A QUASIREGULAR ANALOGUE OF A THEOREM
OF HARDY AND LITTLEWOOD

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ABSTRACT. Suppose that f is analytic in the unit disk. A theorem of Hardy and Littlewood relates the Hölder continuity of f over the unit disk to the growth of the derivative. We prove here a quasiregular analogue of this result in certain domains in n-dimensional space. We replace values of the derivative with a local integral average. In the process we generalize a result on the continuity of quasiconformal mappings due to Nakki and Palka. We also present another proof of the relationship between the growth of the derivative and quasiregular mappings in BMO.

Hardy and Littlewood prove the following result in [9, p. 426, Theorem 40 and p. 427, Theorem 41]. (See also [4, p. 74].)

1.1. Theorem. *Suppose that f is analytic in $D = \{z \mid |z| < 1\}$ and $0 < \alpha \leq 1$. If there exists a constant C_1 such that*

$$(1.1) \qquad |f'(z)| \leq C_1(1 - |z|)^{\alpha-1}$$

for all $z \in D$, then f has a continuous extension to $\overline{D} = \{z \mid |z| \leq 1\}$ and

$$(1.2) \qquad |f(z_1) - f(z_2)| \leq C_2|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \overline{D}$ and some constant C_2 which depends only on C_1 and α.

Conversely, if there exists a constant C_2 such that (1.2) holds for all $z_1, z_2 \in D$, then (1.1) holds for C_1 depending only on C_2 and α.

The main result of this paper generalizes a quasiconformal version of Theorem 1.1, due to Astala and Gehring [1, Theorems 1.9 and 3.17] to a quasiregular version (Theorem 1.2) involving a somewhat larger class of moduli of continuity than t^α, $0 < \alpha \leq 1$.

We assume throughout that $\Omega \subset R^n$ is an open connected set whose boundary, $\partial\Omega$, is nonempty. Also $B(x, R)$ is the open ball centered at $x \in \Omega$ with radius equal to $R > 0$ and $d(x, \partial\Omega)$ is the Euclidean distance between x and $\partial\Omega$. If $B \subset R^n$ is a ball, then σB, $\sigma > 0$, denotes the ball with the same center as B and with radius equal to σ times that of B. When $E \subset R^n$, $|E|$ denotes the n-dimensional Lebesgue measure of E and $\overline{E}$ its closure in R^n.

When $f: \Omega \to R^n$ is differentiable, we denote its Jacobi matrix by Df and the norm of the Jacobi matrix as a linear transformation by $|Df|$. When Df exists a.e. we denote the local Dirichlet integral of f at $x \in \Omega$ by

$$D_f(x) = \left(\frac{1}{|B|} \int_{B/2} |Df|^n \right)^{1/n}.$$

Here $B = B(x, d(x, \partial\Omega))$. When the measure is omitted from an integral, as here, integration with respect to n-dimensional Lebesgue measure is assumed.

A continuous increasing function $\lambda(t): [0, \infty) \to [0, \infty)$ is a majorant if $\lambda(0) = 0$ and if $\lambda(t_1 + t_2) \leq \lambda(t_1) + \lambda(t_2)$ for all $t_1, t_2 \geq 0$.

Lip_λ-extension domains are a wide class of domains discussed in §3. Quasiregular mappings are discussed in §2.

We prove the following result in §4.

1.2. Theorem. *Suppose that f is K-quasiregular in a Lip_λ-extension domain Ω. If there exists a constant C_1 such that*

$$(1.3) \qquad\qquad D_f(x) \leq C_1 \frac{\lambda(d(x, \partial\Omega))}{d(x, \partial\Omega)}$$

for all $x \in \Omega$, then f has a continuous extension to $\overline{\Omega}$ and

$$(1.4) \qquad\qquad |f(x_1) - f(x_2)| \leq C_2 \lambda(|x_1 - x_2| + d(x_1, \partial\Omega))$$

for all $x_1, x_2 \in \overline{\Omega}$. Here C_2 is a constant which depends only on C_1, K, n, λ and Ω.

Conversely if there exists a constant C_2 such that (1.4) holds for all $x_1, x_2 \in \Omega$, then (1.3) holds for all $x \in \Omega$ with C_1 depending only on C_2, K, n, λ and Ω.

If f is analytic and if $\lambda(t) = t^\alpha$, then the conditions (1.1) and (1.3) are equivalent.

Simple examples show that the term $d(x_1, \partial\Omega)$ cannot in general be omitted. For example $f(x) = x|x|^{a-1}$ with $a = K^{1/(1-n)}$ is K-quasiconformal in $B = B(0, 1)$, $D_f(x)$ is bounded over $x \in B$ yet $f \in \text{Lip}_a(B)$ (see [1, Remark 3.12]). However, by suitably modifying a theorem of Näkki and Palka [13] to the quasiregular case, we obtain the following result.

1.3. Theorem. *Suppose that f is K-quasiregular in a Lip_λ-extension domain Ω where $\lambda(t) = t^\alpha$ and $0 < \alpha \leq K^{1/(1-n)}$.*

If there exists a constant C_1 such that

$$(1.5) \qquad\qquad D_f(x) \leq C_1 d(x, \partial\Omega)^{\alpha-1}$$

for all $x \in \Omega$, then f has a continuous extension to $\overline{\Omega}$ and

$$(1.6) \qquad\qquad |f(x_1) - f(x_2)| \leq C_2 |x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in \overline{\Omega}$.

Here C_2 depends only on C_1, K, n, α, and Ω.

The converse follows from Theorem 1.2.

In the case that f is quasiconformal, Astala and Gehring prove Theorem 1.2, with $\lambda(t) = t^\alpha$, and Theorem 1.3 with the operator $a_f(x)$ in place of $D_f(x)$ [1, Theorems 1.9, 3.17, and 3.13]. Here, with $B = B(x, d(x, \partial\Omega))$,

$$a_f(x) = \exp\left\{\frac{1}{n|B|}\int_B \log J_f\right\}$$

where J_f is the determinant of the Jacobi matrix Df. When f is quasi-conformal, $\log J_f$ is integrable over each ball $B \subset \Omega$. If $n = 2$ and f is conformal, then $\log J_f$ is harmonic and $a_f(x) = |f'(x)|$. When f is quasi-conformal $a_f(x)$ and $D_f(x)$ are equivalent (Lemma 2.7) and because of this, Theorems 1.2 and 1.3 reduce to the results in [1]. As the example z^n shows, the ratio $D_f(x)/a_f(x)$ may depend on the local topological index and is in general unbounded for quasiregular f.

We also prove a result corresponding to Theorem 1.2 in the case $\alpha = 0$. Although Theorem 1.4 follows from a result of Vuorinen [18, p. 104, Theorem 4.29] the proof given here is similar to the proof of Theorem 1.2.

If $f = (f_1, f_2, \ldots, f_n) : \Omega \to R^n$, then we write, with $B = B(x, d(x, \partial\Omega))$,

$$D_{f_j}(x) = \left(\frac{1}{|B|}\int_{B/2} |\nabla f_j|^n\right)^{1/n}.$$

We write $\|f\|_*$ for the BMO norm of f over Ω (see §6).

1.4. Theorem. *Suppose that* $f = (f_1, f_2, \ldots, f_n)$ *is K-quasiregular in* Ω. *If there exists a constant* C_1 *so that*

$$(1.7) \qquad\qquad D_{f_j}(x) \le C_1 d(x, \partial\Omega)^{-1}$$

for some $1 \le j \le n$ *and all* $x \in \Omega$, *then*

$$(1.8) \qquad\qquad \|f_j\|_* \le C_2 < \infty$$

where C_2 *depends only on* C_1, K, *and* n.

Conversely, if (1.6) holds for some constant C_2 *and some* $1 \le j \le n$, *then (1.5) holds with* C_1 *depending only on* C_2, K, *and* n.

Theorem 1.4 gives a simple proof of Corollary 3 in [10, p. 280], which states that the BMO-norms of the components of a quasiregular mapping are equivalent.

2.1. Quasiregular mappings. We denote by $W_n^1(\Omega)$ the Sobolev space of functions $f : \Omega \to R^m$ which are L^n-integrable over Ω and have L^n-integrable distributional first derivatives over Ω. $W_{n,\mathrm{loc}}^1(\Omega) = \bigcap W_n^1(\Omega')$ where the intersection is over all Ω' compactly contained in Ω.

2.2. Definition. A function $f : \Omega \to R^n$ is K-quasiregular in $\Omega \subset R^n$, $1 \le K < \infty$, if
(a) $f \in W_{n,\mathrm{loc}}^1(\Omega)$,
(b) $|Df|^n/K \le J_f \le Kl(Df)^n$ a.e. in Ω where $l(Df) = \inf\{|Dfs| \mid |s| = 1\}$.

When $n = 2$, f is 1-quasiregular if and only if it is an analytic function. A homeomorphism in R^n is quasiregular if and only if it is quasiconformal in the usual sense. For information on quasiregular mappings see [3, 12, and 20].

We list here some preliminary results.

2.3. Proposition. *Suppose that* $B \subset R^n$ *is a ball and* $s > n$. *If* $f \in W_s^1(B)$, *then there is a constant* C, *depending only on* n, *such that*

$$(2.1) \qquad |f(x_1) - f(x_2)| \le \frac{C_s}{s-n} |B|^{(s-n)/sn} \left(\int_B |Df|^s \right)^{1/s}$$

for all $x_1, x_2 \in B$.

A proof of Proposition 2.3 can be found in [3, p. 268, Lemma 1.7].

2.4. Proposition. *If* $f \in W_n^1(B)$, *where* $B \subset R^n$ *is a ball, then*

$$(2.2) \qquad \left(\frac{1}{|B|} \int_B |f - f_B|^n \right)^{1/n} \le 2 \operatorname{diam} B \left(\frac{1}{|B|} \int_B |Df|^n \right)^{1/n}.$$

Here $\operatorname{diam} B$ *is the Euclidean diameter of* B *and* f_B *is the average value of* f *over* B.

This result is a special case of Lemma 1.5, p. 266, in [3].

Proposition 2.5, which appears in [3, p. 285, Theorem 5.1], shows that Proposition 2.3 can be applied to quasiregular mappings. For quasiconformal mappings Proposition 2.5 is due to Gehring [7, p. 274, Theorem 1].

2.5. Proposition. *If* f *is* K-*quasiregular in* Ω, *then there are constants,* $s > n$ *and* $C < \infty$, *which depend only on* n *and* K, *such that* $f \in W_{s,\operatorname{loc}}^1(\Omega)$ *and*

$$(2.3) \qquad \left(\int_F |Df|^s \right)^{1/s} \le C d(F, \partial\Omega)^{(n-s)/s} \left(\int_\Omega |Df|^n \right)^{1/n}$$

for all compact sets $F \subset \Omega$. *Here* $d(F, \partial\Omega)$ *is the Euclidean distance between* F *and* $\partial\Omega$.

A proof of the next result can be found in [10, p. 277, (5.3)].

2.6. Proposition. *If* $f = (f_1, f_2, \ldots, f_n)$ *is* K-*quasiregular in* Ω *and if* B *is a ball with* $\sigma B \subset \Omega$, $\sigma > 1$, *then there exists a constant* C, *depending only on* n, *such that*

$$(2.4) \qquad \left(\int_B |Df|^n \right)^{1/n} \le CK \frac{\sigma}{\sigma - 1} \left(\frac{1}{|\sigma B|} \int_{\sigma B} |f_j - a|^n \right)^{1/n}$$

for all $a \in R$ *and all* $j = 1, 2, \ldots, n$.

The following lemma shows that Theorem 1.2 is equivalent to the result of Astala and Gehring [1, Theorems 1.9 and 3.7] if f is quasiconformal and $\lambda(t) = t^\alpha$.

2.7. Lemma. *If* f *is* K-*quasiconformal in* Ω, *then there exists a constant* C, *which depends only on* n *and* K, *such that*

$$\frac{1}{C} a_f(x) \le D_f(x) \le C a_f(x)$$

for all $x \in \Omega$.

Proof. Since f is quasiconformal, $\| \log J_f \|_* < \infty$ [15, p. 261, Theorem 1]. The John-Nirenberg Theorem [15, p. 260, Lemma 1] and [5, p. 230, Theorem

2.1] implies that there exist positive constants a, b, and C_1, which depend only on n and the BMO-norm $\|\log J_f\|_*$, so that

$$(2.5) \qquad \left(\frac{1}{|R|}\int_R J_f^a\right)^{1/a} \leq C_1 \left(\frac{1}{|R|}\int_R J_f^{-b}\right)^{-1/b}$$

for all balls R with $\overline{R} \subset \Omega$ [15, p. 263, Lemma 3]. Next, by a result of Gehring [7, p. 271, Lemma 4], there exists a constant C_2, depending only on n and K, such that

$$(2.6) \qquad \left(\frac{1}{|R|}\int_R |Df|^n\right)^{1/n} \leq C_2 \left(\frac{1}{|R|}\int_R |Df|\right)$$

for all balls R with $\overline{R} \subset \Omega$. It follows from Hölder's inequality (see [10, Remark on p. 272] that we can improve the exponent in the reverse Hölder inequality (2.6) to obtain

$$(2.7) \qquad \left(\frac{1}{|R|}\int_R |Df|^n\right)^{1/n} \leq C_3(n,r) \left(\frac{1}{|R|}\int_R |Df|^r\right)^{1/r}$$

for all balls R with $\overline{R} \subset \Omega$ and all r, $0 < r \leq n$. Fix $x \in \Omega$ and let $B = B(x, \frac{1}{2}d(x, \partial\Omega))$. Using the dilatation inequality Definition 2.2(a), (2.7) with $r = na$ and (2.5) we obtain

$$(2.8) \quad \begin{aligned} &\left(\frac{1}{|B|}\int_B |Df|^n\right)^{1/n} \leq C_4 \left(\frac{1}{|B|}\int_B |Df|^{na}\right)^{1/(na)} \\ &\leq C_4 K^{1/n} \left(\frac{1}{|B|}\int_B J_f^a\right)^{1/(na)} \leq C_5 \left(\frac{1}{|B|}\int_B J_f^{-b}\right)^{-1/(nb)}. \end{aligned}$$

From Jensen's inequality for convex functions [5, p. 34] we obtain

$$(2.9) \qquad \left(\frac{1}{|2B|}\int_{2B} J_f^{-b}\right)^{-1/b} \leq \exp\left(\frac{1}{|2B|}\int_{2B}\log J_f\right).$$

If $\|u\|_* < \infty$ and B_0 and B_1 are balls with $B_0 \subset B_1 \subset \Omega$, then

$$(2.10) \qquad \left|\frac{1}{|B_0|}\int_{B_0} u - \frac{1}{|B_1|}\int_{B_1} u\right| \leq \frac{e}{2}\left(\log\frac{|B_0|}{|B_1|}+1\right)\|u\|_*.$$

See Lemma 5.10 in [2]. Combining (2.8) and (2.9) and applying (2.10) with $B_0 = B$, $B_1 = 2B$ and $u = \log J_f$ we obtain

$$D_f(x) \leq \exp\left(\frac{e}{2n}(n\log 2 + 1)\|\log J_f\|_*\right) C_5 a_f(x).$$

Next using the inequality $J_f \leq |Df|^n$ a.e. and Jensen's inequality we obtain

$$(2.11) \qquad \exp\left(\frac{1}{n|B|}\int_B \log J_f\right) \leq D_f(x).$$

Combining (2.10) and (2.11),

$$a_f(x) \leq \exp\left(\frac{e}{2n}(n\log 2 + 1)\|\log J_f\|_*\right) D_f(x).$$

3.1. Lip_λ-classes and Lip_λ-extension domains.

Suppose that $f: E \to R^m$, $E \subset R^n$. If there exists a constant C such that

$$(3.1) \qquad |f(x_1) - f(x_2)| \leq C\lambda(|x_1 - x_2|)$$

for all x_1, $x_2 \in E$, then we write $f \in \mathrm{Lip}_\lambda(E)$. We denote the smallest such C by $\|f\|_\lambda$. If $\lambda(t) = t^\alpha$ we write $\mathrm{Lip}_\alpha(E)$ and $\|f\|_\alpha$. If there exists a constant C such that

$$(3.2) \qquad |f(x_1) - f(x_2)| \le C\lambda(|x_1 - x_2| + d(x_1, \partial\Omega))$$

for all x_1, $x_2 \in E$, then we write $f \in \mathrm{Lip}_\lambda^*(E)$ and denote the smallest such C by $\|f\|_\lambda^*$. We similarly define $\mathrm{Lip}_\alpha^*(E)$ and $\|f\|_\alpha^*$.

If (3.1) holds for all x_1, $x_2 \in \Omega$ with $|x_1 - x_2| \le \frac{1}{2}d(x_1, \partial\Omega)$ we write $f \in \mathrm{loc}\,\mathrm{Lip}_\lambda(\Omega)$ and $\|f\|_{\lambda, \mathrm{loc}}$. Similarly for (3.2) we write $f \in \mathrm{loc}\,\mathrm{Lip}_\lambda^*(\Omega)$ and $\|f\|_{\lambda, \mathrm{loc}}^*$. In [11, Theorem 2.17] Lappalainen shows that if (3.1) holds whenever $|x_1 - x_2| \le ad(x_1, \partial\Omega)$ for some $a \le 1$, then $f \in \mathrm{loc}\,\mathrm{Lip}_\lambda(\Omega)$. The proof shows that a similar result also holds for $\mathrm{loc}\,\mathrm{Lip}_\lambda^*(\Omega)$.

3.2. Definition. $\Omega \subset R^n$ *is a* Lip_λ-*extension domain if* $f \in \mathrm{Lip}_\lambda(\Omega)$ *whenever* $f \in \mathrm{loc}\,\mathrm{Lip}_\lambda(\Omega)$. *In other words there exists a constant* b *such that* $\|f\|_\lambda \le b\|f\|_{\lambda, \mathrm{loc}}$ *for all* $f : \Omega \to R^m$.

A domain Ω is a Lip_λ-extension domain if and only if there exists a constant M such that each pair x_1, $x_2 \in \Omega$ can be joined by a continuous curve $\gamma \subset \Omega$ which satisfies

$$(3.3) \qquad \int_\gamma \frac{\lambda(d(\gamma(s), \partial\Omega))}{d(\gamma(s), \partial\Omega)}\, ds \le M\lambda(|x_1 - x_2|)$$

(see [11, p. 23, Theorem 4.2]). Here ds is the element of arclength. These domains were first introduced by Gehring and Martio in the case that $\lambda(t) = t^\alpha$ and called Lip_α-extension domains. For certain $\lambda(t)$, the class of Lip_λ-extension domains is large. All uniform domains are Lip_λ-extension domains if and only if there is a constant A such that

$$\int_0^\delta \frac{\lambda(t)}{t}\, dt \le A\lambda(\delta)$$

for all $0 \le \delta < \infty$ (see [11, p. 28, Theorem 4.17] and [8, p. 204, Theorem 2.2]. In particular, if $\lambda(t) = t^\alpha$, all balls, half-spaces and quasiballs are Lip_α-extension domains for all $0 < \alpha \le 1$.

3.3. Lemma. *If* Ω *is a* Lip_λ-*extension domain and if* $f \in \mathrm{loc}\,\mathrm{Lip}_\lambda^*(\Omega)$, *then* $f \in \mathrm{Lip}_\lambda^*(\Omega)$.

The proof is similar to the proof of the characterization of Lip_α-extension domains in [8] and Lip_λ-extension domains in [11]. We include the proof here for completeness.

Proof. Fix x_1, $x_2 \in \Omega$ and let γ be a curve joining x_1 to x_2 in Ω which satisfies (3.3). Choose balls $B(y_i, r_i) = \{x \mid |x - y_i| < r_i\}$ as follows. Set $y_1 = x_2$, $r_1 = \frac{1}{4}d(y_1, \partial\Omega)$ and $l_1 = \max\{s \in [0, l] \mid \gamma(s) \in \overline{B}(y_1, r_1)\}$ where l is the length of γ. Suppose that y_i, r_i and l_i have been chosen for $i = 1, 2, \ldots, k$ and $l_k < l$. Set $y_{k+1} = \gamma(l_k)$, $r_{k+1} = \frac{1}{4}d(y_{k+1}, \partial\Omega)$ and $l_{k+1} = \max\{s \in [0, l] \mid \gamma(s) \in \overline{B}(y_{k+1}, r_{k+1})\}$. When $l_k = l$, the process stops and we write $y_{k+1} = x_1$.

First suppose that $|x_1 - x_2| < |y_k - y_{k+1}| = |y_k - x_1|$. Then $|x_1 - x_2| \leq \frac{1}{4}d(y_k, \partial\Omega)$. Hence

$$d(x_1, \partial\Omega) \geq d(y_k, \partial\Omega) - |x_1 - y_k| \geq \tfrac{3}{4}d(y_k, \partial\Omega)$$

and so $|x_1 - x_2| \leq \frac{1}{3}d(x_1, \partial\Omega)$. Since $f \in \text{loc Lip}_\lambda^*(\Omega)$,

$$|f(x_1) - f(x_2)| \leq \|f\|_{\lambda, \text{loc}}^* \lambda(|x_1 - x_2| + d(x_1, \partial\Omega)).$$

Hence we can suppose that

(3.4) $$|y_k - y_{k+1}| \leq |x_1 - x_2|.$$

Next

(3.5)
$$|f(x_1) - f(x_2)| \leq \sum_{i=1}^{k} |f(y_i) - f(y_{i+1})|$$

$$\leq \|f\|_{\lambda, \text{loc}}^* \sum_{i=1}^{k} \lambda(|y_i - y_{i+1}| + d(y_i, \partial\Omega)).$$

Let $l_0 = 0$ and set $A_i = \{s \in [l_{i-1}, l_i] \mid \gamma(s) \in \overline{B}(y_i, r_i)\}$ for $i = 1, 2, \ldots,$ $k - 1$. $A_i \subset [l_{i-1}, l_i]$ is a closed set and $|A_i| \geq r_i = |y_i - y_{i+1}|$. Here $|A_i|$ is the one-dimensional Lebesgue measure of A_i. Also for each $s \in A_i$,

$$d(\gamma(s), \partial\Omega) \leq |\gamma(s) - y_i| + d(y_i, \partial\Omega) \leq r_i + 4r_i = 5r_i.$$

Hence, since $\lambda(t)/t$ is decreasing,

(3.6) $$\frac{\lambda(5r_i)}{r_i} \leq \frac{5\lambda(d(\gamma(s), \partial\Omega))}{d(\gamma(s), \partial\Omega)}$$

when $s \in A_i$. Using (3.5), (3.4), (3.6), and (3.3) we obtain

$$|f(x_1) - f(x_2)| \leq \|f\|_{\lambda, \text{loc}}^* \sum_{i=1}^{k} \lambda(|y_i - y_{i+1}| + d(y_i, \partial\Omega))$$

$$\leq \|f\|_{\lambda, \text{loc}}^* \left\{ \sum_{i=1}^{k-1} \lambda(5|y_i - y_{i+1}|) + \lambda(|y_k - y_{k+1}| + d(y_k, \partial\Omega)) \right\}$$

$$\leq \|f\|_{\lambda, \text{loc}}^* \left\{ \sum_{i=1}^{k-1} \frac{\lambda(5r_i)}{r_i} |A_i| + \lambda(2|x_1 - x_2| + d(x_1, \partial\Omega)) \right\}$$

$$\leq \|f\|_{\lambda, \text{loc}}^* \left\{ 5 \sum_{i=1}^{k-1} \int_{A_i} \frac{\lambda(d(\gamma(s), \partial\Omega))}{d(\gamma(s), \partial\Omega)} ds + 2\lambda(|x_1 - x_2| + d(x_1, \partial\Omega)) \right\}$$

$$\leq 5\|f\|_{\lambda, \text{loc}}^* \left\{ \int_{\gamma} \frac{\lambda(d(\gamma(s), \partial\Omega))}{d(\gamma(s), \partial\Omega)} ds + \lambda(|x_1 - x_2| + d(x_1, \partial\Omega)) \right\}$$

$$\leq 5M\|f\|_{\lambda, \text{loc}}^* \lambda(|x_1 - x_2| + d(x_1, \partial\Omega)).$$

4.1.. Proof of Theorem 1.2. First suppose that (1.3) holds for all $x \in \Omega$. Since f is K-quasiregular there exists $s > n$, depending only on n and K, such that

$f \in W^1_{s,\text{loc}}(\Omega)$. Fix x_1, $x_2 \in \Omega$ with $|x_1 - x_2| = \frac{1}{4}d(x_1, \partial\Omega)$. We apply (2.1) with $B = B(x_1, \frac{1}{4}d(x_1, \partial\Omega))$, and (2.3) with $F = \overline{B}$ and $\Omega = 2B$ to obtain

$$|f(x_1) - f(x_2)| \leq \frac{Cs}{s-n}|B|^{(s-n)/(sn)}\left(\int_B |Df|^s\right)^{1/s}$$

$$\leq C_1(n, K)\left(\int_{2B} |Df|^n\right)^{1/n}$$

$$\leq C_1(n, K)C_2|x_1 - x_2|D_f(x_1)$$

$$\leq 4C_1(n, K)C_2\lambda(d(x_1, \partial\Omega))$$

$$\leq 16C_1(n, K)C_2\lambda(|x_1 - x_2|).$$

Now assume that $|x_1 - x_2| < \frac{1}{4}d(x_1, \partial\Omega)$. Let $R_1 = \frac{1}{4}d(x_1, \partial\Omega)$ and $R_2 = \frac{1}{4}d(x_2, \partial\Omega)$. Now $R_2 \leq R_1 + \frac{1}{4}|x_1 - x_2| \leq \frac{5}{4}R_1$ and $R_2 \geq R_1 - \frac{1}{4}|x_1 - x_2| \geq R_1 - |x_1 - x_2|$. Hence there exists a point x_3 with $x_3 \in \partial B(x_1, R_1) \cap \partial B(x_2, R_2)$. So we obtain $|f(x_1) - f(x_2)| \leq |f(x_1) - f(x_3)| + |f(x_2) - f(x_3)| \leq C(\lambda(|x_1 - x_3|) + \lambda(|x_2 - x_3|))$ by applying (4.1) to the pairs x_1, x_3 and x_2, x_3. It follows that

$$|f(x_1) - f(x_2)| \leq C(\lambda(d(x_1, \partial\Omega)) + \lambda(|x_1 - x_2| + d(x_1, \partial\Omega)))$$

and so $f \in \text{loc Lip}^*_\lambda(\Omega)$. Since Ω is a Lip_λ-extension domain it follows from Lemma 3.3 that $f \in \text{Lip}^*_\lambda(\Omega)$. Next if $x_0 \in \partial\Omega$ and $\{x_j\}$ is a sequence in Ω with $\lim_{j\to\infty} x_j = x_0$ we have

$$|f(x_j) - f(x_k)| \leq C\lambda(|x_j - x_k| + d(x_j, \partial\Omega)).$$

Hence f tends to a well-defined limit at x_0 and (1.4) is satisfied for all x_1, $x_2 \in \overline{\Omega}$.

Next suppose that (1.4) holds for all x_1, $x_2 \in \overline{\Omega}$. Fix $x \in \Omega$. We apply (2.3) with $B = B(x, \frac{1}{2}d(x, \partial\Omega))$, $\sigma = \frac{3}{2}$ and $a = f(x)$ to obtain

$$(4.1) \qquad D_f(x) \leq \frac{C_1(n)K}{d(x, \partial\Omega)}\left(\frac{1}{|B|}\int_{3B/2} |f - f(x)|^n\right)^{1/n}.$$

Using (1.4) we get

$$(4.2) \qquad \left(\frac{1}{|B|}\int_{3B/2} |f - f(x)|^n\right)^{1/n} \leq \frac{3}{2}\lambda\left(\sup_{y \in 3B/2} |x - y| + d(x, \partial\Omega)\right)$$

$$\leq \frac{3}{2}\lambda(2d(x, \partial\Omega)) \leq 3\lambda(d(x, \partial\Omega)).$$

(1.3) follows by combining (4.1) and (4.2).

5.1. Proof of Theorem 1.3.

We use the following definitions and results. A pair of sets $E = (A, C)$ in R^n is called a condenser when A is open and $C \subset A$ is compact. The condenser E is bounded if A is bounded. Its conformal capacity is defined by

$$\text{cap } E = \inf_u \int_{R^n} |\nabla u|^n$$

where the infimum is taken over all infinitely differentiable u in A with compact support in A and $u(x) \geq 1$ for $x \in C$. If follows from the definition that if $A' \subset A$ and $C \subset C'$, then

$$(5.1) \qquad \operatorname{cap}(A, C) \leq \operatorname{cap}(A', C').$$

For more information concerning condensers see [12, pp. 24–28; 20, pp. 81–102 and 17]. We next define the spherical symmetrization, E^*, of an open or closed set $E \subset R^n$, about the ray $L = \{t\xi \mid 0 \leq t < \infty,\ \xi$ some point of $\partial B(0, 1)\}$ as follows. We let $S^{n-1}(r) = \partial B(0, r)$. We call sets of the form $B(y, \rho) \cap S^{n-1}(r)$, with $y \in S^{n-1}(r)$ and $0 \leq \rho < \infty$, caps of $S^{n-1}(r)$. We define E^* by the conditions:

$$S^{n-1}(r) \cap E^* = \begin{cases} \varnothing, & \text{if and only if } S^{n-1}(r) \cap E = \varnothing, \\ S^{n-1}(r), & \text{if and only if } S^{n-1}(r) \subset E. \end{cases}$$

Otherwise, $S^{n-1}(r) \cap E^*$ is the cap of the sphere $S^{n-1}(r)$ such that

  (i) the center of the cap is the point $S^{n-1}(r) \cap L$.
  (ii) $m_{n-1}(S^{n-1}(r) \cap E^*) = m_{n-1}(S^{n-1}(r) \cap E)$. Here m_{n-1} is the $(n-1)$-dimensional Hausdorff measure on $S^{n-1}(r)$.
  (iii) the cap $S^{n-1}(r) \cap E^*$ is open or closed according as E is open or closed.

Here $0 \leq r < \infty$. For the following result see Gehring [6, p. 505, Theorem 1] and Sarvas [17, p. 522, Theorem 7.5].

5.2. Theorem. *If $E = (A, C)$ is a condenser let C^* be the spherical symmetrization of C about the negative x_1-axis and $A^* = R^n \backslash B$ where B is the spherical symmetrization of $R^n \backslash A$ in the positive x_1-axis, then (A^*, C^*) is a condenser and*

$$\operatorname{cap}(A^*, C^*) \leq \operatorname{cap}(A, C).$$

Theorem 1.3 follows from Theorem 1.2 and the next result. The quasiconformal version of Theorem 5.3 appears in Näkki and Palka [13, p. 379, Theorem 1]. Our proof is similar to their proof.

5.3. Theorem. *Suppose that f is K-quasiregular in $\Omega \subset R^n$ and continuous on $\overline{\Omega}$. If there exist constants α, $0 < \alpha \leq K^{1/(1-n)}$, and $C_1 < \infty$ such that*

$$(5.2) \qquad |f(x) - f(y)| \leq C_1 |x - y|^\alpha$$

for all $x \in \overline{\Omega}$ and $y \in \partial\Omega$, then $f \in \operatorname{Lip}_\alpha(\overline{\Omega})$ with $\|f\|_\alpha$ depending only on C_1, n, K, and α. If Ω is bounded and (5.2) holds for some $0 < \alpha \leq 1$, then $f \in \operatorname{Lip}_\beta(\overline{\Omega})$ with $\beta = \min(\alpha, K^{1/(1-n)})$.

Proof. Assume that f is not constant. Fix x and y with $f(x) \neq f(y)$. First suppose that $|x - y| \geq \frac{1}{2}d(x, \partial\Omega)$. Choose $z \in \partial\Omega$ such that $|x - z| = d(x, \partial\Omega)$. Using (5.2) we obtain

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(y) - f(z)| \leq 6^\alpha C_1 |x - y|^\alpha.$$

Next suppose that $|x - y| < d(x, \partial\Omega)/2$. Let $B = B(x, d(x, \partial\Omega)/2)$ and $C = \overline{B}(x, |x - y|)$. Then $E = (B, C)$ is a bounded condenser with capacity

$$(5.3) \qquad \operatorname{cap} E = \omega_{n-1} \left(\log \frac{d(x, \partial\Omega)}{2|x - y|} \right)^{1-n}.$$

Here ω_{n-1} is the $(n-1)$-dimensional measure of $\partial B(0,1)$. Since f is quasiregular and nonconstant, f is continuous and open in Ω. Hence $F = (f(B), f(C))$ is a bounded condenser in $f(\Omega)$ and $\partial f(B) \subset f(\partial B)$. Since $f(B)$ is bounded, $R^n \backslash f(B)$ contains a, necessarily unique, unbounded component C_∞ with $\partial C_\infty \neq \varnothing$. Let $\xi \in \partial C_\infty$. Since $\partial C_\infty \subset \partial f(B)$, $\xi \in \partial f(B)$. Since $\partial f(B) \subset f(\partial B)$, there is a point $z \in \partial B$ such that $f(z) = \xi$. Hence $f(x), f(y) \in f(C)$ and $f(x) \neq f(y)$, while $f(z) \in C_\infty$. Now the capacity of F is preserved under a similarity transformation T. With $\tilde{f} = T \circ f$ we assume that $\tilde{f}(x) = 0$ and $\tilde{f}(y) = (-1, 0, \dots, 0) = -e_1$. Next let $\tilde{f}(C)^*$ be the spherical symmetrization of $\tilde{f}(C)$ about the negative x_1-axis and $\tilde{f}(B)^* = R^n \backslash D$ where D is the spherical symmetrization of $R^n \backslash \tilde{f}(B)$ in the positive x_1-axis. By Theorem 5.2

$$(5.4) \qquad \mathrm{cap}(\tilde{f}(B)^*, \tilde{f}(C)^*) \leq \mathrm{cap}(\tilde{f}(B), \tilde{f}(C)).$$

We denote by $R_T(t)$ the Teichmüller condenser $(R^n \backslash L_2, L_1)$ where L_1 is the line segment from $-e_1$ to 0 and L_2 is the ray on the x_1-axis from te_1 to ∞. If $t = |\tilde{f}(z)|$, then $L_1 \subset \tilde{f}(C)^*$ and $\tilde{f}(B)^* \subset R^n \backslash L_2$. Hence by (5.1)

$$(5.5) \qquad \mathrm{cap}\, R_T(|\tilde{f}(z)|) \leq \mathrm{cap}(\tilde{f}(B)^*, \tilde{f}(C)^*).$$

We also have the following lower bound for the capacity of the Teichmüller condenser (see [6, p. 518, Lemma 8 and 20, p. 89, Lemma 7.22]).

$$(5.6) \qquad \mathrm{cap}\, R_T(t) \geq \omega_{n-1}[\log \lambda(t+1)]^{1-n}.$$

Here λ is a constant which depends only on n. Moreover since f is K-quasiregular in Ω

$$(5.7) \qquad \mathrm{cap}\, F \leq K \,\mathrm{cap}\, E$$

[12, p. 29, Theorem 7.1]. Notice that $|\tilde{f}(z)| = |f(z) - f(x)|/|f(y) - f(x)|$. Combining (5.6), (5.5), (5.4), (5.7), and (5.3) we obtain

$$(5.8) \qquad \begin{aligned} \omega_{n-1} &\left[\log \lambda \left(1 + \frac{|f(z) - f(x)|}{|f(y) - f(x)|} \right) \right]^{1-n} \leq \mathrm{cap}(\tilde{f}(B), \tilde{f}(C)) \\ &= \mathrm{cap}(f(B), f(C)) \leq K \omega_{n-1} \left[\log \frac{d(x, \partial \Omega)}{2|x-y|} \right]^{1-n}. \end{aligned}$$

Next choose $x_0 \in \partial \Omega$ such that $|x - x_0| = d(x, \partial \Omega)$. We have the estimate

$$(5.9) \qquad \begin{aligned} |f(y) - f(x)| &+ |f(z) - f(x)| \\ &\leq C_1(2|x_0 - x|^\alpha + |z - x_0|^\alpha + |y - x_0|^\alpha) \\ &\leq 6 C_1 d(x, \partial \Omega)^\alpha. \end{aligned}$$

Using (5.9) we can rewrite (5.8),

$$|f(x) - f(y)| \leq 6 \lambda C_1 2^{\alpha'} d(x, \partial \Omega)^{\alpha - \alpha'} |x - y|^{\alpha'}$$

where $\alpha' = K^{1/(1-n)}$. If $\alpha \leq \alpha'$, then since $|x - y| < d(x, \partial \Omega)/2$ we have $f \in \mathrm{Lip}_\alpha(\overline{\Omega})$. Otherwise, if Ω is bounded, then $f \in \mathrm{Lip}_{\alpha'}(\Omega)$ since $d(x, \partial \Omega) \leq$ Euclidean diameter of Ω.

Theorem 5.3 also gives the following.

5.4. **Corollary.** *If f is K-quasiregular in a bounded domain $\Omega \subset R^n$ and if (1.5) is satisfied for all $x \in \Omega$ with $0 < \alpha \le 1$, then $f \in \mathrm{Lip}_\beta(\overline{\Omega})$ where $\beta = \min(\alpha, K^{1/(1-n)})$.*

5.5. *Remark.* Because the components of a quasiregular mapping satisfy $|\nabla f_i| \le K|\nabla f_j|$ a.e., Theorem 1.3 implies a special case of the main result in [14, p. 705, Theorem 3.8]. Namely the components belong to the same Lipschitz class.

6.1. **Bounded mean oscillation.** For $f : \Omega \to R^m$ we denote the BMO-norm by

$$\|f\|_* = \sup_{B \subset \Omega} \frac{1}{|B|} \int_B |f - f_B|.$$

Here the supremum is taken over all balls $B \subset \Omega$ and f_B is the average value of f over B, $(1/|B|)\int_B f$. In the case that u is harmonic in Ω, $\|u\|_* < \infty$ if and only if $|\nabla u(x)| \le Cd(x, \partial\Omega)^{-1}$ for all $x \in \Omega$ and consequently, if and only if $D_u(x) \le Cd(x, \partial\Omega)^{-1}$ for all $x \in \Omega$. Theorem 1.4 is a generalization of this result. Theorem 6.2 is a special case of the main result in ([18], see inequality (1.4)). In the case that Ω is a half space, it appears in [16, p. 4, Hilfssatz 2].

6.2. **Theorem.** *If $f : \Omega \to R^m$ satisfies*

$$(6.1) \qquad \sup_{2B \subset \Omega} \frac{1}{|B|} \int_B |f - f_B| < \infty$$

where the supremum is taken over all balls B with $2B \subset \Omega$, then $\|f\|_* < \infty$.

The next result follows from the John-Nirenberg Theorem [5, p. 233, Corollary 2.3 and 16, p. 32].

6.3. **Theorem.** *If $f : \Omega \to R^m$, and if $1 \le p < \infty$, then there exists a constant C, depending only on n and p, such that*

$$(6.2) \qquad \left(\frac{1}{|B|} \int_B |f - f_B|^p \right)^{1/p} \le C\|f\|_*$$

for all balls $B \subset \Omega$.

6.4. **Proof of Theorem 1.4.** Assume that (1.5) holds for all $x \in \Omega$. By Theorem 6.2 it is sufficient to show that f satisfies (6.1). Let $x_0 \in \Omega$ and $B = B(x_0, \frac{1}{2}d(x_0, \partial\Omega))$. Using Hölder's inequality and (2.2) we obtain

$$\frac{1}{|B|} \int_B |f_j - (f_j)_B| \le \left(\frac{1}{|B|} \int_B |f_j - (f_j)_B|^n \right)^{1/n} \le 2 \, \mathrm{diam}\, B D_{f_j}(x_0).$$

Since $\mathrm{diam}\, B = d(x_0, \partial\Omega)$, (1.7) gives

$$\frac{1}{|B|} \int_B |f_j - (f_j)_B| \le 2C_1$$

and (6.1) follows.

Next assume that (1.8) holds. Applying (2.4) with $\sigma = 2$,

$$(6.3) \qquad \left(\frac{1}{|B|} \int_B |\nabla f_j|^n \right)^{1/n} \le \frac{C(n, K)}{\mathrm{diam}\, B} \left(\frac{1}{|2B|} \int_{2B} |f_j - a|^n \right)^{1/n}$$

for all $a \in R$. Choosing $a = (f_j)_{2B}$ in (6.3) and $p = n$ in (6.2) we obtain

$$D_{f_j}(x_0) \le \frac{C(n, K)\|f\|_*}{d(x_0, \partial\Omega)} \le C_2 C(n, K) d(x_0, \partial\Omega)^{-1}.$$

Theorem 1.4 provides another proof of the following result which appears in [10, p. 280, Corollary 3].

6.5. Corollary. *If* $f = (f_1, f_2, \ldots, f_n)$ *is K-quasiregular in* Ω, *then the* BMO-*norms* $\|f_j\|_*$, $j = 1, 2, \ldots, n$, *are equivalent.*

Proof. The operators $D_{f_j}(x)$ are equivalent since $|\nabla f_j(x)| \le K|\nabla f_i(x)|$ a.e. for all $i, j = 1, 2, \ldots, n$.

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