

THE STRUCTURE OF SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION

KUO-SHUNG CHENG AND TAI-CHIA LIN

ABSTRACT. We give a complete classification of solutions of the elliptic equation $\Delta u + K(x)e^{2u} = 0$ in \mathbb{R}^n , $n \geq 3$, for some interesting cases of K .

1. INTRODUCTION

In this paper we shall investigate the structure of the set of all entire solutions of the equation

$$(1.1) \quad \Delta u + Ke^{2u} = 0$$

in \mathbb{R}^n , $n \geq 3$, where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ and $K \neq 0$ is a given locally Hölder continuous function on \mathbb{R}^n . We shall consider mainly two types of solutions: $u_c(x)$ denotes a solution of (1.1) satisfying

$$(1.2) \quad u_c(x) \rightarrow c \quad \text{as } |x| \rightarrow \infty$$

and $U(x)$ denotes the "maximal solution"

$$(1.3) \quad U(x) = \sup\{u(x) \mid u \text{ is a solution of (1.1)}\}.$$

In this paper we shall consider mainly three questions:

- (a) When do solutions u_c and U exist?
- (b) What are the relations between K and U ?
- (c) When do we have $u = u_c$ or $u = U$ for every solution u of (1.1)?

Equation (1.1) arises from physics and geometry. When $n = 3$ and $K(x) = 1/1 + |x|^2$, (1.1) is known as Eddington's equation, which was used to model globular clusters of stars. When $n = 2$, (1.1) is known as the conformal Gaussian curvature equation on \mathbb{R}^2 . Let (M, g) be a Riemannian manifold of dimension 2 and K be a given function on M . The following question has been raised: can we find a new metric g_1 on M such that K is a Gaussian curvature of g_1 and g_1 is conformal to g (that is, $g_1 = \varphi g$ for some positive function φ on M)? If we write $\varphi = e^{2u}$, then this is equivalent to the problem of solving the elliptic equation

$$(1.4) \quad \Delta_g u - k + Ke^{2u} = 0$$

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on M , where Δ_g and k are the Laplace-Betrami operator and the Gaussian curvature on M in the g -metric respectively. In case M is compact, equation (1.4) has been considered by many authors and we refer the reader to the monograph by Kazdan [K] for details and references. In case M is complete and noncompact, the first natural case seems to be $M = \mathbb{R}^2$ and g is the usual Euclidean metric. In this case, equation (1.4) reduces to equation (1.1) with $n = 2$. We refer the reader to Cheng and Ni [CN1] for details and references about equation (1.1) when $n = 2$.

The first nonexistence result concerning equation (1.1) seems due to Oleinik [O]. She proved

Theorem A. *Suppose that $K \leq 0$ on \mathbb{R}^n and $|K(x)| \geq C|x|^{-p}$ for $|x|$ large for some constant $p < 2$ and some constant $C > 0$. Then (1.1) possesses no entire solution.*

The first existence result concerning equation (1.1) seems due to Ni [N1]. He obtained

Theorem B. *Suppose that $|K(x)| \leq C|x_1|^{-p}$ for $|x_1|$ large and uniformly in x_2 for some $p > 2$, where $x = (x_1, x_2)$ and $x_1 \in \mathbb{R}^m$, $x_2 \in \mathbb{R}^{n-m}$, $m \geq 3$. Then (1.1) possesses infinitely many bounded solutions.*

Later on, Kusano and Oharu [KO] slightly extended Ni's result. They obtained

Theorem C. *Suppose that $|K(x)| \leq \tilde{K}(|x_1|)$ on \mathbb{R}^n and $\int_0^\infty t\tilde{K}(t) dt < \infty$, where $x = (x_1, x_2)$ is as in Theorem B. Then (1.1) possesses infinitely many bounded solutions.*

For a nonexistence result, Cheng and Lin [CL, Theorem 5.3] improved Oleinik's result. They proved

Theorem D. *Suppose that $K \leq 0$ on \mathbb{R}^n and $|K(x)| \geq C|x|^{-2}(\log|x|)^{-1}$ for $|x|$ large for some constant $C > 0$. Then (1.1) possesses no entire solution.*

From the results of Theorems C and D, it seems natural to conjecture that if $K \leq 0$ on \mathbb{R}^n , $|K(x)| \geq \tilde{K}(|x|)$ and $\int_0^\infty t\tilde{K}(t) dt = \infty$, then (1.1) possesses no entire solution. Interestingly enough, this conjecture is not true. Lin and Cheng [LC] constructed a solution for equation (1.1) with $K = K(|x|) \leq 0$ satisfying

$$\int_0^\infty tK(t) dt = -\infty.$$

If we try to investigate the structure of the set of solutions for equation (1.1), then Theorems B and C are not enough. Thus along this direction, we try first to find the type of solution u_c satisfying (1.2). The following theorem answers parts of the existence question of the type of solution u_c .

Theorem 1.1. *Suppose that K satisfies one of the following conditions:*

- (i) $|x| \leq \tilde{K}(|x|)$ on \mathbb{R}^n and $\int_0^\infty t\tilde{K}(t) dt < \infty$,
- (ii) $\int_{\mathbb{R}^n} |K(y)|/|y|^{n-2} dy < \infty$ and

$$M(x)|x|^2(g(x))^{2/(n-2)} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where

$$(1.5) \quad M(x) = \sup_{|y-x| < |x|/2} |K(y)|$$

and

$$(1.6) \quad g(x) = \int_{|y-x| < |x|/2} \frac{|K(y)|}{|y|^{n-2}} dy.$$

Then we have

- (α) There exists a constant $c_0 \in \mathbb{R}$, such that for every $c \in (-\infty, c_0]$, (1.1) possesses a solution u_c satisfying (1.2).
- (β) If, in addition, $K \leq 0$ on \mathbb{R}^n , then for every $c \in (-\infty, \infty)$, (1.1) possesses a unique solution u_c satisfying (1.2).
- (γ) If, in addition, $K \geq 0$ on \mathbb{R}^n , then there exists a constant $c_0 \in \mathbb{R}$, such that for every $c \in (-\infty, c_0)$, (1.1) possesses a solution u_c satisfying (1.2) and (1.1) possesses no solution u_c satisfying (1.2) for $c > c_0$.
- (δ) Let u be a bounded solution of (1.1). Then there exists a constant $c \in \mathbb{R}$ such that $u = u_c$, that is, $u(x) \rightarrow c$ as $|x| \rightarrow \infty$.

A few words about this long theorem are in order. First of all, we do not divide \mathbb{R}^n into $\mathbb{R}^m \times \mathbb{R}^{n-m}$ as in Theorems B and C. But from the proof of Theorem (1.1) in §3 it is easy to see that some of the results can be formulated in this form. Next, we note that condition (i) is the same as in Theorem C. But even in this case, our conclusions are much more clear and complete than that of Theorem C. Finally, we need to give some explanation about condition (ii). From a slightly deep inspection of condition (i), it would be very nice if the condition

$$(1.7) \quad \int_{\mathbb{R}^n} \frac{|K(y)|}{|y|^{n-2}} dy < \infty$$

could imply the existence of the type of solution u_c . Unfortunately, this is not necessarily true. Our proof can not be carried out if K concentrates on a sequence of points which tends to infinity. The other condition in (ii),

$$(1.8) \quad M(x)|x|^2(g(x))^{2/(n-2)} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

is a way to prevent such a concentration of K . On the other hand, in condition (i), since $|K(x)| \leq \tilde{K}(|x|)$, K can concentrate only on a sequence of spheres, not on a sequence of points. This “spread” of K is enough to guarantee the existence of solution u_c .

Are the solutions of (1.1) all of the type u_c ? The answer is in general no. For some K , there is another type of solution U , the “maximal solution.” The following theorem gives a fairly complete answer to the existence question of the maximal solution U of (1.1).

Theorem 1.2. *Suppose that*

- (i) $K \leq 0$ on \mathbb{R}^n and there exists a sequence of bounded smooth simply connected domains $\{\Omega_i\}$ such that $\mathbb{R}^n = \bigcup_{i=1}^{\infty} \Omega_i$, $\bar{\Omega}_i \subset \Omega_{i+1}$, and $K < 0$ on $\partial\Omega_i$, $i = 1, 2, \dots$, and
- (ii) equation (1.1) possess a solution, say v , on \mathbb{R}^n .

Then the function $U(x)$ (the maximal solution) defined by (1.3) is well defined on \mathbb{R}^n and is a solution of (1.1) on \mathbb{R}^n .

In view of Theorem 1.1, the conditions in Theorem 1.2 are quite mild. For example, if $K \leq 0$ on \mathbb{R}^n and $C/|x|^p \geq |K(x)| > 0$ for $|x|$ large for some constant $p > 2$, then condition (i) of Theorem 1.2 holds trivially and condition (ii) also holds due to Theorem 1.1. Is condition (i) necessary for the existence of the maximal solution U ? We do not know the answer to this question. But we conjecture that if $K(x) > 0$ somewhere or there exists a curve extending to infinity such that $K = 0$ on this curve, then the maximal solution U does not exist.

Since the type of solution u_c has nice asymptotic behavior (1.2), we are inclined to find the asymptotic behavior of the maximal solution U . If we can find some explicit relations between the functions K and U , then we can easily obtain the asymptotic behavior of U from that of K . Unfortunately, we obtain no such explicit relation.

In the following we use the notation " $f \sim g$ at ∞ " to denote that "there exist two positive constants C_1 and C_2 such that $C_1 f \geq g \geq C_2 f$ at ∞ ."

We shall consider the following possible asymptotics for K and U .

$$(1.9) \quad K(x) \sim -|x|^{-p}(\log|x|)^{-\beta} \quad \text{at } \infty, \quad p > 2, \beta \in \mathbb{R},$$

$$(1.10) \quad K(x) \sim -|x|^{-2}(\log|x|)^{-\beta} \quad \text{at } \infty, \quad \beta > 1,$$

$$(1.11) \quad K(x) \sim -|x|^{-q}(\log|x|)^{p-k} \exp[-2(\log|x|)^p] \quad \text{at } \infty, \\ p > 1, q \in \mathbb{R}, k \in \mathbb{R},$$

$$(1.12) \quad K(x) \sim -|x|^{\alpha-p} \exp(-2|x|^\alpha) \quad \text{at } \infty, \quad \alpha > 0, p \in \mathbb{R},$$

$$(1.13) \quad U(x) = \frac{p-2}{2} \log|x| + \frac{\beta}{2} \log(\log|x|) + O(1) \quad \text{at } \infty,$$

$$(1.14) \quad U(x) = \frac{\beta-1}{2} \log(\log|x|) + O(1) \quad \text{at } \infty,$$

$$(1.15) \quad U(x) = (\log|x|)^p + \frac{p-2}{2} \log|x| \\ + \frac{k-1}{2} \log(\log|x|) + O(1) \quad \text{at } \infty,$$

$$(1.16) \quad U(x) = |x|^\alpha + \frac{p-2}{2} \log|x| + O(1) \quad \text{at } \infty.$$

The following theorem answers only parts of the question (b) in the beginning of this section.

Theorem 1.3. *Suppose that $K \leq 0$ on \mathbb{R}^n and has one of the asymptotic behaviors (1.9), (1.10), (1.11) and (1.12). Then the maximal solution of U of (1.1) exists and has the asymptotic behaviors (1.13), (1.14), (1.15) and (1.16) respectively.*

Now we come to the question (c) in the beginning of this section. In other words, can we classify all the solutions of equation (1.1)? If $K(x) > 0$ somewhere, then we do not know whether the type of solution u_c is unique or not for a given c and whether the maximal solution U exists or not. Thus if $K(x) > 0$

somewhere, we are still very far from the complete classification of all solutions for equation (1.1).

If $K \leq 0$ on \mathbb{R}^n , for some K , we can find other types of solutions besides the two types u_c and U . For example, let $K(x) = -2ne^{-2|x|^2}$. Let $h(x)$ be any harmonic function satisfying $h(x) \leq \frac{1}{2}|x|^2$ for all $x \in \mathbb{R}^n$. Let $u = h + w$. Then w satisfies

$$(1.17) \quad \Delta w + K(x)e^{2h(x)}e^{2w} = 0.$$

Since $K(x)e^{2h(x)}$ satisfies any one of the two conditions in Theorem 1.1, we conclude that there exists a unique solution w_c of (1.17) satisfying $w_c(x) \rightarrow c$ as $|x| \rightarrow \infty$ for every real c . Thus for this K , we have a family of solutions $u = h + w_c$, $c \in \mathbb{R}$.

If K has some symmetry properties, for example, K is radially symmetric, from the above example, we can not expect that all solutions of (1.1) also have the same symmetry in general. But if we consider only symmetric solutions, then maybe we can classify all symmetric solutions.

Suppose now that $K \leq 0$ on \mathbb{R}^n , K is radially symmetric and K has one of the asymptotics (1.9), (1.10), (1.11) and (1.12). Then from Theorem 1.1, equation (1.1) possesses a unique solution u_c satisfying (1.2) for every $c \in \mathbb{R}$. Since K is radially symmetric, $u_c(Qx)$ is also a solution of (1.1) satisfying (1.2), where $Q \in O(n)$ is a rotation of \mathbb{R}^n . From uniqueness, we have $u_c(Qx) = u_c(x)$ for any $Q \in O(n)$. Hence u_c must be also radially symmetric. Also, from Theorem 1.2, the maximal solution U exists and U is radially symmetric by the same arguments as above. It is natural to ask: Are u_c and U the only radial solutions for equation (1.1)? The following theorem answers this question.

Theorem 1.4. *Suppose that $K \leq 0$ on \mathbb{R}^n , K is radially symmetric and K has one of the asymptotics (1.9), (1.10), (1.11) and (1.12). Let u be a radial solution of equation (1.1). Then either $u = u_c$ for some $c \in \mathbb{R}$ or $u = U$.*

Suppose now that $K \leq 0$ and K has one of the asymptotics (1.9), (1.10), (1.11) and (1.12). Then from Theorems 1.1, 1.2 and 1.3, we know that (1.1) possesses a unique solution u_c satisfying (1.2) for every $c \in \mathbb{R}$ and that the maximal solution U also exists and has the asymptotics (1.13), (1.14), (1.15) and (1.16) respectively. Are u_c and U the only solutions for (1.1)? For K satisfying (1.9) and (1.10), the answer is yes.

Theorem 1.5. *Suppose that $K \leq 0$ on \mathbb{R}^n and K has one of the asymptotics (1.9) and (1.10). Let u be a solution of (1.1). Then either $u = u_c$ for some $c \in \mathbb{R}$ or $u = U$.*

If K decays fast, for example, $K(x) = -2ne^{-2|x|^2}$, then as we already noted, equation (1.1) possesses another family of solutions $u = h + w_c$. But we know that if h is a nonconstant harmonic function on \mathbb{R}^n , then h has no lower bound. Thus if we only consider solutions which have a lower bound, then maybe u_c and U are again the only solutions. We have

Theorem 1.6. *Suppose that $K \leq 0$ on \mathbb{R}^n and K has one of the asymptotics (1.11) and (1.12). Let u be a solution of (1.1) with a lower bound. Then either $u = u_c$ for some $c \in \mathbb{R}$ or $u = U$.*

Finally, we shall consider the case K has compact support. The following theorem is interesting.

Theorem 1.7. *Suppose that $K \leq 0$ on \mathbb{R}^n and K has compact support. Then we have*

(i) *For every harmonic function h on \mathbb{R}^n and every constant $c \in \mathbb{R}$, (1.1) possesses a unique solution satisfying*

$$(1.18) \quad u(x) - h(x) - c \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

(ii) *Let u be a solution of (1.1). Then there exist a harmonic function h and a constant c such that u satisfies (1.18).*

Our main method of proofs is similar to that of Cheng and Ni [CN1, CN2]. For more background material, we refer the reader to Ni [N1, N2].

The organization of this paper is as follows. In §2, we shall sketch the proof of Theorem 1.2. We do not give a complete proof of Theorem 1.2 here because it is almost the same as in Cheng and Ni [CN1]. In §3, we shall give a detailed proof of Theorem 1.1. We shall prove Theorems 1.5, 1.6 and 1.7 in §5.

We would like to thank the referee for his constructive suggestion about the exposition of this paper.

2. EXISTENCE OF MAXIMAL SOLUTIONS

We refer the reader to Cheng and Ni [CN1] for details of the proof. We only sketch the proof here.

Sketch of the proof of Theorem 1.2. First we consider the following boundary value problem

$$(2.1) \quad \begin{cases} \Delta u + Ke^{2u} = 0 & \text{in } \Omega_i, \\ u(x) \rightarrow \infty & \text{as } x \rightarrow \partial\Omega_i. \end{cases}$$

We use barrier method to prove the existence of a solution u_i of (2.1) for each $i = 1, 2, \dots$. Here the assumptions (i) and (ii) of Theorem 1.2 are crucial. By the maximum principle, we prove that $\{u_i\}$ is a monotonically decreasing sequence of functions. Hence by usual elliptic regularity arguments, we prove that u_i converges monotonically to a solution U of (1.2) which is the maximal solution. Q.E.D.

We also list some important and useful properties of the maximal solutions here. We refer the reader to Cheng and Ni [CN1] for details of the proofs.

Theorem 2.1. *Suppose that the hypothesis of Theorem 1.2 hold and that $\tilde{K} \leq K \leq 0$ on \mathbb{R}^n . Let U (\tilde{U} respectively) be the maximal solution of $\Delta u + Ke^{2u} = 0$ ($\Delta u + \tilde{K}e^{2u} = 0$ respectively) on \mathbb{R}^n . Then $\tilde{U} \leq U$.*

Theorem 2.2. *Suppose that K is radially symmetric. Then the maximal solution U (if exists) is also radially symmetric.*

Theorem 2.3. *Suppose that $K \leq 0$ in \mathbb{R}^n and that U is the maximal solution of (1.1) on \mathbb{R}^n . Let u be an arbitrary solution of (1.1) on \mathbb{R}^n . Then either $u < U$ everywhere on \mathbb{R}^n or $u = U$ on \mathbb{R}^n .*

3. EXISTENCE OF SOLUTIONS OF THE TYPE u_c

In this section we shall give a detailed proof of Theorem 1.1. We decompose the proof into several lemmas.

Lemma 3.1. *Suppose that K satisfies the hypothesis (ii) of Theorem 1.1. Then the function v defined by*

$$(3.1) \quad v(x) = \frac{1}{2(2-n)w_n} \int_{\mathbb{R}^n} \frac{K(y)}{|x-y|^{n-2}} dy$$

is well defined on \mathbb{R}^n and satisfies

$$(3.2) \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

and

$$(3.3) \quad \Delta v(x) = K(x), \quad x \in \mathbb{R}^n,$$

where

$$w_n = \int_{|y|<1} dy.$$

Proof. We write

$$v(x) = C_n \int_{|y|\leq 2|x|} \frac{K(y)}{|x-y|^{n-2}} dy + C_n \int_{|y|\geq 2|x|} \frac{K(y)}{|x-y|^{n-2}} dy,$$

where $C_n = 1/n(2-n)w_n$. Now

$$\left| C_n \int_{|y|\geq 2|x|} \frac{|K(y)|}{|x-y|^{n-2}} dy \right| \leq |C_n| \int_{|y|\geq 2|x|} \frac{2^{n-2}|K(y)|}{|y|^{n-2}} dy.$$

Hence v is well defined on \mathbb{R}^n . Equation (3.3) is a consequence of (3.1) and the assumption that K is Hölder continuous. Thus it remains to prove (3.2). Let $\varepsilon > 0$ be given. Choose N and $R_\varepsilon > 2N$, such that

$$(3.4) \quad 4^{n-2} \int_{|y|\geq N} \frac{|K(y)|}{|y|^{n-2}} dy < \varepsilon,$$

$$(3.5) \quad \frac{(2N)^{n-2}}{|x|^{n-2}} \int_{\mathbb{R}^n} \frac{|K(y)|}{|y|^{n-2}} dy < \varepsilon \quad \text{for } |x| > R_\varepsilon,$$

$$(3.6) \quad \frac{n}{2} \left(\frac{3}{2}\right)^{2-4/n} w_n^{1-2/n} (M(x)|x|^2 g(x)^{2/(n-2)})^{(n-2)/n} < \varepsilon, \quad |x| > R_\varepsilon.$$

Then for $|x| > R_\varepsilon$, we have

$$\begin{aligned} \int \frac{|K(y)|}{|x-y|^{n-2}} dy &= \int_{|y|<|x|/2} + \int_{\substack{|x|/2<|y|<2|x| \\ |y-x|>|x|/2}} + \int_{|y-x|<|x|/2} + \int_{|y|>2|x|} \\ &= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned}$$

We now estimate I_1 , I_2 , I_3 and I_4 .

$$\begin{aligned}
 I_1(x) &= \int_{|y| < |x|/2} \frac{|K(y)|}{|x-y|^{n-2}} dy \\
 &= \left(\int_{|y| < N} + \int_{N < |y| < |x|/2} \right) \frac{|K(y)|}{|y-x|^{n-2}} dy \\
 &\leq \int_{|y| < N} \frac{2^{n-2}|K(y)|}{|x|^{n-2}} dy + \int_{N < |y| < |x|/2} \frac{2^{n-2}|K(y)|}{|x|^{n-2}} dy \\
 &\leq \frac{(2N)^{n-2}}{|x|^{n-2}} \int_{|y| < N} \frac{|K(y)|}{|y|^{n-2}} dy + \int_{N < |y| < |x|/2} \frac{|K(y)|}{|y|^{n-2}} dy \\
 &\leq \frac{(2N)^{n-2}}{|x|^{n-2}} \int_{\mathbb{R}^n} \frac{|K(y)|}{|y|^{n-2}} dy + \int_{|y| > N} \frac{|K(y)|}{|y|^{n-2}} dy \\
 &< 2\varepsilon \quad (\text{from (3.4) and (3.5)}).
 \end{aligned}$$

$$\begin{aligned}
 I_2(x) &= \int_{\substack{|x|/2 < |y| < 2|x| \\ |y-x| > |x|/2}} \frac{|K(y)|}{|x-y|^{n-2}} dy \\
 &\leq \int_{\substack{|x|/2 < |y| < 2|x| \\ |y-x| > |x|/2}} \frac{2^{n-2}|K(y)|}{|x|^{n-2}} dy \\
 &\leq 4^{n-2} \int_{\substack{|x|/2 < |y| < 2|x| \\ |y-x| > |x|/2}} \frac{|K(y)|}{|y|^{n-2}} dy \\
 &\leq 4^{n-2} \int_{|y| > N} \frac{|K(y)|}{|y|^{n-2}} dy < \varepsilon \quad (\text{from (3.4)}).
 \end{aligned}$$

$$\begin{aligned}
 I_4(x) &= \int_{|y| > 2|x|} \frac{|K(y)|}{|x-y|^{n-2}} dy \\
 &\leq 2^{n-2} \int_{|y| > 2|x|} \frac{|K(y)|}{|y|^{n-2}} dy \\
 &\leq 4^{n-2} \int_{|y| > N} \frac{|K(y)|}{|y|^{n-2}} dy < \varepsilon \quad (\text{from (3.4)}).
 \end{aligned}$$

Finally we estimate $I_3(x)$. The estimate of $I_3(x)$ is much more involved. It is here we have to use (1.8). For $\rho \in [0, |x|/2]$, we define the function $\eta = \eta(\rho)$ by the following ordinary differential equation

$$(3.7) \quad \begin{cases} \left(\frac{\eta(\rho)}{\rho}\right)^{n-1} \frac{d\eta(\rho)}{d\rho} = \frac{1}{nw_n} \int_{|z|=1} \frac{1}{M(x)} |K(x + \rho z)| dS_z, \\ \eta(0) = 0. \end{cases}$$

Here we of course assume that $M(x) > 0$. If $M(x) = 0$, then $I_3 = 0$. We do not have to estimate $I_3(x)$.

From (1.5), since

$$\frac{1}{nw_n} \int_{|z|=1} dS_z = 1,$$

we have

$$\left(\frac{\eta(\rho)}{\rho}\right)^{n-1} \frac{d\eta(\rho)}{d\rho} \leq 1, \quad 0 \leq \rho \leq \frac{|x|}{2}.$$

Hence we have

$$(3.8) \quad \eta(\rho) \leq \rho \quad \text{for } 0 \leq \rho \leq \frac{|x|}{2}.$$

Let

$$(3.9) \quad \varepsilon(x) = \eta\left(\frac{|x|}{2}\right).$$

Then from (3.7) we have

$$\begin{aligned} (3.10) \quad \frac{1}{n}(\varepsilon(x))^n &= \int_0^{\varepsilon(x)} \eta^{n-1} d\eta \\ &= \int_0^{|x|/2} \rho^{n-1} \left[\frac{1}{nw_n} \int_{|z|=1} \frac{1}{M(x)} |K(x + \rho z)| dS_z \right] d\rho \\ &= \frac{1}{nw_n M(x)} \int_{|y-x| < |x|/2} |K(y)| dy \\ &\leq \frac{1}{nw_n M(x)} \left(\frac{3}{2}|x|\right)^{n-2} g(x). \end{aligned}$$

Now we can estimate $I_3(x)$.

$$\begin{aligned} I_3(x) &= \int_{|y-x| < |x|/2} \frac{|K(y)|}{|y-x|^{n-2}} dy \\ &= \int_0^{|x|/2} \rho^{n-1} \left[\rho^{2-n} \int_{|z|=1} |K(x + \rho z)| dS_z \right] d\rho \\ &= nw_n M(x) \int_0^{\varepsilon(x)} \rho^{2-n} \eta^{n-1} d\eta \quad (\text{from (3.7)}) \\ &\leq nw_n M(x) \int_0^{\varepsilon(x)} \eta d\eta \quad (\text{from (3.8)}) \\ &= \frac{n}{2} w_n M(x) (\varepsilon(x))^2 \\ &\leq \frac{n}{2} \left(\frac{3}{2}\right)^{2-4/n} w_n^{1-2/n} [M(x)|x|^2 g(x)^{2/(n-2)}]^{(n-2)/n} \\ &< \varepsilon \quad (\text{from (3.6)}). \end{aligned}$$

Hence we have

$$\int_{\mathbb{R}^n} \frac{|K(y)|}{|y-x|^{n-2}} dy = I_1(x) + I_2(x) + I_3(x) + I_4(x) \leq 5\varepsilon$$

for $|x| > R_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves that

$$\int_{\mathbb{R}^n} \frac{|K(y)|}{|y-x|^{n-2}} dy \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Thus (3.2) is proved. Q.E.D.

Lemma 3.2. *Suppose that $\tilde{K} = \tilde{K}(r) \geq 0$ on $0 \leq r < \infty$ and satisfies $\int_0^\infty r\tilde{K} dr < \infty$. Then the function*

$$(3.11) \quad v(x) = \frac{1}{n(2-n)w_n} \int_{\mathbb{R}^n} \frac{\tilde{K}(|y|)}{|y-x|^{n-2}} dy$$

is well defined on \mathbb{R}^n and satisfies

$$(3.12) \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

and

$$(3.13) \quad \Delta v(x) = \tilde{K}(|x|) \quad \text{on } \mathbb{R}^n.$$

Proof. From the proof of Lemma 3.1, the only thing we have to check is

$$(3.14) \quad I_3(x) = \int_{|y-x| < |x|/2} \frac{\tilde{K}(|y|)}{|x-y|^{n-2}} dy \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

We use spherical coordinates. Let $x = |x|z_0$ and $y = |y|z$, where $|z| = 1 = |z_0|$. Let $|y| = r$. Then

$$\begin{aligned} I_3(x) &= \int_{|y-x| < |x|/2} \frac{\tilde{K}(|y|)}{|x-y|^{n-2}} dy \\ &= \int_{|x|/2}^{\frac{3}{2}|x|} r^{n-1} \tilde{K}(r) \left[\int_{|z| \in \Phi(r)} \frac{1}{|rz - |x|z_0|^{n-2}} dS_z \right] dr, \end{aligned}$$

where $\Phi(r) \subseteq \{z \mid |z| = 1\}$. it is not difficult to prove that there exists a constant $C > 0$ such that for $|y-x| < |x|/2$,

$$|y-x| = |rz - |x|z_0| \geq C|x|z - |x|z_0| = C|x||z - z_0|.$$

Hence,

$$\begin{aligned} C^{n-2}I_3(x) &\leq \int_{|x|/2}^{\frac{3}{2}|x|} r^{n-1} \tilde{K}(r) \left[\int_{z \in \Phi(r)} \frac{dS_z}{|x|^{n-2}|z - z_0|^{n-2}} \right] dr \\ &\leq \left(\int_{|x|/2}^{\frac{3}{2}|x|} r^{n-1} \tilde{K}(r) \frac{dr}{|x|^{n-2}} \right) \left(\int_{|z|=1} \frac{dS_z}{|z - z_0|^{n-2}} \right) \\ &\leq \left(\frac{3}{2} \right)^{n-2} \left(\int_{|x|/2}^\infty r \tilde{K}(r) dr \right) \left(\int_{|z|=1} \frac{dS_z}{|z - z_0|^{n-2}} \right) \end{aligned}$$

Now,

$$\int_{|z|=1} \frac{dS_z}{|z - z_0|^{n-2}}$$

is a finite number and

$$\int_{|x|/2}^\infty r \tilde{K}(r) dr \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Thus $I_3(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Q.E.D.

Lemma 3.3. *Suppose that $K_1 \geq K_2 \geq 0$ on \mathbb{R}^n and K_1 satisfies the hypotheses in Theorem 1.1. Suppose that the equation $\Delta u + K_1 e^{2u} = 0$ has a solution u*

satisfying $u(x) \rightarrow c$ as $|x| \rightarrow \infty$. Then the equation $\Delta v + K_2 e^{2v} = 0$ also possesses a solution v satisfying $v(x) \rightarrow c$ as $|x| \rightarrow \infty$.

Proof. Since $u(x) \rightarrow c$ as $|x| \rightarrow \infty$, u is a bounded super harmonic function. On the other hand, K_1 satisfies the hypotheses in Theorem 1.1. Hence

$$\int_{\mathbb{R}^n} \frac{1}{|x|^{n-2}} K_1(x) e^{2u(x)} dx < \infty.$$

From [HK, Theorem 3.20], we have

$$(3.15) \quad u(x) = c - C_n \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} K_1(y) e^{2u(y)} dy,$$

where $C_n = 1/n(2 - n)w_n$. Let $u_0 \equiv c$ and

$$(3.16) \quad u_{m+1} = c - C_n \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} K_1(y) e^{2u_m(y)} dy, \quad m = 0, 1, 2, \dots$$

Then it is easy to see that

$$(3.17) \quad u_{m+1} \geq u_m \text{ on } \mathbb{R}^n, \quad m = 0, 1, 2, \dots,$$

and

$$(3.18) \quad u_m \leq u \text{ on } \mathbb{R}^n, \quad m = 0, 1, 2, \dots$$

Now let $v_0 \equiv c$ and

$$v_{m+1} = c - C_n \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} K_2(y) e^{2v_m(y)} dy, \quad m = 0, 1, 2, \dots$$

From the assumption $K_1 \geq K_2 \geq 0$ on \mathbb{R}^n , we have

$$u_m \geq v_m \text{ on } \mathbb{R}^n \quad \text{for all } m = 0, 1, 2, \dots$$

and

$$v_{m+1} \geq v_m \text{ on } \mathbb{R}^n, \quad m = 0, 1, 2, \dots$$

Hence $\{v_m\}$ is a monotonically increasing sequence of functions which is bounded by u on \mathbb{R}^n . Hence $\{v_m\}$ converges to a function v on \mathbb{R}^n and v satisfies

$$(3.19) \quad v(x) = c - C_n \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} K_2(y) e^{2v(y)} dy.$$

This proves that v is a solution of the equation $\Delta u + K_2 e^{2v} = 0$ on \mathbb{R}^n and $v(x) \rightarrow c$ as $|x| \rightarrow \infty$. Q.E.D.

Lemma 3.4. *Let*

$$v(r) = \gamma - \frac{1}{2} \log(1 + \alpha r^2), \quad r \geq 0.$$

Then v satisfies

$$(3.20) \quad v''(r) + \frac{n-1}{r} v'(r) + \bar{K}(r) e^{2v} = 0,$$

where $\bar{K}(r) = e^{-2\gamma} \alpha [(n-2) + 2(1 + \alpha r^2)^{-1}]$.

Proof. Direct calculation yields the lemma. Q.E.D.

Lemma 3.5. *Suppose that $K \geq 0$ on \mathbb{R}^n and that (1.1) has a solution u_0 with $u_0(x) \rightarrow c_0$ as $|x| \rightarrow \infty$, for some $c_0 \in \mathbb{R}$. Then for every $c < c_0$, (1.1) possesses a solution with $u(x) \rightarrow c$ as $|x| \rightarrow \infty$.*

Proof. u_0 is a superharmonic function with $u_0(x) \rightarrow c_0$ as $|x| \rightarrow \infty$. By the maximum principle, $u_0(x) \geq c_0$ for all $x \in \mathbb{R}^n$. Let $c < c_0$, $u_-(x) = c$ and $u_+(x) = u_0(x) - (c_0 - c)$. Then

$$\begin{aligned} \Delta u_+ + Ke^{2u_+} &= \Delta u_0 + Ke^{2(u_0-c_0)+2c} \\ &= -Ke^{2u_0} + Ke^{2u_0-2(c_0-c)} \\ &= Ke^{2u_0}(-1 + e^{-2(c_0-c)}) \leq 0, \end{aligned}$$

and

$$\Delta u_- + Ke^{2u_-} = Ke^{2c} \geq 0.$$

Hence u_+ and u_- are respectively a supersolution and subsolution of (1.1) satisfying $u_+ \geq u_-$ on \mathbb{R}^n . Hence there exists a solution u of (1.1) satisfying $u_+ \geq u \geq u_-$ on \mathbb{R}^n . Since $u_+(x) \rightarrow c$ and $u_-(x) \rightarrow c$ as $|x| \rightarrow \infty$, we conclude that $u(x) \rightarrow c$ as $|x| \rightarrow \infty$. Q.E.D.

Lemma 3.6. *Suppose that $K \geq 0$ on \mathbb{R}^n and that (1.1) possesses a solution u with $u(x) \rightarrow c$ as $|x| \rightarrow \infty$. Then there exists a constant c_0 , depending only on K , such that $c \leq c_0$.*

Proof. Without loss of generality, we assume that $K(0) > 0$. Since K is locally Hölder continuous, there exist constants $\varepsilon > 0$ and $\delta > 0$ such that

$$(3.21) \quad K(x) \geq \varepsilon \quad \text{for } |x| \leq \delta.$$

Now choose $\alpha_0 > 0$ and $\eta > 0$ such that

$$(3.22) \quad K_\alpha(r) = \eta[(n-2) + 2(1 + \alpha r^2)^{-1}] \leq \varepsilon$$

for all $\alpha \geq \alpha_0$ and $r \in [0, \delta]$. Let

$$(3.23) \quad e^{-2\gamma} \cdot \alpha = \eta, \quad \alpha \geq \alpha_0.$$

Consider the following ordinary differential equation

$$(3.24) \quad \begin{cases} u'' + \frac{n-1}{r}u' + K_\alpha(r)e^{2u} = 0, & 0 < r < \delta, \\ u'' + \frac{n-1}{r}u' = 0, & \delta \leq r < \infty, \\ u(0) = \gamma, \quad u'(0) = 0, & \alpha \geq \alpha_0. \end{cases}$$

From Lemma 3.4, the solution u_α of (3.24) is

$$(3.25) \quad u_\alpha(r) = \begin{cases} \gamma - \frac{1}{2} \log(1 + \alpha r^2) & \text{for } 0 \leq r < \delta, \\ \gamma - \frac{1}{2} \log(1 + \alpha \delta^2) - \frac{1}{n-2} \alpha \delta^n (1 + \alpha \delta^2)^{-1} (\delta^{2-n} - r^{2-n}) & \text{for } \delta \leq r < \infty. \end{cases}$$

Hence

$$(3.26) \quad u_\alpha(\infty) = \lim_{r \rightarrow \infty} u_\alpha(r) = \gamma - \frac{1}{2} \log(1 + \alpha \delta^2) - \frac{1}{n-2} \alpha \delta^2 (1 + \alpha \delta^2)^{-1}.$$

Using (3.23), we have

$$(3.27) \quad u_\alpha(\infty) = \frac{1}{2} \log\left(\frac{\alpha}{\eta}\right) - \frac{1}{2} \log(1 + \alpha \delta^2) - \frac{1}{n-2} \alpha \delta^2 (1 + \alpha \delta^2)^{-1}.$$

Let

$$c_0 = \max \left\{ a_0, \sup_{\alpha \geq \alpha_0} u_\alpha(\infty) \right\}.$$

Then from Lemma 3.3, we conclude that $c \leq c_0$. Q.E.D.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. First we shall prove that there exists a function v satisfying

$$(3.28) \quad \Delta v = |K| \quad \text{on } \mathbb{R}^n,$$

$$(3.29) \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

But from Lemma 3.1, the existence of v is assured if K satisfies the hypothesis (ii). Thus it remains to prove (3.28) and (3.29) if K satisfies the hypothesis (i). But from Lemma 3.2, there exists a function \tilde{v} , such that, $\Delta \tilde{v}(x) = \tilde{K}(|x|)$ and $\tilde{v}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. From the maximum principle, $\tilde{v}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Thus

$$\Delta \tilde{v}(x) - |K(x)| = \tilde{K}(|x|) - |K(x)| \geq 0.$$

Hence \tilde{v} and 0 are respectively a subsolution and a supersolution of (3.28). Thus there exists a function v satisfying (3.28) and (3.29). Thus under the hypotheses of Theorem 1.1, we can find a function v satisfying (3.28) and (3.29). By the maximum principle, $v(x) \leq 0$ on \mathbb{R}^n . Let

$$c_0 = \inf_{x \in \mathbb{R}^n} v(x).$$

For $c \in (-\infty, c_0]$, let $u_+ = -v + c$ and $u_- = \beta v + c$, $\beta \geq e^{2c}$. Then we have

$$\begin{aligned} \Delta u_+ + Ke^{2u_+} &= -\Delta v + Ke^{-2v+2c} = -|K| + Ke^{-2v}e^{2c} \\ &\leq |K|[-1 + e^{2(c-c_0)}] \leq 0 \end{aligned}$$

and

$$\begin{aligned} \Delta u_- + Ke^{2u_-} &= \beta|K| + Ke^{2c}e^{2\beta v} \geq \beta|K| - |K|e^{2c}e^{2\beta v} \\ &\geq |K|(\beta - e^{2c}) \geq 0. \end{aligned}$$

Since $u_+ \geq u_-$, $u_+(x) \rightarrow c$ as $|x| \rightarrow \infty$ and $u_-(x) \rightarrow c$ as $|x| \rightarrow \infty$, we conclude that there exists a solution u_c of (1.1) satisfying $u_+ \geq u_c \geq u_-$ and hence $u_c(x) \rightarrow c$ as $|x| \rightarrow \infty$. This proves (α) .

Now suppose that in addition $K \leq 0$ on \mathbb{R}^n . Then as above, there exists a function v satisfying

$$(3.30) \quad \Delta v = -K, \quad v \leq 0 \quad \text{on } \mathbb{R}^n.$$

and

$$(3.31) \quad v(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Let $c \in (-\infty, \infty)$ be a given number. Let $u_+(x) = c$ and $u_-(x) = \alpha v + c$ with $\alpha > e^{2c}$. Then $u_+ \geq u_-$ on \mathbb{R}^n and

$$\begin{aligned} \Delta u_+ + Ke^{2u_+} &= Ke^{2c} \leq 0, \\ \Delta u_- + Ke^{2u_-} &= \alpha(-K) + Ke^{2\alpha v+2c} \geq (-K)(\alpha - e^{2c}) \geq 0. \end{aligned}$$

Thus there exists a solution u_c of (1.1) satisfying $u_+ \geq u_c \geq u_-$ and $u_c(x) \rightarrow c$ as $|x| \rightarrow \infty$. We shall leave the proof of uniqueness of u_c to a general theorem in §4. This proves (β) .

Suppose now that $K \geq 0$ on \mathbb{R}^n . Then from part (α) , there exists a constant c_1 , such that, for every $c \in (-\infty, c_1]$, (1.1) possesses a solution u_c with $u_c(x) \rightarrow c$ as $|x| \rightarrow \infty$. Let

$$A = \{c \mid (1.1) \text{ possesses a solution } u_c \text{ with } u_c(x) \rightarrow c \text{ as } |x| \rightarrow \infty\}.$$

Then $(-\infty, c_1] \subset A$. From Lemma 3.6, A is bounded from above. Let $c_0 = \sup A$. Then from Lemma 3.5, we have $A = (-\infty, c_0]$ or $A = (-\infty, c_0)$. This proves (γ) .

Finally let u be a bounded solution of (1.1). As in the beginning of this proof, there exists a function v satisfying $\Delta v = -Ke^{2u}$ on \mathbb{R}^n and $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Let $w = u - v$. We have $\Delta w = 0$. Hence w is a bounded harmonic function on \mathbb{R}^n . Hence $w \equiv \text{constant} = c$ for some $c \in \mathbb{R}$. This proves (δ) . Q.E.D.

4. PROOFS OF THEOREMS 1.3 AND 1.4

In this section we shall prove Theorems 1.3 and 1.4. But first we give a uniqueness theorem as promised in the proof of Theorem 1.1 (β) in §3.

Theorem 4.1. *Suppose that $K \leq 0$ on \mathbb{R}^n . Let u and v be two solutions of (1.1) satisfying $|u(x) - v(x)| \rightarrow 0$ as $|x| \rightarrow \infty$. Then $u = v$ on \mathbb{R}^n .*

Proof. Let $w = (u - v)^2$. Then

$$\begin{aligned} \Delta w &= 2D(u - v) \cdot D(u - v) + 2(u - v)(\Delta u - \Delta v) \\ &\geq 2(u - v)(-K)(e^{2u} - e^{2v}) \geq 0. \end{aligned}$$

Thus w is a nonnegative subharmonic function on \mathbb{R}^n satisfying $w(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Hence $w = 0$ on \mathbb{R}^n . Q.E.D.

Now we prove Theorem 1.3 first.

Proof of Theorem 1.3. Suppose that K satisfies equation (1.9). Then we can find $K_1 = K_1(|x|)$ such that $K(x) \leq K_1(|x|) \leq 0$ for all $x \in \mathbb{R}^n$ and $K_1(r) = -C_1 r^{-p}(\log r)^{-\beta}$ for $r \geq R_0$ for some large R_0 . Let U_1 be the maximal solution of $\Delta u + K_1 e^{2u} = 0$ on \mathbb{R}^n . Then from Theorems 1.1 and 1.2, U_1 exists and $U \leq U_1$ on \mathbb{R}^n from Theorem 2.1. Furthermore, U_1 is a radial function from Theorem 2.2.

So now let

$$U_1(r) = w(r) + \frac{p-2}{2} \log r + \frac{\beta}{2} \log \log r \quad \text{for } r \geq R_0.$$

Then w satisfies

$$\begin{aligned} (4.1) \quad w'' + \frac{n-1}{r} w' &= C_1 r^{-2} \left[e^{2w} - C_1^{-1} \left(\frac{(p-2)(n-2)}{2} + \frac{\beta(n-2)}{2 \log r} - \frac{\beta}{2(\log r)^2} \right) \right]. \end{aligned}$$

We can choose R_0 so large that

$$(4.2) \quad C_2 \geq \frac{(p-2)(n-2)}{2} + \frac{\beta}{2} \frac{n-2}{\log r} - \frac{\beta}{2(\log r)^2} \geq C_3 > 0,$$

for $r \geq R_0$ for some constants C_2 and C_3 . Let $w_1 = \frac{1}{2} \log(C_2/C_1)$. We shall prove that there exists a constant $R_1 \geq R_0$, such that, $w(r) \leq w_1 + 1$ for all $r \geq R_1$. Suppose that this is not true. Then either (a) $w(r) \geq w_1 + 1$ and $w'(r) < 0$ for all $r \geq R_0$ or (b) there exists $R_2 \geq R_0$ such that $w(R_2) \geq w_1 + 1$ and $w'(R_2) \geq 0$. If (a) holds, then from (4.1), we have

$$(r^{n-1}w'(r))' \geq \frac{C_1}{r^2} r^{n-1} [e^{2(w_1+1)} - e^{2w_1}].$$

Hence we have

$$r^{n-1}w'(r) \geq R_0^{n-1}w'(R_0) + C_1[e^{2w_1}(e^2 - 1)] \frac{1}{n-2} (r^{n-2} - R_0^{n-2}).$$

Thus

$$w'(r) \geq C_4 r^{-1} + O(1)r^{-(n-1)}$$

for some $c_4 > 0$. This contradicts to the fact that $w'(r) < 0$ for all $r \geq R_0$. Hence (a) does not hold.

Suppose that (b) holds. Then for $r \geq R_2$ and $(r - R_2)$ small, we have $w(r) \geq w_1 + \frac{1}{2}$ and hence

$$(r^{n-1}w'(r))' \geq C_1 r^{n-3} [e^{2(w_1+\frac{1}{2})} - e^{2w_1}] = C_5 r^{n-3} > 0$$

Hence

$$w'(r) \geq w'(R_2)R_2^{n-1}r^{-(n-1)} + C_6(r^{n-2} - R_2^{n-2})r^{-(n-1)} > 0.$$

This proves that $w'(r) > 0$ for $r > R_2$ and $(r - R_2)$ small. Hence $w'(r) > 0$ for all $r > R_2$ and $w(r) \geq w_1 + 1$ for all $r \geq R_2$. From (4.1), we have

$$(4.3) \quad \begin{cases} w'' + \frac{n-1}{r}w' \geq \frac{C_7}{r^2}e^{2w}, & r > R_2, \\ w(R_2) \geq w_1 + 1, & w'(R_2) \geq 0. \end{cases}$$

From Cheng and Lin [CL], (4.3) possesses no solution w on $[R_2, \infty)$. Hence (b) does not hold also. Thus $w(r) \leq w_1 + 1$ for r large. This proves that

$$(4.4) \quad U(x) \leq U_1(|x|) \leq \frac{p-2}{2} \log|x| + \frac{\beta}{2} \log \log|x| + C$$

for $|x|$ large for some C .

Now let

$$U_2(r) = \frac{p-2}{4} \log(1+r^2) + \frac{\beta}{2} \log \log(a+r^2) - b,$$

where a and b are two positive constants to be determined later. $U_2(|x|)$ satisfies

$$(4.5) \quad \Delta U_2 + K_2 e^{2U_2} = 0 \quad \text{on } \mathbb{R}^n,$$

where $K_2(r) = -F(r)G(r)$ and

$$F(r) = (1+r^2)^{-(p-2)/2} [\log(a+r^2)]^{-\beta} e^{2b},$$

$$G(r) = \left\{ \frac{p-2}{2} \frac{n+(n-2)r^2}{(1+r^2)^2} + \beta \left[\frac{(n+(n-2)r^2) \log(a+r^2) - 2r^2}{(a+r^2)^2 (\log(a+r^2))^2} \right] \right\}.$$

It is easy to see that $K_2(r) \sim -r^{-p}(\log r)^{-\beta}$ at ∞ . We choose a so large that $K_2(r) < 0$ for all $r \geq 0$ and then choose b so large that $K_2(|x|) \leq K(x)$ for all $x \in \mathbb{R}^n$. From Theorem 2.1, $U_2(|x|) \leq U(x)$. Thus

$$(4.6) \quad \frac{p-2}{2} \log|x| + \frac{\beta}{2} \log \log|x| - C \leq U_2(x) \leq U(x)$$

for $|x|$ large for some C .

Combing (4.4) and (4.6) we prove (1.13).

Now suppose that K satisfies (1.10). We shall prove that U satisfies (1.14). Since the method of the proof is similar to that of (1.13), we shall only indicate the corresponding choices of K_1 , U_1 , K_2 , U_2 . We can choose

$$K_1(r) = -C_1 r^{-2} (\log r)^{-\beta}, \quad \text{for } |x| \text{ large.}$$

Then

$$U(x) \leq U_1(|x|) \leq \frac{\beta-1}{2} \log \log(|x|) + C$$

for $|x|$ large. We choose

$$U_2(r) = \frac{\beta-1}{2} \log \log(a+r^2) - b,$$

$$K_2(r) = -(\beta-1)(a+r^2)^{-1} (\log(a+r^2))^{-\beta} e^{2b} \cdot \left\{ \frac{na + (n-2)r^2}{a+r^2} - \frac{2r^2}{(a+r^2) \log(a+r^2)} \right\}.$$

We can choose a so large that $K_2(r) < 0$ for all $r \geq 0$ and choose b so large that $K_2(|x|) \leq K(x)$ for all $x \in \mathbb{R}^n$. Then

$$\frac{\beta-1}{2} \log \log|x| - C \leq U_2(|x|) \leq U(x) \quad \text{for } |x| \text{ large.}$$

This proves (1.14). Suppose now that K satisfies (1.11). We choose

$$K_1(r) = -C_1 r^{-q} (\log r)^{p-k} e^{-2(\log r)^p} \quad \text{for } r \text{ large.}$$

Then

$$U(x) \leq U_1(|x|) \leq (\log|x|)^p + \frac{q-2}{2} \log|x| + \frac{k-1}{2} \log \log|x| + C$$

for $|x|$ large. We choose

$$U_2(r) = \begin{cases} ar^2 + br^{p+1} + c - d & \text{for } r \leq R_0, \\ (\log r)^p + \frac{q-2}{2} \log r + \frac{k-1}{2} \log \log r - d & \text{if } r \geq R_0, \end{cases}$$

where R_0 is properly chosen, a , b , c , are chosen so that U_2 is a C^2 -function on $(0, \infty)$. Now $U_2(|x|)$ satisfies $\Delta U_2 + K_2 e^{2U_2} = 0$, where K_2 is a negative function which can be calculated. Choose the constant d so that $K_2(|x|) \leq K(x)$ on \mathbb{R}^n . Thus $U_2(|x|) \leq U(x)$. This proves (1.15).

Finally suppose that K satisfies equation (1.12). Then we choose $K_1(r) = -C_1 r^{\alpha-p} e^{-2r^\alpha}$ for r large. Then

$$U(|x|) \leq U_1(|x|) \leq |x|^\alpha + \frac{p-2}{2} \log|x| + C \quad \text{for } |x| \text{ large.}$$

Let

$$U_2(r) = \begin{cases} ar^2 + br^{\alpha+2} + c - d & \text{if } r \leq R_0, \\ r^\alpha + \frac{p-2}{2} \log r - d & \text{if } r \geq R_0, \end{cases}$$

where a, b, c , are chosen such that U_2 is a C^2 -functions on $[0, \infty)$. Then $U_2(|x|)$ satisfies $\Delta U_2 + K_2 e^{2U_2} = 0$ for some negative function K_2 . Choose d so large that $K_2(|x|) \leq K(x) \quad \forall x \in \mathbb{R}^n$. Then $U_2(|x|) \leq U(x)$. This proves (1.16). Q.E.D.

Next we prove Theorem 1.4.

Proof of Theorem 1.4. Let $u = u(r)$ be a radial solution of (1.1). Then u satisfies

$$(4.7) \quad u(r) = u(0) + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r}\right)^{n-2} \right] [-K(s)] e^{2u(s)} ds.$$

Let U be the maximal solution of (1.1). Then $U = U(r)$ satisfies

$$(4.8) \quad U(r) = U(0) + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r}\right)^{n-2} \right] [-K(s)] e^{2U(s)} ds.$$

It is easy to see that $u(0) \leq U(0)$. If $u(0) = U(0)$, then $u = U$ on \mathbb{R}^n by the uniqueness of ordinary differential equation satisfied by u . Suppose that $u(0) < U(0)$. Then $u(r) < U(r)$ and we have from (4.7) and (4.8)

$$\begin{aligned} u(r) &\leq u(0) + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r}\right)^{n-2} \right] [-K(s)] e^{2u(s)} ds \\ &= u(0) + U(r) - U(0). \end{aligned}$$

Hence again from (4.7) and (4.8) we have

$$(4.9) \quad \begin{aligned} u(r) &\leq u(0) + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r}\right)^{n-2} \right] [-K(s)] e^{2u(s)-2(U(0)-u(0))} ds \\ &= u(0) + e^{-2(U(0)-u(0))} [U(r) - U(0)] \\ &\leq O(1) + e^{-2(U(0)-u(0))} U(r). \end{aligned}$$

From Theorem 1.3, we know the asymptotic behaviors of these U 's. Thus from (4.9), we can show that

$$\int_0^\infty s [-K(s)] e^{2u(s)} ds < \infty.$$

From (4.7), we have $u(r) \rightarrow c$ as $r \rightarrow \infty$ for some c . From Theorem 4.1, we conclude that $u = u_c$ for some c . Q.E.D.

5. CLASSIFICATION OF SOLUTIONS

In this section we shall prove Theorems 1.5, 1.6 and 1.7.

Proof of Theorem 1.5. Let u be a solution of (1.1). Suppose that $u \neq U$ on \mathbb{R}^n . Then from Theorem 2.3, $u < U$ on \mathbb{R}^n . Let $w = U - u$. Then w satisfies

$$(5.1) \quad \Delta w - d(x)w = 0,$$

where

$$(5.2) \quad \begin{aligned} d(x) &= -K(x) \frac{e^{2U(x)} - e^{2u(x)}}{U(x) - u(x)} \\ &\leq -K(x) \cdot 2e^{2U(x)} \leq C|x|^{-2} \end{aligned}$$

for $|x|$ large. Let $|y| = R$ and R be large. Consider (5.1) in the ball $B_{R/2}(y) = \{x \in \mathbb{R}^n \mid |x - y| < R/2\}$. Using the Harnack inequality [GT, p. 199], we have

$$(5.3) \quad \sup_{x \in B_{R/8}(y)} w(x) \leq C \inf_{x \in B_{R/8}(y)} w(x)$$

where C is a constant independent of R . Thus there exists a constant $C > 1$ such that

$$(5.4) \quad \sup_{|x|=r} w(x) \leq C \inf_{|x|=r} w(x)$$

for r large.

Now suppose that $w > 0$ is bounded. We have

$$(5.5) \quad \Delta w(x) = -K(x)[e^{2U(x)} - e^{2u(x)}] = -K(x) \frac{[e^{2U(x)} - e^{2u(x)}]}{w^2} w^2.$$

Let M be a bound of w . Then

$$\begin{aligned} -K(x) \frac{e^{2U} - e^{2u}}{w^2} &= -K(x)e^{2u} \left[\frac{e^{2w} - 1}{w^2} \right] \\ &\geq -K(x)e^{2(U(x)-M)} \left[\frac{e^{2w} - 1}{w^2} \right] \\ &\geq C[-K(x)]e^{2U(x)} \end{aligned}$$

for some constant $C > 0$. Thus

$$(5.6) \quad \Delta w \geq C[-K(x)]e^{2U(x)}w^2 \quad \text{on } \mathbb{R}^n.$$

Since $-K(x)e^{2U(x)} \sim |x|^{-2}(\log|x|)^{-1}$ at ∞ , we conclude from [CL] that (5.6) does not possess any solution on \mathbb{R}^n . Hence w is unbounded. Let $\{x_m\}$ be a sequence such that $|x_m| \rightarrow \infty$ as $m \rightarrow \infty$ and $w(x) \rightarrow \infty$ as $m \rightarrow \infty$. From (5.4), we conclude that $\inf_{|x|=|x_m|} w(x) \rightarrow \infty$. Let $\bar{w}(r)$ be the spherical mean of w on $|x| = r$. Then since w is a subharmonic function on \mathbb{R}^n , \bar{w} is a convex function of r^{2-n} . Hence we conclude that $\bar{w}(r) \rightarrow \infty$ as $r \rightarrow \infty$. Again from (5.4), we conclude that $w(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Now w satisfies $\Delta w = -K[e^{2U} - e^{2u}]$. Hence $\bar{w}(r)$ satisfies

$$(5.7) \quad \bar{w}'' + \frac{n-1}{r}\bar{w}' = -\overline{K[e^{2U} - e^{2u}]} = -\overline{Ke^{2U}[1 - e^{-2w}]} \geq C[-Ke^{2U}],$$

where “ $\overline{\quad}$ ” means the spherical average on $|x| = r$.

From the assumptions of K and the asymptotic behaviors of U , we have $-\overline{Ke^{2U}} \geq Cr^{-2}$ or $-\overline{Ke^{2U}} \geq Cr^{-2}(\log r)^{-1}$ for r large. Hence we conclude that $\bar{w}(r) \geq C \log r$ or $\bar{w}(r) \geq C \log \log r$ for r large. By (5.4) again, we have $w(x) \geq C \log|x|$ or $w(x) \geq C \log \log|x|$ for $|x|$ large. Thus $u(x) \leq U(x) - C \log|x|$ or $u(x) \leq U(x) - C \log \log|x|$ for $|x|$ large. Thus

$$\int_{\mathbb{R}^n} \frac{-K(y)}{|y|^{n-2}} e^{2u(y)} dy < \infty.$$

From [HK], u is bounded and hence by Theorem 1.1(δ), $u = u_c$ for some $c \in \mathbb{R}$. Q.E.D.

Proof of Theorem 1.6. Let u be a solution of (1.1) with a lower bound $(-M)$. Let $v = u + M + 1$. Then $v \geq 1$ and satisfies

$$\begin{aligned} \Delta v &= -K(x)e^{2u} = [-K(x)]e^{-2(M+1)}e^{2v} \\ &= [-K(x)]e^{-2(M+1)}\frac{e^{2v}}{v} = d(x)v, \end{aligned}$$

where $d(x) = [-K(x)]e^{-2(M+1)}e^{2v}/v$. For $|x|$ large, if $1 \leq v(x) \leq \frac{1}{2}U(x)$, then

$$d(x) \leq [-K(x)]e^{-2(M+1)}\frac{e^{U(x)}}{1} \leq \frac{C}{|x|^2}.$$

If $v(x) \geq \frac{1}{2}U(x)$, then

$$d(x) \leq [-K(x)]e^{-2(M+1)}\frac{e^{2(U(x)+M+1)}}{\frac{1}{2}U(x)} \leq \frac{C}{|x|^2}.$$

Thus as in the proof of (5.4), we conclude that

$$(5.8) \quad \sup_{|x|=r} v(x) \leq C \inf_{|x|=r} v(x)$$

for r large for some constant $C \geq 1$.

Now suppose that u is bounded also from above. Then from Theorem 1.1(β), (δ), $u = u_c$ for some $c \in \mathbb{R}$. If u is not bounded from above, then there exists a sequence $\{x_m\}$ such that $|x_m| \rightarrow \infty$ as $m \rightarrow \infty$ and $u(x_m) \rightarrow \infty$ as $m \rightarrow \infty$. Hence $v(x_m) \rightarrow \infty$. From (5.8) and the same arguments as in the proof of Theorem 1.5, we conclude that $\lim_{|x| \rightarrow \infty} v(x) = \infty$. Hence $u(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Let $K_2 = K_2(r)$ be a function such that $K_2(|x|) \leq K(x)$ for all $x \in \mathbb{R}^n$ and $|K_2(|x|)| \sim |K(x)|$ for $|x|$ large. Let $v_c(r)$ and $V(r)$ be the solution of

$$(5.9) \quad v'' + \frac{n-1}{r}v' + K_2(r)e^{2v} = 0$$

satisfying $v_c(r) \rightarrow c$ as $r \rightarrow \infty$ and $v_c(r) \rightarrow V(r)$ as $c \rightarrow \infty$. From Theorem 1.4, these are the radial solutions of $\Delta v + K_2e^{2v} = 0$ on \mathbb{R}^n .

Now by the maximum principle, it is easy to see that $v_c(|x|) \leq u(x)$, $x \in \mathbb{R}^n$. Hence $V(|x|) \leq u(x)$, $x \in \mathbb{R}^n$. This proves that $u(x) \geq U(x) - C$, $x \in \mathbb{R}^n$, for some C . As in the proof of Theorem 1.5, we conclude that $u = U$. Q.E.D.

Finally, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. Let h be a harmonic function on \mathbb{R}^n . Consider the problem

$$(5.10) \quad \Delta w + (Ke^{2h})e^{2w} = 0 \quad \text{on } \mathbb{R}^n.$$

From Theorem 1.1(β), (5.10) possesses a unique solution w_c with $w_c(x) \rightarrow c$ as $|x| \rightarrow \infty$ for every c . Hence $u = h + w_c$ is the unique solution satisfying (1.18). This proves (i).

Now let u be a solution of (1.1). Then since $K \leq 0$ and K has compact support, u is a subharmonic function such that

$$\int_{\mathbb{R}^n} \frac{|+\Delta u(y)|}{|y|^{n-2}} dy = \int_{\mathbb{R}^n} \frac{|K(y)|}{|y|^{n-2}} e^{2u(y)} dy < \infty.$$

From [HK], we conclude that there exist a harmonic function h and a constant c such that (1.18) holds. This proves (ii). Q.E.D.

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INSTITUTE OF APPLIED MATHEMATICS, NATIONAL CHUNG CHENG UNIVERSITY, CHIAYI, TAIWAN 62117, REPUBLIC OF CHINA

INSTITUTE OF APPLIED MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU, TAIWAN 30043, REPUBLIC OF CHINA