

## **$L^p$ ESTIMATES FOR THE X-RAY TRANSFORM RESTRICTED TO LINE COMPLEXES OF KIRILLOV TYPE**

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**ABSTRACT.** Let there be given a piecewise continuous rectifiable curve  $\phi: \mathbf{R} \rightarrow \mathbf{R}^n$ . Let  $G_{1,n}(M_{1,n})$  be the usual Grassmannian (bundle) in  $\mathbf{R}^n$ . Define an  $n$ -dimensional submanifold  $M_\phi(\mathbf{R}^n)$  of  $M_{1,n}$  as the set of all copies of  $G_{1,n}$  along the curve  $\phi$ . Following Kirillov, we know that a nice function  $f(x)$  can be recovered from its X-ray transform  $R_{1,n}f$  on  $M_\phi(\mathbf{R}^n)$  if and only if the curve  $\phi$  intersects almost every affine hyperplane. Define a measure on  $M_\phi(\mathbf{R}^n)$  by  $d\mu = d\mu_x(\pi)d\lambda(x)$ , where  $d\mu_x$  is the probability measure on  $M_{1,n}$  carried by the set of lines passing through the point  $x$  and invariant under the stabilizer of  $x$  in  $O(n)$  and  $d\lambda$  is the usual measure on  $\phi$ . We show that, if  $n > 2$  and  $\phi$  is unbounded, then  $\|R_{1,n}f\|_{L^q(M_\phi(\mathbf{R}^n), d\mu)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$  if and only if  $p = q = n - 1$  and  $\phi$  is line-like, that is,  $\lambda(\phi \cap B(0; R)) = O(R)$ . This result gives a classification of Kirillov line complexes in terms of  $L^p$  estimates.

### 0. OVERVIEW

The Radon transform has found applications in many areas of mathematics. For applications to the study of partial differential equations, the reader is referred to John [14]. A natural generalization, the  $k$ -plane transform, has been widely studied (see Helgason [13]). The 1-plane transform, usually called the X-ray transform, has significance not only in theory but also in practical aspects (see Smith, Solmon and Wagner [19]). By integral geometry in the sense of Radon and Gelfand one means reconstructing a function from knowledge of its integrals over  $k$ -planes. An outstanding problem is to determine how to effectively reconstruct a function from partial information about its  $k$ -plane transform.

In this article we restrict our attention to the X-ray transform for the reason that very little is known about effective reconstruction for the general  $k$ -plane transform. Let  $G_{k,n}$  be the Grassmannian manifold of  $k$ -dimensional subspaces in  $\mathbf{R}^n$  and  $M_{k,n}$  the Grassmannian bundle of affine  $k$ -planes in  $\mathbf{R}^n$ .  $M_{k,n}$  is a bundle over  $G_{k,n}$  with fibre dimension  $n - k$ . Since  $\dim M_{k,n} = (n - k)(k + 1)$  it is intuitively clear that for  $k < n - 1$  there are more functions on  $M_{k,n}$  than on  $\mathbf{R}^n$  so the  $k$ -plane transform of  $f \in C_0^\infty(\mathbf{R}^n)$  defined by

$$R_{k,n}f(\pi, v) = \int_{\pi} f(y + v) d\sigma(y)$$

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is highly overdetermined (that is,  $R_{k,n}$  cannot be onto). Here  $(\pi, v)$  is the standard coordinate system on  $M_{k,n}$  with  $\pi \in G_{k,n}$ , and  $v \in \pi^\perp$ , and  $d\sigma$  denotes the Lebesgue measure on the  $k$ -plane  $\pi$ . In fact, Gelfand and Graev [12] characterizes the range of  $R_{k,n}$  on Schwartz class  $S$  as those elements of  $S(M_{k,n})$  annihilated by a set of differential operators. For illustration, we begin with the X-ray transform  $R_{1,3}f$  on  $M_{1,3}$ . Replace the standard coordinates on  $M_{1,3}$  with local ones by the following parametrization of a line which is not parallel to the  $xy$ -plane:

$$x(t) = \alpha t + \beta, \quad y(t) = \gamma t + \delta, \quad z(t) = t,$$

so  $(\alpha, \beta, \gamma, \delta)$  are local coordinates for  $M_{1,n}$ . Then up to a constant,

$$R_{1,3}f(\alpha, \beta, \gamma, \delta) = \int_{t \in \mathbf{R}} f(\alpha t + \beta, \gamma t + \delta, t) dt$$

and we immediately find by differentiating under the integral that

$$(0-1) \quad \frac{\partial^2 R_{1,3}f}{\partial \alpha \partial \delta} - \frac{\partial^2 R_{1,3}f}{\partial \gamma \partial \beta} = 0.$$

Conversely, in a celebrated paper John [15] was actually the one to first write down (0-1) as a necessary and sufficient condition. Furthermore, his characterization was of X-ray transforms of continuous functions of compact support. Equation (0-1) is called the ultrahyperbolic equation. In [12] Gelfand and Graev have generalized this result to  $M_{k,n}$  where, as might be expected, the analogue of (0-1) is quite complicated. This raises immediately the question of how to reconstruct  $f$  effectively. Classically,  $n$ -dimensional submanifolds  $\Omega^n$  of  $M_{k,n}$  are called  $k$ -plane complexes. The problem is to determine for which  $\Omega^n$  does  $R_{k,n}f|_{\Omega^n}$  determine  $f$ . The answers to these questions are unknown; however, much progress has been made on the X-ray transform by Gelfand and his collaborators. Of particular interest for us is the class given by Kirillov [16] in 1961. Let there be given a piecewise continuous curve  $\phi: \mathbf{R} \rightarrow \mathbf{R}^n$ . Kirillov had shown that a nice function  $f(x)$  can be recovered from its X-ray transform  $R_{1,n}f$  on an  $n$ -dimensional submanifold  $M_\phi(\mathbf{R}^n)$  of  $M_{1,n}$  which is the set of all lines passing through  $\phi$ , if and only if the curve  $\phi$  intersects almost all affine hyperplanes. (We shall refer to such a  $\phi$  as a Kirillov curve and  $M_\phi(\mathbf{R}^n)$  as a Kirillov  $n$ -manifold.)

Our goal in this article is to study the  $L^p$  mapping properties of the X-ray transform  $R_{1,n}f$  restricted to  $M_\phi(\mathbf{R}^n)$ . The first reason is that we would like to know how the size of a function influences the size and smoothness of its restricted X-ray transform as measured by  $L^p$  norms. Secondly, we would like to present a result that describes a classification of certain kinds of admissible  $n$ -dimensional submanifolds in terms of an  $L^p$  mapping property.

### 1. INTRODUCTION

Several authors [3, 5, 6, 7, 21] have studied estimates for the  $k$ -plane transform on the whole of  $M_{k,n}$  in Lebesgue and Sobolev spaces. Oberlin and Stein [17] first obtained a complete estimate for the Radon transform ( $k = n - 1$ ). More recently, Christ [5] has found an almost complete picture for the  $k$ -plane transform, for all  $1 \leq k < n - 1$ .

Our purpose in this article is to study the  $L^p$  mapping properties of the X-ray transform  $R_{1,n}f$  restricted to  $M_\phi(\mathbf{R}^n)$ . A natural question is raised. Is it possible to deduce the consequences from these known results by Sobolev imbedding theorem when we deal with estimates for the X-ray transform restricted to  $M_\phi(\mathbf{R}^n)$ ? If  $f \in L^p(\mathbf{R}^n)$  for  $1 < p \leq 2$ , Strichartz [21] and Calderón [5] have shown that  $R_{k,n}f(\pi, v)$  has  $k/p'$  derivatives in the  $v$ -variables in  $L^p$ . However, in our case we can show that there does not exist any higher order of smoothness in the  $v$ -variables by seeing the details from the following section. Hence the answer is no because we need the order of derivatives up to  $n - 1$ . The method we present here to find  $L^p$  estimates for the X-ray transform restricted to  $M_\phi(\mathbf{R}^n)$  is much more complicated than before although the techniques are not essentially new [5, 6, 7]. However, it relies heavily on geometrical arguments adapted to the line complex  $M_\phi$ .

We now introduce a measure on  $M_\phi(\mathbf{R}^n)$  by

$$du = d\pi ds = d\mu_x(\pi)d\lambda(x)$$

where  $d\mu_x$  is the probability measure on  $M_{1,n}$  carried by the set of lines passing through the point  $x$  and invariant under rotations about  $x$  and  $d\lambda$  is the arc-length measure on  $\phi$ . Throughout this article, we assume that  $\phi$  is piecewise continuous and rectifiable. Our main result is the following

**Theorem.** *If  $n > 2$  and  $\phi$  is unbounded, then*

$$(1-1) \quad \|R_{1,n}f\|_{L^q(M_\phi(\mathbf{R}^n), d\mu)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$$

*if and only if  $p = q = n - 1$  and  $\phi$  is line-like, that is,*

$$\lambda(\phi \cap B(0; R)) = O(R), \quad R \rightarrow \infty.$$

Then, by a definition of “line-like,” this result classifies curves in  $\mathbf{R}^n$  in terms of the existence of  $L^p$  estimates for the restricted X-ray transform into two kinds: “line-like” and “non-line-like.”

The succeeding sections are devoted to the proof of this result. The main idea of the proof is first to show the Theorem is true for  $\phi$  is a line and then apply it to any line-like curve. We thus first establish in §3 some geometrical properties of line-likeness. Then follows in §4 a process of normalization which is related to the one Brascamp, Lieb and Luttinger [1] have provided, and then in §5 we use this to prove a rearrangement inequality for the X-ray transform associated with the measure  $d\mu$  on  $M_\phi(\mathbf{R}^n)$ . This reduces us to dealing with the  $L^p$  estimates on the class of radial decreasing functions. Finally, we proceed to the complete proof of our main results.

The sufficient part of the main theorem is contained in author’s Ph.D. Dissertation [22] and presented here for the sake of completeness. I am indebted to my advisor, Professor Allan Greenleaf, for his continued encouragement and advice. Especially, his improvement of my writing and his corrections of my errors will be deeply appreciated.

## 2. CONSTRUCTION OF MEASURES ON $M_\phi(\mathbf{R}^n)$

There are two fairly natural measures on  $M_\phi(\mathbf{R}^n)$ . Let  $l \in S^{n-1}$ ,  $L = Rl$ , and  $C = C(L)$  is the set of all lines passing through  $L$ . First, parametrize  $L$

by  $sR$ ; at each  $sl$  we have a copy of  $S^{n-1}$ . This will furnish the following measure on  $M_\phi(\mathbf{R}^n)$ :  $d\mu_1 = d\pi ds$ .

Secondly, consider the standard coordinates on  $M_{1,n} : (\pi, v)$ ,  $\pi \in S^{n-1}$ ,  $v \in \omega^\perp$ . In these coordinates, away from  $l$ ,  $C$  is a line bundle over  $S^{n-1}$ : For each  $\pi$ , let

$$v(\pi) = l - (l \cdot \pi)\pi \in \pi^\perp.$$

Then

$$C = \left\{ \left( \pi, t \frac{v(\pi)}{|v(\pi)|} \right) : \pi \in S^{n-1}, t \in \mathbf{R} \right\} \subset M_{1,n}$$

except for a set of lower dimension. We thus obtain the second measure:  $d\mu_2 = dt d\pi$ .

The relationship between  $d\mu_1$  and  $d\mu_2$  is as follows. Let  $(\pi, sl)$  be a typical coordinate for  $C(L)$ . Then the corresponding coordinate in  $M_{1,n}$  is

$$(\pi, sl - (sl \cdot \pi)\pi) = (\pi, s(l - (l \cdot \pi)\pi)) = \left( \pi, s|v(\pi)| \frac{v(\pi)}{|v(\pi)|} \right).$$

This implies that  $t = |v(\pi)|s$ , and thus

$$d\mu_2 = dt d\pi = |v(\pi)|d\pi ds = |v(\pi)|d\mu_1.$$

Note that the factor  $|v(\pi)| = |l - (l \cdot \pi)\pi|$  vanishes to first order at both  $l$  and  $-l$ .

The structure of the second measure  $d\mu_2 = dt d\pi$  on  $C(L)$  allows us to study the estimates by comparing with the known results for the structure of the measure  $d\mu = d\lambda_\pi(v) d\pi$  on  $M_{1,n}$ . Note that the dimension of each fiber in  $M_{1,n}$  is reduced to one in  $C(L)$ . It would seem natural to employ the following trace embedding forms of Sobolev's inequality to deduce the consequences for  $C(L)$  from the known results for  $M_{1,n}$ .

**Sobolev's Inequality** [2]. Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  having the cone property specified by a certain finite cone  $C$ , and let  $\Omega^k$  be the  $k$ -dimensional domain obtained by intersecting  $\Omega$  with a  $k$ -dimensional plane in  $\mathbf{R}^n$ ,  $1 \leq k \leq n$ . Moreover, let  $m \geq 0$ ,  $1 \leq p < \infty$  and  $0 < n - mp < k$ . Then

$$W^{m,p}(\Omega) \rightarrow L^r(\Omega^k)$$

for  $p \leq r \leq \frac{kp}{n-mp}$ .

The constants for these embeddings depend only on  $m, p, n, r$  and the cone  $C$  determining the cone property for  $\Omega$ .

The Strichartz's inequality seems to be applicable in this deduction. In fact, for the X-ray transform, Strichartz's inequality [21] reads as

$$I_{1/p'} R_{1,n} : L^p(\mathbf{R}^n) \rightarrow L^{p'}(L^p)(M_{1,n})$$

for  $1 < p \leq 2$ , that is,

$$(2-1) \quad \int_{G_{1,n}} \|R_{1,n} f(\pi, \cdot)\|_{I_{1/p'}(L^p)}^{p'} d\pi \leq A \|f\|_{L^p(\mathbf{R}^n)}^{p'}$$

where  $I_\gamma$  denotes the Riesz potential of order  $\gamma$  on  $\pi^\perp$ ,

$$(I_\gamma F)^\wedge(\pi, v) = |v|^\gamma \widehat{F}(\pi, v),$$

and  $L^q(L^r)$  denotes the mixed norms of  $M_{1,n}$ ,

$$\|F\|_{L^q(L^r)/(M_{1,n})}^q = \int_{G_{1,n}} \left[ \int_{\pi^\perp} |F(\pi, v)|^r d\lambda_\pi(v) \right]^{q/r} d\pi.$$

It will be helpful to note that  $I_{1/p'}(L^p)$  is the familiar Sobolev space of distribution with generalized derivatives of order exactly  $1/p'$  in  $L^p$ ; this means that

$$I_{1/p'}(L^p) \subset W^{1/p', p}(\mathbf{R}^{n-1})$$

and thus we can without loss of generality consider the trace embedding form of Sobolev inequality

$$(2-2) \quad W^{1/p', p}(\mathbf{R}^{n-1}) \rightarrow L^r(\mathbf{R}),$$

that is,

$$\|R_{1,n}f(\pi, \cdot)\|_{L^r(C(L))} \leq \|R_{1,n}f(\pi, \cdot)\|_{I_{1/p'}(L^p)}$$

to attempt to deduce the consequences for  $C(L)$  from the known results, e.g., Strichartz's inequality (2-1) for  $M_{1,n}$ .

One of the sufficient conditions for the trace imbedding form (2-2),  $p \leq r \leq (1 \cdot p)/((n - 1) - p/p')$  requires that  $r = p/(n - p)$ , i.e.  $r^{-1} = np^{-1} - 1$ , which coincides with the consideration of homogeneity. The other condition  $0 < n - 1 - 1/p' \cdot p < 1$ , however, reduced to  $p > n - 1$  but this in turn becomes an unsolved problem as indicated by Calderón [3] and Strichartz [21]: May we drop the restriction  $p \leq 2$  in Strichartz's inequality (2-1). They even observe that one can not expect any other higher order of smoothness in  $v$ -variable than  $p = 2$  for  $M_{2,n}$ . Nevertheless, we would like to state the following result rather than a conjecture by combining inequality (2-1) with (2-2).

**Theorem 2.1.** *Suppose that inequality (2-1) holds for  $p > 1$ . Then*

$$\|R_{1,n}f\|_{L^q_r(L^r_t)(C(L))} \leq \|f\|_{L^p(\mathbf{R}^n)},$$

where  $r^{-1} = np^{-1} - 1$  with  $p > n - 1$  and  $q \leq p'$ .

However, even if we may solve this problem, the argument above is not totally convincing because we indeed require  $p = q = r = n - 1$  in our main result. Christ [5], as well as Strichartz, has a further result saying, in effect, that since the  $k$ -plane transform of a function satisfies a system of ultrahyperbolic PDEs, it satisfies a better Sobolev embedding than the one for general functions on  $M_{k,n}$ . Therefore, it is still possible to release the requirement  $p > n - 1$  and  $q \leq p'$  before to drop the restriction  $p \leq 2$ .

We now introduce Christ's inequality for the  $k$ -plane transform as follows. For test functions  $F(\pi, v)$  defined on  $M_{k,n}$  define the fiberwise Fourier transform by

$$\widehat{F}(\pi, \eta) = \int_{\pi^\perp} e^{-i(\eta, v)} F(\pi, v) dv$$

for each  $\eta \in \pi^\perp$ . Define  $S_0(M_{k,n})$  to be the set of all  $F$  for which

$$(1) \quad |\partial_\pi^\alpha \partial_v^\beta F(\pi, v)| \leq C(\alpha, \beta, M)(1 + |v|)^{-M},$$

for all  $\alpha, \beta, M$ ,

$$(2) \quad \widehat{F}(\pi, \eta) \equiv 0$$

for all  $|\eta| \leq \delta$  and all  $\pi$ , for some  $\delta = \delta(F) > 0$ , and

$$(3) \quad \widehat{F}(\pi, \eta) = \widehat{F}(\theta, \eta)$$

for all  $\pi, \eta$  such that  $\eta$  is perpendicular to both  $\pi$  and  $\theta$ , in notations,  $\eta \perp \pi$  and  $\eta \perp \theta$ .

This definition is natural because we see that, as in [5, Lemma 3.1],  $f \in S_0(M_{k,n})$  if and only if there exists a test function  $f$  on  $\mathbf{R}^n$  with  $\widehat{f}$  identically zero in a neighborhood of the origin such that  $F = R_{k,n}f$ .

Next define fiberwise fractional integration by

$$D_\gamma F(\pi, v) = \int_{\pi^\perp} F(\pi, v - u)|u|^{\gamma+k-n} du.$$

Then Christ's inequality [5] reads as  $D_{h/p'} : L^{p'}(L^p) \rightarrow L^q(L^r)$ , that is,

$$(2-3) \quad \|D_{k/p'} F\|_{L^q(L^r)(M_{k,n})} \leq C \|F\|_{L^{p'}(L^p)(M_{k,n})}$$

for all  $F \in S_0(M_{k,n})$  if  $n/2 \leq k \leq n - 2$ ,  $n - 1 < kp < n$ ,  $q \leq (n - k)p'$  and  $np^{-1} - (n - k)r^{-1} = k$ .

Christ's inequality seems to be not applicable in our deduction for the X-ray transform because we require  $k \geq n/2$  so far. However, by tracing Christ's proof, if the restriction  $p \leq 2$  in Strichartz's inequality may be dropped, then equality (2-3) also holds for the X-ray transform, i.e., the restriction  $n/2 \leq k \leq n - 2$  may be replaced with  $1 \leq k \leq n - 2$ . But, under this assumption, Christ's inequality is still not applicable in our deduction because  $D_{k/p'}$  is the Riesz potential of negative order  $-k/p'$  on  $\pi^\perp$ . The following scheme for the X-ray transform would seem possible to be used to overcome this difficulty.

**Lemma 2.2.** *Suppose  $F \in S_0(M_{1,n})$  and  $0 < \gamma < n - 1$ . For  $(\omega, t) \in S^{n-1} \times \mathbf{R}$ , let*

$$h(\omega, t) = \int_{\substack{\alpha : \omega \perp \alpha \\ \alpha \in G_{1,n}}} (I_\tau R F_\alpha)(\omega, t) d\lambda_\omega(\alpha),$$

where  $\tau = n - 2 - \gamma$ ,  $I_\tau$  acts in the  $t$ -variable,  $F_\alpha = F(\alpha, \cdot)$  and  $R F_\alpha$  denotes the Radon transform (in  $\mathbf{R}^{n-1} = \{y : y \perp \alpha\}$ ) of  $F_\alpha$ . Then, for any  $q, r \geq 1$ ,

$$\|D_\gamma F\|_{L^q_\pi(L^r_t)(C(L), d\mu_2)} \leq C \|h\|_{L^q_\omega(L^r_t)}.$$

*Proof.* Christ [5, Lemma 3.3] shows that

$$\begin{aligned} D_\gamma F(\pi, v) &= C \int_{\substack{\omega \perp \pi \\ \omega \in S_{n-1}}} \int_{\substack{\alpha : \omega \perp \alpha \\ \alpha \in G_{1,n}}} (I_\tau R F_\alpha)(\omega, \langle v, w \rangle) d\lambda_\omega(\alpha) d\lambda_\pi(\omega) \\ &= C \int_{\substack{\omega \perp \pi \\ \omega \in S_{n-1}}} h(\omega, \langle v, w \rangle) d\lambda_\pi(\omega). \end{aligned}$$

Then, by considering the mixed norms  $L^q_\pi(L^r_t)$  on  $C(L)$ , and noting that  $v(\pi) \in \pi^\perp$ , we have

$$\begin{aligned}
 & \|D_\gamma F\|_{L^q_\omega(L^r_\tau)(C(L), d\mu_2)}^q \\
 &= C \int_{G_{1,n}} \left[ \int_{\mathbf{R}} \left[ \int_{\substack{\omega \perp \pi \\ \omega \in \mathcal{S}_{n-1}}} h\left(\omega, t \left\langle \frac{v(\pi)}{|v(\pi)|}, w \right\rangle\right) d\lambda_\pi(\omega) \right]^r dt \right]^{q/r} d\pi \\
 &= C \int_{G_{1,n}} \left[ \int_{\mathbf{R}} \left[ \int_{\substack{\omega \perp \pi \\ \omega \in \mathcal{S}_{n-1}}} h(\omega, t) d\lambda_\pi(\omega) \right]^r dt \right]^{q/r} d\pi \\
 &= C \int_{G_{1,n}} \left[ \left\| \int_{\substack{\omega \perp \pi \\ \omega \in \mathcal{S}_{n-1}}} h(\omega, \cdot) d\lambda_\pi(\omega) \right\|_{L^r_\tau}^q \right] d\pi \\
 &\leq C \int_{G_{1,n}} \left[ \int_{\substack{\omega \perp \pi \\ \omega \in \mathcal{S}_{n-1}}} \|h(\omega, \cdot)\|_{L^r_\tau} d\lambda_\pi(\omega) \right]^q d\pi \\
 &\leq C \int_{G_{1,n}} \int_{\substack{\omega \perp \pi \\ \omega \in \mathcal{S}_{n-1}}} \|h(\omega, \cdot)\|_{L^r_\tau}^q d\lambda_\pi(\omega) d\pi \\
 &= C \int_{\omega \in \mathcal{S}_{n-1}} \|h(\cdot, \cdot)\|_{L^r_\tau}^q d\omega \\
 &= C \|h\|_{L^q_\omega(L^r_\tau)}^q.
 \end{aligned}$$

The fourth inequality follows from Minkowski’s integral inequality and the fifth inequality holds only for  $q \geq 1$ . Q.E.D.

Therefore, Christ’s estimate [5, Lemma 3.4] on the function  $h$  remains helpful if we may drop the restriction  $p \leq 2$  in Strichartz’s inequality.

**Lemma 2.3.** *Suppose that the inequality (2-1) holds for  $p > 1$ . Let  $F \in L^{p'}(L^p)(M_{1,n})$ ,  $\tau = n - 2 - 1/p'$  and  $n - 1 < p < n$ . Then*

$$\|h\|_{L^q_\omega(L^r_\tau)} \leq C \|F\|_{L^{p'}(L^p)},$$

where  $r^{-1} = np^{-1} - 1$ .

Now, combining Lemma 2.3 with Lemma 2.2 we have

**Theorem 2.4.** *Suppose that inequality (2-1) holds for  $p > 1$ . Then*

$$(2-4) \quad \|D_{1/p'} F\|_{L^q_\omega(L^r_\tau)(C(L))} \leq C \|F\|_{L^{p'}(L^p)},$$

where  $r^{-1} = np^{-1} - 1$  with  $n - 1 < p < n$ .

Finally, by letting  $F = I_{1/p'} R_{1,n} f$  in inequality (2-4), it follows from the expression  $R_{1,n} f = D_\gamma(I_\gamma R_{1,n} f)$  and Strichartz’s inequality that

**Theorem 2.5.** *Suppose that inequality (2-1) holds for  $p > 1$ . Then*

$$\|R_{1,n} f\|_{L^q_\omega(L^r_\tau)(C(L))} \leq \|f\|_{L^p(\mathbf{R}^n)}$$

where  $r^{-1} = np^{-1} - 1$  with  $n - 1 < p < n$  and  $q \leq p'$ .

Therefore, by comparing Theorems 2.1 and 2.5, we conclude that Christ's scheme in our deduction will not be better than the direct application of Sobolev's trace embedding form to Strichartz's inequality.

Thus, up to now, our result cannot be derived from previous work and we need an alternative approach. The method we have found is based on a series of geometrical facts.

### 3. GEOMETRICAL PROPERTIES OF LINE-LIKENESS

Intuitively, a line-like curve  $\phi$  may have some geometrical properties of a line in  $\mathbf{R}^n$ . We will find two such to meet our needs. First all, observe that the orthogonal projection of a line down to almost every hyperplane  $H_{n-1}$  in  $\mathbf{R}^n$  is still a line in  $H_{n-1}$ .

**Theorem 3.1.** *Let  $\phi: \mathbf{R} \rightarrow \mathbf{R}^n$  be a line-like curve. Let  $P_{H_{n-1}}\phi: \mathbf{R} \rightarrow H_{n-1}$  be the orthogonal projection of  $\phi$  onto a hyperplane  $H_{n-1}$ . Then  $P_{H_{n-1}}\phi$  is still a line-like curve in almost every hyperplane  $H_{n-1}$ .*

*Proof.* Let  $H^+$  and  $H^-$  be the positive and negative half-line of  $\mathbf{R}$  respectively. Without loss of generality, we may assume that  $\phi$  is continuous on  $\mathbf{R}$  and both  $\phi(H^+)$  and  $\phi(H^-)$  are unbounded. For simplicity, let

$$\phi = (\phi_1, \phi_2, \dots, \phi_n) \quad \text{and} \quad H_{n-1} = \langle e_1, e_2, \dots, e_{n-1} \rangle.$$

Then

$$(3-1) \quad P_{H_{n-1}}\phi = (\phi_1, \phi_2, \dots, \phi_{n-1}, 0).$$

Moreover, if both  $P_{H_{n-1}}\phi(H^+)$  and  $P_{H_{n-1}}\phi(H^-)$  are unbounded, then, for a constant  $C$  and any positive  $R$ , we obtain from formula (3-1) that

$$R \leq \lambda(P_{H_{n-1}}\phi \cap B(0; R)) \leq \lambda(\phi \cap B(0; R)) \leq CR.$$

In other words,  $P_{H_{n-1}}\phi$  is line-like and the result will follow. We now claim that both  $P_{H_{n-1}}\phi(H^+)$  and  $P_{H_{n-1}}\phi(H^-)$  are unbounded for almost every hyperplane  $H_{n-1}$ . It suffices to show that  $P_{H_{n-1}}\phi(H^+)$  is unbounded for any hyperplane  $H_{n-1} \neq H_{n-1}^*$ , if there is a hyperplane  $H_{n-1}^*$  such that  $P_{H_{n-1}^*}\phi(H^+)$  is bounded. In fact, for any  $H_{n-1} \neq H_{n-1}^*$ , let  $\omega^\perp = H_{n-1}$  and  $\hat{\omega}^\perp = H_{n-1}^*$  and then let  $\theta$  be the angle between  $\omega$  and  $\hat{\omega}$ . Since  $P_{H_{n-1}^*}\phi(H^+)$  is bounded, we may find an infinite right circular cylinder  $C^*$  with bounded radius and having  $t\hat{\omega}$  as its center line such that  $\phi(H^+) \subset C^*$ . Macroscopically,  $C^*$  may be viewed as a line and then have a unbounded projection on  $H_{n-1}$  with any aperture  $\pi/2 - \theta$ , that is,  $P_{H_{n-1}}C^*$  is also an infinite right circular cylinder in the hyperplane  $H_{n-1}$ . It follows that  $P_{H_{n-1}}\phi(H^+)$  is unbounded because  $\phi(H^+) \subset C^*$  is unbounded. We then complete the proof. Q.E.D.

On the other hand, let us investigate the second property of interest. At first glance, intuition almost leads us even to conclude that, if  $\phi$  is line-like, then  $\phi$  is contained in an infinite cylinder. However, this is not always true by considering the following

**Example 3.2.** Define the spiral  $\phi: \mathbf{R}^+ \rightarrow \mathbf{R}^2$  by  $\phi(t) = 2^t e^{i2\pi t}$ . Then  $\phi$  is line-like. This can be seen from the following estimates with respect to dyadic radii:

$$\lambda(B(0; 2^k) \cap \phi) \leq 4 \cdot 2 \cdot 2^{k+1}$$

for  $k = 0, 1, 2, \dots$ .

This example inspires us to guess a more complicated geometrical description of a line-like curve.

For convenience, we need a terminology.

**Definition 3.3.** A piecewise continuous curve  $\phi$  is called globally asymptotic if  $\phi$  is linearly covered by a sequence  $T_k$  of finite right circular cylinders in  $\mathbf{R}^n$  with bounded radii and unbounded heights in the sense that  $\phi \subset \bigcup_{k=1}^\infty T_k$  and every consecutive pair  $T_k$  and  $T_{k+1}$  intersect each other with ends, that is, the bottom of  $T_k$  and the top of  $T_{k+1}$  have a nonempty intersection.

**Theorem 3.4.** Let  $\phi: \mathbf{R} \rightarrow \mathbf{R}^n$  be a line-like curve. Then  $\phi$  is globally asymptotic.

The following lemma fits our needs in the induction step.

**Lemma 3.5.** Let  $\phi: \mathbf{R}^+ \rightarrow \mathbf{R}^2$  be a continuous line-like curve with  $\phi(0) = 0$ . Then  $\phi$  is globally asymptotic.

Before proving this lemma, we introduce some notions. It may be assumed that  $\phi$  is unbounded because, otherwise, the result automatically follows. Then, without loss of generality, we may let  $P_E \phi(\mathbf{R}^+) = \mathbf{R}^+$  where  $E$  is the  $x$ -axis. For simplicity, a normalization  $\phi^*$  of  $\phi$  can be formulated as follows. First all, we may formulate  $\phi_1$  from  $\phi$  to be simple by deleting all possible loops of  $\phi$  and then reflect  $\phi_1$  lying in the lower half-plane with respect to the  $x$ -axis into the upper half-plane to obtain

$$\phi_2(t) = \begin{cases} (P_E \phi_1(t), -P_F \phi_1(t)), & \text{if } P_F \phi_1(t) < 0; \\ \phi_1(t), & \text{otherwise,} \end{cases}$$

where  $F$  is the  $y$ -axis. We again have a simple curve  $\phi_3$  by deleting all possible loops of  $\phi_2$ . Moreover,  $\phi_3$  may be regarded as a polygonal curve having constant length  $L^*$  on each side. Here,  $\phi_3$  is polygonal means in the image of  $\phi_3$  may be expressed as a countable union of line segments. The idea is to approximate the curve by inscribed polygons in terms of the triangle inequality and geometrical properties of line-likeness. Note that  $L^*$  may be any large number, because we have the following observation.

**Lemma 3.6.** The set  $K_\phi = \{\lambda(B(0; k) \cap \phi \cap S_k): k = 1, 2, \dots\}$  is almost bounded, where  $S_k$  is the vertical strip in  $\mathbf{R}^2$  bounded by  $x = k - 1$  and  $x = k$ .

A sequence  $\{x_k\}$  in  $\mathbf{R}^2$  is called almost bounded if there exists a number  $M > 0$ , s.t.,  $|x_k| \leq M$  for all  $k \in \mathbf{Z}^+$  except a set  $B^+ \subset \mathbf{Z}^+$  of upper Banach density zero.

**Definition 3.7.** A set  $S \subset \mathbf{Z}^+$  will be said to have positive upper Banach density if there is a number  $\delta > 0$  such that, for any  $k \in \mathbf{Z}^+$ ,

$$\frac{\#(S \cap \{1, 2, 3, \dots, k\})}{k} > \delta;$$

otherwise,  $S$  is of upper Banach density zero.

*Proof.* Assume by the contrary that  $K_\phi$  is not almost bounded, i.e.,  $K_\phi$  is almost unbounded. Then, without loss of generality, we may assume  $K_\phi$  is monotonic. More precisely, there exists a strictly increasing sequence  $\{t_k\}_{k=0}^\infty$  with  $t_0 = 0$  such that  $P_E\phi(t_k) = k$  and  $\|\phi(t_k) - \phi(t_{k-1})\|$  diverges increasingly.

By the triangle inequality, we have

$$\lambda(B(0; k) \cap \phi \cap S_k) \geq \|\phi(t_k) - \phi(t_{k-1})\|$$

for  $k = 1, 2, \dots$ , and thus,

$$\begin{aligned} \lambda(B(0; k) \cap \phi) &\geq \sum_{i=1}^k \|\phi(t_i) - \phi(t_{i-1})\| \\ &= k\gamma + \sum_{i=1}^k [\|\phi(t_i) - \phi(t_{i-1})\| - \gamma] \end{aligned}$$

for any  $\gamma > 0$ . This implies that, for  $k$  large enough,

$$\frac{\lambda(B(0; k) \cap \phi)}{k} \geq \gamma$$

and a contradiction holds because  $\phi$  is line-like. Q.E.D.

Therefore, by enlarging the strip  $S_k$  to be the strip bounded by  $x = kL^*$  and  $x = (k - 1)L^*$  it is not difficult to see that  $\phi_3$  may be regarded as a simple polygonal curve in the first quadrant with constant length  $L^*$  on each side. Moreover, there exists a strictly increasing sequence  $\{t_k\}_{k=0}^\infty$  with  $t_0 = 0$  such that each node of  $\phi_3$  are labeled as  $\phi_3(t_k)$  for  $k = 0, 1, 2, \dots$ . We now retain the point  $\phi_3(t_k)$  if

$$P_F\phi_3(t_k) = \max_{P_E\phi_3(t) = P_E\phi_3(t_k)} P_F\phi_3(t)$$

and then obtain a new sequence, say,  $\{\phi_3(t_k^*)\}_{k=0}^\infty$  which satisfies the property that  $t_k^* < t_{k+1}^*$  and  $P_E\phi_3(t_k^*) < P_E\phi_3(t_{k+1}^*)$  for  $k = 0, 1, 2, \dots$ . Let  $L^{**}$  be any number greater than  $L^*$  and form a new sequence  $\{\phi_3(t_k^{**})\}_{k=0}^\infty$  from  $\{\phi_3(t_k^*)\}_{k=0}^\infty$  by requiring that either

$$B(\phi_3(t_k^{**}); L^{**}) \cap B(\phi_3(t_{k+1}^{**}); L^{**}) = \phi$$

or

$$(3-2) \quad B(\phi_3(t_k^{**}); L^{**}) \cap B(\phi_3(t_{k-1}^{**}); L^{**}) = \phi$$

for all  $k = 1, 2, 3, \dots$ .

**Definition 3.8.** Let  $\phi: \mathbf{R} \rightarrow \mathbf{R}^n$  be line-like. We call a collection of balls  $\{B(\phi_3(t_k^{**}); L^{**})\}$  satisfying (3-2) a partial chain for the curve  $\phi$  with order  $L^{**}$ . The resulting polygonal curve  $\hat{\phi}$  connecting either from  $\phi_3(t_k^{**})$  to  $\phi_3(t_{k+1}^{**})$  or from  $\phi_3(t_k^{**})$  to  $\phi_3(t_{k-1}^{**})$  by depending on (3-2) respectively for  $k = 0, 1, 2, \dots$  is called an open macroscopic curve for  $\phi$  with order  $L^{**}$ . Analogously, the polygonal curve  $\check{\phi}$  connecting from  $\phi_3(t_k^{**})$  to  $\phi_3(t_{k+1}^{**})$  for  $k = 0, 1, 2, \dots$  is called a closed macroscopic curve for  $\phi$  with order  $L^{**}$ .

**Lemma 3.9.** *Let  $L^{**} > L^*$ . Then an open macroscopic curve  $\hat{\phi}$  for  $\phi$  with order  $L^{**}$  is line-like.*

Obviously, by the triangle inequality, a closed macroscopic curve  $\tilde{\phi}$  for  $\phi$  with order  $L^{**} > L^*$  is line-like. Then Lemma 3.9 follows from the comparison of  $\tilde{\phi}$  and  $\hat{\phi}$ . We leave the details to the reader. Therefore, a normalization  $\phi^*$  of  $\phi$  may be formulated by gluing up each continuous piece of  $\hat{\phi}$  with translation as shown below because such  $\phi^*$  is globally asymptotic if and only if  $\hat{\phi}$  is. In fact, each continuous piece of  $\tilde{\phi}$  may be expressed as  $\overline{\phi_3(t_{2k}^{**})\phi_3(t_{2k+1}^{**})}$ : the line segment from  $\phi_3(t_{2k}^{**})$  to  $\phi_3(t_{2k+1}^{**})$  for  $k = 1, 2, \dots$  and then we may construct  $\phi^*$  by induction

$$\begin{aligned} \phi^* = & \overline{\phi_3(t_0^{**})\phi_3(t_1^{**})} \cup [\phi_3(t_1^{**}) - \phi_3(t_2^{**})] + \overline{\phi_3(t_2^{**})\phi_3(t_3^{**})} \cup \dots \\ & \cup \sum_{k=0}^n [\phi_3(t_{2k+1}^{**}) - \phi_3(t_{2k+2}^{**})] + \overline{\phi_3(t_{2n+2}^{**})\phi_3(t_{2n+3}^{**})} \cup \dots \end{aligned}$$

Now, let  $x = P_E\phi^*(t)$  and  $f_\phi^*(x) = P_F\phi^*(t)$ , then the following property of  $\phi^*$  is easy to check.

**Lemma 3.10.** *The induced relation  $f_\phi^*$  is well defined on  $\mathbf{R}^+$ . Moreover  $f_\phi^*$  is positive, continuous and the graph of  $f_\phi^*$  is polygonal.*

This lemma implies that  $\phi^*$  may be coordinately parametrized by the positive  $x$ -axis. We now claim that  $\phi^*$  is globally asymptotic and so is  $\phi$ . First all, observe that there exists a strictly increasing sequence  $\{\hat{t}_k\}_{k=0}^\infty$  with  $t_0 = 0$  such that  $\{P_E\phi^*(\hat{t}_{k+1}) - P_E\phi^*(\hat{t}_k)\}_{k=0}^\infty$  diverges increasingly and

$$\phi^*(\hat{t}_k) \in B(0; P_E\phi^*(\hat{t}_k)) \quad \text{for } k = 0, 1, 2, \dots$$

Secondly, let

$$d_{k,1}^1 = \max_{\hat{t}_k \leq t \leq \hat{t}_{k+1}} d(\phi^*(t), L_{k,1}^1),$$

where  $L_{k,1}^1$  is understood to be the line from  $\phi^*(\hat{t}_k)$  to  $\phi^*(\hat{t}_{k+1})$ . Then let  $\phi^*(t_{k,1}^1)$  be the point which attains this maximum and is nearest to the middle line between  $\phi^*(\hat{t}_k)$  and  $\phi^*(\hat{t}_{k+1})$ . Here, we call  $\phi^*(\hat{t}_k)$  and  $\phi^*(\hat{t}_{k+1})$  the generators for  $\phi^*(t_{k,1}^1)$ . Inductively, consider first the segment  $\overline{\phi^*(\hat{t}_k)\phi^*(\hat{t}_{k+1})}$ . Then, as above, we obtain a point  $\phi^*(t_{k,2}^1)$  which attains the maximum

$$d_{k,2}^1 = \max_{\hat{t}_k \leq t \leq \hat{t}_{k+1}} d(\phi^*(t), L_{k,2}^1)$$

where  $L_{k,2}^1$  is the line from  $\phi^*(\hat{t}_k)$  to  $\phi^*(t_{k,1}^1)$  and  $\phi^*(t_{k,2}^1)$  is the nearest point to the middle line between  $\phi^*(\hat{t}_k)$  and  $\phi^*(t_{k,1}^1)$ . Similarly, we have such kind of point  $\phi^*(t_{k,2}^2)$  for the segment  $\overline{\phi^*(t_{k,1}^1)\phi^*(\hat{t}_{k+1})}$ . Repeat this process, we then obtain a sequence of points  $\phi^*(t_{k,j}^i)$ ,  $1 \leq i \leq 2^{j-1}$ , where the step number  $j$  of our process is to be determined later. Next, we connect the point  $\phi^*(t_{k,j}^i)$  to its generators respectively, for  $1 \leq i \leq 2^{j-1}$ . The resulting curve is called the zigzag curve of  $j$ th step in the  $k$ -region. Finally, we denote by  $\alpha_{k,j}^i$  the angle between the segment connecting  $\phi^*(t_{k,j}^i)$  to its left generator  $L_{k,j}^i$  (according to the projection onto the  $x$ -axis) and the segment connecting these

two generator. Similarly, we have the angle  $\beta_{k,j}^i$  for its right generator  $R_{k,j}^i$ .  
Let

$$\hat{\alpha}_{k,j}^i = \max\{\alpha_{k,j}^i, \beta_{k,j}^i\} \quad \text{and} \quad \hat{\beta}_{k,j}^i = \min\{\alpha_{k,j}^i, \beta_{k,j}^i\}.$$

An observation of plane geometry follows:

**Lemma 3.11.** *Both  $\hat{\alpha}_{k,j}^i$  and  $\hat{\beta}_{k,j}^i$  are decreasing in  $j$ .*

We now return to the proof of Lemma 3.5. Let  $L$  be any fixed large number. Then, for  $k$  large enough, we may obtain a process of  $j$ -step such that

$$\min_{1 \leq i \leq 2^j - 1} \{ \max\{ \|\phi^*(t_{k,j}^i) - L_{k,j}^i\|, \|\phi^*(t_{k,j}^i) - R_{k,j}^i\| \} \} \geq L.$$

Moreover, in the  $k$ -region, the geodesic curve  $\phi_k^*$  of  $j$ th step has the properties:

$$\begin{aligned} \lambda(\phi_k^*) &\approx \left( \prod_{l=1}^j \sec \hat{\alpha}_{k,l}^i \right) \|\phi^*(\hat{t}_k) - \phi^*(\hat{t}_{k+1})\| \\ (3-3) \qquad &\approx \left( \prod_{l=1}^j \sec \hat{\beta}_{k,l}^i \right) \|\phi^*(\hat{t}_k) - \phi^*(\hat{t}_{k+1})\|. \end{aligned}$$

In terms of Lemma 3.11, this forces that both  $\hat{\alpha}_{k,j}^i$  and  $\hat{\beta}_{k,j}^i$  tends to zero as  $k \rightarrow \infty$  because, otherwise, (3-3) will contradict to the line-likeness of  $\phi^*$ . Therefore, we see that, given a right circular cylinder with radius one and length  $L$ , there is a zigzag curve  $\phi_k^*$  lies in this cylinder such that  $\phi_k^*$  has nonempty intersections with both ends of the cylinder. This implies that  $\phi^*$  is globally asymptotic and so is  $\phi$ .  $\square$

We now extend our Lemma 3.5 to  $\mathbf{R}^n$  space. Assume by induction step Lemma 3.5 is true for  $\mathbf{R}^{n-1}$ . Then consider the projection  $P_i\phi$  of  $\phi$  onto the hyperplane  $\langle e_i \rangle^\perp$ . By induction assumption  $P_i\phi$  goes through a sequence  $\{C_k(P_i\phi)\}$  of arbitrary long open tubes in  $\langle e_i \rangle^\perp$  with constant radius. Since there is only one projection to be possibly bounded, we may assume  $P_n\phi$  be this projection. Now consider the product  $C_k(P_i\phi) \times \langle e_i \rangle$  and the intersection

$$C(\phi) = \bigcap_{i=1}^n \left\{ \bigcup_{k=1}^\infty [C_k(P_i\phi) \times \langle e_i \rangle] \right\}.$$

Then  $C(\phi)$  meets our needs and then completes the proof of Theorem 3.4.  $\square$

#### 4. PROCESS OF NORMALIZATION

We would like to establish a rearrangement inequality in terms of a specified process of normalization. In what follows, we will employ a fixed point theorem of contractive type. The Steiner symmetrization is an important notion in setting up this process in the  $n$ -space.

**Definition 4.1.** Let  $K$  and  $V$  be a bounded measurable set and a hyperplane of  $\mathbf{R}^s$  respectively. Then the set  $K_V^*$  is called the Steiner symmetrization of  $K$  with respect to  $V$ , if, for every line  $L$  perpendicular to  $V$ ,  $K_V^* \cap L$  is the symmetrical rearrangement of  $K \cap L$  with respect to reflection to  $V$ . Moreover, let  $\Gamma^s$  be the set of all bounded measurable sets in  $\mathbf{R}^s$ . Then the Steiner operator  $S_V$  on  $\Gamma^s$  with respect to a given hyperplane  $V$  is defined as  $S_V(K) = K_V^*$ .

Clearly,  $S_V$  is measure-preserving. We now describe a general process which normalizes any set  $K$  in  $\Gamma^s$  into the ball centered at the origin with the same measure as  $K$  in terms of Steiner operator.

**Definition 4.2.** Let  $K \in \Gamma^s$ , and let  $S$  be the ball centered at the origin with measures  $\lambda_s(S) = \lambda_s(K)$ . Then a process of normalization of  $K$  is a sequence of sets  $K_j \in \Gamma^s$  where  $K_0 = K$  and  $K_{j+1}$  is obtained from  $K_j$  by a sequence of iterated measure-preserving operators such that

$$\lim_{j \rightarrow \infty} \lambda_s(K_j \Delta S) = 0.$$

A known process of normalization is the following

**Brascamp-Lieb-Luttinger process** [1]. We specify the sequence  $K_j$  of sets in Definition 4.2 by induction. Given  $K_j$  choose a hyperplane  $V_1$ , such that

$$\lambda_s(K_{jV_1}^* \Delta S) < \inf_V \lambda_s(K_{jV}^* \Delta S) + j^{-1}.$$

Then construct  $K_{j+1}$  from  $K_j$  by  $s$  consecutive Steiner symmetrizations with respect to a sequence of hyperplanes  $V_1, V_2, \dots, V_n$  in  $\mathbf{R}^s$  (beginning with  $V_1$  specified above) whose orthogonal complements are pairwise orthogonal. In that way, we have

$$\lambda_s(K_{j+1} \Delta S) < \inf_V \lambda_s(K_{jV}^* \Delta S) + j^{-1}.$$

This process can be applied to prove a rearrangement inequality for any rotational invariant integral operator. However, it does not meet our need because we cannot apply it directly to the integral operator corresponding to the X-ray transform  $R_{1,n}f$  on  $M_\phi$  which is not invariant under the full rotation group.

We now construct another process of normalization. Before the construction, we need

**Definition 4.3.** Let  $\Gamma_k^s$  be the set of all bounded sets in  $\mathbf{R}^s$  with measure  $k$ . Let  $\varphi$  be a line passing through the origin, which can be identified as a point  $\omega$  in the real projective space  $\mathbf{RP}^{s-1}$ . Then, define  $\Gamma_k^s(\varphi)$  to be the subset of  $\Gamma_k^s$  consisting of element whose intersection with each affine hyperplane  $W_{\varphi(t)}$  perpendicular to  $\varphi$  at point  $\varphi(t)$  is a ball  $B_{\varphi(t)}$  in  $W_{\varphi(t)}$  with radius  $r(t)$ , where  $\varphi(t)$  is the linear parametrization of  $\varphi$  with  $\varphi(0) = 0$  and  $r(t)$  is a decreasing even function of  $t$ .

In what follows, the process of normalization can be constructed by induction on  $s$ . Beginning with  $s = 2$ , some interesting geometrical properties will be established.

The element in  $\Gamma_k^s(\varphi)$  are called regular sets with axis  $\varphi$ . We will first describe an intermediate process that transforms a regular set into a finite cylinder.

**Definition 4.4.** Let  $K \in \Gamma_k^2$ . Define the regularizer operator  $R_\varphi: \Gamma_k^2 \rightarrow \Gamma_k^2$  by

$$R_\varphi(K) = S_{W_0} \circ S_\varphi(K).$$

For each  $K \in \Gamma_k^2$ , let  $\hat{t}$  be the largest number such that  $\varphi(\hat{t}) \in K$ . Then, regarding  $\varphi$  as center line and  $\varphi(0)$  as center point, let  $\hat{C}$  be the cylinder having height  $2\|\varphi(\hat{t})\|$  and volume  $k$ .

The difference between  $K$  and  $\hat{C}$  inspires us to define some operators.

**Definition 4.5.** Let  $d(K) = K - \widehat{C}$  and  $D(K) = K - d(K)$ . Let  $H_\varphi^+$  and  $H_\varphi^-$  be the two half-space divided by  $W_0$  such that  $\varphi(\hat{t}) \in H_\varphi^+$  and  $\varphi(-\hat{t}) \in H_\varphi^-$ . Let

$$\hat{d}(K) = \{\varphi(-\hat{t}) + d(K) \cap H_\varphi^+\} \cup \{\varphi(\hat{t}) + d(K) \cap H_\varphi^-\}.$$

Then define the alternating-shift operator  $G_\varphi: \Gamma_k^2(\varphi) \rightarrow \Gamma_k^2$  by

$$G_\varphi(K) = \hat{d}(K) \cup D(K).$$

Moreover, we define the cylinderizer operator  $C_\varphi: \Gamma_k^2(\varphi) \rightarrow \Gamma_k^2$  by  $C_\varphi(K) = R_\varphi \circ G_\varphi(K)$ .

An interesting question is raised: Does the iterated sequence  $\{C_\varphi^i(K)\}$  converge to the cylinder  $\widehat{C}$ ?

Now consider the set  $\Psi = \{C_\varphi^i(K)\}_{i=0}^\infty \cup \widehat{C}$  and define on  $\Psi$  by  $d(A, B) = \lambda_2(A\Delta B)$  for all  $A, B \in \Psi$ . With respect to this metric, the regularizer operator  $R_\varphi$  and the alternating-shift operator  $G_\varphi$  are continuous on their respective domains. This implies that the cylinderizer operator is continuous on  $\Psi$ .

More precisely, we will show that

$$(4-1) \quad d(C_\varphi(K), C_\varphi(\widehat{C})) < d(K, \widehat{C})$$

for any  $K \neq \widehat{C}$ . It follows that  $C_\varphi$  possesses at most a fixed point and thus, to answer the above question, it suffices to show that the iterated sequence  $\{C_\varphi^i\}$  converges because  $C_\varphi$  is continuous on  $\Psi$  and thus

$$C_\varphi \left( \lim_{i \rightarrow \infty} (C_\varphi^i(K)) \right) = \lim_{i \rightarrow \infty} C_\varphi^i(K).$$

However, inequality (4-1) is easily to be established by the following observation:

$$d(C_\varphi^{i+1}(K)) \subset d(C_\varphi^i(K)), \quad D(C_\varphi^i(K)) \subset D(C_\varphi^{i+1}(K))$$

for  $i = 0, 1, 2, \dots$ .

Finally, since the sequence  $\{D(C_\varphi^i(K))\}$  is bounded by  $\widehat{C}$ , we conclude that the iterated sequence  $\{C_\varphi^i(K)\}$  converges to  $\widehat{C}$ , thus we obtain an intermediate process which transforms a regular set  $K$  in  $\Gamma_k^2(\varphi)$  into a finite cylinder  $\widehat{C}$  with the same measure  $k$ . This correspondence may be formulated as an operator  $T_\varphi: \Gamma_k^2(\varphi) \rightarrow \Gamma_k^2(\varphi)$  defined by  $T_\varphi(K) = \widehat{C}$ .

Such an operator  $T_\varphi$  is introduced for the reason of simplicity in our process of normalization. In fact, the set  $R_\varphi \circ S_E \circ T_\varphi(K)$  should be a convex symmetrical body and thus we may reduce all elements in  $\Gamma_k^2(\varphi)$  to be symmetrical convex.

Now let  $K \in \Gamma_k^2(\varphi)$  and  $E = \langle e_1 \rangle$ . Define the normalizer operator  $N_\varphi: \Gamma_k^2(\varphi) \rightarrow \Gamma_k^2(\varphi)$  by

$$N_\varphi(K) = R_\varphi \circ S_E(K).$$

We thus form a process of normalization by setting up  $K_0 = R_\varphi \circ S_E \circ T_\varphi(K)$  and  $K_{j+1} = N_\varphi(K_j)$ , for  $j = 1, 2, \dots$ .

As before, define a metric on  $\Gamma_k^2(\varphi)$  by  $d(A, B) = \lambda_2(A\Delta B)$  for all  $A, B$  in  $\Gamma_k^2(\varphi)$ . An interesting question is also raised: Does the iterated sequence  $\{N_\varphi^i(K_0)\}$  converge to the ball  $S$  for any  $K_0 \in \Gamma_k^2(\varphi)$ ?

We introduce the notion of fixed-point theorem of contraction type which is due to Edelstein.

**Definition 4.6.** Let  $(\Omega, d)$  be a metric space. A self-mapping  $T: \Omega \rightarrow \Omega$  is called nonexpansive if

$$d(T(x), T(y)) \leq d(x, y)$$

for all  $x, y \in \Omega$ , and is called strict contraction if

$$d(T(x), T(y)) < d(x, y)$$

for all  $x, y \in \Omega$  with  $x \neq y$ .

**Edelstein's fixed-point theorem** [9]. Let  $(\Omega, d)$  be a metric space. Suppose that  $T: \Omega \rightarrow \Omega$  is strict contraction and there is a point  $x_0 \in \Omega$  such that the iterated sequence  $\{T^i(x_0)\}$  has a cluster point  $z$  in  $\Omega$ . Then  $T^i(x_0) \rightarrow z$ , the unique fixed point of  $T$ .

We therefore have to deal with the following two lemmas which meet the sufficient condition of Edelstein's fixed-point theorem.

**Definition 4.7.** Let  $\omega$  be a unit vector in the complex plane with  $\omega^4 \neq 1$ . The line  $\varphi$  associated with  $\omega$  in Definition 4.3 is called rational if there exists a positive integer  $l$  such that  $\omega$  is identified as one of the  $l$ th roots of unity in the complex plane determined by  $\varphi$  and the projection  $\Pi_E(\varphi)$  of  $\varphi$  into  $E$ , where  $E = \langle e_1 \rangle$ ; otherwise,  $\varphi$  is called nonrational.

**Definition 4.8.** An element  $A \in \Gamma_k^2$  is called a regular body if  $0 \in A$  and  $A$  can be radially divided into a finite number of congruent subsets.

**Lemma 4.9.** The normalizer operator  $N_\varphi$  on  $\Gamma_k^2(\varphi)$  is a strict contraction if and only if  $\varphi$  is nonrational.

Before proving this lemma, we note without proof that  $N_\varphi$  is nonexpansive for any  $\varphi$ .

*Proof. Necessity.* Assume that  $\varphi$  is rational. Then we may find a regular  $l$ -polygon  $\tilde{P} \in \Gamma_k^2(\varphi)$  such that  $N_\varphi(\tilde{P}) = \tilde{P}$ . In fact, let  $P$  be the convex hull of the  $l$ th roots of unity. Then there exists a positive number  $\alpha$  such that  $\lambda_2(\alpha P) = k$ , and thus let  $\tilde{P} = \alpha P$ . This implies that

$$d(N_\varphi(\tilde{P}), N_\varphi(S)) = d(\tilde{P}, S),$$

a contradiction.

*Sufficiency.* Since  $N_\varphi$  is nonexpansive, we may assume that there are two distinct elements  $A$  and  $B$  in  $\Gamma_k^2(\varphi)$  such that

$$d(N_\varphi(A), N_\varphi(B)) = d(A, B).$$

We claim that

$$\lambda_1(\Pi_E(A \setminus B) \cap \Pi_E(B \setminus A)) = 0.$$

In fact, suppose not. Let

$$Q = \Pi_E(A \setminus B) \cap \Pi_E(B \setminus A).$$

Denote by  $L(v)$  the straight line perpendicular to  $E$  through  $v \in E$ . Let  $A(v) = A \cap L(v)$  and  $B(v) = B \cap L(v)$ , then if  $v \in Q$ , neither  $A(v) \subset B(v)$  nor  $B(v) \subset A(v)$  therefore

$$\lambda_1(A_E^*(v)\Delta B_E^*(v)) = |\lambda_1(A(v)) - \lambda_1(B(v))| < \lambda_1(A(v)\Delta B(v))$$

whenever  $v \in Q$ . We thus have, by  $\lambda_1(Q) > 0$ ,

$$\lambda_2(A_E^* \Delta B_E^*) < \lambda_2(A \Delta B)$$

because, in general, for all  $v \in E$ , we have

$$\lambda_1(A_E^*(v) \Delta B_E^*(v)) \leq \lambda_1(A(v) \Delta B(v)).$$

This implies that

$$d(N_\varphi(A), N_\varphi(B)) < d(A, B),$$

a contradiction.

Hence we have  $\lambda_1(Q) = 0$  and then  $Q$  is empty because we may assume that both  $A$  and  $B$  have piecewise smooth boundary. Note that  $Q$  is empty if and only if both  $A$  and  $B$  are  $S_E$ -invariant. We now claim that  $Q$  is empty if and only if both  $A$  and  $B$  are regular bodies in the plane. This implies that  $\varphi$  is rational and the result will follow.

We consider  $A$  only. Let  $\Omega_0$  be one of the sectors of  $A$  bounded by  $\varphi$  and  $E$ . Without loss of generality, we may assume  $\arg \omega \in (0, \frac{\pi}{2})$  and  $\Omega_0$  is located in the first quadrant. Because  $A$  is  $S_E$ -invariant, we may form  $\Omega_1$  by reflecting  $\Omega_0$  with respect to  $E$ . We now take the union  $\Omega = \Omega_0 \cup \Omega_1$  and then obtain the sector  $\tilde{\Omega}$  by reflecting  $\Omega$  with respect to  $\varphi$ . We claim that  $\tilde{\Omega}$  is just the counterclockwise rotation of  $\Omega$  by the angle  $2 \arg \omega$ . In fact, divide  $\tilde{\Omega}$  into two equal portions, say,  $\tilde{\Omega}_0$  and  $\tilde{\Omega}_1$  which are clockwise labeled. Then  $\tilde{\Omega}_0$  and  $\tilde{\Omega}_1$  are the counterclockwise rotation of  $\Omega_0$  and  $\Omega_1$  by the angle  $2 \arg \omega$  respectively. Repeating this process, we conclude that  $A$  is a regular body. This completes the proof. Q.E.D.

However, for higher dimension  $s > 2$ , this argument does not apply. For instance, assume that  $\varphi$  is rational in  $\mathbf{R}^3$ . Then we may not find any regular set  $P$  in  $\Gamma_k^3(\varphi)$  such that  $N_\varphi(P) = P$ . This implies that  $N_\varphi$  must be a strict contraction. Indeed, we have

**Lemma 4.10.** *Let  $s > 2$ . Then the normalizer operator  $N_\varphi$  on  $\Gamma_k^s(\varphi)$  is a strict contraction if  $\varphi$  is nondegenerate.*

Here, we call a line  $\varphi$  is nondegenerate if  $\varphi$  is neither perpendicular to  $E$  nor lying on  $E$ .

*Proof.* We start with  $s = 3$ . Any element  $P$  in  $\Gamma_k^3(\varphi)$  looks like a symmetric shuttle with  $\varphi$  as center line. A geometric observation shows that we may not find such  $P$  with  $N_\varphi(P) = P$  if  $\varphi$  is nondegenerate. Assume by induction that Lemma 4.10 is true for  $s - 1$ . We claim that  $N_\varphi$  is a strict contraction on  $\Gamma_k^s(\varphi)$ . Let  $H_\varphi$  be the hyperplane in  $\mathbf{R}^s$  containing  $\varphi$  and perpendicular to  $E$ . By induction assumption, we have

$$d(N_\varphi(A \cap H_\varphi), N_\varphi(B \cap H_\varphi)) < d(A \cap H_\varphi, B \cap H_\varphi)$$

whenever  $A, B \in \Gamma_k^s(\varphi)$  with  $A \neq B$ . Then, the convexity of metric  $d$  reveals that, for some small  $\delta > 0$ ,

$$d(N_\varphi(A_\delta), N_\varphi(B_\delta)) < d(A_\delta, B_\delta),$$

where

$$A_\delta = \{x \in \mathbf{R}^s : d(x, H_\varphi) \leq \delta, x \in A\}$$

and

$$B_\delta = \{x \in \mathbf{R}^s : d(x, H_\varphi) \leq \delta, x \in B\}.$$

Since  $N_\varphi$  is nonexpansive, we then conclude that  $N_\varphi$  is a strict contraction on  $\Gamma_k^s(\varphi)$  and then the result follows.

**Lemma 4.11.**  $\{N_\varphi^i(K_0)\}$  possesses a cluster point for any  $K_0 \in \Gamma_k^2(\varphi)$ .

*Proof.* It suffices to show that there exists a subsequence  $\{N_\varphi^{i_i}(K_0)\}$  and a set  $M \in \Gamma_k^2(\varphi)$  such that

$$\lim_{i \rightarrow \infty} \lambda_2(N_\varphi^{i_i}(K_0)\Delta M) = 0.$$

Let  $\chi_i$  be the characteristic function of  $N_\varphi^i(K_0)$ . Then,  $x \in N_\varphi^i(K_0)$  implies  $y \in N_\varphi^i(K_0)$  if  $|y_p| \leq |x_p|$ ,  $p = 1, 2$ , and therefore

$$\int_{\mathbf{R}} |\chi_i(x_1 + y_1, x_2) - \chi_i(x_1, x_2)| dx_1 \leq 2|y_1|$$

and

$$\int_{\mathbf{R}} |\chi_i(x_1, x_2 + y_2) - \chi_i(x_1, x_2)| dx_2 \leq 2|y_2|.$$

Note that all  $N_\varphi^i(K_0)$  are contained in some ball of radius  $R$  centered at the origin. This implies that

$$\int_{\mathbf{R}^2} |\chi_i(x + y) - \chi_i(x)| dx \leq 2(2R)(|y_1| + |y_2|).$$

In other words,

$$\lim_{y \rightarrow 0} \int_{\mathbf{R}^2} |\chi_i(x + y) - \chi_i(x)| dx = 0$$

uniformly on  $i$ . Hence the family of functions  $\{\chi_i\}$  is conditionally compact [8] in  $L^1(\mathbf{R}^2)$  and the result follows.

Therefore, by Edelstein's fixed-point theorem, we conclude that  $N_\varphi^i(K_0) \rightarrow S$  for any  $K_0 \in \Gamma_k^2(\varphi)$ . We then proceed our construction by induction. Assume that the process of normalization in Definition 4.2 holds for  $\Gamma^{s-1}$ . We construct the desired sequence of sets  $K_j \in \Gamma^s$  as follows.

Given  $K \in \Gamma_k^s$ . Let  $\{t_l\}$  be the set of rational points in  $\mathbf{R}$ . Let  $\{K_j^0\}$  be the resulting sequence in the affine hyperplane  $W_{\varphi(t_0)}$  associated with a double sequence  $\{V_{i,j}^0\}$  of  $s - 2$  dimensional affine hyperplanes in  $W_{\varphi(t_0)}$  such that

$$\lim_{j \rightarrow \infty} \lambda_{s-1}(K_j^0 \Delta B_{\varphi(t_0)}) = 0.$$

Now form the hyperplane  $W_{i,j}^0$  in  $\mathbf{R}^s$  containing both  $\varphi$  and  $V_{i,j}^0$  and take iteratively the Steiner operator  $S_{W_{i,j}^0}$  on  $K$  under the double index  $i, j$ . Denote by  $K^0$  the resulting set in  $\Gamma_k^s$ . Begin again with  $K^0$  and  $t_1$ , we also have the resulting set  $K^1$ . Repeat this process for  $l = 2, 3, \dots$ , and pass to the limits, we obtain an element  $\tilde{K}$  whose intersection with each affine hyperplane  $W_{\varphi(t)}$  perpendicular to  $\varphi$  at point  $\varphi(t)$  is a ball in  $W_{\varphi(t)}$  with radius  $r(t)$ .

Now, for higher dimension  $s = 3, 4, \dots$ , define the regularizer operator  $R_\varphi: \Gamma_k^s \rightarrow \Gamma_k^s(\varphi)$  by  $R_\varphi(K) = S_{W_0}(\tilde{K})$ . Moreover, define the normalizer operator  $N_\varphi: \Gamma_k^s(\varphi) \rightarrow \Gamma_k^s(\varphi)$  by  $N_\varphi(K) = R_\varphi \circ S_E(K)$ . Our construction of process

of normalization proceeds as follows: Given  $K \in \Gamma_k^s$ . Let  $K_0 = R_\phi \circ S_E \circ T_\phi(K)$  and  $K_{j+1} = N_\phi(K_j)$  for  $j = 0, 1, 2, \dots$ , where  $T_\phi$  is the natural generalization from the case  $s = 2$  to higher dimension and  $E = \langle e_1, e_2, \dots, e_{s-1} \rangle$ . Note that each  $K_j$  is convex. Then, by using the same arguments as in the case  $s = 2$ , one can verify without difficulty that  $K_j \rightarrow S$ . This completes our construction.

5. REARRANGEMENT INEQUALITY

Following Christ's technique, a more complicated rearrangement inequality is established. For simplicity, let  $\phi$  be the  $e_1$ -axis. We are now ready to state a rearrangement inequality for the X-ray transform  $R_{1,n}$  associated with the measure  $d\mu$  on  $M_\phi(\mathbb{R}^n)$ .

**Theorem 5.1.** For  $n \geq 3$ ,

$$\|R_{1,n}f\|_{L^{n-1}(M_\phi(\mathbb{R}^n), d\mu)} \leq C\|R_{1,n}f^*\|_{L^{n-1}(M_\phi(\mathbb{R}^n), d\mu)},$$

where  $f \in C_0(\mathbb{R}^n)$  and  $f^*$  is the radial decreasing rearrangement of  $f$ .

Before proving this result, note that

$$\begin{aligned} \|R_{1,n}f\|_{L^{n-1}(M_\phi(\mathbb{R}^n), d\mu)}^{n-1} &= \iint [R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x) \\ &= \iiint f(y)[R_{1,n}f(\pi(x, y))]^{n-2} d\mu_x(\pi)|x - y|^{1-n} dy d\lambda(x). \end{aligned}$$

This suggests an introduction of the following multilinear operator  $A = A_\phi$  defined by

$$\begin{aligned} A(f_1, f_2, \dots, f_n) &= \iint f_1(x_1) \prod_{j=2}^n \int_{\pi(x_1, x_0)} f_j(x_j) d\lambda_\pi(x_j) |x_1 - x_0|^{1-n} dx_1 d\lambda(x_0), \end{aligned}$$

where  $f_1$  is the characteristic function of the ball centered at the origin with radius  $R$  large enough, and  $d\lambda_\pi$  is the usual measure on the line  $\pi(x_1, x_0)$  connecting from  $x_1$  to  $x_0$ .

We claim that

$$A(f_1, f_2, \dots, f_n) \leq CA(f_1^*, f_2^*, \dots, f_n^*)$$

for  $R$  large enough, and thus, in particular,

$$\begin{aligned} &\iint R_{1,n}f_1(\pi)[R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x) \\ &\leq \iint R_{1,n}f_1^*(\pi)[R_{1,n}f^*(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x). \end{aligned}$$

Let  $R$  again be large enough such that  $B(0; \frac{R}{2})$  contains both the supports of  $f$  and  $f^*$ , then, by using the fact that  $\phi$  is a straight line and  $n \geq 3$ ,

$$\begin{aligned} &\iint R_{1,n}f_1(\pi)[R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x) \\ &\simeq \iint [R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x) \end{aligned}$$

and the same equality is also applied to  $f^*$ . This implies that

$$\begin{aligned} R \iint [R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi)d\lambda(x) \\ \leq CR \iint [R_{1,n}f^*(\pi)]^{n-1} d\mu_x(\pi)d\lambda(x) \end{aligned}$$

and thus Theorem 5.1 follows.

Now, let  $x_j = (z_j, t_j) \in \mathbf{R}^{n-1} \times \mathbf{R}$  and

$$\begin{aligned} A_\varepsilon(f_1, f_2, \dots, f_n) \\ = \iint f_1(x_1) \prod_{j=2}^n \int \varepsilon^{1-n} f_j(x_j) \chi_\varepsilon(x_j) dx_j |x_1 - x_0|^{1-n} dx_1 d\lambda(x_0), \end{aligned}$$

where  $\chi_\varepsilon(x_j)$  denotes the characteristic function of the set  $\{x_j: d(x_j, \pi(x_1, x_0)) \leq \varepsilon\}$ . Then

$$\begin{aligned} A(f_1, f_2, \dots, f_n) &= \lim_{\varepsilon \rightarrow 0} A_\varepsilon(f_1, f_2, \dots, f_n) \\ &= \lim_{\varepsilon \rightarrow 0} \iiint f_1(z_1, t_1) \prod_{n=2}^n \int f_j(z_j, t_j) \\ &\quad \cdot \int_{\Omega} \varepsilon^{1-n} \chi_\varepsilon(x_j) d\beta(x_j) d\alpha(z_j) |x_1 - x_0|^{1-n} dz_1 dt_1 d\lambda(x_0), \end{aligned}$$

where for each  $1 < j \leq n$ ,  $t_j$  is determined by the requirement that  $x_j \in \pi(x_1, x_0)$ ,  $d\alpha(z_j)$  is the measure on the line  $\pi((z_1, 0), (z_0, 0))$ , and  $d\beta$  is the measure on the hyperplane

$$\Omega = \{x + (z_j, 0): ((z_j - z_0, 0), x) = 0\}.$$

Note that

$$\int_{\Omega} \varepsilon^{1-n} \chi_\varepsilon(x_j) d\beta(x_j) \approx \varepsilon^{1-n} \cdot \frac{|x_1 - x_0|}{|z_1 - z_0|} \cdot \varepsilon \cdot \varepsilon^{n-2}.$$

This implies that

$$\begin{aligned} (5-1) \quad A(f_1, f_2, \dots, f_n) \\ = \iint \left( \int \left( \int \prod_{j=1}^n f_j(z_j, t_j) dt_1 \right) d\alpha(z_1) \cdots d\alpha(z_n) \right) \\ \times |z_1 - z_0|^{1-n} dz_1 d\lambda(x_0). \end{aligned}$$

We now divide the proof of Theorem 5.1 into three steps.

*Step A.* Recall that a nonnegative, measurable function  $f^*(x|V)$  on  $\mathbf{R}^n$  is called the Steiner symmetrization of  $f(x)$  with respect to a fixed hyperplane  $V$ , if  $f^*(x|V)$  is the radial decreasing rearrangement of  $f(x)$  along each line perpendicular to  $V$ . We claim that

**Lemma 5.2.** For any fixed hyperplane  $V_\phi$  containing  $\phi$ ,

$$A(f_1, f_2, \dots, f_n) \leq A(f_1^*(\cdot|V_\phi), f_2^*(\cdot|V_\phi), \dots, f_n^*(\cdot|V_\phi)).$$

Without loss of generality, we may choose an orthogonal coordinate system in  $\mathbf{R}^n$  such that the  $x_n$ -axis is perpendicular to  $V_\phi$  and assume that each  $f_j$

is the characteristic function of a bounded measurable set  $B_j$  with piecewise smooth boundary in  $\mathbf{R}^n$ , that is,  $f_j = \chi_{B_j}$ . Thus, by (5-1), it suffices to show that

$$(5-2) \quad \int \prod_{j=2}^n f_j(z_j, t_j) dt_1 \leq \int \prod_{j=2}^n f_j^*((z_j, t_j)|V_\phi) dt_1,$$

where  $z_0, z_1, \dots, z_n$  are fixed. The following two easy geometrical lemmas play a key role in the proof of (5-2).

**Lemma 5.3.** *Given a point  $p$ , two distinct lines  $\pi_0: x = a$  and  $\pi: x = b$  in  $\mathbf{R}^2$  with  $p \notin \pi_a$  and  $p \notin \pi_b$ . Then, for any two measurable sets  $C, D$  in the line  $\pi_a$  with measures  $\lambda_{\pi_a}(C) \leq \lambda_{\pi_b}(D)$ , we have  $\lambda_{\pi_a}(C') \leq \lambda_{\pi_b}(D')$ , where*

$$C' = \{\pi_a \cap \pi(x, p) : x \in C\} \quad \text{and} \quad D' = \{\pi_b \cap \pi(x, p) : x \in D\}.$$

By a standard approximation argument we may assume that these sets to be a finite union of disjoint compact intervals and then the above lemma follows from a simple geometrical argument.

Using Lemma 5.3, inequality (5-2) follows from the following

**Lemma 5.4.** *Given a point  $p$  and  $k$  distinct lines  $\pi_{b_j}: x = b_j$  in  $\mathbf{R}^2$  such that  $p$  lies in the  $x$ -axis and  $p \notin \pi_{p_j}$  for each  $1 \leq j \leq k$ . Then, for any measurable set  $\tilde{B}_j$  in  $\pi_{b_j}$ ,*

$$\begin{aligned} \lambda_{\pi_{b_1}}(\tilde{B} \equiv \{x \in \pi_{b_1} : \pi(x, p) \text{ meets each } \tilde{B}_j\}) \\ \leq \lambda_{\pi_{b_1}}(\tilde{B}^* \equiv \{x \in \pi_{b_1} : \pi(x, p) \text{ meets each } \tilde{B}_j^*\}) \end{aligned}$$

where  $\tilde{B}_j^*$  is the symmetrical rearrangement of  $\tilde{B}_j$ .

In fact, following the notions in Lemma 5.3, identify  $\pi_a$  with the set  $\{(z_2, t_2) : t_2 \in \mathbf{R}\}$  and identify  $\pi_b$  with the set  $\{(z_1, t_1) : t_1 \in \mathbf{R}\}$ . Then, by letting  $C = \tilde{B}$  and  $D = \tilde{B}^*$ , we have

$$\lambda_{\pi_b}(C') = \int \prod_{j=2}^n f_j(z_j, t_j) dt_1$$

and

$$\lambda_{\pi_b}(D') = \int \prod_{j=2}^n f_j^*((z_j, t_j)|V_\phi) dt_1.$$

Therefore, inequality (5-2) follows from Lemmas 5.3 and 5.4. This completes the proof of Lemma 5.2.

We now state the main result for Step A.

**Theorem 5.5.** *For  $1 \leq j \leq n$ , let  $f_j^\phi$  be the function such that, for any point  $p_j \in \phi$ ,  $f_j^\phi|_{W_{p_j}}$  is the radial decreasing rearrangement of  $f_j|_{W_{p_j}}$  in the affine hyperplane  $W_{p_j}$  perpendicular to  $\phi$  at point  $p_j$ . Then*

$$A(f_1, f_2, \dots, f_n) \leq A(f_1^\phi, f_2^\phi, \dots, f_n^\phi).$$

This result follows from Lemma 5.2 and the process of constructing  $\tilde{K}$  in §4 by proceeding simultaneously on  $j$ .

**Step B.** We now claim that

**Lemma 5.6.** *Let  $W_0$  be the hyperplane perpendicular to  $\phi$ . Then*

$$A(f_1, f_2, \dots, f_n) \leq A(f_1^*(\cdot|W_0), f_2^*(\cdot|W_0), \dots, f_n^*(\cdot|W_0)).$$

Let  $x_j = (t_j, z_j) \in \mathbf{R} \times \mathbf{R}^{n-1}$ . As the preceding arguments, formula (5-1) becomes

$$\begin{aligned} & A(f_1, f_2, \dots, f_n) \\ &= \iint \left( \int \prod_{j=1}^n f_j(t_j, z_j) dt_1 dt_0 d\alpha(z_2) \cdot d\alpha(z_n) |z_1|^{1-n} \right) dz_1. \end{aligned}$$

Therefore, to prove Lemma 5.6, it suffices to show that

$$\left( \prod_{j=2}^n f_j(t_j, z_j) dt_1 dt_0 \leq \int \prod_{j=2}^n f_j^*((t_j, z_j)|W_0) \right) dt_1 dt_0,$$

that is, for  $R$  large enough,

$$\begin{aligned} & \int f_0(t_0, 0) f_1(t_1, z_1) \prod_{j=2}^n f_j(t_j, z_j) dt_1 dt_0 \\ (5-3) \quad & \leq \int f_0((t_0, 0)|W_0) f_1((t_1, z_1)|W_0) \prod_{j=2}^n f_j^*((t_j, z_j)|W_0) dt_1 dt_0, \end{aligned}$$

where, both  $f_0$  and  $f_1$  are the characteristic function of the ball centered at the origin with radius  $R$ .

To finish the proof of inequality (5-3), recall the notion of Steiner convexity. Let  $d, k$  be positive integers and  $x_1, \dots, x_d \in \mathbf{R}^k$ . A subset  $E \subset (\mathbf{R}^k)^d$  is said to be Steiner convex if for every orthonormal basis  $(\mu_1, \dots, \mu_k)$  of  $\mathbf{R}^k$  and every  $t \in (\mathbf{R}^{k-1})^d$ , the subset

$$\{(x_1, \dots, x_d) \in E: (\langle x_1, \mu_1 \rangle, \dots, \langle x_1, \mu_{k-1} \rangle), \dots, (\langle x_d, \mu_1 \rangle, \dots, \langle x_d, \mu_{k-1} \rangle) = t\}$$

is convex and balanced in the sense that it is Steiner symmetric with respect to reflection to the hyperplane  $\langle \mu_1, \dots, \mu_{k-1} \rangle$ .

Moreover, a real-valued function  $K$  on  $(\mathbf{R}^k)^d$  is said to be Steiner convex if the level sets  $\{x: K(x) > \lambda\}$  is Steiner convex for all  $\lambda \in \mathbf{R}$ .

We now state a variant of Brascamp, Lieb and Luttinger's rearrangement inequality, which is due to Christ [6].

**Lemma 5.7.** *Suppose that  $l, d, k$  are positive integer,  $x = (x_1, \dots, x_d) \in (\mathbf{R}^k)^d$  and  $K_1, \dots, K_l$  are Steiner convex functions on  $(\mathbf{R}^k)^d$ . Then for any nonnegative functions  $f_1, \dots, f_d$  on  $\mathbf{R}^k$ ,*

$$\int \prod_{j=1}^d f_j(x_j) \prod_{i=1}^l K_i(x) dx_1 \cdots dx_d \leq \int \prod_{j=1}^d f_j^*(x_j) \prod_{i=1}^l K_i(x) dx_1 \cdots dx_d.$$

Now return to the proof of inequality (5-3). Let  $l_j$  be a linear function of  $t_j, t_0, t_1$  with coefficients depending on  $z_j$  for  $2 \leq j \leq n$  such that  $x_j \in \pi(x_0, x_1)$

if and only if  $l_j(t_j, t_0, t_1) = 0$ . Then, by Lemma 5.7, one sees that

$$\begin{aligned} & \int f_0(t_0, 0) f_1(t_1, z_1) \prod_{j=2}^n f_j(t_j, z_j) dt_1 dt_0 \\ &= \lim_{\varepsilon \rightarrow 0} \int f_0(t_0, 0) f_1(t_1, z_1) \prod_{j=2}^n f_j(t_j, z_j) \varepsilon^{1-n} \\ & \quad \times \prod_{i=2}^n \chi_{|l_i(t_i, t_0, t_1)| \leq \varepsilon}(t_i) dt_0 \cdots dt_n \\ &= \int f_0(t_0, 0) f_1(t_1, z_1) \prod_{j=2}^n f_j^*((t_j, z_j) | W_0) dt_1 dt_0. \end{aligned}$$

This completes the proof of (5-3) and thus of Lemma 5.6.

We now state the main result for Step B in terms of Theorem 5.5 and Lemma 5.6.

**Theorem 5.8.** *Let  $f_j \equiv \chi_{B_j}$  and  $\tilde{f}_j^\phi \equiv \chi_{R_\phi(B_j)}$ . Then*

$$A(f_1, f_2, \dots, f_n) \leq A(\tilde{f}_1^\phi, \tilde{f}_2^\phi, \dots, \tilde{f}_n^\phi).$$

The process can be described intuitively as follows: Assume each  $f_i$  is the characteristic function of a bounded measurable set  $B_j$ . Then, following from Theorem 5.8,  $B_j$  is suitably transformed by the regularize operator into a regular set which is more easily handled.

*Step C.* We now apply the process of normalization discussed in §4 to handle the regular sets.

Let  $\varphi$  be any line passing through the origin. Define

$$\begin{aligned} & A_\varphi(f_1, f_2, \dots, f_n) \\ &= \iint \left( \int \left( \int \prod_{j=1}^n f_j(z_j, t_j) dt_1 \right) d\alpha(z_2) \cdot d\alpha(z_n) \right) \\ & \quad \times |z_1 - z_0|^{1-n} dz_1 d\lambda(x_0) \\ (5.4) \quad &= \iint \left( \int \left( \int \prod_{j=2}^n f_j(z_j, t_j) d\alpha(z_2) \cdots d\alpha(z_n) \right) f_1(z_1, t_1) dt_1 \right) \\ & \quad \times |z_1 - z_0|^{1-n} dz_1 d\lambda(x_0), \end{aligned}$$

where  $d\lambda(x_0)$  is the measure on  $\varphi$ . Then, by a rigid motion, we have the analogous inequality:

$$(5-5) \quad A_\varphi(f_1, f_2, \dots, f_n) \leq A_\varphi(\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n),$$

where  $f_j = \chi_{B_j}$  and  $\tilde{f}_j = \chi_{R_\varphi(B_j)}$ .

Equality (5-5) shows that we may reduce a bounded measurable set to a regular set. Furthermore, we may transform a regular set into a finite cylinder. The details are as follows.

First note that the operator  $A_\varphi(f_1, f_2, \dots, f_n)$  is multilinear. Now regard  $R_\varphi(B_j)$  as  $K_j$  in Definition 4.5. Let

$$E_j^+ = d(K_j) \cap H_\varphi^+, \quad F_j^+ = \varphi(-\hat{t}_j) + d(K_j) \cap H_\varphi^+,$$

and

$$E_j^- = d(K_j) \cap H_\varphi^-, \quad F_j^- = \varphi(\hat{t}_j) + d(K_j) \cap H_\varphi^-.$$

Then

$$A_\varphi(\chi_{E_1^\pm}, \chi_{E_2^\pm}, \dots, \chi_{E_n^\pm}) = A_\varphi(\chi_{F_1^\pm}, \chi_{F_2^\pm}, \dots, \chi_{F_n^\pm})$$

because  $A_\varphi$  is translation-invariant along the  $\varphi$ -axis and thus

$$A_\varphi(\chi_{d(K_1)}, \chi_{d(K_2)}, \dots, \chi_{d(K_n)}) = A_\varphi(\chi_{\hat{d}(K_1)}, \chi_{\hat{d}(K_2)}, \dots, \chi_{\hat{d}(K_n)})$$

because

$$\chi_{\hat{d}(K_j)} = \chi_{F_j^+} + \chi_{F_j^-}.$$

We then conclude that

$$A_\varphi(\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n) \leq A_\varphi(\hat{f}_1, \hat{f}_2, \dots, \hat{f}_n),$$

where  $\hat{f}_j = \chi_{G_\varphi(K_j)}$ . Moreover, we have

$$A(\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n) \leq A(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n),$$

where  $\bar{f}_j = \chi_{C_\varphi(K_j)}$ .

Repeating this process, we then transform a bounded measurable set into a finite cylinder

$$A_\varphi(f_1, f_2, \dots, f_n) \leq A_\varphi(\check{f}_1, \check{f}_2, \dots, \check{f}_n),$$

where  $\check{f}_j = \chi_{T_\varphi(K_j)}$ .

Now define

$$N\check{f}_j \equiv \chi_{N_\varphi \circ T_\varphi(K_j)},$$

where  $\varphi$  is a nondegenerate line which is chosen to satisfy the following inequalities:

$$\begin{aligned} (1 + \frac{1}{4})\|\varphi(\hat{t}_j(N_\varphi \circ T_\varphi(K_j)))\| \\ \leq \|\varphi(\hat{t}_j(T_\varphi(K_j)))\| \leq (1 + \frac{1}{2})\|\varphi(\hat{t}_j(N_\varphi \circ T_\varphi(K_j)))\| \end{aligned}$$

for each  $j = 1, 2, \dots, n$ . By a geometrical argument, it is not difficult to show that, for fixed  $x_1$  and  $x_0$

$$\int \check{f}_j(z_j, t_j) d\alpha(z_j) \leq (1 + \frac{1}{2}) \int \check{f}_j(z_j, t_j) d\alpha(z_j)$$

where  $S\check{f}_j = \chi_{S_E \circ T_\varphi(K_j)}$  for  $j = 1, 2, \dots, n$ . It then follows from equality (5-4) that

$$A_\varphi(\check{f}_1, \check{f}_2, \dots, \check{f}_n) \leq (1 + \frac{1}{2})A_\varphi(S\check{f}_1, S\check{f}_2, \dots, S\check{f}_n).$$

Furthermore, by using the regularizer operator, we have

$$A_\varphi(\check{f}_1, \check{f}_2, \dots, \check{f}_n) \leq (1 + \frac{1}{2})A_\varphi(N\check{f}_1, N\check{f}_2, \dots, N\check{f}_n).$$

We now start with  $K_j^0 \equiv R_\varphi \circ S_E \circ T_\varphi(K_j)$  for the process of normalization. Note that  $K_j^0$  must be a convex body and so are  $K_j^i \equiv N_\varphi^i(K_j^0)$  for  $i = 1, 2, 3, \dots$ . It is not difficult to show that

$$A_\varphi(N\check{f}_1, N\check{f}_2, \dots, N\check{f}_n) \leq (1 + \frac{1}{4})A_\varphi(Nf_1^0, Nf_2^0, \dots, Nf_n^0),$$

where  $Nf_j^0 \equiv \chi_{N\varphi(K^0)}$ . Repeat this process of normalization, we conclude from §3.4 that

$$(5-6) \quad A_\varphi(f_1, f_2, \dots, f_n) \leq \prod_{i=1}^\infty (1 + 2^{-i}) A_\varphi(f_1^*, f_2^*, \dots, f_n^*).$$

We now state the main result for Step C.

**Theorem 5.9.** For  $n \geq 3$ ,

$$A(f_1, f_2, \dots, f_n) \leq CA(f_1^*, f_2^*, \dots, f_n^*),$$

where  $f_1$  is the characteristic function of the ball centered at the origin with radius large enough and  $f_2, \dots, f_n \in C_0(\mathbf{R}^n)$ .

*Proof.* First note that

$$A(f_1, f_2, \dots, f_n) = A_\varphi(\check{f}_1, \check{f}_2, \dots, \check{f}_n),$$

where  $\check{f}_j$  is obtained from  $f_j$  by the rigid motion such that  $\phi$  is transformed onto  $\varphi$ . The result follows from (5-6) and the fact that

$$A_\varphi(\check{f}_1^*, \check{f}_2^*, \dots, \check{f}_n^*) = A(f_1^*, f_2^*, \dots, f_n^*).$$

### 6. $L^p$ ESTIMATES FOR THE X-RAY TRANSFORM ON RADIAL DECREASING FUNCTIONS VIA $d\mu$

Having already given the rearrangement inequality for the X-ray transform associated with the measure  $d\mu$  on  $M_\varphi(\mathbf{R}^n)$ , we need only to deal with the  $L^p$  estimates on the class of all radial decreasing functions for the restricted X-ray transform. The method we use is heavily dependent on an inequality due to Drury [8].

**Theorem 6.1.** Let  $n > 2$ . Then

$$\|R_{1,n}f^*\|_{L^{n-1}(M_\varphi(\mathbf{R}^n), d\mu)} \leq C\|f^*\|_{L^{n-1}(\mathbf{R})},$$

where  $f \in C_0(\mathbf{R}^n)$ .

Before proving this result, we need some facts. Let us define, for  $x \in \mathbf{R}^n$ ,  $1 \leq a < n$  and  $g \in C_0(\mathbf{R}^n)$ ,

$$S_a g(x) = \left\{ \int |R_{1,n}g(\pi)|^a d\mu_x(\pi) \right\}^{1/a}.$$

Then, for  $R > 0$ , we have the following inequality which is due to Drury [8].

$$(6-1) \quad S_a g(x) \leq R^{1/a} \{|g|^a * \theta_R(x)\}^{1/a} + R^{-(n-a)/a} \|g\|_{L^a(\mathbf{R}^n)},$$

where  $\theta_R(z) = |z|^{1-n}$  if  $|z| < R$ ;  $\theta_R(z) = 0$  otherwise. By definition, we have

$$\begin{aligned} \|R_{1,n}g\|_{L^{n-1}(M_\varphi(\mathbf{R}^n), d\mu)}^{n-1} &= \iint [R_{1,n}g(\pi)]^{n-1} d\mu_x(\pi) d\lambda(x) \\ &= \int [S_{n-1}g(x)]^{n-1} d\lambda(x). \end{aligned}$$

Moreover, decompose

$$S_{n-1}g(x) \equiv S'_{n-1}g(x) + S''_{n-1}g(x),$$

where

$$S'_{n-1}g = S_{n-1}g \cdot \chi_{\{|x| \leq 1\}} \quad \text{and} \quad S''_{n-1}g = S_{n-1}g \cdot \chi_{\{|x| > 1\}}.$$

Then, by Minkowski inequality,

$$(6-2) \quad \|R_{1,n}g\|_{L^{n-1}(M_\phi(\mathbf{R}^n), d\mu)} \leq \|S'_{n-1}g\|_{L^{n-1}(\phi, d\lambda)} + \|S''_{n-1}g\|_{L^{n-1}(\phi, d\lambda)}.$$

We establish some lemmas to prove Theorem 6.1.

**Lemma 6.2.** *Let  $a \geq 1$ . Then*

$$\|S_a g\|_{L^\infty(\phi, d\lambda)} \leq C \|g\|_{L^n(\mathbf{R}^n)}$$

for all radial decreasing function  $g \in C_0(\mathbf{R}^n)$ .

*Proof.* First note that  $\|S_a g\|_{L^\infty(\phi, d\lambda)} = S_a g(0)$ . By the given condition, we have  $S_a g(0) = S_b g(0)$  for any  $b \geq 1$ . In particular,  $S_a g(0) = S_n g(0)$  and thus

$$\|S_a g\|_{L^\infty(\phi, d\lambda)} = \|S_n g\|_{L^\infty(\phi, d\lambda)} \leq R^{(n-1)/n} \{g^n * \theta_R(0)\}^{1/n} + \|g\|_{L^n(\mathbf{R}^n)}$$

for  $R > 0$  by inequality (6-1). The result follows by letting  $R \rightarrow 0$ .

**Lemma 6.3.** *Let  $n > 2$ . Then*

$$\|S'_{n-1}g\|_{L^{n-1}(\phi, d\lambda)} \leq C \|g\|_{L^{n-1}(\mathbf{R}^n)}$$

for all radial decreasing function  $g \in C_0(\mathbf{R}^n)$ .

*Proof.* By the well-known property of the distribution, we have

$$\|S'_{n-1}g\|_{L^{n-1}(\phi, d\lambda)}^{n-1} = (n-1) \int_0^\infty \tau^{n-2} \alpha'(\tau) d\tau,$$

where

$$\alpha'(\tau) \equiv \lambda\{x : S'_{n-1}g(x) > \tau\}.$$

Now let

$$E'_{\tau/2} \equiv \{x : R^{(n-2)/(n-1)} \{g^{n-1} * \theta_R(x)\}^{1/(n-1)} > \frac{\tau}{2} \text{ and } |x| \leq 1\}$$

and  $\alpha'_R(\tau) \equiv \lambda(E'_{\tau/2})$ . Then if we assume that, for  $a < (n-1)/n$ ,

$$(n-1-na)^{1/(na-a)} R^{-1/(n-1)} \|g\|_{L^{n-1}(\mathbf{R}^n)} = \frac{\tau}{2},$$

we have

$$\begin{aligned} \alpha'(\tau) &\leq \alpha'_R(\tau) \\ &\leq \left(\frac{\tau}{2}\right)^{-a} \int_{E'_{\tau/2}} R^{(n-2)/(n-1)} \{g^{n-1} * \theta_R(x)\}^{a/(n-1)} d\lambda(x) \\ &= \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \int_0^1 \{g^{n-1} * \theta_R(re_1)\}^{a/(n-1)} r^a r^{-a} dr \\ &\leq C \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \left\{ \int_0^1 \{g^{n-1} * \theta_R(re_1) r^{n-1}\} dr \right\}^{a/(n-1)} \\ &= C \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \|g^{n-1} * \theta_R\|_{L^1(\mathbf{R}^n)}^{a/(n-1)} \\ &\leq C \left(\frac{\tau}{2}\right)^{-a} R^a \|g\|_{L^{n-1}(\mathbf{R}^n)}^a \\ &\leq C \tau^{-na} \|g\|_{L^{n-1}(\mathbf{R}^n)}^{na}. \end{aligned}$$

Therefore, by Lemma 6.2,

$$\|S'_{n-1}g\|_{L^{n-1}(\phi, d\lambda)}^{n-1} \leq C \|g\|_{L^n(\mathbf{R}^n)}^{n-1-na} \|g\|_{L^{n-1}(\mathbf{R}^n)}^{na}.$$

The result follows by letting  $a \rightarrow (n-1)/n$ .

**Lemma 6.4.** *Let  $n > 2$ . Then*

$$\|S''_{n-1}g\|_{L^{n-1}(\phi, d\lambda)} \leq C \|g\|_{L^{n-1}(\mathbf{R}^n)}$$

for all radial decreasing function  $g \in C_0(\mathbf{R}^n)$ .

*Proof.* As before, we have

$$\|S''_{n-1}g\|_{L^{n-1}(\phi, d\lambda)}^{n-1} = (n-1) \int_0^\infty \tau^{n-2} \alpha''(\tau) d\tau,$$

where

$$\alpha''(\tau) \equiv \lambda\{x: S''_{n-1}g(x) > \tau\}.$$

Now let

$$E''_{\tau/2} \equiv \{x: R^{(n-2)/(n-1)}\{g^{n-1} * \theta_R(x)\}^{1/(n-1)} > \frac{\tau}{2} \text{ and } |x| > 1\}$$

and  $\alpha''_R(\tau) \equiv \lambda(E''_{\tau/2})$ . Then if we assume that, for  $a > (n-1)/n$ ,

$$(n-1-na)^{1/(nab-ab)} R^{-1/(n-1)} \|g\|_{L^{n-1}(\mathbf{R}^n)} \leq \frac{\tau}{2}, \quad 0 < b < 1,$$

we may let

$$(n-1-na)^{1/(nab-ab)} R^{-1/b(n-1)} \|g\|_{L^{n-1}(\mathbf{R}^n)} = \frac{\tau}{2}$$

that is,

$$R = (n-1-na)^{1/a} \frac{\tau^{b(n-1)}}{2} \|g\|_{L^{n-1}(\mathbf{R}^n)}^{b(n-1)}.$$

and we have

$$\begin{aligned} \alpha''(\tau) &\leq \alpha''_R(\tau) \\ &\leq \left(\frac{\tau}{2}\right)^{-a} \int_{E''_{\tau/2}} R^{(n-2)/(n-1)} \{g^{n-1} * \theta_R(x)\}^{a/(n-1)} d\lambda(x) \\ &= \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \int_1^\infty \{g^{n-1} * \theta_R(re_1)\}^{a/(n-1)} r^a r^{-a} dr \\ &\leq C \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \left\{ \int_1^\infty \{g^{n-1} * \theta_R(re_1)\}^{n-1} dr \right\}^{a/(n-1)} \\ &= C \left(\frac{\tau}{2}\right)^{-a} R^{(n-2)a/(n-1)} \|g^{n-1} * \theta_R\|_{L^1(\mathbf{R}^n)}^{a/(n-1)} \\ &\leq C \left(\frac{\tau}{2}\right)^{-a} R^a \|g\|_{L^{n-1}(\mathbf{R}^n)}^a \\ &\leq C \tau^{-a} \tau^{(1-n)ba} \|g\|_{L^{n-1}(\mathbf{R}^n)}^a \|g\|_{L^{n-1}(\mathbf{R}^n)}^{(n-1)ba}. \end{aligned}$$

Therefore, by Lemma 6.2,

$$\|S''_{n-1}G\|_{L^{n-1}(\phi, d\lambda)}^{n-1} \leq C \|g\|_{L^n(\mathbf{R}^n)}^{n-1-a+(1-n)ba} \|g\|_{L^{n-1}(\mathbf{R}^n)}^a \|g\|_{L^{n-1}(\mathbf{R}^n)}^{(n-1)ba}$$

because, given  $0 < b < 1$ , there are  $a$ 's such that  $a > (n-1)/n$  and  $a < (n-1)/(1+(n-1)b)$ .

The result follows by letting  $a \rightarrow (n-1)/n$  and  $b \rightarrow 1$ .

We thus complete the proof of Theorem 6.1 in terms of inequality (6-2).

7. PROOF OF THE THEOREM

We are now ready to give the proof of our main result.

*Proof of necessity.* (1) By homogeneity, let  $f$  be the characteristic function of the ball  $B(0; r)$  and let  $\phi_r = \lambda(B(0; r) \cap \phi)$ . Then  $\|f\|_{L^p(\mathbb{R}^n)} \simeq r^{np^{-1}}$  and  $\|R_{1,n}f\|_{L^q(M_\phi(\mathbb{R}^n), d\mu)} \geq r\phi_r^{q^{-1}}$ . Therefore, inequality (1-1) forces that, for all  $r > 0$ ,  $\phi_r^{q^{-1}} \leq Cr^{np^{-1}-1}$ . Without loss of generality, let  $\phi(0) = 0$ . Then  $\phi_r \geq r$  for all  $r > 0$  and thus we have  $q^{-1} = np^{-1} - 1$ .

(2) Moreover, assume that  $\phi_r \neq O(r)$ , that is,  $\limsup_{r \rightarrow \infty} r^{-1}\phi_r = \infty$  a contradiction holds because

$$[r^{-1}\phi_r]^{q^{-1}} \leq Cr^{np^{-1}-q^{-1}-1}.$$

Thus, we have  $\phi_r = O(r)$ , that is,  $\phi$  is line-like.

(3) Therefore, by Theorem 3.4,  $\phi$  is linearly covered by a sequence of finite right circular cylinder with bounded radii and unbounded heights. Now we claim that  $p = q = n - 1$ . Without loss of generality, assume that  $\phi$  is linearly covered by a sequence  $\{C_k(\phi)\}$  of finite right circular cylinder with unbounded heights  $H_k$  and constant radii one. Let  $f_k$  be the characteristic function  $C_k(\phi)$ . Then

$$\|R_{1,n}f_k\|_{L^q(M_\phi(\mathbb{R}^n), d\mu)} \geq H_k^{1/q} + H_k^{(2-n+q)/q}$$

and  $\|f_k\|_{L^p(\mathbb{R}^n)} \simeq H_k^{1/p}$ . Hence, inequality (1.1) yields that  $\frac{1}{p} \geq \frac{1}{q}$  and  $\frac{1}{p} \geq \frac{q+2-n}{q} = 1 + \frac{1}{q} + \frac{1-n}{q} = \frac{n}{p} + \frac{1-n}{q}$ . This verifies that  $p = q = n - 1$ .

We now turn to prove the converse part of the theorem, having already established this in the case when  $\phi$  is a straight line.

*Proof of sufficiently.* Let  $V_k = \langle e_1, \dots, e_k, 0, \dots, 0 \rangle$  and  $\phi_n = \phi$ . Define by induction  $\phi_k$  be the natural projection  $P_k$  of  $\phi_{k+1}$  into  $V_k$  for  $1 \leq k \leq n - 1$ . Then, by Theorem 3.1 and a finite steps of rigid motions, we may assume that the standard coordinate system  $\{e_1, e_2, \dots, e_n\}$  guarantees that each  $\phi_k$  is line-like. Therefore, it suffices to show that

$$\|R_{1,n}f\|_{L^{n-1}(M_{\phi_{k+1}}(\mathbb{R}^n), d\mu)} \leq C\|R_{1,n}f^*(\cdot|\langle e_{k+1} \rangle^\perp)\|_{L^{n-1}(M_{\phi_k}(\mathbb{R}^n), d\mu)}.$$

Since each  $\phi_k$  is line-like, we need only to show that

$$\int [R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) \leq \int [R_{1,n}f^*(\cdot|\langle e_{k+1} \rangle^\perp)(\pi)]^{n-1} d\mu_{P_k(x)}(\pi)$$

for all  $x \in \phi_{k+1}$ . That is, by a perpendicular translation,

$$\int [R_{1,n}f(\pi)]^{n-1} d\mu_x(\pi) \leq \int [R_{1,n}f^*(\cdot|x + \langle e_{k+1} \rangle^\perp)(\pi)]^{n-1} d\mu_x(\pi).$$

Without loss of generality, we show that, for  $k = n - 1$ ,

$$(7-1) \quad \int [R_{1,n}f(\pi)]^{n-1} d\mu_0(\pi) \leq \int [R_{1,n}f^*(\cdot|\langle e_n \rangle^\perp)(\pi)]^{n-1} d\mu_0(\pi).$$

Let

$$B(f_1, \dots, f_n) = \int f_1(x_1) \prod_{j=2}^n \int_{\pi(x_1, 0)} f_j(x_j) d\lambda_\pi(x_j) |x_1|^{1-n} dx_1,$$

where  $f_1 \equiv \chi_{B(0, R)}$ .

We claim that

$$(7-2) \quad B(f_1, \dots, f_n) \leq B(f_1^*(\cdot|e_n)^\perp), \dots, f_n^*(\cdot|e_n)^\perp)$$

and then verify (7-1) for  $R$  large enough. In fact, following the derivations and notations of equality (5-1), we have, for  $R$  large enough,

$$B(f_1, \dots, f_2) = \int \left( \int \left( \int \prod_{j=1}^n f_j(z_j, t_j) dt_1 \right) d\alpha(z_2) \cdots d\alpha(z_n) \right) |z_1|^{1-n} dz_1.$$

Thus, as the same arguments in Step A of §5, it suffices to show that

$$\int \prod_{j=2}^n f_j(z_j, t_j) dt_1 \leq \int \prod_{j=2}^n f_j^*((z_j, t_j)|e_n)^\perp dt_1.$$

This inequality readily follows because it is really the same one as inequality (5-2). Therefore, inequality (7-2) is established and so is inequality (7-1). In other words, we have reduced the proof from the case when  $\phi$  is line-like into the case when  $\phi$  is a straight line. This completes the proof of our main theorem.

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