the main theorem (Theorem 5.1). The new complication in proving Theorem 5.1 is that ergodicity precludes choosing a single point in each leaf of a foliation, so finding the appropriate cutoff function becomes more difficult. Thus we devote one section (§4) to proving a result (Lemma 4.7) about cutoff functions. With Lemma 4.7, the proof of the main theorem is routine.

The foliations we consider here will be the measurable foliations of Zimmer [Z1, Z2]. These include any C^∞-foliation of a manifold, although we specialize here to the case of foliations with a holonomy invariant system of transverse measures and with finite total volume. On manifolds, these kinds of foliations have been extensively studied.

The lack of positive superharmonic functions and the lack of convex functions are basic tools in the study of finite volume manifolds of positive Ricci curvature. It is our hope that these tools can be set up and used in a foliation context as well.

I would like to thank R. Schoen for showing me a proof of Proposition 0.2. It is the basis of this paper. Conversations with A. Freire have been very helpful as well. I would also like to thank G. Stuck for pointing out a number of mistakes in the original version of this paper.

1. BASIC DEFINITIONS

All Borel spaces we consider will be standard, i.e., Borel isomorphic to $[0, 1]$. A measure space will always be a standard Borel space with a σ-finite Borel measure.

For the remainder of the paper, we fix a positive integer n (to denote the dimension of the leaves of the foliation we study). Let D denote the unit ball in $\mathbf{R}^n$.

If R is an equivalence relation on a set S, then, for all $s \in S$, we let $[s]_R$ denote the equivalence class of s. If $A \subseteq S$, then the *R-saturation of A* is

denoted $[A]_R := \bigcup_{a \in A} [a]_R$. We say that R is *countable* if every equivalence class of R is countable.

Definition 1.1. If R is a countable equivalence relation on a Borel space B, then we say that R is *Borel* if its graph $R \subseteq B \times B$ is a Borel subset. A Borel automorphism $f: B \to B$ is called an *automorphism of* R if $(b, f(b)) \in R$, for all $b \in B$.

Definition 1.2. A countable Borel equivalence relation R on a finite measure space T is said to be *measure preserving* if every automorphism of R is measure preserving.

Definition 1.3. A *flow box* consists of

(i) a countable measure preserving Borel equivalence relation R on a finite measure space T; and

(ii) a measurable map $t \to \omega_t$ from T to the space of all Riemannian metrics on D. Let μ_t be the volume density associated to ω_t, and define a measure μ on $T \times D$ by

$$\mu(A) := \int_T \mu_t\{d \in D | (t, d) \in A\}\, dt,$$

for all measurable $A \subseteq T \times D$. The resulting measure space is denoted $T \times_\omega D$.

In the next definition, no measurability assumption is made on the map $\mathscr{R}$. In fact, Definition 1.5 makes precise what it means for the map $\mathscr{R}$ to be "measurable".

Definition 1.4. Let $\mathscr{F}$ be a Borel equivalence relation on a measure space M. Let $\mathscr{R}: L \mapsto \mathscr{R}(L)$ be an association of an n-dimensional C^∞-Riemannian manifold structure to each equivalence class $L \subseteq M$. A *foliation chart for* $(M, \mathscr{F}, \mathscr{R})$ consists of

(i) a flow box $(T, R, t \mapsto \omega_t)$; and

(ii) a measure preserving Borel injection $u: T \times_\omega D \to M$ such that (a) for all $d, d' \in D$, for all $t, t' \in T$,

$$(t, t') \in R \Leftrightarrow (u(t, d), u(t', d')) \in \mathscr{F},$$

such that (b), for every equivalence class $L \subseteq M$, $u(T \times \{0\}) \cap L$ has no accumulation points in the topological space L and such that (c) for all $t \in T$, with $L_t := [u(t, 0)]_{\mathscr{F}}$, the map

$$d \mapsto u(t, d): (D, \omega_t) \to (L_t, \mathscr{R}(L_t))$$

is an isometry.

Definition 1.5. Let $\mathscr{F}$ be a Borel equivalence relation on a finite measure space M. Let $\mathscr{R}$ assign an n-dimensional C^∞-Riemannian manifold structure to each equivalence class of $\mathscr{F}$. Then $(M, \mathscr{F}, \mathscr{R})$ is a *finite volume oriented measurable Riemannian foliation* if there exists a countable collection $\mathscr{A} :=$ $\{(T_i, R_i, t \mapsto \omega_t^i, u_i)\}_{i=1, 2, \dots}$ of foliation charts such that

(i) $\bigcup_i u_i(T_i \times R_i)$ contains a.e. equivalence class; and

(ii) for every i, for every $t \in T_i$, $u_i(t, \cdot)$ is an orientation preserving C^∞-isometry of (D, ω_t^i) onto an open subset of the Riemannian manifold $[u_i(t, 0)]_{\mathscr{F}}$.

Such a collection $\mathscr{A}$ is called an *atlas*. Equivalence classes are called leaves.

Now, if f is a function on M, then it makes sense to speak of f being C^∞ along leaves. For such functions, we have a well-defined leafwise gradient ∇f and a leafwise Laplacian Δf.

2. Partitions of unity

Let $(M, \mathscr{F}, \mathscr{R})$ be a finite volume oriented measurable Riemannian foliation.

Definition 2.1. If $\mathscr{A} = \{(T_i, R_i, t \mapsto \omega_t^i, u_i)\}_i$ is an atlas, then a sequence $f_i \colon M \to [0, 1]$ is called a *locally finite partition of unity subordinate to* $\mathscr{A}$ if
  (i) $\sum_i f_i = 1$ a.e. on M;
  (ii) each f_i is measurable on M and C^∞ along leaves;
  (iii) for all i, $f_i = 0$ on $M \backslash u_i(T_i \times D_i)$;
  (iv) for all i, for all $t \in T_i$, $d \mapsto f_i(t, d) \colon D \to \mathbf{R}$ has compact support.
  (v) if L is a leaf and $K \subseteq L$ is compact, then $f_i|K \equiv 0$, for all but finitely many i.

Lemma 2.2. *There exists an atlas* $\mathscr{A}$ *for* $(M, \mathscr{F}, \mathscr{R})$ *which has a locally finite partition of unity subordinate to it.*

Proof. Let $\mathscr{B} = \{(U_j, S_j, t \mapsto \alpha_t^j, v_j)\}_{j=1,2,\ldots}$ be an atlas for $(M, \mathscr{F}, \mathscr{R})$.

Let $g_k \colon D \to [0, 1]$, $k = 1, 2, \ldots$, be a locally finite partition of unity for the open ball $D \subseteq \mathbf{R}^n$ such that every g_k has compact support. (Recall that "local finiteness" means that, for any compact $K \subseteq D$, $g_k|D \equiv 0$, for all but finitely many k.)

Let $r_1, r_2, \ldots$ be a strictly increasing sequence of positive real numbers tending toward 1. For $l = 1, 2, \ldots$, let $D_l \subseteq D$ be the open ball about 0 of radius r_l. For each l, let $h_l \colon D \to [0, 1]$ be a C^∞ function satisfying $h_l \equiv 0$ on D_l and $h_l \equiv 1$ on $D \backslash D_{l+1}$.

Let $\pi_2^j \colon T_j \times D \to D$ denote projection onto the second coordinate. For all positive integers j, k, l, define $g_k^j, h_l^j \colon M \to [0, 1]$ by

$$g_k^j(m) := \begin{cases} (g_k \circ \pi_2^j \circ v_j^{-1})(m), & \text{if } m \in v_j(U_j \times D_j), \\ 0, & \text{otherwise}, \end{cases}$$

$$h_l^j(m) := \begin{cases} (h_l \circ \pi_2^j \circ v_j^{-1})(m), & \text{if } m \in v_j(U_j \times D_j), \\ 1, & \text{otherwise}. \end{cases}$$

Then, for all $m \in M$, there exists j, k such that $g_k^j(m) \neq 0$. However, for some values of m, there might exist infinitely many such pairs (j, k).

For all j, k, let $\overline{g}_k^j := g_k^j h_1^{j-1} h_2^{j-2} \ldots h_{j-1}^1$. Then for all $m \in M$ there exists j, k such that $\overline{g}_k^j(m) \neq 0$. Further, around each m, there is a neighborhood N in the topological space $[m]_{\mathscr{F}}$ such that $\overline{g}_k^j|N \equiv 0$ for all but finitely many pairs (j, k). Thus, for every compact subset K in any leaf, $\overline{g}_k^j|K \equiv 0$, for all but finitely many (j, k).

We now renumber. Let $i \mapsto (\varphi(i), \psi(i)) \colon \mathbf{Z} \to \mathbf{Z}^2$ be a bijection. Define

$$\overline{f}_i := g_{\psi(i)}^{\varphi(i)}, \quad T_i := U_{\varphi(i)}, \quad R_i := S_{\varphi(i)},$$

$$t \mapsto \omega_t^i := t \mapsto \alpha_t^{\varphi(i)}, \quad u_i := v_{\varphi(i)}.$$

Let $\mathscr{A} := \{(T_i, R_i, t \mapsto \omega_t^i, u_i)\}_i$ and normalize: let $f_i := \overline{f}_i / \sum_i \overline{f}_i$. $\square$

3. FORMS

Definition 3.1. If ζ is a k-form on a Riemannian manifold L, then, for all $l \in L$, we define

$$\|\zeta\|_l := \max |\langle \zeta, e_1 \wedge \cdots \wedge e_k \rangle_l|,$$

where $e_1, \ldots, e_k$ ranges over orthonormal collections of k vectors in the tangent space $T_l L$.

Let $*$ denote the Hodge-$*$ operator on an oriented n-dimensional Riemannian manifold L. Then, for any n-form λ on L, for any $l \in L$,

$$\|\lambda\|_l = |(*\lambda)(m)| = |\langle \lambda, e_1 \wedge \cdots \wedge e_n \rangle_l|,$$

for any orthonormal frame $e_1, \ldots, e_n$ in $T_l L$.

Lemma 3.2. *If η is an $(n-1)$-form on an n-dimensional Riemannian manifold L and if $\varphi: L \to \mathbf{R}$ is C^∞, then*

$$\|d\varphi \wedge \eta\|_l \leq |(\nabla \varphi)(l)| \cdot \|\eta\|_l,$$

for all $l \in L$.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal frame in $T_l L$ such that $\langle d\varphi, e_i \rangle_l = 0$, for all $i = 2, \ldots, n$. Then

$$
\begin{aligned}
\|d\varphi \wedge \eta\|_l &= |\langle d\varphi \wedge \eta, e_1 \wedge \cdots \wedge e_n \rangle_l| \\
&= |\langle d\varphi, e_1 \rangle_l| \cdot |\langle \eta, e_2 \wedge \cdots \wedge e_n \rangle_l| \\
&\leq |(\nabla \varphi)(l)| \cdot \|\eta\|_l. \quad \square
\end{aligned}
$$

It makes sense to speak of a *measurable k-form on* $(M, \mathscr{F}, \mathscr{R})$, i.e., a system of k-forms (one on each leaf) which becomes measurable when pulled back into any flow box. Since all the leaves are oriented, we have a well-defined leafwise Hodge-$*$ operator,

$$*: \{\text{measurable } k\text{-forms}\} \to \{\text{measurable } (n-k)\text{-forms}\}.$$

We also have a well-defined leafwise exterior derivative d.

Let $(M, \mathscr{F}, \mathscr{R})$ be a finite volume oriented measurable Riemannian foliation. If ζ is a measurable k-form on $(M, \mathscr{F}, \mathscr{R})$ and if $m \in M$, then restricting ζ to the Riemannian manifold $[m]_{\mathscr{F}}$ gives meaning to $\|\zeta\|_m := \|\zeta|[m]_{\mathscr{F}}\|_m$.

Definition 3.3. If ζ is a measurable k-form on $(M, \mathscr{F}, \mathscr{R})$, then we say that ζ is *bounded* if there exists a constant $K \in \mathbf{R}$ such that $\|\zeta\|_m \leq K$, for all $m \in M$.

If λ is a measurable n-form on $(M, \mathscr{F}, \mathscr{R})$, then we say that λ is *integrable* if the function $m \mapsto \|\lambda\|_m : M \to \mathbf{R}$ is in $L^1(M)$. In this case, we define

$$\int_M \lambda := \int_M (*\lambda) \, d\mu,$$

where μ denotes the measure on M.

Since M is a finite measure space, any bounded n-form is integrable.

Now let $\{f_i\}_{i=1,2,\ldots}$ be a locally finite partition of unity subordinate to some atlas $\mathscr{A}$ for $(M, \mathscr{F}, \mathscr{R})$, see Lemma 2.2.

Lemma 3.4. *If η is a measurable $(n-1)$-form on $(M, \mathscr{F}, \mathscr{R})$, if i is a positive integer and if $d(f_i\eta)$ is integrable, then*

$$\int_M d(f_i\eta) = 0.$$

Proof. We may pull the form $f_i\eta$ back to the ith flow box, integrate along leaves, then integrate the resulting function along the transversal T_i.

Since f_i is compactly supported along the leaves (when pulled back into the ith flow box, see Definition 2.1, condition (iv)), the leafwise integrals are of compactly supported exact n-forms. They are therefore all equal to 0, and the lemma follows. □

4. Cutoff functions

Let $(M, \mathscr{F}, \mathscr{R})$ be a finite volume oriented measurable Riemannian foliation such that a.e. leaf is complete. Let μ denote the (finite) measure on M. For $A \subseteq M$ of positive measure, we define

$$\mu_A(S) := \frac{\mu(S \cap A)}{\mu(A)},$$

for all measurable $S \subseteq M$.

Definition 4.1. If $\varepsilon > 0$ and $A \subseteq M$ has positive measure, then an (ε, A)-*cutoff function* is a measurable function $\varphi: M \to [0, 1]$ which is C^∞ along leaves and which satisfies
  (i) $\mu_A\{\varphi = 1\} > 1 - \varepsilon$; and
  (ii) $|\nabla\varphi| < \varepsilon$ on M.

The product rule implies that the product of an (ε, A)-cutoff function and an (ε', A)-cutoff function is an $(\varepsilon + \varepsilon', A)$-cutoff function.

If $S \subseteq M$ and $m \in M$, then we define

$$d(m, S) := \inf\{d_m(m, s) | s \in S \cap [m]_{\mathscr{F}}\},$$

where d_m denotes distance in the Riemannian manifold $[m]_{\mathscr{F}}$. If $S \cap [m]_{\mathscr{F}} = \varnothing$, then we set $d(m, S) = +\infty$.

Fix, for the remainder of this section, a C^∞ function $s: \mathbf{R} \to \mathbf{R}$ such that
  (i) $s(t) = 0$, for all $t \leq 1$;
  (ii) $s(t) = 1$, for all $t \geq 3$; and
  (iii) $0 \leq s'(t) < 0.98$, for all $t \in \mathbf{R}$.
Extend s to $\mathbf{R} \cup \{+\infty\}$ by setting $s(+\infty) = 1$.

Lemma 4.2. *If $S \subseteq M$ is measurable and $\varepsilon > 0$, then there exists a measurable, leafwise C^∞ function $\varphi: M \to \mathbf{R}$ such that:*
  (a) $\varphi|S \equiv 0$;
  (b) $|\nabla\varphi| < \varepsilon$ on M; and
  (c) $\varphi(m) = 1$ for all $m \in M$ satisfying $d(m, S) > 4/\varepsilon$.

Proof. Let $\psi: M \to \mathbf{R}$ be defined by $\psi(m) := s(\varepsilon d(m, S))$. Then ψ satisfies conclusions (a) and (c). In fact, ψ satisfies
  (a') $\psi(m) = 0$, for all $m \in M$ satisfying $d(m, S) < 1/\varepsilon$; and
  (c') $\psi(m) = 1$, for all $m \in M$ satisfying $d(m, S) > 3/\varepsilon$.

Unfortunately $m \mapsto d(m, S)$ is not differentiable, so we cannot just set $\varphi = \psi$. However $m \mapsto d(m, Sj)$ is leafwise Lipschitz of norm 1, and so ψ is leafwise Lipschitz of norm $< (.98)\varepsilon$. We now smooth ψ in each leaf of each flow box and sum over a partition of unity to get φ:

Let $b_1, b_2, \cdots \in C_C^\infty(\mathbf{R}^n)$ be a sequence of "bump" functions approximating the Dirac mass at 0. Specifically, we assume

(A) for all k, $b_k \geq 0$ on $\mathbf{R}^n$ and $\int b_k = 1$; and

(B) $r_k := \max\{d_{\mathbf{R}^n}(0, p) | p \in \operatorname{supp} b_k\}$ satisfies $r_k < 1/\varepsilon$, for all k, and also satisfies $r_k \to 0$ as $k \to \infty$.

Let $\{f_i\}_i$ and $\mathscr{A} := \{(T_i, R_i, t \mapsto \omega_t^i, u_i)\}_i$ be as in Lemma 2.2. Let $\psi_i^t(d) := \psi(u_i(t, d))$, for all i, $t \in T_i$, $d \in D$. For all $m \in M$, define

$$J(m) := \{i | f_i(m) \neq 0\}, \quad n(m) := |J(m)|,$$

where $|\cdot|$ denotes cardinality.

Fix i and $t \in T_i$. Let $D' := \{d \in D | f_i(u_i(t, d)) = 0\}$. Then D' is compact. Let $\omega := \omega_t^i$. Let d_ω denote the distance function in (D, ω) and let ∇_ω denote the gradient on (D, ω). Let

$$N := \max\{n(u_i(d', t)) | d' \in D'\}, \quad G := \max\{|\nabla_\omega f_i(u_i(d', t))| | d' \in D'\}.$$

Since $\{f_i\}_i$ is locally finite, $N < \infty$.

Fix all $x, y \in D'$ and fix $v \in \mathbf{R}^n$ such that $\|v\| = 1$. Then, for all $\eta > 0$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow x + tv, y + tv \in D, \quad \frac{d_\omega(x + tv, y + tv)}{d_\omega(x, y)} < 1 + \eta.$$

By compactness of D' and of the unit sphere in $\mathbf{R}^n$, we may, given η, choose δ independent of x, y and v. Taking $\eta = 1/98$, we conclude that there exists $\delta_0 > 0$ such that

$$x, y \in D', \|v\| < \delta_0 \Rightarrow x + v, y + v \in D, \quad \frac{d_\omega(x + v, y + v)}{d_\omega(x, y)} < \frac{99}{98}.$$

Let k be the smallest positive integer satisfying

(C) $D' + \operatorname{supp} b_k \subseteq D$;

(D) if $x, y \in D'$ and $v \in \operatorname{supp} b_k$, then

$$\frac{d_\omega(x + v, y + v)}{d_\omega(x, y)} < \frac{99}{98};$$

and

(E) $\varphi_t^i := \psi_t^i * b_k : D' \to \mathbf{R}$ satisfies $|\varphi_t^i - \psi_t^i| < \varepsilon/100GN$ on D'.

Note: (C) and (D) above are obtained by choosing k so large that $r_k < \delta_0$.

Now fix i and let $t \in T_i$ be variable. Define $\varphi_i : M \to \mathbf{R}$ by $\varphi_i(u_i(t, d)) := \varphi_t^i(d)$ and $\varphi_i \equiv 0$ on $M \backslash u_i(T \times D)$. Since ψ is leafwise Lipschitz of norm $< (.98)\varepsilon$, it follows from (D) that φ_i is leafwise Lipschitz of norm $< (.99)\varepsilon$. Further, φ_i is leafwise C^∞, by standard properties of convolution, so $|\nabla \varphi_i| < (.99)\varepsilon$ on M. Finally, from (E), we see that, for all $m \in \operatorname{supp} f_i$,

$$(1) \qquad |\varphi_i(m) - \psi(m)| < \frac{\varepsilon}{100|\nabla f_i(m)| n(m)}.$$

Let $\varphi := \sum f_i \varphi_i$. Now $\sum f_i \equiv 1$, so $\sum \nabla f_i \equiv 0$, so $\sum (\nabla f_i)\psi \equiv 0$ on M. From this and the product rule,

$$\nabla \varphi \equiv \sum (\nabla f_i)(\varphi_i - \psi) + \sum f_i \nabla \varphi_i$$

on M. Since $f_i \geq 0$, we have $f_i(m) = 0 \Rightarrow (\nabla f_i)(m) = 0$. Taking absolute values and using the estimate (1), we have, for all $m \in M$,

$$|\nabla \varphi(m)| \leq \sum_{i \in J(m)} |\nabla f_i(m)| \frac{\varepsilon}{|100 \nabla f_i(m)| n(m)} + \sum_i [f_i(m)][(.99)\varepsilon]$$

$$\leq (.01)\varepsilon + (.99)\varepsilon = \varepsilon.$$

Thus conclusion (b) is satisfied.

By (B), every $r_k < 1/\varepsilon$, so conclusion (a) follows from (a'). Similarly, conclusion (c) follows from (c'). $\square$

For the remainder of this section fix $\varepsilon > 0$ and fix some $A \subseteq M$ of positive measure.

Lemma 4.3. *Let $F: M \to \mathbf{R}$ be measurable and bounded on any compact subset of any leaf. Then there exists a constant $K \in \mathbf{R}$ and an (ε, A)-cutoff function φ such that $\varphi F \equiv 0$ on $\{|F| > K\}$. In particular, φF is bounded.*

Proof. Let $G(m) := \sup\{F(m')|m' \in \overline{B}(m, 4/\varepsilon)\}$, where $\overline{B}(m, r)$ denotes the closed ball of radius r about m in the Riemannian manifold $[m]_{\mathscr{F}}$. (Since a.e. leaf is complete, closed balls are a.e. compact, so G is finite a.e.)

Since G is measurable, choose K so large that

$$\mu_A\{G \leq K\} \geq 1 - \varepsilon.$$

Let $S := \{|F| > K\}$ and apply Lemma 4.2. $\square$

Corollary 4.4. *If ζ is a measurable k-form on $(M, \mathscr{F}, \mathscr{R})$, then there exists an (ε, A)-cutoff function φ such that $\varphi\zeta$ is bounded.*

Proof. Let $F(m) := \|\eta\|_m$ in Lemma 4.3. $\square$

Corollary 4.5. *Let $\zeta_1, \ldots, \zeta_N$ be measurable forms on $(M, \mathscr{F}, \mathscr{R})$. Then there exists an (ε, A)-cutoff function φ such that $\varphi\zeta_1, \ldots, \varphi\zeta_N$ are all bounded.*

Proof. By Corollary 4.4, we may choose for each i, an $(\varepsilon/N, A)$-cutoff function φ_i such that $\varphi_i\zeta_i$ is bounded. By the product rule, $\varphi := \varphi_1 \ldots \varphi_N$ is an (ε, A)-cutoff function. The product of a bounded function and a bounded form is a bounded form, so this φ has the required properties. $\square$

Let $\{f_i\}_{i=1,2,\ldots}$ be a locally finite partition of unity subordinate to an atlas $\mathscr{A}$ for $(M, \mathscr{F}, \mathscr{R})$, see Lemma 2.2.

Corollary 4.6. *Let ζ be a measurable k-form on $(M, \mathscr{F}, \mathscr{R})$. Then there exists an (ε, A)-cutoff function φ such that $\varphi f_i \zeta \equiv 0$ on M, for all but finitely many i.*

Proof. Let $F(m) := \max\{i|f_i(m) \neq 0\}$. Since the partition of unity $\{f_i\}_i$ is locally finite (see Definition 2.1(v)), this function is bounded on compact subsets of leaves, and so we may apply Lemma 4.3. $\square$

We now come to the main technical result that we will need:

Lemma 4.7. *Let $A \subseteq M$ have positive measure, let $\varepsilon > 0$ and let η be a measurable $(n-1)$-form on $(M, \mathscr{F}, \mathscr{R})$. Then there exists an (ε, A)-cutoff function φ such that*

(i) $\varphi \, d\varphi \wedge \eta$ *is bounded;*
(ii) $\varphi^2 \, d\eta$ *is bounded;*

(iii) $d(\varphi^2\eta)$ *is bounded (hence integrable); and*

(iv) $\int_M d(\varphi^2\eta) = 0$.

Proof. By Corollary 4.6, choose an $(\varepsilon/2, A)$-cutoff function ψ and a positive integer N such that $\psi f_i\eta \equiv 0$ on M, for all $i > N$.

By Corollary 4.5, choose an $(\varepsilon/2, A)$-cutoff function χ such that

(A) $\chi\eta$ is bounded;

(B) $\chi\, d\eta$ is bounded;

(C) $\chi(f_i\eta)$ is bounded, for all $i = 1, \ldots, N$; and

(D) $\chi d(f_i\eta)$ is bounded, for all $i = 1, \ldots, N$.

Let $\varphi = \chi\psi$; by the product rule, φ is an (ε, A)-cutoff function. Since the product of a bounded function and a bounded form is a bounded form, we conclude

(A') $\varphi\eta = \psi(\chi\eta)$ is bounded;

(B') $\varphi^2\, d\eta = (\chi\psi^2)(\chi\, d\eta)$ is bounded;

(C') $\varphi(f_i\eta) = \psi(\chi f_i\eta)$ is bounded, for all $i = 1, \ldots, N$; and

(D') $\varphi^2 d(f_i\eta) = (\chi\psi^2)(\chi d(f_i\eta))$ is bounded, for all $i = 1, \ldots, N$.

By definition of a cutoff function (Definition 4.1), the function $|\nabla\varphi|$ is bounded. Thus, by Lemma 3.2, if ζ is any bounded measurable $(n-1)$-form, then $d\varphi \wedge \zeta$ is also bounded. Thus

(A'') $\varphi\, d\varphi \wedge \eta = d\varphi \wedge (\varphi\eta)$ is bounded, for all $i = 1, \ldots, N$; and

(C'') $\varphi\, d\varphi \wedge (f_i\eta) = d\varphi \wedge (\varphi f_i\eta)$ is bounded, for all $i = 1, \ldots, N$.

Now conclusion (i) is just (A'') and conclusion (ii) is (B'). Conclusion (iii) is a linear combination of (i) and (ii), since

$$d(\varphi^2\eta) = 2\varphi\, d\varphi \wedge \eta + \varphi^2\, d\eta.$$

By exactly the same argument, (C'') and (D') imply that the form

$$d(\varphi^2 f_i\eta) = 2\varphi\, d\varphi \wedge (f_i\eta) + \varphi^2 d(f_i\eta)$$

is bounded, for $i = 1, \ldots, N$. As bounded implies integrable, Lemma 3.4 yields

$$(1) \qquad \int_M d(\varphi^2 f_i\eta) = 0,$$

for all $i = 1, \ldots, N$. On the other hand, by the choice of ψ, we have for all $i > N$ that $\varphi f_i\eta = \chi(\psi f_i\eta) \equiv 0$ on M. Thus conclusion (iv) follows from summing (1) over $i = 1, \ldots, N$. $\square$

5. HARMONIC FUNCTIONS

We now state and prove the main theorem of this paper.

Theorem 5.1. *Let* $(M, \mathscr{F}, \mathscr{R})$ *be a finite volume oriented measurable Riemannian foliation such that a.e. leaf is complete. Let* $h: M \to (0, \infty)$ *be a measurable function such that, for a.e. leaf* $L \subseteq M$, *the function* $h|L$ *is superharmonic on the Riemannian manifold* L. *Then* $h|L$ *is constant, for a.e. leaf* $L \subseteq M$.

Proof. Let $u := \ln h$ and let $\eta := *du$, where $*$ denotes the leafwise Hodge-$*$ operator. Then $\Delta u + |\nabla u|^2 \leq 0$ a.e. on M.

Assume *for a contradiction* that there exists some $\alpha > 0$ such that

$$A := \{|\nabla u| > \alpha\}$$

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has positive measure. Let $\varepsilon > 0$ be arbitrary. Choose φ as in Lemma 4.7. Then

$$\int_M 2\varphi \, d\varphi \wedge \eta + \varphi^2 \, d\eta = \int_M d(\varphi^2 \eta) = 0.$$

From this and conclusion (i) of Lemma 4.7,

$$(1) \qquad -\int_M \varphi^2 \, d\eta = \int_M 2\varphi \, d\varphi \wedge \eta < \infty.$$

Let λ denote the (oriented) volume n-form on the leaves of $\mathscr{F}$, i.e., $\lambda := *1$. Then,

$$*d\eta = *d * du = \Delta u \le -|\nabla u|^2,$$

a.e. on M, and

$$*(d\varphi \wedge \eta) = *(d\varphi \wedge (*du)) = \nabla \varphi \cdot \nabla u.$$

Thus (1) becomes

$$(2) \qquad \int_M \varphi^2 |\nabla u|^2 \, d\mu \le \int_M 2\varphi \nabla \varphi \cdot \nabla u \, d\mu < \infty,$$

where μ denotes the measure on M.

Now $2ab \le 2(a/\sqrt{2})(b\sqrt{2})$, so

$$2ab \le \left(\frac{a}{\sqrt{2}}\right)^2 + (b\sqrt{2})^2 = \frac{a^2}{2} + 2b^2.$$

Letting $a := \varphi|\nabla u|$ and $b := |\nabla \varphi|$, we obtain

$$2\varphi \nabla \varphi \cdot \nabla u \le 2(\varphi|\nabla u|)|\nabla \varphi| \le \frac{\varphi^2 |\nabla u|^2}{2} + 2|\nabla \varphi|^2.$$

Plugging this into (2) yields

$$(3) \qquad \int_M \varphi^2 |\nabla u|^2 \, d\mu \le \frac{1}{2} \int_M \varphi^2 |\nabla u|^2 \, d\mu + 2 \int_M |\nabla \varphi|^2 \, d\mu.$$

By (2), $\frac{1}{2}\int_M \varphi^2 |\nabla u|^2 \, d\mu < \infty$, so we may subtract this quantity from both sides of (3) and obtain

$$(4) \qquad \frac{1}{2} \int_M \varphi^2 |\nabla u|^2 \, d\mu \le 2 \int_M |\nabla \varphi|^2 \, d\mu.$$

Let $B := \{\varphi = 1\} \cap A$. Since φ is an (ε, A)-cutoff function (Definition 4.1),

$$(1 - \varepsilon)\mu(A) < \mu(B) \quad \text{and} \quad |\nabla \varphi|^2 < \varepsilon^2.$$

Since $B \subseteq A = \{\alpha^2 < |\nabla u|^2\}$,

$$(1 - \varepsilon)\alpha^2 \mu(A) < \alpha^2 \mu(B) < \int_B \alpha^2 \, d\mu$$

$$< \int_B |\nabla u|^2 \, d\mu = \int_B \varphi^2 |\nabla u|^2 \, d\mu \le \int_M \varphi^2 |\nabla u|^2 \, d\mu.$$

So, by (4),

$$\frac{(1 - \varepsilon)\alpha^2 \mu(A)}{2} < 2 \int_M |\nabla \varphi|^2 \, d\mu < 2 \int_M \varepsilon \, d\mu = 2\varepsilon^2 \mu(M),$$

for all $\varepsilon > 0$. As $\mu(A) > 0$ and $\alpha > 0$, this is a *contradiction*, for ε sufficiently small.

Thus $|\nabla u| = 0$ a.e. on M, so, by continuity, u is constant on a.e. leaf. So the same must be true of $h = e^u$. $\quad\square$

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