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SPECTRAL MULTIPLICITY FOR $\mathrm{Gl}_n(R)$

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ABSTRACT. We study the behavior of the cuspidal spectrum of $\Gamma\backslash\mathscr{H}$, where $\mathscr{H}$ is associated to $\mathrm{Gl}_n(R)$ and Γ is cofinite but not compact. By a technique that modifies the Lax-Phillips technique and uses ideas from wave equation techniques, if r is the dimension of $\mathscr{H}$, $N_\alpha(\lambda)$ is the counting function for the Laplacian attached to a Hilbert space H_α, $M_\alpha(\lambda)$ is the multiplicity function, and H_0 is the space of cusp forms, we obtain the following results:

Theorem 1. *There exists a space of functions* H^1, *containing all cusp forms, such that*

$$N'(\lambda) = C_r(\operatorname{Vol} X)\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n+1}}(\log\lambda)^{n-1}).$$

Theorem 2.

$$M_0(\lambda) = O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n+1}}(\log\lambda)^{n-1}).$$

1. INTRODUCTION AND STATEMENT OF RESULTS

The spectrum of the Laplacian on a manifold has been a topic of great interest for many years and many interesting results have been proved. Many of the results, however, require that the manifold be compact, as in general for a noncompact manifold one has continuous spectrum, and among other things one may not use certain analytic techniques, such as the min-max principle or Dirichlet-Neumann bracketing, directly. In the case of a locally symmetric space of finite volume, one may try to use the Selberg trace formula to obtain information about the spectrum. This has, however, only been done successfully for the rank one case and for a handful of other examples, since, due to the complexity of the formula, certain technical difficulties arise and it is difficult to obtain an estimate for the contribution from the Eisenstein series.

In this paper we are able to obtain for locally symmetric spaces associated to $\mathrm{Gl}_n(R)$ a bound on the multiplicity of a certain part of the discrete spectrum, namely the cuspidal spectrum. We will define this below.

We now introduce some notation and state our main theorems.

Let $\mathscr{H}$ denote the homogeneous space associated to $\mathrm{PGl}_n(R)$, let Γ be a discrete subgroup of $\mathrm{Gl}_n(R)$, let $X = \Gamma\backslash\mathscr{H}$, and assume that Γ is such that X has finite invariant volume. We will generally also assume that X is not compact, as stronger results than ours are known in greater generality in this case. Given a Hilbert space of functions on X, denoted by H_a, let $N_a(\lambda)$

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denote the number of independent eigenfunctions of the Laplacian acting on functions in H_a with eigenvalue less than or equal to λ, and let $M_a(\lambda)$ denote the number of such eigenfunctions with eigenvalue equal to λ. Let H_0 denote the space of cuspidal eigenfunctions on X. Finally, let r denote the dimension of X.

Theorem 1. *There exists a space of functions H^1, containing all cusp forms, such that*

$$N'(\lambda) = C_m(\operatorname{Vol} X)\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n+1}}(\log\lambda)^{n-1}).$$

It is easy to see that $r = ((n^2) + n - 2)/2$. Our main result is a corollary of Theorem 1.

Theorem 2.

$$M_0(\lambda) = O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n+1}}(\log\lambda)^{n-1}).$$

We should now make some remarks. First, we would like to obtain the error term $O(\lambda^{\frac{r-1}{2}})$, in analogy with the compact case. Our inability to obtain this estimate is due mainly to the fact that we have to use a cheap estimate in the cusp. This is responsible for the fraction of a power that we have in the estimate that we would not like to have. It is likely that one power of $\log\lambda$ can be removed from our estimate by studying the boundary of certain partitions that we put on X more carefully. Second, when $n = 2$ the theorems still hold except that an extra power of the logarithm must be included. The extra power is due to the fact that the surface area of the boundary of a fundamental domain is finite when it is greater than 2 but grows logarithmically as a function of the y variable when $n = 2$.

As we mentioned before this problem has a long history.

The first basic result is Weyl's law [We] for a bounded domain in R^n. He studies the boundary value problem where functions are assumed to satisfy either Dirichlet of Neumann boundary conditions. His law states that

$$N(\lambda) \sim C_n(\operatorname{Vol} X)\lambda^{\frac{n}{2}}.$$

This result was improved by Courant to give the error term

$$O(\lambda^{\frac{(n-1)}{2}}\log\lambda)$$

[Co, CH]. In Courant's theorem the implied constant depends on the surface area of the boundary of the region.

Weyl proves his law by using Dirichlet-Neumann bracketing and the fact that one may easily find the asymptotic value of $N(\lambda)$ for a cube. The main idea in Courant's improvement of Weyl's law is that one can compute the error term for cubes, and that the more cubes used the larger this cumulative error is, but if one uses many cubes one may ignore the contribution to $N(\lambda)$ from the boundary and create only a small error. The theorem is then proved by balancing the two errors. Balancing one error in the interior of a region and another near the boundary is standard technique for this kind of problem. Our proof uses a similar balancing of errors in certain places.

When studying the Laplacian on a manifold direct analyses such as those described above are generally not the best way to attack the problem. Instead, one generally studies related functions. Let X denote, for a moment, a compact manifold, so that the Laplacian Δ has pure point spectrum. Let ϕ_i denote an orthonormal basis of square integrable eigenfunctions, listed so that the eigenvalues are nondecreasing. Let

$$e(x, y, \lambda) = \sum_{\lambda_i \leq \lambda^2} \phi_i(x)\phi_i(y).$$

Clearly

$$N(\lambda^2) = \int_X e(x, x, \lambda)\, dx$$

with dV the volume element for the Riemannian manifold. $e(x, y, \lambda)$ is generally difficult to study and certain transforms of it are studied instead. For example, one may study the Laplace transform of $e(x, y, \lambda)$. One then has the fundamental solution to the heat equation $v(x, y, t)$. We have

$$v(x, x, t) = \sum_{\lambda_i < \lambda^2} e^{-\lambda_i t}$$

and by studying the singularity of v when t approaches 0, one may, by use of a Tauberian theorem, deduce Weyl's law for a noncompact manifold [MP].

This use of the heat equation does not yield an error term for Weyl's law. Hörmander [Ho] considers the cosine transform of $e(x, y, t)$ given by

$$u(x, y, t) = \int_0^\infty \cos \lambda t \frac{d}{d\lambda} e(x, y, \lambda)\, d\lambda.$$

This gives the fundamental solution of the wave equation. Studying the singularity of its trace (a distribution) at $t = 0$, Hörmander obtains Weyl's law with the error term $O(\lambda^{(n-1)/2})$, where n is the dimension of the manifold. We should also mention a result due to Agmon [Ag] that we will use later. By studying

$$\int_0^\infty \frac{1}{\lambda^2 - z} \frac{d}{d\lambda} e(x, y, \lambda)\, d\lambda$$

Agmon deduces $e(x, y, t) = O(\lambda^n)$, uniformly in x and y as t tends to 0 (λ tends to infinity). We will have more to say about this later.

When the manifold has a boundary the situation is more complicated. The main result that we need is due to Seeley [Se], who obtains Hormander's result for manifolds with boundary, for both the Dirichlet and Neumann problems. We will use a slightly weaker result, also due to Seeley, that is valid for manifolds whose boundary is piecewise smooth such that a cone can be inserted into the manifold at any singularity of the boundary. One may show that the error term is off by one power of $\log \lambda$. To do this, one uses the finite propagation speed of the fundamental solution of the wave equation to allow one to "ignore" the part of the manifold that is "near" the boundary. (Of course "near" must be made precise.) This allows one to use free space estimates in the interior of the manifold. Near the boundary, Agmon's estimate is used. One then balances the estimates. To minimize the error, one lets near mean that one has $d(x, \partial X) > \frac{1}{\lambda}$. Actually Seeley uses a clever argument to show that if one is greater than $\frac{1}{\sqrt{\lambda}}$ from the boundary, then propagation of the wave equation only hits the boundary at most once. This is how he obtains the same bound as Hörmander and is why some restrictions on the boundary must be imposed, at least for the Neumann problem.

When the manifold is noncompact the situation is considerably more difficult. In general, the presence of continuous spectrum will make Weyl's law false. The Laplacian on R^n gives an obvious example. In the case of a finite volume, noncompact manifold, things are more subtle. From now on we will restrict ourselves to the case of X being a finite volume locally symmetric space of the form $\Gamma\backslash\mathscr{H}$, where $\mathscr{H}$ is the homogeneous space attached to $\mathrm{Gl}_n(R)$ and Γ is a discrete group. The techniques that are developed also work, however, for general symmetric spaces.

In the classical case $n = 2$ corresponding to the upper half-plane, a technique for handling the continuous spectrum was developed by Lax and Phillips. They created a modified space of functions on which the Laplacian Δ has pure discrete spectrum. This space contains the space of cusp forms (to be described later) and they obtain Weyl's law on this space. In particular, this gives an upper bound for $N_0(\lambda)$, the space of cusp forms. Their proof uses Dirichlet-Neumann bracketing, the min-max principle, and some elementary approximations that reduce one to the well-known Euclidean case. Actually Lax and Phillips also interpret the part of the space that does not correspond to cusp forms, in terms of the scattering matrix. We will not discuss this point further, however.

In general, lower bounds are difficult to obtain for $N(\lambda)$. For X associated to a Lie group G, Donnelly [Do] studies the automorphic heat equation and deduces that $N_0(\lambda)$ is less than or equal to the asymptotic estimate that one expects from Weyl's law. He modifies the heat equation in the cusp to account for the cuspidal condition. He also creates a space on which the Laplacian has discrete spectrum, and uses Neumann bracketing in his argument. As he is not trying to get a lower bound, he does not have to use Dirichlet bracketing, and, as he uses the heat equation, he is not able to derive an error term.

We now describe the ideas used in our proof. We start by generalizing the Lax-Phillips method to obtain a Hilbert space of functions on X, on which the Laplacian has pure point spectrum. This space contains all cusp forms. We may use Dirichlet-Neumann bracketing. In a compact region of the space, we add boundary surfaces to which we give either Dirichlet or Neumann boundary conditions. Within any one of the "cubes" created we use the fact that we are in a symmetric space to obtain a uniform estimate in the interior of the cube for $e(x, y, t)$. Near the boundaries, we use Agmon's estimate and we account for the change in the implied constant as the cubes vary. In the cusps of the manifold, we simply use trivial upper and lower bounds. The lower bound is 0 and the upper bound is Weyl's law bound. (The latter bound is not actually trivial. We call it trivial as we are making no attempt to get an error term.) The upper bound may be obtained by Donnelly's method, or it may be obtained by a modification of the Lax-Phillips method, using elementary approximations. At this point we have an estimate with error term in the compact part of the manifold and we have upper (lower) bounds for the Neumann (Dirichlet) problem in the cusp. The larger the piece of the manifold in which we use our compact estimates, the larger the total error for the compact part, but the smaller the error arising from the cusp. Balancing these two errors proves the theorem. It should also be remarked that if one does not want to obtain an error term, one can use this method to obtain an elementary proof of Donnelly's theorem by modifying the classical techniques used by Lax and Phillips to create approximations to the min-max problem in Euclidean space. (Actually a few minor complications occur, but we do not need this for our results and will not remark on it further.)

We now describe the contents of the rest of the paper. In §2 we introduce additional notation and several more definitions. We also give some technical results, relating to certain sets and operators in a coordinate system that we have introduced. §3 contains the proof of the theorems. We first introduce spaces of functions and show that the Laplacian has pure point spectrum. We do this by studying certain Fourier expansions that are introduced. It should be mentioned that Donnelly's technique could also be used here. We next prove the needed theorems about the compact manifolds with boundary. A key problem is that certain estimates must be kept uniform, or at least their behavior must be controlled. We use the fact that we are working with a symmetric space to make certain estimates uniform. For other estimates, related to Agmon's constant, we control certain estimates involving compact regions that approach the cusp by combining certain geometric considerations with explicit formulas. After this is done we are able to prove our theorem. What remains to be done is to decide, in terms of a parameter that has been introduced, where one should start to use the cusp estimate and where one should use the compact contribution's estimate. Balancing the errors proves the theorem. We close the paper by pointing out that the arguments hold in more general situations, and by giving results for some cases of interest.

2. Definitions, notation, and preliminary results

In this section, we introduce some notation that has not yet been introduced, and make certain definitions that we will need. We also present certain known results that will be needed later.

We will first introduce a coordinate system for $\mathscr{H}$. It is well known that any matrix g in the group $\mathrm{Gl}_n(R)$ may be decomposed as $g = nak$, where

$$a = \begin{bmatrix} y_1 & \cdots & y_{n-1} & & & \\ & & y_1 & \cdots & y_{n-2} & \\ & & & \cdots & & y_1 \\ & & & & \ddots & & 1 \end{bmatrix}, \qquad y_i > 0, \tag{2.1a}$$

$$n = \begin{bmatrix} 1 & x_{1,2} & \cdots & x_{1,n} \\ & 1 & \cdots & \\ & & \ddots & x_{n-1,n} \\ & \cdots & & 1 \end{bmatrix}, \tag{2.1b}$$

and k is orthogonal. This is generally known as the Iwasawa decomposition. The strange choice of coordinates will make certain later results easier to state. We may choose the x's and y's above as coordinates for $\mathscr{H}$. We may think of $\mathscr{H}$ as an equivalence class of matrices.

Given a matrix g, it acts on $\mathscr{H}$ as follows. For a point τ in $\mathscr{H}$, take any element of the equivalence class of matrices, and by abuse of notation call it τ. Consider the matrix multiplication $g\tau$. By rewriting this in terms of the Iwasawa decomposition, we obtain a unique element of $\mathscr{H}$.

For our purposes it is easier not to work with X, but with a subset of $\mathscr{H}$, denoted by $\mathscr{D}$, that will be a fundamental domain for Γ. By a fundamental domain we will mean an open set $\mathscr{D}$ such that

$$\gamma \circ \mathscr{D} \cap \gamma' \circ \mathscr{D} = \varnothing, \quad \text{unless } \gamma = \gamma', \tag{2.2i}$$

$$\bigcup_{\gamma\in\Gamma} \gamma\overline{\mathscr{D}} = \mathscr{H}. \tag{2.2ii}$$

It is a general fact that such sets do exist. They are certainly not unique.

When X is not compact, but is of finite volume, any fundamental domain is not a compact subset of $\mathscr{H}$, but has a finite number of cusps where the fundamental domain approaches the boundary of $\mathscr{H}$. The fundamental domain may be divided into regions for our purposes, as in Lax-Phillips, such that one region is compact and all of the other regions contain exactly one cusp. In this way we may reduce our problem to the case of one cusp. We may also assume that we are working with a full rank cusp, as other cases are essentially lower dimensional problems. We may assume that the cusp is located at infinity, in other words all of the y variables are at infinity. By arithmeticity and the solution of the congruence subgroup problem [Ma, Zi], we may assume that we are studying a congruence subgroup and so the cusp looks essentially like the cusp for $\mathrm{Sl}_n(Z)$. We now describe a fundamental domain for this group, due to Grenier [Gr]. In future arguments we will often assume for notational simplicity that our cusp has this shape, although all that needs to be changed for the general case, as far as the cusp is concerned, is a change in the constants that appear in the inequalities describing the domain. This fundamental domain is particularly well suited for our purposes, although of course the result does not depend on the choice of fundamental domain. First we note that another Iwasawa decomposition of a matrix in $\mathrm{Gl}_n(R)$ is

$$g = \begin{pmatrix} v & \\ & W \end{pmatrix} \begin{bmatrix} 1 & {}^t x \\ & I_{n-1} \end{bmatrix}, \tag{2.3}$$

where $(a)[b]$ means ${}^t bab$, ${}^t b$ being the transpose of the matrix b, v is a positive real number, W is in $\mathrm{Gl}_{n-1}(R)$, and $x \in R^{n-1}$. Changing between coordinates is tedious. We will later state in our coordinates the information we actually need.

Lemma 2.4. *A fundamental domain for* $\mathrm{Gl}_n(Z)$ *is given by the set of elements satisfying*

$$V < v[a + {}^t xc] + W[c], \quad \textit{for } a \in Z,\ c \in Z^{n-1} - 0, \tag{2.5i}$$

$$W \textit{ is in a fundamental domain for } \mathrm{Gl}_{n-1}(R), \tag{2.5ii}$$

$$0 < x_1 < \tfrac{1}{2}, \qquad |x_i| < \tfrac{1}{2} \quad \textit{for } i = 2, 3, \ldots. \tag{2.5iii}$$

We remark that v being small and the inductive procedure give $y_i > \frac{\sqrt{3}}{2}$.

A particular consequence of this lemma of importance to us is that if all the y variables are sufficiently large, then only the inequalities of the form $0 < x < \frac{1}{2}$ involve the x variables. Thus the fundamental domain has a box shape at infinity [Te]. We now refine this notion.

Let M_j denote a matrix with $n-1-j$ columns and j rows. Let P_j denote matrices of the form

$$P_j = \begin{pmatrix} I & M_j \\ 0 & I \end{pmatrix}, \tag{2.6}$$

so, for example, if $n = 4$

$$P_2 = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

Each P_j forms an abelian group. Also, if y_j is sufficiently large, the x variables contained in P_j inside the fundamental domain will have a box shape, in other words, only the obvious inequalities are needed.

We now give some facts about analysis on $\mathscr{H}$. We only state the results in terms of our coordinates, although some of them are easier to show by using partial Iwasawa decompositions. We only state what is needed for our proofs.

The invariant volume element is given by

$$dv = \frac{d(\mathrm{Eucl})}{y_1^{n_1} \cdots y_{n-1}^{n_{n-1}}}, \tag{2.8}$$

where $n_i \geq n$. Further, if one considers a Euclidean cube with sides in the y variables of length one and containing a fundamental domain for the x variables, we have that as $y_1 \cdots y_{n-1} \to \infty$ the ratio of the surface area to the volume of the cube is $O(y_1 \cdots y_{n-1})$.

The Laplacian on $\mathscr{H}$ in our coordinates decomposes into several positive semidefinite operators. The terms involving derivatives with respect to y variables is one such term. The others are terms involving partial derivatives with respect to x variables that all lie in the same column. Implicit in this statement is that no other mixed partials exist. We now must know the spectral decomposition of the space of square integrable functions on the fundamental domain. This space may be decomposed as

$$L^2(\mathscr{D}) = L_c^2(\mathscr{D}) \oplus L_0^2(\mathscr{D}) \oplus L_r^2(\mathscr{D}), \tag{2.9}$$

where c represents the part of the space corresponding to continuous spectrum, in other words, the contribution due to the Eisenstein series, r corresponds to functions that are not cuspidal but are square integrable, and 0 corresponds to the cusp forms, which we shall define shortly. We must first give several other definitions.

Definition 2.10. A differential operator D is an invariant differential operator if

$$D\phi(\gamma \circ \tau) = D(\tau)|_{\gamma\circ\tau}$$

for all functions ϕ and for all $\gamma \in \mathrm{Gl}_n(R)$. The Laplacian is one of $n-1$ independent such D.

Next, given any square integrable function, we define the following Fourier expansions.

Definition 2.11. For the parabolic P_i, which for its M_j forms an $m \times (n-m)$ matrix, we can define a Fourier expansion by

$$\phi(\tau) = \sum_{a \in Z^{m(n-m)}} c e^{2\pi i \operatorname{Tr}({}^t a p)}, \tag{2.12}$$

where the p are the variables in the parabolic, and the constant c depends on all other variables.

We will use this definition at several points in this paper. Our first use is to define a cusp form.

Definition 2.13. A cusp form is a function in $L^2(\mathscr{D})$ such that it is an eigenfunction for all invariant differential operators, and, for all Fourier expansions, the zeroth coefficient is identically zero.

Our emphasis in the rest of the paper is on the set of cuspidal eigenfunctions. It should also be noted that for our purposes we do not actually need to consider cusp forms. Instead we could consider eigenfunctions of the Laplace-Beltrami operator that satisfy the cuspidal condition at infinity. It is conceivable that there is no difference between these problems, but for certain spaces current technology can yield a better result if one considers a true cusp form.

3. Proof of the theorems

As we noted before, the space $H = L^2(\mathscr{D})$ has continuous spectrum with respect to the Laplacian. This makes certain classical arguments not directly usable. We will remedy this situation by creating several function spaces such that they have pure point spectrum for the Laplacian. We will first create one such space, and its elements will also include in it all true cusp forms. We then create families of auxiliary spaces by adding Dirichlet or Neumann conditions at certain boundaries introduced to the fundamental domain. These spaces will clearly also have pure discrete spectrum, and so we will be able to use Dirichlet-Neumann bracketing. We will also need to introduce function spaces over certain compact regions of the fundamental domain. In the sequel we will denote the space of cusp forms by H_0.

Before introducing the function spaces we wish to interpret the Laplacian on a set of functions on the fundamental domain in terms of a quadratic form. First consider C^2 compactly supported functions on $\mathscr{H}$ that satisfy the appropriate boundary conditions for the fundamental domain's boundaries. This will eventually be completed to get the Laplacian on the full Hilbert space of square integrable functions. If boundary surfaces have been added to $\mathscr{D}$ we may instead impose along these surfaces the condition that the functions vanish (Dirichlet) or have vanishing normal derivative (Neumann) along the boundaries. Also if a surface divides $\mathscr{D}$ into a compact and noncompact part we may impose the condition that all Fourier coefficients vanish in the noncompact part (after the boundary).

We consider the following quadratic form:

$$C(\phi, \psi) = \int_{\mathscr{D}} (\nabla\phi\nabla\psi + \phi\psi)\, dV. \tag{3.1}$$

Of course a particular consequence of our definition (3.1) is

$$C(\phi) = \int_{\mathscr{D}} (|\nabla\phi|^2 + |\phi|^2)\, dV. \tag{3.2}$$

The completion of our space of core functions with respect to the quadratic form will be considered to be the Laplacian plus the identity. It is easy to see by integration by parts that for sufficiently smooth functions

$$\int_{\mathscr{D}} |\nabla\phi|^2\, dV = \int_{\mathscr{D}} \phi\Delta\phi\, dV. \tag{3.3}$$

We will generally work with the quadratic form.

We now introduce the function spaces that we will use in the rest of the paper. We begin by introducing the space that will contain all of the cusp forms, and will have pure point spectrum with respect to the Laplacian. Fix a number k. Consider the set of functions as above such that if $\max_i\, y_i > k$ in the fundamental domain, then all the cuspidal conditions hold. We will let H' denote the completion of this set of functions with respect to the quadratic form C. This then gives us the Laplacian acting on the space. (By abuse of notation, we can keep using C for the quadratic form even though it is acting on different function spaces.)

Proposition 3.4. *The Laplacian Δ, acting on H^1, has pure point spectrum.*

The proof of this proposition follows from standard functional analysis after we have proved the following proposition.

Proposition 3.5. *The unit ball in H^1 is precompact in L^2.*

Proof. The Rellich criterion gives us that the ball is locally compact in L^2, so the real work is in the cusp. We must show that for any ϕ contained in $\mathscr{B}$

$$\lim_{j\to\infty} \int_{\mathscr{D}\cap\{\min y_i > j\}} |\phi|^2\, dV = 0 \tag{3.6}$$

uniformly.

The integral is clearly less than

$$\lim_{j\to\infty} \sum_i \int_{\mathscr{D}\cap\{y_i > j\}} |\phi|^2\, dV. \tag{3.7}$$

We will show that each of these integrals tends to zero uniformly. We use each of the Fourier expansions with respect to a P_i. We first consider y_1 going to infinity and the Fourier expansion with respect to P_1. We change our notation from before and write the Fourier expansion as

$$\phi(\tau) = \sum_{\vec{\alpha}} \phi_{\vec{\alpha}}(\tilde{\tau}) e^{2\pi i \vec{\alpha}\cdot\vec{x}}, \tag{3.8}$$

where $\vec{x} = (x_{1,n}, \dots, x_{n-1,n})$, $\tilde{\tau}$ denotes the variables not in the parabolic, and $\vec{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$. The cuspidal condition implies that

$$\phi_{\vec{\alpha}}(\tilde{\tau}) = 0.$$

We now refine our orthogonal decompositon. We first decompose the space as

$$H^1 = L_1 \oplus L_0, \tag{3.9}$$

where L_1 consists of functions ϕ_1 such that the single integral with respect to $x_{1,n}$ vanishes.

For the first of these spaces we have the expansion

$$\phi_1(\tau) = \sum_{\alpha_1 \neq 0} \phi_{\vec{\alpha}}(\tilde{\tau}) e^{2\pi i \vec{\alpha} \cdot \vec{x}}. \tag{3.10}$$

We now consider the quadratic forms effect on this expansion. We have

$$\begin{aligned} \int_{\mathscr{D} \cap \{y_1 > j\}} |\phi_1|^2 \, dV &\ll \int_{\mathscr{D}' \cap \{y_i > j\}} \sum_{\alpha_1 \neq 0} \alpha_1^2 |\phi_{\vec{\alpha}}(\tilde{\tau})|^2 \, dV' \\ &= \int_{\mathscr{D} \cap \{y_i > j\}} \left| \frac{\partial \phi_1}{\partial x_{1,n}} \right|^2 dV \\ &\ll \frac{1}{j^2} \int_{\mathscr{D}} |\nabla \phi|^2 \, dV \ll \frac{1}{j^2}. \end{aligned} \tag{3.11}$$

Here dV' means dV, where the integration with respect to $x_{1,n}$ is excluded, and $\mathscr{D}'$ is the appropriately restricted domain.

This is because there is a term in the gradient of the form

$$y_1^2 \left| \frac{\partial \phi_1}{\partial x_{1,n}} \right|^2$$

and the totality of all other terms are positive and may be ignored for our purposes. Hence for this space we have our result.

For the other half of (3.9), we refine it, giving

$$L_0 = L_{01} \oplus L_{00}, \tag{3.12}$$

where functions in L_{01} are such that the integral in the $x_{2,n}$ variables vanishes.

We now proceed as before. We need only note that as $\phi_{\vec{\alpha}}(\tilde{\tau})$ vanishes for all functions in this space, with $\alpha_1 = 0$, there will be a strictly positive term for the gradient here as well. For the first parabolic we continue in this manner.

To study the behavior of the integral as y_2 tends to infinity we use the Fourier expansion with respect to the parabolic P_2. Our procedure is similar to the procedure described above. We first refine the spectral decomposition into a term where the Fourier coefficient with respect to the upper right-hand x variable vanishes and one where it does not. In the second case our argument is as above. In the first case we refine the expansion again, this time with our interest being with the x variable below it in the same column. We proceed down the column in this way, until we have used all variables relevant to this parabolic. We then move to the top variable of the column to the left. We proceed as before. The key point in all of this is if all of the previously considered variables have a vanishing Fourier coefficient term, then the variable under consideration has a term in the gradient that is positive. Also, not all of the variables can have vanishing Fourier coefficient by the cuspidal condition that holds in the cusp. We then move to the third parabolic to study the behavior as y_3 tends to infinity and continue in this manner until all parabolics have been studied. This completes the proof of Proposition 3.5 and hence gives the proof of Proposition 3.4.

It is clear that H' contains H_0. We thus want to obtain an estimate for the counting function for H'. We will first have to introduce many auxiliary function spaces. Either these spaces cover only a compact region of the fundamental domain, or are obtained by adding Dirichlet or Neumann boundary conditions to H'. Thus in either case we will clearly be working with spaces that have pure discrete spectrum for the Laplacian.

Definition 3.13. Let $H_{D,h}$ denote the completion of the space of functions that satisfy the cuspidal condition for $\max_i y_i > k$ (after k) that vanish if

$$\prod_{y_i>1} y_i \geq h.$$

Let $H_{N,h}$ denote the completion of the set of functions that satisfy the cuspidal conditions after k, and Neumann conditions along

$$\prod_{y_i>1} y_i = h.$$

For a "cube" in the fundamental domain (for small values of our parameters, we choose the shape that keeps us in the fundamental domain) with sides for the y variables of Euclidean length one and bottom corner for the y variables, let

$$(y_1, y_2, \dots, y_{n-1}) = \vec{y}$$

denote the bottom corner of the cube. By this we mean the y variables are minimized. For simplicity, if we are not at the bottom of the fundamental domain, we assume the y_i are integers. Let $H_{D,\vec{y}}$ denote the space of L^2 functions in the cube that satisfy Dirichlet boundary conditions and the cuspidal conditions. Let $H_{N,\vec{y}}$ denote the analogous space for the Neumann problem. Let $H_{D,k}$ or $H_{N,k}$ denote the space with Dirichlet (Neumann) conditions along the boundary in the region $\max_i y_i \leq k$. Let $H_{N,h,\infty}$ denote the space of cuspidal functions with Neumann conditions defined for

$$\prod_{y_i>1} y_i > h.$$

Remark. If $\Gamma \neq \mathrm{Sl}_n(Z)$, only minor changes are needed, both here and in previously used Fourier expansions.

The following proposition is clear from Dirichlet-Neumann bracketing.

Proposition 3.14. *Choose "cubes" in the fundamental domain so that they cover the fundamental domain for*

$$\prod_{y_i>1} y_i \leq h.$$

Call the collection chosen $\mathscr{C}$; *then*

$$N_{D,k}(\lambda) + \sum_{\mathscr{C}} N_{D,\vec{y}}(\lambda) \leq N_{D,h}(\lambda) \leq N'(\lambda) \leq N_{N,h}(\lambda)$$
$$\leq N_{N,k}(\lambda) + \sum_{\mathscr{C}} N_{N,\vec{y}}(\lambda) + N_{N,h,\infty}(\lambda).$$

We now must analyze the right- and left-hand contributions of the previous inequality. The easiest term to handle is the contribution due to the compact parts of the space. For this contribution we need merely quote Seeley's result and note that for this situation we are studying a fixed compact portion of the fundamental domain.

Lemma 3.15.

$$N_{N,k}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}),$$
$$N_{D,k}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}),$$

where V_s *is the volume of the region under consideration.*

Remark. By abuse of notation, we will use V_s for all of these volumes. They of course sum to the volume of $\mathscr{D}$. Only the volume of the region after h will concern us. It is $O(h^{1-n}\log h)$.

We will next concern ourselves with the contributions to the various cubes that are considered. The first thing to note is that we may in these compact cubes in the Dirichlet case ignore the cuspidal conditions when we are concerned with asymptotic results. (We could produce a similar result for the Neumann case but it is not needed.) A similar result appears in the work of Lax and Phillips.

The reason that we may ignore the cuspidal condition is that the space of functions on the cube with Dirichlet boundary conditions may be orthogonally decomposed into a space that is cuspidal, in the sense that we may study one of its Fourier expansions and decompose the space into functions for which the zeroth Fourier coefficient vanishes and those for which only the zeroth Fourier coefficient is nonzero. Functions belonging to the second class of functions are really functions on a space of lower dimension and we may obtain through a Weyl's argument (or by Euclidean approximations as in Lax and Phillips) that they give a lower order contribution.

We now study the contribution to the counting function of a given cube, with either Dirichlet or Neumann boundary conditions. Of course not all of these regions are cubes but the only thing that we need is the fact that there are only a finite number of different geometric shapes that need be considered. We will give our argument in such a way that it applies to any of the cases considered.

For each cube, the spectral function $e(x, y, t)$ (we abuse notation here by not using subscripts) is such that in the interior of the cube, in other words in the region where the distance to the boundary is greater than t, we may, as we are working with a symmetric space, assume that we are dealing with the free space problem. Because of this we may obtain the free space estimate

$$|e(x, y, t) - C\lambda^r| \ll \left(\frac{1}{t^r} + \frac{\lambda^{r-1}}{t}\right). \tag{3.16}$$

Near the boundary, in other words when the above estimate does not hold, we use Agmon's estimate. The constant in his estimate depends on the cone condition for the domain. This constant will become worse linearly with the inverse of the volume of the region under consideration. However, as this estimate is only used near the boundary, it is only used in a region of volume equal to the portion of the cube that is within t of the boundary. This region is approximately proportional to the surface area of the cube. Combining all of this we get that the error caused by Agmon's estimate grows by the ratio of the surface area to the volume of the cube. As noted before, this ratio is $O(y_1 \dots y_{n-1})$, and so we have, following the work of Seeley, the following lemma.

Lemma 3.17.

$$N_{N,\vec{y}}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(y_1 \cdots y_{n-1}\lambda^{\frac{r-1}{2}} \log\lambda),$$
$$N_{D,\vec{y}}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(y_1 \cdots y_{n-1}\lambda^{\frac{r-1}{2}} \log\lambda).$$

Before finally proving our theorems, we must consider the noncompact contributions that we will need. For the Dirichlet problem there is really nothing to say. We are looking for a lower bound in this case, and clearly the contribution of the cusp is nonnegative. This will be all that we need. For the Neumann case, the following rather easy estimate will suffice for our purposes. This estimate follows from Donnelly's work, but can also be derived by an argument that is a variant of the Lax and Phillips argument. In both cases it is essential that one work with functions that are cuspidal in the cusp. Donnelly clearly uses this in his heat equation argument. In a Lax-Phillips argument the key point is that the constant function is not allowed. Then one finds from an easy upper bound estimate of additional cubes into the space above the line b, that eventually there is no contribution at all. We state a lemma that handles the Neumann case. What we of course have is that the Weyl upper bound holds in the cusp.

Lemma 3.18.

$$N_{N,h,\infty}(\lambda) = O(V_s\lambda^{\frac{r}{2}}) = O(V_s\lambda^{\frac{r}{2}}h^{1-n}\log h).$$

We now state a lemma that handles the totality of the compact contributions.

Lemma 3.19.

$$\sum_{\mathscr{C}} N_{N,\vec{y}}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}h^2(\log h)^{n-2}\log\lambda),$$
$$\sum_{\mathscr{C}} N_{D,\vec{y}}(\lambda) = CV_s\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}h^2(\log h)^{n-2}\log\lambda).$$

The proof of this lemma simply consists of summing up the various contributions that have already been studied.

The above results lead us to the following proposition.

Proposition 3.20.

$$N'_{N,h}(\lambda) = C\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}h^2(\log h)^{n-2}\log\lambda + \lambda^{\frac{r}{2}}(\log h)h^{1-n}),$$
$$N'_{D,h}(\lambda) = C\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}h^2(\log h)^{n-2}\log\lambda + \lambda^{\frac{r}{2}}(\log h)h^{1-n}).$$

The proof of Theorem 1 now follows by choosing the parameter to minimize the error.

For completeness we now prove Theorem 2. Since

$$N'(\lambda+1) - N'(\lambda-1) = C((\lambda+1)^{\frac{r}{2}} - (\lambda-1)^{\frac{r}{2}}) + O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n+1}}(\log\lambda)^{n-1}),$$

we have

$$M_0(\lambda) < N'(\lambda+1) - N'(\lambda-1). \tag{3.21}$$

Simple algebra now proves Theorem 2.

We should note several extensions of the above theorems. First, we need not assume that we have a locally symmetric space but that we merely have a manifold such that its cusps are isometric to a symmetric space. Second, the technique described is quite general, and one does not have to work with

$Gl_n(R)$. For example, the same sort of technique, when applied to products of half-spaces (Hilbert modular groups being important examples of this case) or to Siegel's upper half-plane, yield the following results.

Theorem. *If H is the product of symmetric spaces of possibly different dimension, and $n_0 \geq 3$ is the minimum dimension of the half-spaces, then*

$$N'(\lambda) = C_r(\operatorname{Vol} X)\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{n_0+1}}(\log\lambda)^{n_0-1}).$$

If $G = \mathrm{Sp}_n(R)$, then

$$N'(\lambda) = C_r(\operatorname{Vol} X)\lambda^{\frac{r}{2}} + O(\lambda^{\frac{r-1}{2}}\lambda^{\frac{1}{2n+1}}(\log\lambda)^{n-1}).$$

For the general symmetric space the technique yields an improvement over Donnelly's upper bound for the cuspidal counting function.

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