

A SHORT PROOF OF ZHELUDEV'S THEOREM

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ABSTRACT. We give a short proof of Zheludev's theorem that states the existence of precisely one eigenvalue in sufficiently distant spectral gaps of a Hill operator subject to certain short-range perturbations. As a by-product we simultaneously recover Rofe-Beketov's result about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. Our methods are operator theoretic in nature and extend to other one-dimensional systems such as perturbed periodic Dirac operators and weakly perturbed second order finite difference operators. We employ the trick of using a selfadjoint Birman-Schwinger operator (even in cases where the perturbation changes sign), a method that has already been successfully applied in different contexts and appears to have further potential in the study of point spectra in essential spectral gaps.

Our main hypothesis reads:

(I) Let $V \in L^1_{\text{loc}}(\mathbb{R})$ be real-valued and of period $a > 0$, and suppose $W \in L^1(\mathbb{R}, (1 + |x|) dx)$ to be real-valued, $W \neq 0$ on a set of positive Lebesgue measure.

Given V , one defines the Hill operator H_0 in $L^2(\mathbb{R})$ as the form sum of the Laplacian in $L^2(\mathbb{R})$,

$$(1) \quad -\frac{d^2}{dx^2} \quad \text{on } H^2(\mathbb{R}),$$

and the operator of multiplication by V ,

$$(2) \quad H_0 := -\frac{d^2}{dx^2} \dot{+} V.$$

(To be more precise, since V is not assumed to be continuous, we should define H_0 as a direct integral over reduced operators on $L^2([0, a])$, see [12, §XIII.16].) Similarly, the perturbed Hill operator H_g is defined as the form sum in $L^2(\mathbb{R})$

$$(3) \quad H_g := H_0 \dot{+} gW, \quad g > 0.$$

Standard spectral theory [2, 10, 11, 12] then yields that

$$(4) \quad \sigma(H_0) = \sigma_{\text{ac}}(H_0) = \bigcup_{n \in \mathbb{N}} [E_{2(n-1)}, E_{2n-1}], \\ -\infty < E_0 < E_1 \leq E_2 < E_3 \leq E_4 < \dots,$$

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$$(5) \quad \begin{aligned} \sigma_p(H_0) &= \sigma_{sc}(H_0) = \emptyset, \\ \sigma_{ess}(H_g) &= \sigma_{ac}(H_g) = \sigma(H_0), \quad \sigma_{sc}(H_g) = \emptyset. \end{aligned}$$

The spectral gaps of H_0 (the essential spectral gaps of H_g) are denoted by

$$(6) \quad \rho_0 := (-\infty, E_0), \quad \rho_n := \begin{cases} (E_{2n-1}, E_{2n}), & E_{2n-1} < E_{2n}, \\ \emptyset, & E_{2n-1} = E_{2n}, \quad n \in \mathbb{N}. \end{cases}$$

Moreover one has

$$(7) \quad \sigma_p(H_g) \subset \bigcup_{n \in \mathbb{N}_0} \rho_n$$

and all eigenvalues of H_g are simple. (Here $\sigma(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_{sc}(\cdot)$, and $\sigma_p(\cdot)$ denote the spectrum, absolutely continuous spectrum, singularly continuous spectrum, and point spectrum (the set of eigenvalues) respectively.) Following the usual terminology we call ρ_n an open spectral gap whenever $\rho_n \neq \emptyset$.

The purpose of this paper is to give a short proof of the following theorem that summarizes results of Firsova, Rofe-Beketov, and Zheludev:

Theorem 1 [3, 4, 6, 13, 14, 17, 18]. *Assume Hypothesis (I). Then*

- (i) H_g has finitely many eigenvalues in each open gap ρ_n , $n \geq 0$.
- (ii) H_g has at most two eigenvalues in every open gap ρ_n for n large enough.
- (iii) If $\int_{\mathbb{R}} dx W(x) \neq 0$, H_g , $g > 0$ has precisely one eigenvalue in every open spectral gap ρ_n for n sufficiently large.

Remark 2. Parts (i) and (ii) are due to Rofe-Beketov [13]. Part (iii), under the additional conditions $\text{sgn}(W) = \text{constant}$, $W \in L^1(\mathbb{R}; (1+x^2)dx)$, V piecewise continuous and W bounded is due to Zheludev [17]. In [18] the condition $\text{sgn}(W) = \text{constant}$ has been replaced by $\int_{\mathbb{R}} dx W(x) \neq 0$ but it has been left open as to whether there are one or two eigenvalues in sufficiently distant spectral gaps ρ_n . The present version of (iii) was first proved by Firsova [3, 4] (see also [6]) and Rofe-Beketov [14] on the basis of ODE methods. The case of a perturbed Hill operator on the halfline $(0, \infty)$ has also been studied in [8].

Before we give a short proof of Theorem 1 based on operator theoretic methods we need to prepare various well-known results on Hill operators and establish some further notation.

The Green's function $G_0(z, x, x')$ (the integral kernel of the resolvent $(H_0 - z)^{-1}$) reads

$$(8) \quad \begin{aligned} G_0(z, x, x') &= W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0))^{-1} \\ &\times \begin{cases} \psi_-(z, x, x_0)\psi_+(z, x', x_0), & x \leq x', \\ \psi_+(z, x, x_0)\psi_-(z, x', x_0), & x \geq x', \end{cases} \\ & \quad x_0 \in [0, a], z \in \mathcal{R}. \end{aligned}$$

Here $W(f, g)$ denotes the Wronskian of f and g ,

$$(9) \quad W(f, g)(x) := f(x)g'(x) - f'(x)g(x),$$

and ψ_{\pm} are the Floquet solutions of H_0 defined by

$$(10) \quad \begin{aligned} \psi_{\pm}(z, x, x_0) &:= c(z, x, x_0) + \phi_{\pm}(z, x_0)s(z, x, x_0), \quad z \in \mathcal{R}, x \in \mathbb{R}, \\ \psi_{\pm}(z, x_0, x_0) &= 1, \quad z \in \mathcal{R}, \end{aligned}$$

$$(11) \quad \begin{aligned} \phi_{\pm}(z, x_0) := & \{\Delta(z) \pm [\Delta(z)^2 - 1]^{1/2} \\ & - c(z, x_0 + a, x_0)\}s(z, x_0 + a, x_0)^{-1}, \\ & z \in \mathcal{R}, \end{aligned}$$

where Δ denotes the discriminant (Floquet determinant) of H_0 ,

$$(12) \quad \Delta(z) := [c(z, x_0 + a, x_0) + s'(z, x_0 + a, x_0)]/2, \quad z \in \mathbb{C},$$

and s, c is a fundamental system of distributional solutions of $H_0 f = z f$, $z \in \mathbb{C}$, with

$$(13) \quad \begin{aligned} s(z, x_0, x_0) = 0, \quad s'(z, x_0, x_0) = 1, \\ c(z, x_0, x_0) = 1, \quad c'(z, x_0, x_0) = 0, \quad z \in \mathbb{C}. \end{aligned}$$

Moreover, ψ_{\pm} are meromorphic functions on the two-sheeted Riemann surface \mathcal{R} of $[\Delta(z)^2 - 1]^{1/2}$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C} \setminus \sigma(H_0)$ (or $\mathbb{C} \setminus [\rho(H) \cap \mathbb{R}]$, $\rho(\cdot)$ the resolvent set) in the usual (crosswise) way. \mathcal{R} is assumed to be compactified if only finitely many spectral gaps of H_0 are open, otherwise \mathcal{R} is noncompact. Since we do not need this Riemann surface explicitly in the following considerations we assume that a suitable choice of cuts has been made and omit further details.

We note that s, c , and Δ are entire with respect to $z \in \mathbb{C}$, and Δ and G_0 are independent of the chosen reference point $x_0 \in [0, a]$. Especially, by considering a particular open gap $\rho_n = (E_{2n-1}, E_{2n})$, $n \geq 1$, one can always choose x_0 in such a way that the zeros of $s(z, x_0 + a, x_0)$ (there is precisely one simple zero in each $\overline{\rho_n}$, $n \geq 1$, they constitute the Dirichlet eigenvalues of H_0 restricted to $(x_0, x_0 + a)$) are not at $\partial_{\rho_n} = \{E_{2n-1}, E_{2n}\}$. (This fact is relevant in (11) and will be needed later on in (20).) From now on, when considering a particular gap ρ_n , we always assume that ρ_n is open, i.e., $\rho_n \neq \emptyset$. For simplicity we shall also assume that $E_0 \geq 1$ and for notational convenience we introduce $E_{-1} = 1$ (in order not to distinguish $n = 0$ and $n \geq 1$ in the following).

We also note that

$$(14) \quad W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0)) = -2[\Delta(z)^2 - 1]^{1/2}s(z, x_0 + a, x_0)^{-1}, \quad z \in \mathcal{R},$$

and

$$(15) \quad -2[\Delta(z)^2 - 1]^{1/2}G_0(z, x, x) = s(z, x + a, x), \quad z \in \mathbb{C}, \quad x \in \mathbb{R}.$$

Moreover, restricting z to the upper sheet \mathcal{R}_+ of \mathcal{R} from now on, the Floquet solutions ψ_{\pm} have the particular structure

$$(16) \quad \begin{aligned} \psi_{\pm}(z, x, x_0) = e^{\mp \alpha(z)(x-x_0)} p_{\pm}(\alpha(z), x, x_0), \\ p_{\pm}(\alpha(z), x + a, x_0) = p_{\pm}(\alpha(z), x, x_0), \quad z \in \mathcal{R}_+, \quad x \in \mathbb{R}, \end{aligned}$$

where $\alpha(z)$ is given by

$$(17) \quad \begin{aligned} \alpha(z) := a^{-1} \ln\{\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\}, \quad z \in \mathcal{R}_+, \\ \cosh[\alpha(z)a] = \Delta(z), \quad \sinh[\alpha(z)a] = [\Delta(z)^2 - 1]^{1/2}, \end{aligned}$$

and the branch of $[\Delta(z)^2 - 1]^{1/2}$ on \mathcal{R}_+ is chosen such that

$$(18) \quad \psi_{\pm}(z, \cdot, x_0) \in L^2(0, \pm\infty), \quad z \in \mathcal{R}_+ \setminus \sigma(H_0).$$

α (resp. $\alpha - \pi i$) is positive on open gaps ρ_{2n} (resp. ρ_{2n+1}), $n \in \mathbb{N}_0$, and monotonic near E_0, E_{4n-1}, E_{4n} (resp. E_{4n-3}, E_{4n-2}), $n \in \mathbb{N}$.

We also note the asymptotic relations

$$(19) \quad s(\lambda, x_0 + a, x_0) \underset{\lambda \uparrow \infty}{=} \lambda^{-1/2} \sin[\lambda^{1/2} a] + O(\lambda^{-1}),$$

and [18]

$$(20) \quad \begin{aligned} & p_{\pm}(\alpha(E_{r(n)}), x, x_0)^2 \\ & \underset{n \rightarrow \infty}{=} \frac{1}{2} \left[1 + \frac{a^2}{4n^2 \pi^2} \frac{c'(E_{r(n)}, x_0 + a, x_0)}{s(E_{r(n)}, x_0 + a, x_0)} \right] \\ & \quad \cdot \{ 1 - \cos[(4n\pi/a)(x - x_0) + 2\delta_{r(n)}] + O(n^{-1}) \}, \end{aligned}$$

$$(21) \quad \delta_{r(n)} := \arctan \left\{ (2n\pi/a) \left| \frac{s(E_{r(n)}, x_0 + a, x_0)}{c'(E_{r(n)}, x_0 + a, x_0)} \right|^{1/2} \right\},$$

$$r(n) = 4n - 1, 4n,$$

and similarly for the odd open gaps ρ_{2n+1} , $n \in \mathbb{N}_0$. (In order to avoid that $s(E_{r(n)}, x_0 + a, x_0) = 0$ in (20), we tacitly made use of the fact that we may choose $x_0 = x_0(n)$ appropriately without affect Δ and the Green's function G_0 in (8). Such a choice will always be assumed in the following.)

Given these preliminaries we can split the Green's function G_0 into two parts as follows. For simplicity we only consider even open gaps ρ_{2n} , $n \in \mathbb{N}_0$, in details. The analysis for odd gaps ρ_{2n+1} , $n \in \mathbb{N}_0$, is completely analogous.

$$(22) \quad \begin{aligned} G_0(\lambda, x, x') &= -[s(\lambda, x_0 + a, x_0)/2 \sinh \alpha(\lambda)] p_{(4n-1)}(\alpha(E_{4n}), x, x_0) \\ & \quad \cdot p_{(4n-1)}(\alpha(E_{4n}), x', x_0) + R_0(\lambda, x, x'), \\ p_{(4n-1)}(\alpha(E_{4n}), x, x_0) &:= p_{+ (4n-1)}(\alpha(E_{4n}), x, x_0) \\ &= p_{- (4n-1)}(\alpha(E_{4n}), x, x_0), \quad x \in \mathbb{R}, \end{aligned}$$

for $\lambda \in [E_{4n} - \varepsilon_n, E_{4n}]$ ($\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n]$) with $\varepsilon_n > 0$ sufficiently small, $n \in \mathbb{N}_0$. One has the bound [13, 17]

$$(23) \quad |R_0(\lambda, x, x')| \leq C |E_{4n-1}|^{-1/2} (1 + |x| + |x'|),$$

$$\lambda \in \overline{\rho_{2n}}, \alpha(\lambda) \in [0, \varepsilon_n], x, x' \in \mathbb{R},$$

with C independent of $n \in \mathbb{N}_0$. Since Zheludev [17, 18] relies on the estimate (23), he is forced to assume $W \in L^1(\mathbb{R}; (1 + x^2) dx)$ in order to make the integral kernel $|W(x)|^{1/2} R_0(\lambda, x, x') |W(x')|^{1/2}$ to be the integral kernel of a bounded (in fact Hilbert-Schmidt) operator in $L^2(\mathbb{R})$. In order to avoid this limitation we shall employ instead a device from [1] and use a different splitting of G_0 :

$$(24) \quad \begin{aligned} G_0(z, x, x') &= G_0(z, x_0, x_0)^{-1} G_0(z, x, x_0) G_0(z, x_0, x') \\ & \quad + G_{0, x_0}^D(z, x, x') \\ & := \gamma(z) P_{x_0}(z, x, x') + G_{0, x_0}^D(z, x, x'), \\ \gamma(z) &:= -\{s(z, x_0 + a, x_0)/2 \sinh[\alpha(z)a]\}, \end{aligned}$$

where $G_{0,x_0}^D(z, x, x')$ denotes the integral kernel of the resolvent of the Dirichlet operator H_{0,x_0}^D obtained from H_0 by imposing an additional Dirichlet boundary condition at x_0 . Explicitly we have

$$(25) \quad P_{x_0}(\lambda, x, x') = \begin{cases} \psi_-(\lambda, x, x_0), & x \leq x_0 \\ \psi_+(\lambda, x, x_0), & x \geq x_0 \end{cases} \begin{cases} \psi_-(\lambda, x', x_0), & x' \leq x_0 \\ \psi_+(\lambda, x', x_0), & x' \geq x_0 \end{cases},$$

$$\lambda \in \overline{\rho_{2n}}, \quad n \in \mathbb{N}_0,$$

and, similar to (3.7) in [1],

$$(26) \quad |G_{0,x_0}^D(\lambda, x, x')| \leq C|E_{2n-1}|^{-1/2}|x_{<}| \leq C|E_{2n-1}|^{-1/2}|x|^{1/2}|x'|^{1/2},$$

$$\lambda \in \overline{\rho_{2n}}, \quad n \in \mathbb{N}_0, \quad \alpha(\lambda) \geq 0 \text{ small enough,}$$

where C is independent of n and

$$(27) \quad |x_{<}| := \begin{cases} 0, & x \leq x_0 \leq x' \text{ or } x' \leq x_0 \leq x, \\ \min(|x - x_0|, |x' - x_0|) & \text{otherwise.} \end{cases}$$

In order to derive (26) one separately considers the four regions $x \leq x' \leq x_0$, $x' \leq x \leq x_0$, $x_0 \leq x' \leq x$, $x_0 \leq x \leq x'$ (the cases $x \leq x_0 \leq x'$, $x' \leq x_0 \leq x$ being trivial) and uses the mean value theorem to bound

$$(28) \quad |p_+(\alpha(\lambda), y, x_0) - p_-(\alpha(\lambda), y, x_0)| \leq D\alpha(\lambda)|y - x_0|,$$

$$\lambda \in \overline{\rho_{2n}}, \quad \alpha(\lambda) \geq 0 \text{ small enough,}$$

with D independent of $n \in \mathbb{N}_0$.

Finally, we introduce Birman-Schwinger type operators and related quantities. We distinguish three cases and again study even (open) gaps ρ_{2n} , $n \in \mathbb{N}_0$ for simplicity.

(a) $W \leq 0$. We factorize

$$(29) \quad w := |W|^{1/2}, \quad W = -w^2,$$

and define the Birman-Schwinger kernel by

$$(30) \quad k(\lambda) := -gw(H_0 - \lambda)^{-1}w, \quad \lambda \in \rho_{2n}, \quad n \in \mathbb{N}_0, \quad g > 0.$$

Then the selfadjoint Birman-Schwinger kernel satisfies $k(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}))$ ($\mathcal{B}_2(\cdot)$ the set of Hilbert-Schmidt operators) and due to (24)–(26)

$$(31) \quad \begin{aligned} k(\lambda) &= -\gamma(\lambda)gP(\lambda) - gM(\lambda), \quad \lambda \in \rho_{2n}, \\ \gamma(\lambda) &\underset{\lambda \rightarrow E_{4n}}{=} \underset{(4n-1)}{C_{4n}} |\alpha(\lambda)|^{-1}, \quad C_{4n-1} < 0 < C_{4n}, \quad n \in \mathbb{N}_0, \end{aligned}$$

where $P(\lambda)$, $\lambda \in \overline{\rho_{2n}}$, is a positive rank one projection, $M(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}))$, $\lambda \in \overline{\rho_{2n}}$, is selfadjoint, and

$$(32) \quad \|M(\lambda)\| \leq CE_{4n-1}^{-1/2}, \quad \lambda \in \overline{\rho_{2n}}, \quad n \in \mathbb{N}_0,$$

with C independent of n . (One can show that $\alpha(\lambda) \underset{\lambda \rightarrow E_{4n}}{=} \underset{(4n-1)}{d_{4n}} |\lambda - E_{4n}|^{1/2}$)

for some constants $\underset{(4n-1)}{d_{4n}} > 0$.)

(b) $W \geq 0$. Introducing the factorization

$$(33) \quad w := |W|^{1/2}, \quad W = w^2,$$

one defines

$$(34) \quad \hat{k}(\lambda) := gw(H_0 - \lambda)^{-1}w, \quad \lambda \in \rho_{2n}, \quad n \in \mathbb{N}_0, \quad g > 0.$$

Then $\hat{k}(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}))$ and (31) and (32) (with $\gamma \rightarrow -\gamma$) hold again.

(c) $W = W_+ - W_-$, $W_{\pm} > 0$ on sets of positive Lebesgue measure. If necessary, we modify W_{\pm} such that

$$(35) \quad \begin{aligned} W &= W_+ - W_- = \widetilde{W}_+ - \widetilde{W}_-, \\ \widetilde{W}_{\pm} &\geq (1 + x^2)^{-1-\varepsilon}, \quad \varepsilon > 0, \quad \widetilde{W}_{\pm} \in L^1(\mathbb{R}, (1 + |x|)dx), \\ \tilde{w}_{\pm} &:= \widetilde{W}_{\pm}^{1/2}. \end{aligned}$$

Following a device of Simon [16] we define the selfadjoint Birman-Schwinger kernel by

$$(36) \quad \begin{aligned} \tilde{K}(\lambda) &:= g\tilde{w}_+(H_0 - g\widetilde{W}_- - \lambda)^{-1}\tilde{w}_+ \in \mathcal{B}_2(L^2(\mathbb{R})), \\ &\lambda \in \rho_{2n} \setminus \sigma_p(H_0 - g\widetilde{W}_-), \quad n \in \mathbb{N}_0, \quad g > 0. \end{aligned}$$

The fact that $\tilde{K}(\lambda)$ is selfadjoint (as opposed to the usual choice

$$|W|^{1/2} \operatorname{sgn}(W)(H_0 - \lambda)^{-1}|W|^{1/2}),$$

even though W changes sign, will be of crucial importance below. (This trick has also been employed successfully in [7].)

Given all these preliminaries we now turn to the

Proof of Theorem 1. It suffices to treat the even open gaps ρ_{2n} , $n \in \mathbb{N}_0$.

(A) $W \leq 0$. Since

$$(37) \quad \frac{d}{d\lambda}k(\lambda) = -gw(H_0 - \lambda)^{-2}w \leq 0, \quad \lambda \in \rho_{2n}, \quad n \in \mathbb{N}_0,$$

all eigenvalues of $k(\lambda)$ are monotonically decreasing with respect to $\lambda \in \rho_{2n}$. Moreover, by the Birman-Schwinger principle [12], $H_g = H_0 - g|W|$ has an eigenvalue $E^* \in \rho_n$ iff $k(E^*)$ has an eigenvalue -1 of the same multiplicity. Since E^* is necessarily simple, no eigenvalues of $k(\lambda)$ can cross in ρ_n . Because of (31), $k(\lambda)$ has precisely one eigenvalue decreasing from $+\infty$ at E_{4n-1} to $O(E_{4n-1}^{-1/2})$ near E_{4n} and one eigenvalue branch decreasing from $O(E_{4n-1}^{-1/2})$ near E_{4n-1} to $-\infty$ at E_{4n} (assuming n large enough such that $E_{4n-1} \gg 1$). The remaining eigenvalues of $k(\lambda)$ in ρ_{2n} are of order $O(E_{4n-1}^{-1/2})$ for n large enough. Thus choosing n sufficiently large, precisely one eigenvalue of $K(\lambda)$ (the one diverging to $-\infty$) will cross -1 . Since $k(\lambda)$ is compact, only finitely many eigenvalues of $k(\lambda)$ cross -1 in each gap ρ_n . This proves (i) and (iii) for $W \leq 0$.

Since $W \geq 0$ can be dealt with analogously, the only difference being that now $\frac{d}{d\lambda}\hat{k}(\lambda) \geq 0$ on ρ_n and hence the eigenvalues of $\hat{k}(\lambda)$ are monotonically increasing (accounting for no eigenvalue crossing of the line -1 on ρ_0 since $\hat{k}(\lambda) \geq 0$ on ρ_0), we immediately turn to the general case.

(B) $\operatorname{sgn}(W) \neq \text{constant}$.

Throughout the rest of the proof we assume that $\lambda \in \overline{\rho_{2n}}$ with n large enough unless otherwise stated. We start with the elementary identity

$$(38) \quad \begin{aligned} \tilde{K}_-(\lambda) &:= g\tilde{w}_-(H_0 - \tilde{W}_- - \lambda)^{-1}\tilde{w}_- \\ &= -1 + [1 - g\tilde{w}_-(H_0 - \lambda)^{-1}\tilde{w}_-]^{-1} \\ &:= -1 + [1 + \tilde{k}_-(\lambda)]^{-1}, \quad \lambda \in \rho_{2n} \setminus \{E_{2n}^*\}, \end{aligned}$$

where E_{2n}^* denotes the unique eigenvalue of $H_0 - g\tilde{W}_-$ in ρ_{2n} determined in Part A. We note that

$$(39) \quad \tilde{k}_-(\lambda) = -\tilde{\gamma}(\lambda)g\tilde{P}_-(\lambda) - g\tilde{M}_-(\lambda), \quad \lambda \in \rho_{2n},$$

where the selfadjoint rank-one operator $\tilde{P}_-(\lambda)$, $\lambda \in \overline{\rho_{2n}}$, has the integral kernel

$$(40) \quad \left[\int_{\mathbb{R}} dy \tilde{W}_-(y)P_{x_0}(\lambda, y, y) \right]^{-1} \tilde{w}_-(x)P_{x_0}(\lambda, x, x')\tilde{w}_-(x'), \quad \lambda \in \overline{\rho_{2n}},$$

$$(41) \quad \tilde{\gamma}(\lambda) := \gamma(\lambda) \int_{\mathbb{R}} dx \tilde{W}_-(y)P_{x_0}(\lambda, x, x,), \quad \lambda \in \rho_{2n},$$

and $\tilde{M}_-(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}))$, $\lambda \in \overline{\rho_{2n}}$, is selfadjoint with integral kernel

$$(42) \quad \tilde{w}_-(x)G_{0,x_0}^D(\lambda, x, x')\tilde{w}_-(x'), \quad \lambda \in \overline{\rho_{2n}}.$$

Next we introduce the orthogonal projection

$$(43) \quad \tilde{Q}_-(\lambda) := 1 - \tilde{P}_-(\lambda), \quad \lambda \in \overline{\rho_{2n}},$$

and insert (39) into (38). Assuming $\varepsilon_n > 0$ sufficiently small, a straightforward computation (inverting $1 + \text{rank one} + \text{perturbation}$) then yields for the behavior of $\tilde{K}_-(\lambda)$ near the band edges E_{4n-1}, E_{4n} ,

$$(44) \quad \begin{aligned} \tilde{K}_-(\lambda) &= -1 - \tilde{P}_-(\lambda) + [1 - g\tilde{Q}_-(\lambda)\tilde{M}_-(\lambda)\tilde{Q}_-(\lambda)]^{-1} + O(\gamma(\lambda)^{-1}) \\ &= -\tilde{P}_-(\lambda) + \tilde{Q}_-(\lambda)\{[1 - g\tilde{Q}_-(\lambda)\tilde{M}_-(\lambda)\tilde{Q}_-(\lambda)]^{-1} - 1\}\tilde{Q}_-(\lambda) + O(\gamma(\lambda)^{-1}) \\ &= \begin{pmatrix} -1 & O \\ O & [1 - g\tilde{Q}_-(\lambda)\tilde{M}_-(\lambda)\tilde{Q}_-(\lambda)]^{-1} - 1 \end{pmatrix} + O(\gamma(\lambda)^{-1}), \\ &\quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}], \end{aligned}$$

with respect to the decomposition $L^2(\mathbb{R}) = \tilde{P}_-(\lambda)L^2(\mathbb{R}) \oplus \tilde{Q}_-(\lambda)L^2(\mathbb{R})$. (Here the symbol $O(\gamma(\lambda)^{-1})$ denotes a compact operator with norm bounded by $C|\gamma(\lambda)|^{-1}$.) In particular,

$$(45) \quad \|\tilde{K}_-(\lambda)\| = O(1), \quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]$$

for $\varepsilon_n > 0$ sufficiently small. Noticing that

$$(46) \quad \tilde{K}(\lambda) = (\tilde{w}_+/\tilde{w}_-)\tilde{K}_-(\lambda)(\tilde{w}_+/\tilde{w}_-), \quad \lambda \in \overline{\rho_{2n}} \setminus \{E^*\},$$

we infer for the behavior of $\tilde{K}(\lambda)$ near the band edges E_{4n-1}, E_{4n} that

$$(47) \quad \begin{aligned} \tilde{K}(\lambda) &= -\tilde{P}(\lambda) + (\tilde{w}_+/\tilde{w}_-)\tilde{Q}_-(\lambda)\{[1 - g\tilde{Q}_-(\lambda)\tilde{M}_-(\lambda)\tilde{Q}_-(\lambda)]^{-1} - 1\} \\ &\quad \cdot \tilde{Q}_-(\lambda)(\tilde{w}_+/\tilde{w}_-) + O(\gamma(\lambda)^{-1}) \\ &:= -\tilde{P}(\lambda) + \tilde{L}(\lambda) + O(\gamma(\lambda)^{-1}), \\ &\quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]. \end{aligned}$$

Here $\tilde{P}(\lambda)$ has the integral kernel

$$(48) \quad \left[\int_{\mathbb{R}} dy \tilde{W}_-(y) P_{x_0}(\lambda, y, y) \right]^{-1} \tilde{w}_+(x) P_{x_0}(\lambda, x, x') \tilde{w}_+(x'), \quad \lambda \in \overline{\rho_{2n}},$$

and by using a geometric series expansion one checks that $\tilde{L}(\lambda)$ indeed extends to a $\mathcal{B}_2(L^2(\mathbb{R}))$ -operator for $\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]$ with $\varepsilon_n > 0$ sufficiently small. Moreover,

$$(49) \quad \|\tilde{L}(\lambda)\|_{n \rightarrow \infty} = O(E_n^{-1/2}), \quad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}].$$

It remains to study $\tilde{K}(\lambda)$ near E_{2n}^* . By (38) we have

$$(50) \quad \begin{aligned} \tilde{K}_-(\lambda) &= -\tilde{k}_-(\lambda)[1 + \tilde{k}_-(\lambda)]^{-1} \\ &= -\mu_1(\lambda)g[1 + \mu_1(\lambda)g]^{-1}P_1(\lambda) - gR_1(\lambda)[1 + gR_1(\lambda)]^{-1}, \end{aligned} \quad \lambda \in \rho_{2n} \setminus \{E_{2n}^*\},$$

where we used the spectral representation for $\tilde{k}_-(\lambda)$,

$$(51) \quad \begin{aligned} \tilde{k}_-(\lambda) &= \mu_1(\lambda)gP_1(\lambda) + gR_1(\lambda), \\ P_1(\lambda)R_1(\lambda) &= R_1(\lambda)P_1(\lambda) = 0, \quad \lambda \in \rho_{2n}, \end{aligned}$$

with $\mu_1(\lambda)g$ the unique eigenvalue branch of $\tilde{k}_-(\lambda)$ diverging to $-\infty$ as $\lambda \uparrow E_{4n}$, $P_1(\lambda)$ the associated rank one projection onto the corresponding eigenspace, and

$$(52) \quad \|R_1(\lambda)\| \leq C|E_{4n-1}|^{-1/2}, \quad \lambda \in \overline{\rho_{2n}},$$

by (32). By (46), (50) yields an analogous formula for $\tilde{K}(\lambda)$, $\lambda \in \rho_{2n} \setminus \{E_{2n}^*\}$. Given these results one can now finish the proof (similar to Part A). Since

$$(53) \quad \frac{d}{d\lambda} \tilde{K}(\lambda) = g\tilde{w}_+(H_0 - g\tilde{W}_- - \lambda)^{-2}\tilde{w}_+ \geq 0, \quad \lambda \in \rho_n,$$

all eigenvalues of $\tilde{K}(\lambda)$ are monotonically increasing with respect to $\lambda \in \rho_n$. By the Birman-Schwinger principle, $H_g = H_0 + gW$ has an eigenvalue $E^* \in \rho_n$ iff $\tilde{K}(E^*)$ has an eigenvalue -1 with multiplicities preserved. Since H_g has only simple eigenvalues, again no eigenvalue crossing of $\tilde{K}(\lambda)$ occurs in ρ_n . Due to (47), (49), (50), and its analog for $\tilde{K}(\lambda)$, $\tilde{K}(\lambda)$ has precisely one eigenvalue branch $\nu_1(\lambda)$ in (E_{2n}^*, E_{4n}) that is monotonically increasing from $-\infty$ at E_{2n}^* to $O(1)$ near E_{4n} , all other eigenvalues of $\tilde{K}(\lambda)$ in (E_{2n}^*, E_{4n}) being $O(E_{4n-1}^{-1/2})$. Similarly, there is precisely one monotonically increasing eigenvalue branch $\nu_2(\lambda)$ of $\tilde{K}(\lambda)$ in (E_{4n-1}, E_{2n}^*) that is $O(E_{4n-1}^{-1/2})$ near E_{4n-1} and $+\infty$ at E_{2n}^* , and precisely one eigenvalue branch $\nu_3(\lambda)$ that is $O(1)$ near E_{4n-1} and $O(E_{4n-1}^{-1/2})$ near E_{2n}^* , all other eigenvalues of $\tilde{K}(\lambda)$ being $O(E_{4n-1}^{-1/2})$ throughout (E_{4n-1}, E_{2n}^*) . The $O(1)$ branches near E_{4n} are of course due to $\tilde{P}(\lambda)$ in

(47) (see also (48)). Given n sufficiently large we thus have the following distinctions:

- (a) If $\int_{\mathbb{R}} dx W(x) > O$, then (20), (25), and (48) imply that only $\nu_3(\lambda)$ crosses -1 .

- (b) If $\int_{\mathbb{R}} dx W(x) < O$, then (20), (25), and (48) imply that only $\nu_1(\lambda)$ crosses -1 .
- (c) If $\int_{\mathbb{R}} dx W(x) = O$, then $\nu_1(\lambda)$, $\nu_3(\lambda)$ may or may not cross -1 and we have either 0, 1, or 2 eigenvalues in ρ_{2n} .

Since $\tilde{K}(\lambda)$ is compact, only finitely many eigenvalues can cross -1 in each gap ρ_n . This completes the proof of Theorem 1. \square

Since one can replace the phrase “for n large enough” by “ $g > 0$ sufficiently small” in every step of the above proof, Theorem 1 can also be viewed as a “weak-coupling” result in the following sense:

Theorem 3. *Assume Hypothesis (I). Then*

(i) H_g has at most two eigenvalues in every open gap ρ_n , $n \in \mathbb{N}_0$ for $g > 0$ sufficiently small.

(ii) Abbreviate

$$(54) \quad I(E_{2n}) := \int_{\mathbb{R}} dx W(x) p(\alpha(E_{2n}), x, x_0)^2, \quad n \in \mathbb{N}_{(0)},$$

and assume that $g > 0$ is small enough. Then H_g has no eigenvalues in $\rho_n = (E_{2n-1}, E_{2n})$, $n \in \mathbb{N}$ if $I(E_{2n-1}) < 0$ and $I(E_{2n}) > 0$, H_g has precisely one eigenvalue in ρ_n if $I(E_{2n-1}) < 0$ and $I(E_{2n}) < 0$ or $I(E_{2n-1}) > 0$ and $I(E_{2n}) > 0$, and H_g has two eigenvalues in ρ_n if $I(E_{2n-1}) > 0$ and $I(E_{2n}) < 0$. Moreover, H_g has no eigenvalues in $\rho_0 = (-\infty, E_0)$ if $I(E_0) > 0$ and precisely one eigenvalue in ρ_0 if $I(E_0) \leq 0$.

Proof. By the paragraph preceding Theorem 3 we only need to demonstrate the last assertion in the case $I(E_0) = 0$. For that purpose we first prove that $R_0(E_0, x, x')$ (see (22) and (24)) is conditionally positive definite, i. e.,

$$(55) \quad \int_{\mathbb{R}^2} dx dx' W(x) p(\alpha(E_0), x, x_0) R_0(E_0, x, x') W(x') p(\alpha(E_0), x', x_0) > 0$$

$$\text{if } I(E_0) = \int_{\mathbb{R}} dx W(x) p(\alpha(E_0), x, x_0)^2 = 0.$$

(We also note that $R_0(E_0, x, x') = G_{0,x_0}^D(E_0, x, x')$.) In order to prove (55) we invoke the eigenfunction expansion associated with H_0 . Let

$$(56) \quad f(\cdot) = s - \lim_{R \rightarrow \infty} (2\pi)^{-1/2} \int_{|\beta| \leq R} d\beta \hat{f}_{\pm}(\beta) \Psi_{\mp}(\beta, \cdot),$$

$$\hat{f}_{\pm}(\cdot) = s - \lim_{R \rightarrow \infty} (2\pi)^{-1/2} \int_{|y| \leq R} dy f(y) \Psi_{\pm}(\cdot, y), \quad f \in L^2(\mathbb{R}),$$

where

$$(57) \quad \Psi_{\pm}(\beta, x) := a^{1/2} \left[\int_{x_0}^{x_0+a} dy \psi_{-}(z(\beta), y, x_0) \psi_{+}(z(\beta), y, x_0) \right]^{-1/2} \cdot \psi_{\pm}(z(\beta)x, x_0),$$

$$(58) \quad \Psi_{\pm}(-\beta, x) = \Psi_{\mp}(\beta, x) = \overline{\Psi_{\pm}(\beta, x)}, \quad \beta \in \mathbb{R},$$

and

$$(59) \quad \cosh[\beta(z)a] = \Delta(z), \quad \sinh[\beta(z)a] = [\Delta(z)^2 - 1]^{1/2}$$

with $\beta(z)$ an appropriate analytic continuation of $\text{arc sinh}\{[\Delta(z)^2 - 1]^{1/2}\}$ to the Riemann surface \mathcal{R} (see, e.g., [5] for more details). If $f \in L^1(\mathbb{R})$ then the integral for \hat{f}_\pm in (56) becomes an ordinary Lebesgue integral over \mathbb{R} since $\Psi_\pm(\beta, x)$ is uniformly bounded in $x \in \mathbb{R}$. (If $V = 0$ then $\Psi_\pm(\beta, x) = e^{\pm i\beta x}$.) We also note that

$$(60) \quad z(\beta) \underset{\beta \rightarrow 0}{=} E_0 + (2\mathcal{K}_0)^{-1} \beta^2 + O(\beta^4)$$

for some $\mathcal{K}_0 > 0$. Next we define

$$(61) \quad \omega(\cdot) := W(\cdot)p(\alpha(E_0), \cdot, x_0)$$

and compute for $\lambda < E_0$,

$$(62) \quad \int_{\mathbb{R}^2} dx dx' \omega(x) R_0(\lambda, x, x') \omega(x') = \int_{\mathbb{R}^2} dx dx' \omega(x) G_0(\lambda, x, x') \omega(x') \\ = \int_{\mathbb{R}} d\beta |\hat{\omega}_+(\beta)|^2 [z(\beta) - \lambda]^{-1},$$

where we used (22) together with $I(E_0) = 0$ in the first equality and

$$(63) \quad ((H_0 - \lambda)^{-1} \Psi_\pm(\beta(z), x) = [z(\beta) - \lambda]^{-1} \Psi_\pm(\beta(z), x), \\ z(\beta) \geq E_0, \beta \in \mathbb{R},$$

$$(64) \quad (2\pi)^{-1} \int_{\mathbb{R}} dx \Psi_-(\beta, x) \Psi_+(\beta', x) = \delta(\beta - \beta')$$

(in the distributional sense) and the real-valuedness of ω in the second equality. Since $p(\alpha(E_0), x, x_0)$ is uniformly bounded in $x \in \mathbb{R}$ we have

$$\omega \in L^1(\mathbb{R}; (1 + |x|) dx)$$

and hence

$$(65) \quad \infty > \int_{\mathbb{R}^2} dx dx' \omega(x) R_0(E_0, x, x') \omega(x') \\ = \int_{\mathbb{R}} d\beta |\hat{\omega}_+(\beta)|^2 [z(\beta) - E_0] > 0$$

by (23) and the monotone convergence theorem. This proves (55). It remains to go through the proof of Theorem 1 step-by-step. In fact, let E_0^* be the unique eigenvalue of $H_0 \dot{-} g \widetilde{W}_-$ in $\rho_0 = (-\infty, E_0)$ determined by Part A of the proof of Theorem 1. Since (53) remains valid for $n = 0$, and

$$(66) \quad (H_0 \dot{-} g \widetilde{W}_- - \lambda)^{-1} \geq 0 \quad \text{for } \lambda \in (-\infty, E_0^*),$$

we have

$$(67) \quad \tilde{K}(\lambda) \geq 0 \quad \text{for } \lambda \in (-\infty, E_0^*).$$

Thus no eigenvalue branch of $\tilde{K}(\lambda)$ can cross -1 for $\lambda < E_0^*$. In the interval (E_0^*, E_0) there is precisely one eigenvalue branch $\nu_1(\lambda)$ that is monotonically increasing from $-\infty$ at E_0^* to $O(1)$ near E_0 , all other eigenvalues of $\tilde{K}(\lambda)$ being $O(g)$ throughout $[E_0^*, E_0]$. In order to prove that $\nu_1(\lambda)$ actually crosses

-1 for $g > 0$ small enough we next consider $\tilde{K}(E_0) = n - \lim_{\lambda \uparrow E_0} \tilde{K}(\lambda)$. In analogy to (44) one proves

$$(68) \quad \tilde{K}_-(E_0) = -\tilde{P}_-(E_0) + g\tilde{Q}_-(E_0)\tilde{M}_-(E_0)\tilde{Q}_-(E_0) + O(g^2),$$

where $O(g^2)$ denotes a compact operator with norm bounded by Cg^2 . This yields

$$(69) \quad \tilde{K}(E_0) = -\tilde{P}(E_0) + g(\tilde{w}_+/\tilde{w}_-)\tilde{Q}_-(E_0)\tilde{M}_-(E_0)\tilde{Q}_-(E_0)(\tilde{w}_+/\tilde{w}_-) + O(g^2),$$

where $\tilde{P}(E_0)$ is an orthogonal projection with integral kernel (see (22), (25) and (48))

$$(70) \quad \left[\int_{\mathbb{R}} dy \tilde{W}_+(y) p(\alpha(E_0), y, x_0)^2 \right]^{-1} \tilde{w}_+(x) p(\alpha(E_0), x, x_0) p(\alpha(E_0), x', x_0) \tilde{w}_+(x')$$

since $I(E_0) = 0$, and \tilde{M}_-, \tilde{Q}_- have been introduced in (42), (43). A simple computation then yields

$$(71) \quad \begin{aligned} & (\tilde{w}_+ p(\alpha(E_0), \cdot, x_0), \tilde{K}(E_0) \tilde{w}_+ p(\alpha(E_0), \cdot, x_0)) / \|\tilde{w}_+ p(\alpha(E_0), \cdot, x_0)\|^2 \\ & = -1 + g \iint_{\mathbb{R}^2} dx dx' \omega(x) R_0(E_0, x, x') \omega(x') + O(g^2). \end{aligned}$$

By (55) this indeed proves that $\nu_1(\lambda)$ crosses -1 for $g > 0$ sufficiently small. \square

Remark 4. To the best of our knowledge the fact that $R_0(E_0, x, x')$ is conditionally positive definite (in the sense of (55)) and that for $g > 0$ small enough H_g has precisely one eigenvalue in $\rho_0 = (-\infty, E_0)$ if $I(E_0) = 0$ appears to be new. It generalizes a corresponding result of [15] (extended in [9]) in the special case where $V \equiv 0$.

Evidently, our strategy of using a selfadjoint Birman-Schwinger kernel, even if $\text{sgn}(W) \neq \text{constant}$, extends to perturbed one-dimensional periodic Dirac operators and weakly perturbed second-order finite difference operators.

Finally, we remark that Theorem 1, in particular, implies that N -soliton solutions of the Korteweg-de Vries equation relative to a periodic background solution (i.e., relative reflectionless solutions) will in general not decay as $x \rightarrow +\infty$ and $x \rightarrow -\infty$ since by definition they are associated with the insertion of N eigenvalues in the spectral gaps of the period background Hamiltonian.

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