TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 332, Number 1, July 1992

FREE α-EXTENSIONS OF AN ARCHIMEDEAN VECTOR LATTICE AND THEIR TOPOLOGICAL DUALS

ANTHONY J. MACULA

ABSTRACT. Arch denotes the category of Archimedean vector lattices with vector lattice homomorphisms, and α denotes an uncountable cardinal number or the symbol ∞. Arch(α) denotes the category of Arch objects with α-complete Arch morphisms.

In this paper we construct, for each $L \in |\text{Arch}|$, α-complete extensions L' of L that lift Arch morphisms from L to *α-complete* Arch morphisms from L'. Specifically, we construct the *free α-extension* and the *free α-regular* extension of an Arch object L. By virtue of the latter, the full subcategory of α-complete objects, in Arch(α), is epireflective. The proofs work in Boolean algebras and recover the results obtained in [K, Y, and S]. Our proofs are different and, it can be argued, more natural.

$\mathscr{W}$ denotes the category of Arch objects with distinguished weak unit and Arch morphisms that preserve units. We exploit a certain contravariant functor $Y : \mathscr{W} \to$ Comp (the so-called Yosida functor, analogous to the Stone-space functor) from $\mathscr{W}$ to the category of compact Hausdorff spaces with continuous functions, to convert algebraic results in $\mathscr{W}$ to topological results in the topological category *α-SpFi*. Specifically, we show that the Yosida "dual" of the $\mathscr{W}$-free α-regular extension of $C(X)$ is the *α-disconnected α-SpFi monocoreflection* of the compact space X, thereby showing that the full subcategory of α-disconnected spaces, in *α-SpFi*, is monocoreflective.

1. Introduction

α denotes an uncountable cardinal number or the symbol ∞. The meaning of $\alpha = \infty$ will be clear from the context. When we write $\alpha < \infty$ or $|A| < \infty$, where A is a set, we mean that α or $|A|$ is an arbitrary cardinal number.

Arch denotes the category of Archimedean vector lattices with vector lattice homomorphisms. An element $u \in L \in |\text{Arch}|$ is called a *weak unit* if the band (complete ideal) generated by u is all of L [LZ, dJvR]. u is called a *strong unit* if the principal ideal generated by u is all of L. $\mathscr{W}$ ($\mathscr{S}$) denote the category of Arch objects with distinguished weak (strong) unit and unit preserving Arch

Received by the editors December 1, 1989 and, in revised form, May 16, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 06A23, 06B30, 18B30, 54C10, 54G05; Secondary 06B15, 06F15.
Key words and phrases. Free α-regular extension, free α-extension, α-complete vector lattice, α-disconnected space, Yosida space.

0002-9947/92 \$1.00 + \$.25 per page

morphisms. Obviously a strong unit is a weak unit, so $\mathscr{S}$ is a subcategory of $\mathscr{W}$.

An Arch morphism $\varphi : L \to M$ is called α-complete if, for $A \subset L$ with $|A| < \alpha$, we have that $\varphi(\bigvee^L A) = \bigvee^M \varphi[A]$ whenever $\bigvee^L A$ exists in L. Arch(α), $\mathscr{W}(\alpha)$, and $\mathscr{S}(\alpha)$ denote the categories of Arch, $\mathscr{W}$, and $\mathscr{S}$ objects with α-complete morphisms respectively.

An $L \in |\text{Arch}|$ is called α-complete $\bigvee^L A$ exists in L for all bounded above $A \subset L$ with $|A| < \alpha$.

Recall L is called *Dedekind complete* if every subset of L that is bounded above has a supremum in L; thus L is ∞-complete if and only if L is Dedekind complete.

$L \subseteq M$ denotes that L is an Arch subspace of M (i.e., L is a vector lattice subspace of M), while $L \subseteq^\alpha M$ denotes that, in addition to $L \subseteq M$, the inclusion of L into M is an α-complete Arch morphism. As usual, we reserve $\subset$ for ordinary set inclusion.

The next two definitions, as do the definitions of an α-complete morphism and an α-complete object, make sense in $\mathscr{W}$, $\mathscr{S}$, and Boolean algebras $(\equiv \mathscr{BA})$, as well as in Arch.

For $L \subseteq M$, we say that L is α-full subspace in, or of, M if $b = \bigvee^M \{A : A \subset L, |A| < \alpha\}$ implies that $b \in L$ (i.e., $b = \bigvee^L A$). We call an Arch embedding $\varphi : L \hookrightarrow M$ α-full if $\varphi[L]$ is α-full in M.

What we call an α-full subspace of M, Sikorski (in §23 of [S] for Boolean algebras) calls an α-subalgebra of M.

For $L \subseteq M$, we say that L *α-generates* M if M is the smallest α-full subspace of M that contains L. That is, if L' is α-full in M and $L \subseteq L'$, then $L' = M$.

Let L and M be Arch objects and let $\varphi : L \hookrightarrow M$ be an Arch morphism with M α-complete. We say the pair (φ, M) is an Arch *free α-extension* of L if $\varphi[L]$ α-generates M, and each Arch morphism from L into an α-complete Arch object N can be extended to an α-complete Arch morphism from M into N. We say the pair (φ, M) is an Arch *free α-regular extension* of L if φ is an Arch α-complete embedding, $\varphi[L]$ α-generates M, and each α-complete Arch morphism from L into an α-complete Arch object N can be extended to an α-complete Arch morphism from M into N.

Analogously, one obtains the definitions of a $\mathscr{W}$, $\mathscr{S}$, or $\mathscr{BA}$ free α-regular extension and a $\mathscr{W}$, $\mathscr{S}$, or $\mathscr{BA}$ free α-extension by replacing the symbol Arch with $\mathscr{W}$, $\mathscr{S}$, or $\mathscr{BA}$ in the above.

In §3, we indicate that, for each L, there is essentially, one free α-regular extension, and one free α-extension, of L.

Henceforth, L, M, and N denote Arch objects, and maps between them are always considered to be Arch morphisms unless otherwise stated.

For $\alpha < \infty$, Yaqub shows in [Y] that every Boolean algebra has a free α-extension. However, in [Ha], Hales shows that the free Boolean algebra on ω generators does not have a free ∞-extension.

The existence of the free α-regular extension of a Boolean algebra (for $\alpha < \infty$) was first proved by Kerstan in [K], and subsequently, independently by Sikorski and Yaqub in [S_2 and Y] respectively. In [S], Sikorski expands on the idea of free α-regular extensions of Boolean algebras and discusses what he calls $(J, M, \mathfrak{m})$-extensions. Also, since in $\mathscr{BA}$, the injective objects are exactly the ∞-complete Boolean algebras [S, 33.1], the free ∞-regular extension of a Boolean algebra B is the Dedekind completion of B [S]. Moreover, even though there are no injectives in Arch [Co], the ∞-regular extension of an $L \in |\text{Arch}|$ is the Arch Dedekind completion of L [BH_1, M_1].

What we do here in §3, for $\alpha < \infty$, is construct the Arch free α-extension and the Arch free α-regular extension of an Arch object L. The proofs work in $\mathscr{W}$ and $\mathscr{S}$, as well as in $\mathscr{BA}$, and recover the results obtained in [K, Y, and S]. Our proofs are different, and, it can be argued, more natural.

In §4 here we exploit a certain contravariant functor $Y : \mathscr{W} \to \text{Comp}$ (the so-called Yosida functor, analogous to the Stone-space functor) from $\mathscr{W}$ to the category of compact Hausdorff spaces with continuous functions, to convert algebraic results in $\mathscr{W}$ to topological results in the topological category α-$SpFi$. Specifically, we show that the Yosida "dual" of the $\mathscr{W}$-free α-regular extension of $C(X)$ is the α-disconnected α-$SpFi$ *monocoreflection* of the compact space X, thereby showing that the full subcategory of α-disconnected spaces, in α-$SpFi$, is monocoreflective.

We begin a discussion about the Yosida functor and the category α-$SpFi$.

2. The Yosida functor and the category α-$SpFi$

X, Y, and Z denote compact Hausdorff spaces; f, g, and h denote continuous functions; and Comp denotes the category of compact Hausdorff spaces with continuous functions.

We review the Yosida representation theory. For each $L \in |\mathscr{W}|$ there is an associated compact Hausdorff space, $Y(L)$, called the *Yosida space* of L. See [LZ, BKW, and HeR]. Also for each $\varphi : L \to M$, there is associated continuous function $Y(\varphi) : Y(M) \to Y(L)$. See [$\text{HR}_1$]. It turns out that Y is a faithful (contravariant) functor from $\mathscr{W}$ to Comp. The functor Y works very much like the Stone functor from Boolean algebras to Boolean spaces. $Y(L)$, like the Stone space of a Boolean algebra, is a maximal ideal space. The elements of $Y(L)$ are ideals of L that are maximal for the property of not containing the weak unit. If the weak unit is a strong unit then these ideals are the actual maximal ideals of L. The topology on this space is the hull-kernel topology. In fact, if we view a Boolean algebra, B, as a $\mathscr{W}$ object (i.e., if $L(S(B))$ is the locally constant real-valued functions on the Stone space $S(B)$, then $L(S(B_1)) \cong L(S(B_2))$ iff $B_1 \cong B_2$), the Yosida functor can be thought of as an extension of the Stone functor.

The following paragraph comes from [BH_2].

The archetypal $\mathscr{W}$ object is $C(X)$ (the ring of continuous real-valued functions on X with the pointwise sup and inf). The weak unit of $C(X)$ will always be taken to be the constant function $\mathbf{1}$. Note that the weak unit $\mathbf{1}$ is indeed a strong unit and $(C(X), \mathbf{1})$ is an $\mathscr{S}$ object. Let $D(X)$ be the set of extended real-valued continuous functions, $f : X \to [-\infty, +\infty]$, for which $f^{-1}(\mathbf{R})$ is dense in X. In the pointwise order, $D(X)$ is a lattice, but usually fails to be a vector space. For $f, g, h \in D(X)$, we say "$f + g = h$ in $D(X)$" if $f(x) + g(x) = h(x)$ when $x \in f^{-1}(\mathbf{R}) \cap g^{-1}(\mathbf{R}) \cap h^{-1}(\mathbf{R})$ (which is a dense set in X). It may well happen that, for particular $f, g \in D(X)$, there is no $h \in D(X)$ with $f + g = h$ in $D(X)$ (e.g., take $X = [-\infty, +\infty]$, f the obvious extension of $x+\sin x$, and g the extension of $-x$). However, it may well happen that a subset $L \subset D(X)$ has the property that for all $f, g \in L$ there is an $h \in D(X)$ with $f+g=h$ in $D(X)$; if L is also a vector lattice under the pointwise operation in $D(X)$ and the constant function, $\mathbf{1}$, is in L, then we say "$(L, \mathbf{1})$ (or just L) is a $\mathscr{W}$ object in $D(X)$", (e.g., $C(X)$ is a $\mathscr{W}$ object, in $D(X)$). If X has the property that each dense cozero set is C^*-embedded [GJ], then X is called ω_1-quasi-F (or just quasi-F) [DHH, HVW, BHN, M_2]. If X is ω_1-quasi-F, then $(D(X), \mathbf{1}) \in |\mathscr{W}|$. See [HJ].

2.1 **Theorem** (see [BKW, HR1]). (a) *There is a $\mathscr{W}$ isomorphism, $\widehat{\ }: L \to \widehat{L} \subset D(Y(L))$, onto a $\mathscr{W}$ object, $\widehat{L}$ in $D(Y(L))$, with $\hat{w}_L = \mathbf{1}$, and $\widehat{L}$ separates the points of $Y(L)$.*

(b) *If L' is a $\mathscr{W}$ object in $D(X)$ which separates the points of X, and for $a \in L$, if $a \mapsto a'$ is a $\mathscr{W}$ isomorphism from L to L', then there is a homeomorphism $f: X \to Y(L)$ such that $a' = \hat{a} \circ f$ for all $a \in L$.*

2.1(b) is used to recognize Yosida representations.

2.2 **Corollary.** $Y(C(X)) = X$.

Proof. In $D(X)$, $C(X)$ satisfies 2.1(b).

2.3 **Theorem** [HR_1]. *Let $\varphi_i : L \to M$ for $i = 1, 2$.*

(a) *There is a unique continuous function, $Y(\varphi_1): Y(M) \to Y(L)$, such that $\varphi(a)^\frown = \hat{a} \circ Y(\varphi_1)$ for all $a \in L$.*

(b) *Y is a faithful functor, i.e., if $\varphi_1 \neq \varphi_2$, then $Y(\varphi_1) \neq Y(\varphi_2)$.*

(c) *φ_1 is one-to-one if and only if $Y(\varphi_1)$ is onto, and if φ_1 is onto, then $Y(\varphi_1)$ is one-to-one.*

(d) *Let $\gamma: C(X) \to M$ and $M \in |\mathscr{S}|$. Then $Y(\gamma): Y(M) \to X$ is one-to-one if and only if γ is onto.*

Henceforth, L and $\widehat{L}$ are identified.

Thus we will consider $a \in L$ as an extended real-valued function on $Y(L)$.

The next proposition is straightforward.

2.4 **Proposition.** *Let $f: X \to Y$. Define $f': C(Y) \to C(X)$ by $f'(g) = g \circ f$ for $g \in C(Y)$* [GJ, 10.2]. *Then f' is a $\mathscr{W}$ morphism and $Y(f') = f$.*

2.5 **Theorem** (Banach-Stone). *A function $f: X \to Y$ is a homeomorphism if and only if $f': C(Y) \to C(X)$ is an isomorphism in $\mathscr{W}$.*

Proof. $Y(f') = f$. Apply 2.3(c) and (d).

We begin a discussion of the topological category α-$SpFi$. See [BHM, BHN, M_1, M_2, and BH_3].

Let $\mathrm{Coz}(X) = \{f^{-1}(\mathbf{R}\backslash\{0\}) : f \in C(X)\}$. A subset $V \subset X$ is said to be an α-cozero set if

$$V = \bigcup\{U_i : i \in I,\ |I| < \alpha,\ U_i \in \mathrm{Coz}(X)\}.$$

Note that an ω_1-cozero set is a cozero set. Recall that by "$|I| < \infty$" we mean that "$|I|$ is unrestricted," so that every open set is an ∞-cozero set. We denote the collection of α-cozero sets of X by $\mathrm{Coz}_\alpha(X)$. Let $G_\alpha X$ denote the filter base of dense members of $\mathrm{Coz}_\alpha(X)$.

A continuous function $f : X \to Y$ is called an α-*SpFi* morphism if $f^{-1}(G) \in G_\alpha X$ whenever $G \in G_\alpha Y$. We thus have a topological category, denoted α-*SpFi*, which consists of compact Hausdorff spaces and α-*SpFi* morphisms.

Below (2.6) is a cardinal generalization of 4.2(c) of [BH$_1$]. See also [M$_2$]. It tells us that the Yosida functor converts an α-complete $\mathscr{W}$ morphism into an α-*SpFi* morphism, and, that every α-*SpFi* morphism arises from an α-complete $\mathscr{W}$ morphism.

2.6 **Lemma.** *$\varphi : L \to M$ is α-complete if and only if $Y(\varphi) : Y(M) \to Y(L)$ is an α-SpFi morphism. Moreover, a function $f : X \to Y$ is an α-SpFi morphism if and only if f' (2.5) is α-complete.*

From this result it is routine to see that Y restricted to $\mathscr{W}(\alpha)$ is a functor to α-*SpFi*.

Moreover, Y take $\mathscr{W}(\alpha)$ epics to α-*SpFi* monics. Recall that in a category, a morphism e is called epic if $f_1 \circ e = f_2 \circ e$ implies that $f_1 = f_2$, and a morphism m is called *monic* if $m \circ f_1 = m \circ f_2$ implies that $f_1 = f_2$. We have

2.7 **Lemma** [M$_2$]. *$\varphi : L \to M$ is epic in $\mathscr{W}(\alpha)$ if and only if $Y(\varphi) : Y(M) \to Y(L)$ is monic in α-SpFi.*

Note that epics in $\mathscr{W}(\alpha)$ are not always surjective, nor are the monics in α-*SpFi* always injective [BHM].

A space X is said to be α*-disconnected* if the closure of every α-cozero is open. It is obvious that the notions of ∞-disconnected and extremally disconnected (and ω_1-disconnected and basically disconnected) are equivalent.

As the next two statements indicate, the topological concept of α-disconnected has an algebraic counterpart. The first is a cardinal generalization of (3.3) ((a) $\Rightarrow$ (d)) of [BH$_1$]. See also [M$_1$].

2.8 **Lemma.** *Let $L \in |\mathscr{W}|$. If L is α-complete, then $Y(L)$ is α-disconnected.*

The converse of 2.8 is not true [BH$_1$, M$_1$, M$_2$]. However, if we only consider $\mathscr{W}$ objects of the form $C(X)$, for compact X, we get the following well-known result:

2.9 **Theorem** (Stone-Nakano). *$C(X)$ is α-complete if and only if X is α-disconnected.*

Note that one direction of 2.9 follows from 2.8 and the fact that $Y(C(X)) = X$ (2.2).

3. Free α-extensions

3.1 **Proposition** (see [M$_2$]). *Let $\varphi : L \to M$. The following are equivalent.*

(a) *φ is α-complete.*

(b) *There is a $c \in L$ such that whenever $B \subset L$, $|B| < \alpha$, and $c = \bigvee^L B$, then $\varphi(c) = \bigvee^M \varphi[B]$.*

(c) *For $A \subset L$ with $|A| < \alpha$, we have that $\varphi(\bigwedge^L A) = \bigwedge^M \varphi[A]$ whenever $\bigwedge^L A$ exists in L.*

Sometimes, as the next proposition indicates, the first factor of an α-complete map is also α-complete. As usual, $\gamma : M \hookrightarrow N$ denotes that γ is injective.

3.2 **Proposition.** *Let $\varphi: L \to M$ and $\gamma: M \hookrightarrow N$. If $\gamma \circ \varphi$ is α-complete, then φ is α-complete.*

Proof. Let $c \in L$ and suppose $c = \bigvee^L A$ where $A \subset L$ and $|A| < \alpha$. We claim that $\varphi(c) = \bigvee^M \varphi[A]$. Suppose not. Then there is a $b \in M$ such that $\varphi(c) > b > \varphi(a)$ for all $a \in A$ ($>$ means strictly greater than). Because γ is injective we have $\gamma \circ \varphi(c) > \gamma(b) > \gamma \circ \varphi(a)$ for all $a \in A$. But this contradicts the assumption that $\gamma \circ \varphi$ is α-complete. For then $\gamma \circ \varphi(c) = \bigvee^N \gamma \circ \varphi[A]$.

Recall from the introduction the definition of an α-full subspace.

Note, for $L \subseteq M$, it is possible for L to be α-full, but not α-completely embedded, in M.

3.3 **Example.** Let X be a non-ω_1-disconnected Boolean space. Then there is a cozero set U such that $\overline{U}$ is not clopen. Let $L = \{f \in C(X) : f[X \backslash U] = r$ for some $r \in \mathbf{R}\}$. Clearly, L is ∞-full in $C(X)$, but L is not even ω_1-completely embedded in $C(X)$: Since there are clopen sets $\{V_n : n \in \mathbf{N}\}$ and C such that $U = \bigcup_n V_n$ and $U \subseteq C$, it is obvious that the sup of $\{\chi_{V_n}\}$ in $C(X)$ is not $\mathbf{1}$ ($\chi_c \geq \chi_{V_n}$ for all $n \in N$). However, $\bigvee_n^L \chi_{V_n} = \mathbf{1}$ because, if $h \in L$ and $h \geq \chi_{V_n}$ for all $n \in \mathbf{N}$, then $h(x) \geq 1$ for all $x \in U$. Therefore $h(x) \geq 1$ for all $x \in \overline{U}$, but then, since $X \backslash U \cap \overline{U} \neq \varnothing$, it follows that $h \geq \mathbf{1}$.

However, if M is α-complete, then the α-full subspaces of M are easy to identify. See [S, §23]. We have

3.4 **Proposition.** *Let $\varphi: L \hookrightarrow M$ be an embedding and suppose M is α-complete. Then $\varphi[L]$ is α-full in M if and only if both L and φ are α-complete.*

Proof. Verification is straightforward.

3.5 **Proposition.** *Let $L \subseteq M \subseteq N$. If L is α-full in M and M is α-full in N, then L is α-full in N.*

Proof. Let $A \subset L$, $|A| < \alpha$, and $a = \bigvee^N A$. Since M is α-full in N, we have that $a \in M$ and $\bigvee^M A = \bigvee^N A$. Hence, since L is α-full in M, it follows that $\bigvee^L A = \bigvee^M A = \bigvee^N A$.

Let $S \subset L$. The smallest subspace of L that contains S is the intersection of all the subspaces of L that contain S. We denote this subspace by $\langle S \rangle^L$, and we say that S generates $\langle S \rangle^L$ in L. Moreover, $\langle S \rangle^L$ is the smallest subset of L that contains S and is closed under the finite vector lattice operations.

Now for given α, one may ask if there is a smallest α-complete subspace of L that contains S? In general, this question does not always make sense because there may not be any α-complete subspace of L that contain S (e.g., $C([0, 1])$ contains no ω_1-complete subspaces). However, there are always α-full subspaces of L that contain S.

Recall, for $L \subseteq M$, we say that L α-generates M if M is the smallest α-full subspace of M that contains L. That is, if L' is α-full in M and $L \subseteq L'$, then $L' = M$. For $S \subseteq M$ we say S α-generates M if $\langle S \rangle$ α-generates M.

Clearly, an arbitrary intersection of α-full subspaces of M is an α-full subspace of M. Therefore, for $L \subseteq M$, the subspace of M which L α-generates, denoted $\langle L \rangle_\alpha^M$, is the intersection of all the α-full subspaces of M that contain

L. However, this "outside in" description does not provide much information about $\langle L\rangle_\alpha^M$. Is the inclusion of L into $\langle L\rangle_\alpha^M$ epic in some sense? How big is $\langle L\rangle_\alpha^M$? To answer these and other questions about $\langle L\rangle_\alpha^M$ we use an "inside out" construction of $\langle L\rangle_\alpha^M$.

In what follows below, we may assume, without loss of generality, that α is a regular cardinal because: α^+ is always a regular cardinal, and the α-completeness properties of objects and morphisms are equivalent to their respective α^+-completeness properties when α is a singular cardinal.

3.6 **Definition.** Let $L \subseteq M$. Fix an $\alpha < \infty$ and define

$$S_\alpha^M(L) = \left\{\bigvee\nolimits^M A : A \subset L,\, |A| < \alpha,\, \bigvee\nolimits^M A \text{ exists}\right\}.$$

For ordinals $\xi < \alpha$ we recursively define $L(\xi)^M$ (omitting ^{M} when the context is clear) by setting:

$L(0) = L$,
$L(\xi) = \langle S_\alpha(L(\xi - 1))\rangle$ if ξ is not a limit ordinal,
$L(\xi) = \bigcup_{\sigma<\xi} L(\sigma)$ if ξ is a limit ordinal.

Finally, we define $\langle L\rangle_\alpha^M = \bigcup_{\xi<\alpha} L(\xi)$.

Note, we also omit the superscript ^{M} in $\langle L\rangle_\alpha^M$ whenever the context is clear.

3.7 **Lemma.** *Let $L \subseteq M$ and $\alpha < \infty$.*

(a) *$\langle L\rangle_\alpha$ is a subspace of M.*

(b) *If two α-complete morphisms from $\langle L\rangle_\alpha$ agree on L, they are equal. One might say the embedding of L into $\langle L\rangle_\alpha$ is "epic for α-complete morphisms"* [H].

(c) *If $L \subseteq^\alpha M$, then the embedding of L into $\langle L\rangle_\alpha$ is epic in* Arch(α).

(d) $|\langle L\rangle_\alpha| \le |L|^\alpha$.

(e) *If $L \subseteq M \subseteq N$ and M is α-full in N, then $\langle L\rangle_\alpha \subseteq M$.*

(f) *$\langle L\rangle_\alpha$ is α-full in M, and L α-generates $\langle L\rangle_\alpha$.*

(g) *If M is α-complete, then $\langle L\rangle_\alpha$ is α-complete and $\langle L\rangle_\alpha \subseteq^\alpha M$.*

Proof. (a) Clear.

(b) Let $\gamma_i : \langle L\rangle_\alpha \to N$ be α-complete with $i = 1, 2$, and suppose $\gamma_1|L = \gamma_2|L$. We claim $\gamma_1|L(\xi) = \gamma_2|L(\xi)$ for all $\xi < \alpha$, from whence, (b) will follow. We proceed by transfinite induction. The claim is true for $L(0) = L$. Suppose it is true for all ordinals $\sigma < \xi$. If ξ is a limit ordinal, it is clear from the definition of $L(\xi)$ that $\gamma_1|L(\xi) = \gamma_2|L(\xi)$. If ξ is not a limit ordinal, then $L(\xi) = \langle S_\alpha(L(\xi - 1))\rangle$. It suffices to see that $\gamma_1 = \gamma_2$ on the set $S_\alpha(L(\xi - 1))$. Let $b \in S_\alpha(L(\xi - 1))$. Then there is a set $A \subset L(\xi - 1)$ with $|A| < \alpha$ such that $b = \bigvee^{\langle L\rangle_\alpha} A$. So $\gamma_1(b) = \bigvee^N \gamma_1[A] = \bigvee^N \gamma_2[A] = \gamma_2(b)$.

(c) Apply 3.2 and (b) above.

(d) Clearly, $|S_\alpha(L)| < |L|^\alpha$, so it follows that $|\langle S_\alpha(L)\rangle| < |L|^\alpha$. Transfinite induction gives us that for each $\xi < \alpha$, we have $|L(\xi)| < |L|^\alpha$, hence $|\langle L\rangle_\alpha| = |\bigcup_{\xi<\alpha} L(\xi)| \le |L|^\alpha$.

(e) Clearly, if M is α-full in N and $L \subseteq M$, then $\langle S_\alpha^N(L)\rangle^N \subseteq M$. Transfinite induction implies that $L(\xi)^N \subseteq M$ for all $\xi < \alpha$, hence $\langle L\rangle_\alpha^N \subseteq M$.

(f) Let $A \subset \langle L\rangle_\alpha$ with $|A| < \alpha$. For each $a \in A$, $a \in L(\xi_a)$ for some $\xi_a < \alpha$. Therefore, $\sup_A \xi_a = \kappa < \alpha$. It follows that $A \subset L(\kappa)$. So if $b = \bigvee^M A$, then $b \in S_\alpha(L(\kappa + 1)) \subset \langle L\rangle_\alpha$.

To see that L α-generates $\langle L\rangle_\alpha$ let $L \subseteq L' \subseteq \langle L\rangle_\alpha$ and suppose L' is α-full in $\langle L\rangle_\alpha$. Then L' is α-full in M ((c) here). Therefore it follows that $L' = \langle L\rangle_\alpha$ ((e) here).

(g) Apply (f) here then 3.4.

3.8 **Definition.** For a given L, (φ, M) is called a *free α-regular extension* of L if it satisfies the following conditions.

(i) φ M are α-complete.

(ii) $\varphi[L]$ α-generates M.

(iii) For each α-complete $\gamma : L \to N$ with N α-complete, there is a unique α-complete morphism $\tau : M \to N$ such that $\gamma = \mu \circ \varphi$.

We say that (φ, M) is a *free α-extension of L* if it satisfies the following conditions.

(i$'$) M is α-complete.

(ii$'$) $\varphi[L]$ α-generates M.

(iii$'$) For each $\gamma : L \to N$ with N α-complete, there is a unique α-complete morphism $\tau : M \to N$ such that $\gamma = \tau \circ \varphi$.

Note, conditions (iii) and (iii$'$) respectively imply that free α-regular extensions and free α-extensions of L are essentially unique.

Below we construct, for each L, the free α-regular extension and the free α-extension.

For a fixed $\alpha < \infty$ and L, let T be a fixed set with $|T| = |L|^\alpha$. Let $A_\alpha(T)$ be the *set* of all α-complete Arch objects which have T as the underlying set. Let I and J be sets of morphisms defined as follows:

$$I = \{\varphi : L \to M : M \in A_\alpha(T),\ \varphi \text{ is } \alpha\text{-complete}\},$$
$$J = \{\gamma : L \to M : M \in A_\alpha(T)\}.$$

If we let M_φ and M_γ be the codomains of φ and γ respectively, we can see that L is naturally embedded in each of the products $\prod_I M_\varphi$ and $\prod_J M_\gamma$ by means of the evaluation map, e.g., $e : L \hookrightarrow \prod_I M_\varphi$ defined by $e(a)_\varphi = \varphi(a)$ for all $a \in L$. Since the operations in Arch products are coordinatewise it is straightforward to see that $\prod_I M_\varphi$ and $\prod_J M_\gamma$ are α-complete, $L \subseteq^\alpha \prod_I M_\varphi$, and $L \subseteq \prod_J M_\gamma$. Define

$$FR_\alpha L = \langle L\rangle_\alpha \subseteq^\alpha \prod_I M_\varphi \quad \text{and} \quad F_\alpha L = \langle L\rangle_\alpha \subseteq \prod_J M_\gamma.$$

Note that the products in Arch and $\mathscr{W}$ are the set-theoretic products with coordinatewise operations, and the product in $\mathscr{S}$, of a set of $\mathscr{S}$-objects M_i, is obtained by first forming the product in Arch, and then taking the principal ideal generated by the element (u_i), where u_i is the strong unit of M_i.

3.9 **Theorem.** *Let i be the inclusion of L in $FR_\alpha L$ and i' be the inclusion of L in $F_\alpha L$. Then $(i, FR_\alpha L)$ and $(i', F_\alpha L)$ are the free α-regular extension and the free α-extension of L respectively.*

Proof. We prove that $(i, FR_\alpha L)$ is the free α-regular extension of L. The proof that $(i', F_\alpha L)$ is the free α-extension of L is similar.

Since $FR_\alpha L$ is α-complete, $F_\alpha L \subseteq^\alpha \prod_I M_\varphi$, and L α-generates $FR_\alpha L$ (3.7(g), (f)), it suffices to see that any α-complete morphism $\varphi : L \to M$, with M α-complete, can be extended to an α-complete morphism $\overline{\varphi} : \prod_I M_\varphi \to$

M. We will show that $\overline{\varphi}$ is essentially a projection out of $\prod_I M_\varphi$. (Note, projections are ∞-complete.)

Let $\varphi : L \to M$ be α-complete and let $M_{\varphi'} = \langle \varphi[L] \rangle_\alpha^M$. Since $M_{\varphi'}$ is α-complete and $|M_{\varphi'}| < |L|^\alpha$ (3.7(g), (d)), $M_{\varphi'}$ is isomorphic to some member of $A_\alpha(T)$. Therefore we can consider the morphism $\varphi' : L \to M_{\varphi'}$, where $\varphi'(a) = \varphi(a)$ for $a \in L$, to be a member of I (3.2). Moreover, since $M_{\varphi'} \subseteq^\alpha M$ (3.7(g)), we need only to extend φ to an α-complete morphism $\overline{\varphi} : \prod_i M_\varphi \to M_{\varphi'}$. This is easily done by taking $\overline{\varphi} = \pi_{\varphi'}$, where $\pi_{\varphi'}$ is the φ' projection out of the product $\prod_I M_\varphi$. See Figure 3.1.

$$\begin{tikzcd}
\prod_I M_\varphi \\
{}^{\alpha}\cup\!| \quad \pi_{\varphi'} \\
FR_\alpha L \\
\varphi : L \arrow[u] \arrow[r] & M_{\varphi'} = \langle \varphi[L] \rangle_\alpha^M \subseteq^\alpha M
\end{tikzcd}$$

FIGURE 3.1

Recall in an abstract category $\mathscr{B}$, a full subcategory $\mathscr{A}$ is called an *epireflective* subcategory of $\mathscr{B}$ if, for each $B \in |\mathscr{B}|$, there is an $A_B \in |\mathscr{A}|$ and an epimorphism in $\mathscr{B}$, $e : B \to A_B$, such that for each $\mathscr{B}$ morphism, $f : B \to A$, to an $A \in |\mathscr{A}|$, there exists a (necessarily) unique $\mathscr{A}$ morphism, $\overline{f} : A_B \to A$, satisfying $f = \overline{f} \circ e$. (e, A_B) is called the $\mathscr{A}$ epireflection of B. (Note, epireflections are essentially unique.)

The existence of the free α-regular extension and 3.7(c) together imply:

3.10 **Theorem.** *In* $\mathrm{Arch}(\alpha)$*, full subcategory of α-complete objects is epireflective, and, for each L, $(i, FR_\alpha L)$ is the α-complete epireflection of L in* $\mathrm{Arch}(\alpha)$.

Note all the results (in, and about, Arch and $\mathrm{Arch}(\alpha)$) of this section have analogs in $\mathscr{W}$, $\mathscr{W}(\alpha)$, $\mathscr{S}$, $\mathscr{S}(\alpha)$, $\mathscr{BA}$ and, $\mathscr{BA}(\alpha)$. Moreover, the proofs, are essentially identical to those in Arch and $\mathrm{Arch}(\alpha)$. Most importantly, for each $\mathscr{W}$ (and $\mathscr{S}$) object, there is, in $\mathscr{W}$ (and $\mathscr{S}$), a free α-regular extension and a free α-extension. And, as in $\mathrm{Arch}(\alpha)$, we have the following:

3.11 **Theorem.** *In $\mathscr{W}(\alpha)$ [and $\mathscr{S}(\alpha)$], the full subcategory of α-complete objects is epireflective, and, for each $L \in |\mathscr{W}|$ $[L \in |\mathscr{S}|]$, the $\mathscr{W}$ $[\mathscr{S}]$ free α-regular extension $(i, \mathscr{W} - FR_\alpha L)$ $[(i, \mathscr{S} - FR_\alpha L)]$ is the α-complete epireflection of L in $\mathscr{W}(\alpha)$ $[\mathscr{S}(\alpha)]$.*

4. α-DISCONNECTED α-$SpFi$ COREFLECTIONS

Consider the $\mathscr{W}$-free α-regular extension $(i, FR_\alpha C(X))$, and the $\mathscr{W}$-free α-extension $(i', F_\alpha C(X))$, of $C(X)$. Let $m_\alpha X = Y(FR_\alpha C(X))$ and $M_\alpha X =$

$Y(F_\alpha C(X))$. $(m_\alpha X, Y(i))$ and $(M_\alpha X, Y(i'))$ are preimages of X. See Figures 4.1 and 4.2.

$$\begin{array}{ccc} C(X) & \overset{i}{\hookrightarrow} & FR_\alpha C(X) \\ X = Y(C(X)) & \xleftarrow{Y(i)} & Y(FR_\alpha C(X)) = m_\alpha X \end{array}$$

FIGURE 4.1

$$\begin{array}{ccc} C(X) & \overset{i'}{\hookrightarrow} & F_\alpha C(X) \\ X = Y(C(X)) & \xleftarrow{Y(i')} & Y(F_\alpha C(X)) = M_\alpha X \end{array}$$

FIGURE 4.2

Let $m_\alpha = Y(i)$ and $M_\alpha = Y(i')$. We have the following:

4.1 **Theorem.** *Let $\alpha < \infty$.*

(a) *$m_\alpha X$ and $M_\alpha X$ are α-disconnected.*

(b) *In α-$SpFi$, m_α is monic.*

(c) *If $f : Y \to X$ is an α-$SpFi$ and Y is α-disconnected, then there is a unique α-$SpFi$ morphism $\overline{f} : Y \to m_\alpha X$ such that $f = m_\alpha \circ \overline{f}$.*

(d) *If $h : Y \to X$ is continuous and Y is α-disconnected, then there is a unique α-$SpFi$ morphism $\overline{h} : Y \to M_\alpha X$ such that $h = M_\alpha \circ \overline{h}$.*

Proof. (a) $FR_\alpha C(X)$ and $F_\alpha C(X)$ are α-complete so $m_\alpha X$ and $M_\alpha X$ are α-disconnected (2.8).

(b) i is epic in $\mathscr{W}(\alpha)$; thus m_α is monic in α-$SpFi$ (2.7).

(c) $f' : C(X) \to C(Y)$ is an α-$SpFi$ morphism (2.4 and 2.6). And since $C(Y)$ is α-complete (2.9), there is a unique α-complete $\mathscr{W}$ morphism $\overline{f}' : FR_\alpha C(X) \to C(Y)$ such that $f' = \overline{f}' \circ i$. Hence $f = Y(f') = Y(\overline{f}' \circ i) = Y(i) \circ Y(\overline{f}') = m_\alpha \circ Y(\overline{f}')$. Take $\overline{f} = Y(\overline{f}')$. $\overline{f}$ is an α-$SpFi$ morphism (2.6).

(d) The proof of (d) is similar to that of (c).

Let $\mathscr{B}$ be a category and $\mathscr{A}$ a subcategory of $\mathscr{B}$. We call $\mathscr{A}$ a monocoreflective subcategory of $\mathscr{B}$ if, for each $B \in |\mathscr{B}|$, there is a $A_B \in |\mathscr{A}|$ and a monic in $\mathscr{B}$, $m_B : A_B \to B$, such that for each $\mathscr{B}$ morphism, $f : A \to B$, from an $A \in |\mathscr{A}|$, there is a (necessarily) unique $\mathscr{A}$ morphism, $\overline{f} : A \to A_B$, satisfying $f = m_B \circ \overline{f}$. (A_B, m_B) is called an $\mathscr{A}$ monocoreflection of B. (Note, monocoreflections are essentially unique.)

4.2 **Theorem.** *In α-$SpFi$, the full subcategory of α-disconnected spaces is monocoreflective, and for each X, $(m_\alpha X, m_\alpha)$ is the α-disconnected monocoreflection of X in α-$SpFi$.*

Proof. Apply 4.1(b) and (c).

4.3 *Remark.* Let $\mathrm{Ba}(X)$ be the σ-algebra of Baire sets of X and let $\mathscr{Z}(X)$ be the σ-ideal generated by the nowhere dense zero-sets of X. In [BH$_3$] it is shown that $M_{\omega_1}X$ is the Stone space of $\mathrm{Ba}(X)$, and $m_{\omega_1}X$ is the Stone space of the quotient $\mathrm{Ba}(X)/\mathscr{Z}(X)$. Also, for X, the Stone space of the Borel sets modulo the meager Borel sets is called the absolute or Gleason cover, EX, of X

[PW], and $m_\infty X \cong EX$ [BHM, W]. See also [M_1, M_2]. Can $M_\alpha X$ and $m_\alpha X$, for general α, be represented in a similar fashion? I do not know the answer for arbitrary X, however, if X is α-cozero complemented, the answer is "yes" for $m_\alpha X$ [M_3]. X is α-cozero complemented if, for each $U \in \mathrm{Coz}_\alpha(X)$, there is a $V \in \mathrm{Coz}_\alpha X$ such that $U \cup V$ is dense in X, and $U \cap V = \varnothing$.

References

[B] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. Amer. Math. Soc., Providence, R. I., 3rd ed., 2nd printing, 1973.

[BH_1] R. N. Ball and A. W. Hager, *Epicompletion of Archimedean l-groups and vector lattices with weak unit*, J. Austral. Math. Soc. Ser. A **48** (1990), 25–56.

[BH_2] ——, *Epicomplete archimedean l-groups and vector lattices*, Trans. Amer. Math. Soc. **322** (1990), 459–478.

[BH] ——, *Characterization of epimorphisms in Archimedean l-groups and vector lattices*, Chapter 8 Lattice-Ordered Groups, Advances and Techniques, (A. Glass and W. C. Holland, eds.), Kluwer, Dordrecht, 1989.

[BH_4] ——, *Application of spaces with filters to Archimedean l-groups*, Proc. Conf. on Ordered Algebraic Structures, Curaçao, 1988 (J. Martinez, ed.), Kluwer, Dordrecht, 1989.

[BHM] R. N. Ball, A. W. Hager, and A. J. Macula, *An α-disconnected space has no proper monic preimage*, Topology Appl. (to appear).

[BHN] R. N. Ball, A. W. Hager, and C. Neville, *The κ-ideal completion of an Archimedean l-group and the κ-quasi-F cover of a compact space*, General Topology Appl., Proc. Northeast Topology Conf., Wesleyan Univ., 1988 (R.M. Shortt, ed.), Marcel Dekker, New York, 1990, pp. 7–50.

[BKW] A. Bigard, K. Keimel and S. Wolfenstein, *Groupes et anneaux réticules*, Lecture Notes in Math., vol. 608, Springer-Verlag, Berlin, Heidelberg, and New York, 1977.

[C] H. B. Cohen, *The κ-extremally disconnected spaces as projectives*, Canad. J. Math. **16** (1964), 253–260.

[Co] P. Conrad, *Minimal vector lattice covers*, Bull. Austral. Math. Soc. **4** (1971), 35–39.

[DHH] F. Dashiell, A. W. Hager and M. Henriksen, *Order-Cauchy completions of rings and vector lattices of continuous functions*, Canad. J. Math. **32** (1980), 657–685.

[dJvR] E. DeJonge and A. C. M. Van Rooij, *Introduction to Riesz spaces*, Math. Centre Tracts, no. 78, Amsterdam, 1977.

[G] A. M. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482–489.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

[H] A. W. Hager, *Boolean algebras with α-complete morphisms*, unpublished manuscript.

[HR_1] A. W. Hager and L. C. Robertson, *Representing and ringifying a Riesz space*, Symposia Math. **21** (1977), 411–431.

[HR_2] ——, *On imbedding into a ring of an Archimedean lattice-ordered group*, Canad. J. Math. **31** (1979), 1–8.

[Ha] A. W. Hales, *On the non-existence of free complete boolean algebras*, Fund. Math. **54** (1964), 45–66.

[HJ] M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. **50** (1961), 73–94.

[HVW] M. Henriksen, J. Vermeer, and R. G. Woods, *Quasi-F covers of Tychonoff spaces*, Trans. Amer. Math. Soc. **303** (1987), 779–803.

[HS] H. Herrlich and G. Strecker, *Category theory*, Allyn and Bacon, Boston, Mass., 1973.

[HeR] E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* I, Springer-Verlag, Berlin, 1963.

[K] J. Kerstan, *Tensorielle Erweiterungen distributiver Verbände*, Math. Nachr. **22** (1960), 1–20.

[LZ] W. Luxemburg and A. Zaanen, *Riesz spaces*, Vol. I, North-Holland, Amsterdam, 1971.

[M_1] A. J. Macula, Thesis, Wesleyan Univ., 1989.

[M_2] ——, *α-Dedekind complete Archimedean vector lattices vs, α-quasi-F spaces*, Topology Appl. (to appear).

[M_3] ——, *Monic sometimes means α-irreducible*, General Topology and its Appl., Vol. 134, (S. J. Andima et al., ed.), Marcel Dekker, New York, 1991, pp. 239–260.

[PW] J. R. Porter and R. G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, New York, 1988.

[S] R. Sikorski, *Boolean algebra*, 3rd ed., Springer-Verlag, Berlin, 1969.

[S_2] ——, *On extensions and products of boolean algebras*, Fund. Math. **53** (1963), 99–116.

[W] R. G. Woods, *Covering properties and coreflective subcategories*, Proc. CCNY Conf. on Limits 1987, Ann. N. Y. Acad. Sci. (to appear).

[Y] K. Yosida, *On the representation of a vector lattice*, Proc. Imp. Acad. Tokyo **18** (1942), 339–342.

DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457