WEAK TOPOLOGIES FOR THE CLOSED SUBSETS
OF A METRIZABLE SPACE

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ABSTRACT. The purpose of this article is to propose a unified theory for topologies on the closed subsets of a metrizable space. It can be shown that all of the standard hyperspace topologies—including the Hausdorff metric topology, the Vietoris topology, the Attouch-Wets topology, the Fell topology, the locally finite topology, and the topology of Mosco convergence—arise as weak topologies generated by families of geometric functionals defined on closed sets. A key ingredient is the simple yet beautiful interplay between topologies determined by families of gap functionals and those determined by families of Hausdorff excess functionals.

1. INTRODUCTION

From the point of view of analysis, the favorite topology for the (nonempty) closed and bounded subsets of a metric space $\langle X, d \rangle$—especially the closed and bounded convex subsets of a normed linear space—is the Hausdorff metric topology. For A and B closed and bounded, the *Hausdorff distance* between them is defined by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$
$$= \inf\{\varepsilon > 0 : A \subset S_\varepsilon[B] \text{ and } B \subset S_\varepsilon[A]\},$$

where $S_\varepsilon[F]$ is the *ε-enlargement* of the set F, i.e., $S_\varepsilon[F] = \{x \in X : d(x, F) < \varepsilon\}$. Denoting the *Hausdorff excess* [CV] $\sup_{b \in B} d(b, A)$ of B over A by $e_d(B, A)$, we may write $H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}$.

The Hausdorff distance so defined makes sense for arbitrary closed sets as well, and yields an infinite valued metric on the nonempty closed subsets $CL(X)$ of X [CV, KT]. For closed sets we have the formula [Co]

$$H_d(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|,$$

so that Hausdorff metric convergence of a sequence of closed sets $\langle A_n \rangle$ to A amounts to the uniform convergence of $\langle d(\cdot, A_n) \rangle$ to $d(\cdot, A)$.

But there is a problem with this metric in the more general setting: it is obviously too strong. In the plane, one would like the sequence of lines $\langle L_n \rangle$ where L_n has slope $1/n$ and y-intercept 0 to converge to the x-axis. This, of course, fails for the Hausdorff metric induced by the Euclidean metric. Various weaker convergence notions/topologies for closed sets have been considered over the past thirty years, with variable success: Kuratowski convergence and the associated Fell topology; Wijsman convergence (pointwise convergence of $\langle d(\cdot, A_n) \rangle$ to $d(\cdot, A)$) and the associated Wijsman topology; Mosco convergence and the associated Mosco topology. Only recently has a completely acceptable replacement (at least in the convex case) for the Hausdorff metric been investigated: the metrizable topology of uniform convergence of $\langle d(\cdot, A_n) \rangle$ to $d(\cdot, A)$ on bounded subsets of X. Given $x_0 \in X$, a local base for this topology [Be2, BDC, AP, ALW] at $A \in CL(X)$ consists of all sets of the form

$$\Sigma_n[A] \equiv \{F \in CL(X): F \cap S_n[x_0] \subset S_{1/n}[A] \text{ and } A \cap S_n[x_0] \subset S_{1/n}[F]\}$$
$$(n \in Z^+).$$

In the setting of convex analysis, this topology reduces to the Hausdorff metric topology for closed and bounded convex sets [BL1], is stable with respect to duality [Be3, Pe], and is well suited for approximation and optimization. In view of its seminal study in [AW], we call this the *Attouch-Wets topology* τ_{aw_d}, although it has been often called the bounded Hausdorff topology [AP, Pe].

In the last few years, a significant development in the study of topologies on the closed subsets of a metric space has been the presentation of many basic topologies as weak topologies. Given a topology τ on $CL(X)$, one seeks a family $\{\psi_i: i \in I\}$ of extended real functionals on $CL(X)$ such that τ is the weakest topology for which each ψ_i is continuous. Here are some typical results within this general framework.

If X is a metrizable space and $\{d_i: i \in I\}$ is the family of all compatible metrics, then the Vietoris topology [Mi, KT, En] is the weakest topology τ on $CL(X)$ such that for each $x \in X$ and $i \in I$, $A \to d_i(x, A)$ is τ-continuous [BLLN, Theorem 3.1]. Thus the Vietoris topology is the weak topology determined by $\{d_i(x, \cdot): x \in X \text{ and } i \in I\}$. The Fell topology [At, Fe, KT] on $CL(X)$ is Hausdorff if and only if X is locally compact, and in this context there exists a compatible metric d (specifically, one such that each closed d-ball that is a proper subset of X is compact) such that the Fell topology is the Wijsman topology determined by d, i.e., the weak topology generated by $\{d(x, \cdot): x \in X\}$ [Be4, Theorem 2]. The topology of Mosco convergence on the closed convex subsets of a Banach space X, compatible with Mosco convergence [At, Mo, BF] of sequences [Be1, Theorem 3.1] is Hausdorff if and only if X is reflexive [BB]. In this case, the Mosco topology is the weak topology generated by the family of functionals $\{D_d(\cdot, K): K \text{ weakly compact and convex}\}$ [Be1, Theorem 3.3], where d is the distance functional associated with the norm of X, and

$$D_d(A, B) = \inf\{d(a, b): a \in A, b \in B\} = \inf_{b \in B} d(b, A),$$

is the *gap* between two closed sets A and B relative to the metric d. Moreover, there exists a renorming of X with associated metric d (one such that the dual norm has the Kadec property) such that the Mosco topology is generated by

$\{d(x, \cdot): x \in X\}$ [BF, Be7]. If X is an arbitrary normed linear space, then the weak topology on the closed convex sets determined by $\{D_d(\cdot, C): C$ closed and convex$\}$ coincides with the weak topology determined by $\{d(x, \cdot): x \in X\} \cup \{s(y, \cdot): y \in X^*\}$, where $s(y, A) = \sup\{\langle y, a \rangle: a \in A\}$ is the value of the support functional for the set A at y [Be5]. The weak topology determined by support functionals alone on the closed convex sets alone is studied in [SZ1].

Here, we systematically study topologies on $CL(X)$ induced by gap functionals and excess functionals where one set argument is fixed, varying both the set and metric over prescribed classes. In particular, we show that the Hausdorff metric and Attouch-Wets topologies both fit within this framework in a similar way. One point of departure for this work is the d-proximal topology τ_{δ_d} on $CL(X)$ introduced in [BLLN], which may be defined as the weak topology on $CL(X)$ determined by the family of gap functionals $\{D_d(\cdot, F): F \in CL(X)\}$. A second point of departure is a certain weakening σ_d of τ_{aw_d}, considered recently in the context of convex analysis and optimization, by two different sets of authors [AAB, SP].

Definition. The *bounded d-proximal topology* σ_d on $CL(X)$ has as a local base at $A \in CL(X)$ all sets of the form

$$\Phi_A[n; a_1, a_2, \ldots, a_k] \equiv \{F \in CL(X): F \cap S_n[x_0] \subset S_{1/n}[A],$$
$$\text{and } \forall i \leq k, \ d(a_i, F) < 1/n\}$$

where $\{a_1, a_2, \ldots, a_k\}$ is a finite subset of A and $n \in Z^+$.

As we shall see, this topology is the weak topology determined by $\{D_d(\cdot, B): B \in CL(X) \text{ and } B \text{ bounded}\}$, a result which has been obtained independently and concurrently in sequential form by Sonntag and Zalinescu [ZS2] (private communication). We then form natural duals for the d-proximal and bounded d-proximal topologies, both in terms of their local presentations and in terms of the lattice-theoretic approach to hyperspaces as promoted by Levi and Lechicki [FLL, LL], and show that these topologies are weak topologies determined by Hausdorff excess functionals. Putting these together yields the Hausdorff metric and Attouch-Wets topologies.

2. Preliminaries

As stated in §1, $CL(X)$ will denote the nonempty closed subsets of a metric space $\langle X, d \rangle$. We need to review some basic facts about *hyperspace topologies*, i.e., topologies on $CL(X)$. In view of the results mentioned in §1, a basic topology on $CL(X)$ is the *Wijsman topology* τ_{W_d} [Wi, Co, FLL, LL, BLLN, Be6], which is the weakest topology τ on $CL(X)$ such that for each $x \in X$, the functional $A \to d(x, A)$ is τ-continuous. This topology is a function space topology, in that τ_{W_d} is the topology that $CL(X)$ inherits from $C(X, R)$, equipped with the topology of pointwise convergence, under the identification $A \leftrightarrow d(\cdot, A)$. Similarly, the Hausdorff metric (resp. Attouch-West) topology is the topology that $CL(X)$ inherits from $C(X, R)$, equipped with the topology of uniform convergence (resp. uniform convergence on bounded sets), under the identification $A \leftrightarrow d(\cdot, A)$.

Another basic class of hyperspace topologies are the hit-and-miss topologies. To introduce these, we need some notation. For $E \subset CL(X)$, we introduce the

following subsets of $CL(X)$:

$$E^- \equiv \{A \in CL(X): A \cap E \neq \varnothing\};$$
$$E^+ \equiv \{A \in CL(X): A \subset E\};$$
$$E^{++} \equiv \{A \in CL(X): \text{ there exists } \varepsilon > 0 \text{ with } S_\varepsilon[A] \subset E\}.$$

A set in E^- hits E, whereas a set in E^+ misses E^c. A set in E^{++} really misses E^c! Using this notation, we list some standard hit-and-miss topologies:

(1) The *Vietoris topology* τ_V [Mi, KT, En] on $CL(X)$ has as a subbase all sets of the form V^- where V is open, and all sets of the form W^+ where W is open;

(2) The *Fell topology* τ_F [Fe, KT] on $CL(X)$ has as a subbase all sets of the form V^- where V is open, and all sets of the form W^+ where W has compact complement;

(3) For X a normed linear space, the *Mosco topology* τ_M [Be1, BB, Be7] on the weakly closed sets has as a subbase all sets of the form V^- where V is norm open, and all sets of the form W^+ where W has weakly compact complement.

Notice that in (2) and (3), $W^+ = W^{++}$ for the given class of sets W. If in (1), we replace W^+ by W^{++}, as W runs over the open sets, we obtain the so-called *d-proximal topology* τ_{δ_d} studied in [BLLN]. This topology has a presentation as a weak topology, alluded to in §1: the *d*-proximal topology is the weakest topology τ on $CL(X)$ such that for each $F \in CL(X)$, $A \to D_d(A, F)$ is τ-continuous [BLLN, Theorem 3.2]. Analogously, the Mosco topology is the weakest topology τ on the weakly closed subsets of a reflexive Banach space such that for each weakly compact set K, $A \to D_d(A, K)$ is τ-continuous, where d is the metric induced by the norm [Be1, Theorem 3.3]. These two results are special cases of a general phenomenon that we now describe.

Definition. Let Ω be a class of nonempty closed subsets of a metric space $\langle X, d \rangle$. We say that Ω is *stable under enlargements* if for each $A \in \Omega$ and $\alpha > 0$, we have $\mathrm{cl}\, S_\alpha[A] \in \Omega$.

Within $CL(X)$, we distinguish these classes:

$$K(X) = \text{the nonempty compact subsets};$$
$$CLB(X) = \text{the nonempty closed and bounded subsets};$$

Evidently, $CLB(X)$ and $CL(X)$ are stable under enlargements, as is $K(X)$ provided closed and bounded subsets of X are compact. In a normed linear space, the convex sets, the connected sets, and the starshaped sets are stable under enlargements. If the space is reflexive, then the weakly compact sets also have this property.

We need to consider semicontinuity of functionals defined on hyperspaces. Let T be a topological space. Recall that $f: T \to [-\infty, \infty]$ is called *lower semicontinuous* provided for each $\alpha \in R$, $\{t: f(t) \leq \alpha\}$, is a closed subset of T. We call f *upper semicontinuous* provided $-f$ is lower semicontinuous.

Theorem 2.1. *Let $\langle X, d \rangle$ be a metric space, and let Ω be a class of closed subsets that is stable under enlargements and that contains the singleton subsets of X. Let Π be a subset of $CL(X)$. Then the topology τ_1 on Π having as*

a subbase all sets of the form V^- where V is open, and all sets of the form $(E^c)^{++}$ where $E \in \Omega$, is the weakest topology τ on Π such that for every $E \in \Omega$, $A \to D_d(A, E)$ is τ-continuous.

Proof. Let τ_{weak} be the weak topology so described. We first show $\tau_{\text{weak}} \supset \tau_1$. Suppose $A \in \Pi$. If $A \in V^-$ where V is open, we can find $a \in A$ and $\varepsilon > 0$ with $S_\varepsilon[a] \subset V$. Then $\{F \in \Pi: d(a, F) < \varepsilon\} = \{F \in \Pi: D_d(F, \{a\}) < \varepsilon\}$ is a τ_{weak}-neighborhood of A, and $\{F \in \Pi: d(a, F) < \varepsilon\} = S_\varepsilon[A]^- \subset V^-$. On the other hand, for each $E \in \Omega$, we have $(E^c)^{++} = \{F \in \Pi: D_d(F, E) > 0\} \in \tau_{\text{weak}}$. This proves $\tau_{\text{weak}} \supset \tau_1$.

For the other inclusion, it suffices to show that $F \to D_d(F, E)$ is τ_1-continuous on Π for each $E \in \Omega$. For upper semicontinuity, fix $A \in \Pi$ and $\varepsilon > 0$. Pick $a \in A$ with $d(a, E) < D_d(A, E) + \varepsilon/2$. Then if $F \in (S_{\varepsilon/2}[a])^- \cap \Pi$, we have $D_d(F, E) < D_d(A, E) + \varepsilon$. Lower semicontinuity of the gap functional holds if $D_d(A, E) = 0$. Otherwise, write $D_d(A, E) = \alpha > 0$ and let $\varepsilon \in (0, \alpha)$ be arbitrary. Then since Ω is stable under enlargements, $((\text{cl}\, S_{\alpha - \varepsilon}[E])^c)^{++}$ is a τ_1-neighborhood of A, and if $F \in ((\text{cl}\, S_{\alpha - \varepsilon}[E])^c)^{++}$, we obtain $D_d(F, E) \geq \alpha - \varepsilon$. $\square$

For emphasis, we state as corollaries these special cases.

Corollary 2.2. *Let $\langle X, d \rangle$ be a metric space. Then the d-proximal topology on $CL(X)$, having as a subbase all sets of the form V^- where V is open, and all sets of the form W^{++} where W is open, is the weakest topology τ on $CL(X)$ such that $A \to D_d(A, F)$ is τ-continuous for each $F \in CL(X)$.*

Corollary 2.3. *Let $\langle X, d \rangle$ be a reflexive Banach space. Then the Mosco topology τ_M on the nonempty weakly closed subsets of X, having as a subbase all sets of the form V^- where V is norm open, and all sets of the form W^+ where W has weakly compact complement, is the weakest topology τ on the weakly closed subsets such that $A \to D_d(A, K)$ is τ-continuous for each weakly compact subset K of X.*

Proof. By weak lower semicontinuity of the norm, for each $\alpha > 0$, we have $\text{cl}\, S_\alpha[K] = \{x \in X: d(x, K) \leq \alpha\}$ weakly closed whenever K is weakly compact. Thus by reflexivity, weak compactness of K yields weak compactness of $\text{cl}\, S_\alpha[K]$. Finally, if A is weakly closed, and K is weakly compact and $A \cap K \neq \varnothing$, then $A \in (K^c)^{++}$, again by weak lower semicontinuity of the norm. $\square$

3. LOCAL AND GLOBAL PRESENTATIONS OF σ_d

Lemma 3.1. *Let $\langle X, d \rangle$ be a metrizable space, and let $A \in CL(X)$. The following families also constitute local bases for the bounded d-proximal topology σ_d at A:*

(i) *All sets of the form $\Theta_A[B; \varepsilon; a_1, a_2, \ldots, a_k] = \{F \in CL(X): F \cap B \subset S_\varepsilon[A], \text{ and } \forall i \leq k, \ d(a_i, F) < \varepsilon\}$, where $\{a_1, a_2, \ldots, a_k\} \subset A$ and B is bounded;*

(ii) *All sets of the form $\Lambda_A[B; \varepsilon; x_1, x_2, \ldots, x_k] = \{F \in CL(X): \forall x \in B, \ d(x, A) - \varepsilon < d(x, F) \text{ and } \forall i \leq k, \ d(x_i, F) < d(x_i, A) + \varepsilon\}$, where $\{x_1, x_2, \ldots, x_k\} \subset X$ and B is bounded.*

Proof. Let $\{a_1, a_2, \ldots, a_k\} \subset A$ and let $n \in Z^+$ be given. With $B = S_n[x_0]$ and $\varepsilon = 1/n$, we have $\Theta_A[B; \varepsilon; a_1, a_2, \ldots, a_k] = \Phi_A[n; a_1, a_2, \ldots, a_k]$. Now let B be bounded, $\varepsilon > 0$, and $\{a_1, a_2, \ldots, a_k\} \subset A$ be given. If $F \in \Lambda_A[B; \varepsilon; a_1, a_2, \ldots, a_k]$, then for every $x \in F \cap B$, we have $d(x, A) - \varepsilon < d(x, F) = 0$ so that $x \in S_\varepsilon[A]$. This proves that $\Lambda_A[B; \varepsilon; a_1, a_2, \ldots, a_k] \subset \Theta_A[B; \varepsilon; a_1, a_2, \ldots, a_k]$. Finally, let B be bounded, $\varepsilon > 0$, and points $\{x_1, x_2, \ldots, x_k\} \subset X$ be given. Choose for each $i \leq k$ a point $a_i \in A$ with $d(x_i, a_i) < d(x_i, A) + \varepsilon/2$. There exists $m \in Z^+$ such that $1/m < \varepsilon$ and $B \subset S_m[x_0]$. Choose $m_0 > m$ so large that $F \cap S_{m_0}[x_0] \neq \varnothing$ for each $F \in \Phi_A[m_0; a_1, a_2, \ldots, a_k]$ (we may for example, take $m_0 > m$ so large that $S_{m_0}[x_0] \supset S_1[a_1]$). Set $n = 2m + m_0 + 1$. We claim that $\Phi_A[n; a_1, a_2, \ldots, a_k] \subset \Lambda_A[B; \varepsilon; x_1, x_2, \ldots, x_k]$.

Fix $F \in \Phi_A[n; a_1, a_2, \ldots, a_k]$. For each $i \leq k$, we have $d(a_i, F) < 1/n < 1/2m < \varepsilon/2$ so that

$$d(x_i, F) \leq d(x_i, a_i) + d(a_i, F) < d(x_i, A) + \varepsilon.$$

For each $x \in S_m[x_0]$ take $z_x \in F$ with $d(x, z_x) < d(x, F) + 1/3m$. Since F hits $S_{m_0}[x_0]$, we have

$$d(x_0, z_x) \leq d(x_0, x) + d(x, z_x) < m + (m + m_0 + 1) = n.$$

Since $F \cap S_n[x_0] \subset S_{1/n}[A]$, there exists $a_x \in A$ with $d(a_x, z_x) < 1/n < 1/3m$. Thus, we have

$$d(x, A) \leq d(x, a_x) \leq d(x, z_x) + d(z_x, a_x)$$
$$< d(x, F) + \frac{1}{3m} + \frac{1}{3m} < d(x, F) + \frac{2}{3m}.$$

This proves that $\forall x \in S_m[x_0]$, we have $d(x, A) - 1/m < d(x, F)$, whence $\forall x \in B$, $d(x, A) - \varepsilon < d(x, F)$. This proves that $F \in \Lambda_A[B; \varepsilon; x_1, x_2, \ldots, x_k]$. $\square$

We intentionally work with the different presentations of σ_d throughout this paper, as the situation dictates. The next two results are representative scenarios in this regard.

Corollary 3.2. *Let $\langle X, d \rangle$ be a metric space. Then $\sigma_d \supset \tau_{W_d}$.*

Proof. Fix $A \in CL(X)$. Given $x \in X$ and $\varepsilon > 0$, $\Lambda_A[\{x\}; \varepsilon; x] = \{F \in CL(X): d(x, A) - \varepsilon < d(x, F) < d(x, A) + \varepsilon\} = \{F \in CL(X): |d(x, F) - d(x, A)| < \varepsilon\}$. $\square$

As τ_{W_d} is Hausdorff, the same must be true for the stronger σ_d.

Corollary 3.3. *Let $\langle X, d \rangle$ be a metric space. Then*

   (i) $\sigma_d \subset \tau_{aw_d}$ *on $CL(X)$;*
   (ii) *If $\langle A_\lambda \rangle$ is a net in $CL(X)$ σ_d-convergent to a totally bounded closed set A, then $A = \tau_{aw_d}$-$\lim A_\lambda$.*

Proof. (i) Fix $x_0 \in X$, and fix $A \in CL(X)$, and let $\Phi_A[n; a_1, a_2, \ldots, a_k]$ be a σ_d-neighborhood of A. Choose $m > n$ so large that $\{a_1, a_2, \ldots, a_k\} \subset S_m[x_0]$. We clearly have $\Sigma_m[A] \subset \Phi_A[n; a_1, a_2, \ldots, a_k]$.

(ii) Fix $x_0 \in X$ and $n \in Z^+$. By total boundedness, choose $\{a_1, a_2, a_3, \ldots, a_k\}$ in A with $A \subset S_{1/2n}[\{a_1, a_2, a_3, \ldots, a_k\}]$. We then have $\Sigma_n[A] \supset \Phi_A[2n; a_1, a_2, \ldots, a_k]$. Thus, $\langle A_\lambda \rangle$ must be in $\Sigma_n[A]$ eventually, so that $A = \tau_{aw_d}$-$\lim A_\lambda$. $\square$

It can be shown that pairwise coincidence of the topologies τ_{aw_d}, σ_d, and τ_{W_d} occurs if and only if bounded subsets of X are totally bounded. This fact, as well as necessary and sufficient conditions for first countability, second countability, and metrizability, as well as a description of the properties of the underlying metric that determine the bounded d-proximal topology, are presented in [BL2]. As we will introduce several new topologies in this paper, it would be distracting (and tedious) to pursue a complete analysis here.

We now turn to global presentations of σ_d.

Theorem 3.4. *Let $\langle X, d \rangle$ be a metric space. The topology σ_d is the weakest topology τ on $CL(X)$ such that for each closed and bounded subset B of X, the gap functional $A \to D_d(A, B)$ is τ-continuous on $CL(X)$.*

Proof. For $B \in CLB(X)$ fixed, we first show that $F \to D_d(F, B)$ is σ_d-continuous. Fix $A \in CL(X)$ and let $\varepsilon > 0$. There exists $a \in A$ such that $d(a, B) < D_d(A, B) + \varepsilon/2$. Now $\Theta_A[B; \varepsilon/2; a]$ is a σ_d-neighborhood of A, and $D_d(F, B) < D_d(A, B) + \varepsilon$ for each F in the neighborhood. This proves σ_d-upper semicontinuity of $F \to D_d(F, B)$ at A. Lower semicontinuity is obvious if $D_d(A, B) = 0$; so, suppose $D_d(A, B) = \alpha > 0$. Let $\varepsilon \in (0, \alpha)$ and $a_1 \in A$ be arbitrary. We claim that for each F in the σ_d-neighborhood $\Lambda_A[S_{\alpha-\varepsilon}[B]; \varepsilon; a_1]$ of A, we have $D_d(F, B) \geq \alpha - \varepsilon$.

Fix $F \in \Theta_A[S_{\alpha-\varepsilon}[B]; \varepsilon; a_1]$. We claim that $F \cap S_{\alpha-\varepsilon}[B]$ is empty. Otherwise, taking $x \in F \cap S_{\alpha-\varepsilon}[B]$, we have $x \in S_\varepsilon[A]$, so that $A \cap S_\varepsilon[S_{\alpha-\varepsilon}[B]] \neq \varnothing$, contradicting $D_d(A, B) = \alpha$. This means that $D_d(F, B) \geq \alpha - \varepsilon$, establishing σ_d-lower semicontinuity of the gap functional at A. Thus if τ_{weak} is the weakest topology τ on $CL(X)$ such that $F \to D_d(F, B)$ is τ-continuous for each closed and bounded set B, we have $\sigma_d \supset \tau_{\mathrm{weak}}$.

To show that $\sigma_d \subset \tau_{\mathrm{weak}}$, we show that if $\langle A_\mu \rangle$ is a net in $CL(X)$ τ_{weak}-convergent to A, then $A = \sigma_d\text{-}\lim A_\mu$. To this end, let $\Theta_A[B; \varepsilon; a_1, a_2, \ldots, a_k]$ be a σ_d-neighborhood of A, where B is bounded and $\{a_1, a_2, \ldots, a_n\} \subset A$. For each $i \leq k$ we have

$$0 = D_d(A, \{a_i\}) = \lim_\mu D_d(A_\mu, \{a_i\}) = \lim_\mu d(a_i, A_\mu).$$

Thus, there exists an index μ_0 such that for each $\mu \geq \mu_0$ and each $i \leq k$, we have $d(a_i, A_\mu) < \varepsilon$. It remains to show that eventually, $A_\mu \cap B \subset S_\varepsilon[A]$. Suppose to the contrary that $A_\mu \cap B \not\subset S_\varepsilon[A]$ for each μ in some cofinal set of indices M. For each $\mu \in M$, pick $x_\mu \in A_\mu \cap B$ with $d(x_\mu, A) \geq \varepsilon$. Then $B' = \mathrm{cl}\{x_\mu : \mu \in M\}$ is closed and bounded and $D_d(A, B') \geq \varepsilon$. However, $\lim_\mu D_d(A_\mu, B') \geq \varepsilon$ is impossible, since $D_d(A_\mu, B') = 0$ frequently. This contradicts $A = \tau_{\mathrm{weak}}\text{-}\lim A_\mu$. Thus, eventually, $A_\mu \cap B \subset S_\varepsilon[A]$ must hold, and we have shown that eventually $A_\mu \in \Theta_A[B; \varepsilon; a_1, a_2, \ldots, a_k]$ must hold. $\square$

One consequence of Theorem 3.4 is that the topology σ_d is completely regular, for any weak topology induced by a family of functions into uniform spaces has a natural compatible uniformity. We also remark that in the last theorem, there is really no need to require that the sets B be closed as well as bounded, since for any B, we have $D_d(A, B) = D_d(A, \mathrm{cl}\, B)$.

Gap functionals determined by a fixed closed argument need not be σ_d-continuous or even τ_{aw_d}-continuous, as the following example shows.

Example. In the plane with the usual metric, let $F = \{(x, y): y = 1\}$, let $A_n = \{(x, y): y = x/n\}$ and let $A = \{(x, y): y = 0\}$. Then $A = \tau_{aw_d}\text{-}\lim A_n$ and $D_d(F, A) = 1$, whereas for each n, $D_d(F, A_n) = 0$.

Corollary 3.5. *Let X be a reflexive Banach space. Then the bounded d-proximal topology when restricted to the weakly closed nonempty subsets of X is finer than the Mosco topology.*

Proof. The Mosco topology is generated by a smaller family of gap functionals, namely those determined by the weakly closed and norm bounded subsets of X. □

It can be shown that the bounded d-proximal topology coincides with the Mosco topology if and only if the underlying space is finite dimensional [BL2].

As a result of Theorems 2.1 and 3.4, we may represent σ_d as a hit-and-miss topology.

Theorem 3.6. *Let $\langle X, d \rangle$ be a metric space. Then a subbase for the bounded d-proximal topology consists of all sets of the form V^- where V is open, and all sets of the form $(B^c)^{++}$, where B is closed and bounded.*

In view of Theorems 3.4 and 3.6, the topology σ_d is indeed an analogue of the d-proximal topology τ_{δ_d} introduced in [BLLN]. Now τ_{δ_d} is the weakest topology on $CL(X)$ such that $A \to \rho(x, A)$ is continuous, where ρ ranges over the metrics that define the same uniformity as d and x ranges over X [BLLN, Theorem 3.7]. Put somewhat differently, $\tau_{\delta_d} = \sup\{\tau_{W_\rho}: \rho$ is uniformly equivalent to $d\}$, where the supremum is taken in the lattice of hyperspace topologies. Does σ_d admit such a presentation? Our next result resolves this affirmatively.

Theorem 3.7. *Let $\langle X, d \rangle$ be a metric space and let $\Delta = \{\rho: \rho$ is a metric uniformly equivalent to d that determine the same bounded sets as $d\}$. Then σ_d is the weak topology on $CL(X)$ determined by $\{\rho(x, \cdot): x \in X, \rho \in \Delta\}$.*

Proof. Let τ_{weak} be the specified weak topology. Clearly, if d and ρ are uniformly equivalent, then for each $A \subset X$, we have $A_d^{++} = A_\rho^{++}$. Thus, if in addition, d and ρ determine the same bounded sets, then Theorem 3.6 guarantees that $\sigma_d = \sigma_\rho$. Since by Corollary 3.2, $\tau_{W_\rho} \subset \sigma_\rho$, we obtain $\tau_{\text{weak}} \subset \sigma_d$.

For the other inclusion, we recall that for each open V, the set V^- belongs to each Wijsman topology [FLL, Proposition 2.1]. So, it remains to show that $(B^c)_d^{++} \in \tau_{\text{weak}}$, whenever B is a closed and bounded subset of X. We dispose of some special cases. If $B = X$, then $(B^c)_d^{++} = \varnothing$, which is in each Wijsman topology. If $B = \{x\}$ for some x, then with respect to each ρ, $(B^c)_\rho^{++} = (B^c)_d^{++} = \{F \in CL(X): \rho(x, F) > 0\}$, which again is in τ_{W_ρ} for each ρ. It remains to consider the case that B is not a singleton, and $B \neq X$.

Fix $A \in (B^c)_d^{++}$; we will produce $\rho \in \Delta$, $y_0 \in X$, and $\delta > 0$ such that

$$A \in \{F \in CL(X): \rho(y_0, A) - \delta < \rho(y_0, F)\} \subset (B^c)_d^{++}.$$

This would show that $(B^c)_d^{++}$ contains a τ_{weak}-neighborhood of each of its points.

Our metric $\rho: X \times X \to [0, \infty)$ will be of the form

$$\rho(x, y) = \alpha d(x, y) + |d(x, B) - d(y, B)|,$$

where $\alpha > 0$. That ρ and d define the same uniformities follows from the d-uniform continuity of $x \to d(x, B)$. Since $\rho(x, y) \leq (\alpha + 1)d(x, y)$, d-bounded sets are ρ-bounded, and since $d(x, y) \leq \rho(x, y)/\alpha$, ρ-bounded sets are d-bounded. Thus, $\rho \in \Delta$. Recalling that B consists of at least two points, the choice of α we make is $\alpha = D_d(A, B)/4 \operatorname{diam} B$. Fix $y_0 \in B$ and set $\delta = \alpha d(y_0, A)$. We intend to show that if $F \in CL(X)$ and $\rho(y_0, A) - \delta < \rho(y_0, F)$, then $F \in (B^c)_d^{++}$.

First note that

$$\rho(y_0, A) = \inf_{a \in A} \alpha d(y_0, a) + d(a, B) \geq \alpha d(y_0, A) + D_d(A, B) = D_d(A, B) + \delta.$$

Thus, $\rho(y_0, F) > D_d(A, B)$. This means that for each $x \in F$, we have

$$(*) \qquad \alpha d(x, y_0) + d(x, B) > D_d(A, B).$$

We consider two cases for $x \in F$: (i) $d(x, y_0) \leq 2 \operatorname{diam} B$; (ii) $d(x, y_0) > 2 \operatorname{diam} B$. In the first case, by the choice of α, we get $\alpha d(x, y_0) \leq \frac{1}{2} D_d(A, B)$, so by $(*)$ we have $d(x, B) > \frac{1}{2} D_d(A, B)$. In the second case, since $y_0 \in B$, we have $d(x, B) \geq \operatorname{diam} B$. Thus,

$$D_d(F, B) \geq \min\{\tfrac{1}{2} D_d(A, B), \operatorname{diam} B\},$$

and we have $F \in (B^c)_d^{++}$. This proves that $(B^c)_d^{++} \in \tau_{\text{weak}}$ for each closed and bounded subset B of X, and we conclude that $\sigma_d \subset \tau_{\text{weak}}$. $\square$

One might guess from the previous results that there is a complete analogy between the d-proximal topology and the bounded d-proximal topology σ_d, upon replacing closed sets by closed and bounded sets in any theorem valid for τ_{δ_d}. Surprisingly, this is not the case, as we now show.

Theorem 3.8. *Let $\langle X, d \rangle$ be a metric space. Then the d-proximal topology is the weak topology on $CL(X)$ determined by the family of excess functionals $\{e_d(\cdot, F): F \in CL(X)\}$.*

Proof. Let τ_{weak} be the weak topology determined by $\{e_d(\cdot, F): F \in CL(X)\}$. Fix A_0 and F in $CL(X)$. Upper semicontinuity of the excess functional $e_d(\cdot, F)$ at A_0 occurs if $e_d(A_0, F) = \infty$. Otherwise, $(S_\varepsilon[A_0])^{++}$ is a τ_{δ_d}-neighborhood of A_0, and for each closed subset A in this neighborhood, we have $e_d(A, F) \leq e_d(A_0, F) + \varepsilon$. Lower semicontinuity occurs if $e_d(A_0, F) = 0$. Otherwise, noting that $e_d(A_0, F) = \infty$ is possible, let $\alpha < e_d(A_0, F)$ be arbitrary, and choose $\varepsilon > 0$ with $\alpha + \varepsilon < e_d(A_0, F)$. Choose $a_0 \in A_0$ with $d(a_0, F) > \alpha + \varepsilon$. Then $S_\varepsilon[a_0]^-$ is a τ_{δ_d}-neighborhood of A_0, and for each $A \in S_\varepsilon[a_0]^-$, we have $e_d(A, F) > \alpha$. This proves τ_{δ_d}-continuity of such an excess functional, so that $\tau_{\delta_d} \supset \tau_{\text{weak}}$.

For the other inclusion, suppose $A_0 \in V^-$ with V open in X. Then there exists $a_0 \in A_0$ and $\varepsilon > 0$ such that $S_\varepsilon[a_0] \subset V$. With $F = \{x \in X: d(x, a_0) \geq \varepsilon\}$ we have

$$A_0 \in \{A \in CL(X): e_d(A, F) > 0\} \subset V^-.$$

If $A_0 \in V^{++}$ with V open, there exists $\varepsilon > 0$ with $S_\varepsilon[A_0] \subset V$. We then have

$$A_0 \in \{A \in CL(X): e_d(A, A_0) < \varepsilon\} \subset V^{++}.$$

Together, these yield $\tau_{\delta_d} \subset \tau_{\text{weak}}$. $\square$

Example. For closed and bounded sets B, the functional $A \to e_d(A, B)$ need not be σ_d-continuous or even τ_{aw_d}-continuous on $CL(X)$. For example, on the line with the usual metric, let $B = \{0\}$. Then $\{0\} = \tau_{aw_d}\text{-}\lim\{0, n\}$, $\lim_{n \to \infty} e_d(\{0, n\}, B) = \infty$, whereas $e_d(\{0\}, B) = 0$.

We do not intend to study the weak topology determined by $\{e_d(\cdot, B): B \in CLB(X)\}$, for such excess functionals fail to separate unbounded sets, and a non-Hausdorff topology results. It may be useful to study this weak topology on $CLB(X)$, but we do not consider subspaces of $CL(X)$ with induced topologies here.

4. The Attouch-Wets and Hausdorff metric topologies as weak topologies

Of the three local presentations of the topology σ_d we now concentrate on the one involving sets of the form $\Lambda_A[B; \varepsilon; x_1, x_2, \ldots, x_k]$, for it corresponds naturally to a combination of upper and lower halves of hyperspace topologies in the style of [FLL] or [AAB], as we now explain.

Recall that the Wijsman topology associated with a fixed metric d is the weakest topology on $CL(X)$ such that for each $x \in X$, $d(x, \cdot): CL(X) \to [0, \infty)$ is continuous. We may split this into its lower and upper halves [FLL, LL]:

$$\tau_{W_d}^- = \text{the weakest topology on } CL(X) \text{ such that } \forall x, d(x, \cdot)$$
$$\text{is upper semicontinuous; } \tau_{W_d}^+ = \text{the weakest topology on } CL(X)$$
$$\text{such that } \forall x, d(x, \cdot) \text{ is lower semicontinuous.}$$

Note that the lower (resp. upper) half corresponds to upper (resp. lower) semicontinuity of distance functionals! A local base for $\tau_{W_d}^-$ (resp. $\tau_{W_d}^+$) at $A \in CL(X)$ consists of all sets of the form $\{F \in CL(X): \forall i \leq k, d(x_i, F) < d(x_i, A) + \varepsilon\}$ (resp. $\{F \in CL(X): \forall i \leq k, d(x_i, F) > d(x_i, A) - \varepsilon\}$), where $\{x_1, x_2, \ldots, x_k\}$ is a finite subset of X and $\varepsilon > 0$. On the other hand, the Attouch-Wets topology splits into $\tau_{aw_d}^-$ and $\tau_{aw_d}^+$, where a local base for $\tau_{aw_d}^-$ (resp. $\tau_{aw_d}^+$) at A consists of all sets of the form $\{F \in CL(X): \forall x \in B, d(x, F) < d(x, A) + \varepsilon\}$ (resp. $\{F \in CL(X): \forall x \in B, d(x, A) - \varepsilon < d(x, F)\}$), where B is an arbitrary bounded subset of X, and $\varepsilon > 0$. By Lemma 3.1, $\sigma_d = \tau_{W_d}^- \vee \tau_{aw_d}^+$, where the supremum is taken in the lattice of hyperspace topologies. A natural dual for σ_d is the *dual bounded d-proximal topology* σ_d^* given by $\sigma_d^* = \tau_{W_d}^+ \vee \tau_{aw_d}^-$. In view of the above remarks, a local base for σ_d^* at $A \in CL(X)$ consists of all sets of the form

$$\Lambda_A^*[B; \varepsilon; x_1, x_2, \ldots, x_k] = \{F \in CL(X): \forall x \in B, d(x, F) < d(x, A) + \varepsilon,$$
$$\text{and } \forall i \leq k, d(x_i, A) - \varepsilon < d(x_i, F)\},$$

where $\varepsilon > 0$, $\{x_1, x_2, \ldots, x_k\} \subset X$, and $B \subset X$ is bounded.

The proof of the following fact is left to the reader (see the proof of Lemma 3.1).

Lemma 4.1. *Let* $\langle X, d \rangle$ *be a metric space. Then a local base for the topology* σ_d^* *at* $A \in CL(X)$ *consists of all sets of the form*

$$\Psi_A^*[B\,; \varepsilon\,; x_1, x_2, \ldots, x_k] = \{F \in CL(X)\colon B \cap A \subset S_\varepsilon[F], \text{ and}$$
$$\forall i \leq k, \ d(x_i, A) - \varepsilon < d(x_i, F)\},$$

where B *is a bounded subset of* X, $\varepsilon > 0$, *and* $\{x_1, x_2, \ldots, x_k\}$ *is a finite subset of* X.

Theorem 4.2. *Let* $\langle X, d \rangle$ *be a metric space. Then the topology* σ_d^* *is the weakest topology* τ *on* $CL(X)$ *such that for each* $B \in CLB(X)$, $A \to e_d(B, A)$ *is* τ-*continuous on* $CL(X)$.

Proof. We first show that each excess functional is σ_d^*-continuous. Fix $B \in CLB(X)$. We write $\alpha = e_d(B, A) = \sup_{b \in B} d(b, A)$. Note that α is finite since B is bounded. Let $\varepsilon > 0$ be arbitrary, and choose $b_0 \in B$ such that $d(b_0, A) > \alpha - \varepsilon/2$. We claim that if $F \in \Psi_A^*[S_{\alpha+\varepsilon}[B]\,; \varepsilon/2\,; b_0]$, then $|e_d(B, A) - e_d(B, F)| \leq \varepsilon$.

Let $b \in B$ be arbitrary. There exists $a \in A$ with $d(b, a) < d(b, A) + \varepsilon/2$. This means that $a \in S_{\alpha+\varepsilon}[B]$, so there exists $x \in F$ with $d(x, a) < \varepsilon/2$. Thus,

$$d(b, F) \leq d(b, x) < d(b, a) + \varepsilon/2 < d(b, A) + \varepsilon,$$

so that

$$e_d(B, F) = \sup_{b \in B} d(b, F) \leq \sup_{b \in B} d(b, A) + \varepsilon = \alpha + \varepsilon.$$

On the other hand, we have

$$e_d(B, F) = \sup_{b \in B} d(b, F) \geq d(b_0, F) > d(b_0, A) - \varepsilon/2 > \alpha - \varepsilon.$$

This establishes σ_d^*-continuity of the excess.

Next let τ_{weak} be the weakest topology on $CL(X)$ such that for each closed and bounded set B, $A \to e_d(B, A)$ is continuous. First, notice that for fixed $x \in X$, $A \in CL(X)$, and $\varepsilon > 0$,

$$\{F \in CL(X)\colon d(x, F) > d(x, A) - \varepsilon\}$$
$$= \{F \in CL(X)\colon e_d(\{x\}, F) > e_d(\{x\}, A) - \varepsilon\}.$$

This proves that $\tau_{W_d}^+ \subset \tau_{\text{weak}}$. If $\tau_{aw_d}^- \not\subset \tau_{\text{weak}}$, then there exists a net $\langle A_\lambda \rangle$ in $CL(X)$ convergent to $A \in CL(X)$ in τ_{weak} that fails to $\tau_{aw_d}^-$-converge to A. By Lemma 4.1, there exists a bounded set B, $\varepsilon > 0$, and a cofinal set of indices M in the underlying directed set such that for each $\lambda \in M$, there exists $a_\lambda \in A \cap B$ with $d(a_\lambda, A_\lambda) \geq \varepsilon$. Then $B_0 = \text{cl}\{a_\lambda\colon \lambda \in M\}$ is a bounded subset of A, $e_d(B_0, A) = 0$, and $\lim \sup e_d(B_0, A_\lambda) \geq \varepsilon$. This contradicts the continuity of $F \to e_d(B_0, F)$ at $F = A$ with respect to τ_{weak}. Thus, $\tau_{aw_d}^- \subset \tau_{\text{weak}}$, so that $\sigma_d^* = \tau_{W_d}^+ \vee \tau_{aw_d}^- \subset \tau_{\text{weak}}$. □

The example following Theorem 3.4 shows equally well that $A \to e_d(F, A)$ need not be τ_{aw_d}-continuous for a fixed closed set argument F. Thus such an excess functional need not be σ_d^*-continuous. For gap functionals, the weak topology determined by $\{D_d(B, \cdot)\colon B \in CLB(X)\}$ is unchanged if we replace d by a uniformly equivalent metric ρ with the same bounded sets. This is not the case for the weak topology determined by $\{e_d(B, \cdot)\colon B \in CLB(X)\}$.

Example. If d is the zero-one metric on Z^+ and ρ is the metric defined by $\rho(1, i) = 2$ for $i > 1$ and $\rho(i, j) = 1$ for $1 < i < j$, then $\sigma_d^* \neq \sigma_\rho^*$. To see this, for $n \in Z^+$, let $A_n = \{1, n + 1, n + 2, \dots\}$ and let $A = \{1\}$. It is easy to check that for each $i \in Z^+$, we have $\lim_{n \to \infty} d(i, A_n) = d(i, A)$ so that $A = \tau_{W_d}\text{-}\lim A_n$. Also, for each $\varepsilon > 0$ and $n \in Z^+$ we have $A \subset S_\varepsilon[A_n]$ so that $A = \tau_{\text{aw}_d}^-\text{-}\lim A_n$. Together, these yield $A = \sigma_d^*\text{-}\lim A_n$. But $\rho(3, A) \neq \lim_{n \to \infty} \rho(3, A_n)$, so that $\langle A_n \rangle$ fails to converge to A in σ_ρ^* or even in τ_{W_ρ}.

Since $\tau_{W_d}^+$ is weaker than $\tau_{\text{aw}_d}^+$ and $\tau_{W_d}^-$ is weaker than $\tau_{\text{aw}_d}^-$, we see that $\sigma_d \vee \sigma_d^* = \tau_{\text{aw}_d}$. Combining Theorems 4.2 and 3.4, we get this characterization of the Attouch-Wets topology as a weak topology.

Theorem 4.3. *Let* $\langle X, d \rangle$ *be a metric space. Then the Attouch-Wets topology on* $CL(X)$ *is the weakest topology* τ *on* $CL(X)$ *such that for each closed and bounded subset* B *of* X, *both* $A \to D_d(B, A)$ *and* $A \to e_d(B, A)$ *are* τ-*continuous.*

Theorem 4.4. *Let* $\langle X, d \rangle$ *be a metric space. Then the Attouch-Wets topology* τ_{aw_d} *on* $CL(X)$ *is the weakest topology* τ *on* $CL(X)$ *such that for each metric* ρ *uniformly equivalent to* d *that determines the same bounded sets as* d, *and for each closed and bounded subset* B *of* X, $A \to e_\rho(B, A)$ *is* τ-*continuous.*

Proof. Let Δ be the class of metrics described in the proof of Theorem 3.7. If $\rho \in \Delta$, then $\tau_{\text{aw}_d} = \tau_{\text{aw}_\rho}$ [BDC, Theorem 3.2], so that by Theorem 4.3, $A \to e_\rho(B, A)$ is τ_{aw_d}-continuous for each $\rho \in \Delta$ and each closed and bounded set B. On the other hand, the weak topology must contain the weak topology determined by $\{e_\rho(\{x\}, \cdot): x \in X, \ \rho \in \Delta\} = \{\rho(x, \cdot): x \in X, \ \rho \in \Delta\}$, which by Theorem 3.7, is σ_d. By Theorem 4.2, the weak topology also contains σ_d^*; so, it contains σ_d^*; so, it contains $\sigma_d \vee \sigma_d^* = \tau_{\text{aw}_d}$. $\square$

Of course, we may also split the Hausdorff metric topology into its upper and lower parts (see, e.g., [KT, p. 39]). A local base for $\tau_{H_d}^+$ (resp. $\tau_{H_d}^-$) at $A \in CL(X)$ consists of all sets of the form $\{F \in CL(X): F \subset S_\varepsilon[A]\}$ (resp. $\{F \in CL(X): A \subset S_\varepsilon[F]\}$), where $\varepsilon > 0$. Dualizing the d-proximal topology $\tau_{\delta_d} = \tau_{W_d}^- \vee \tau_{H_d}^+$ to get the *dual d-proximal topology* $\tau_{W_d}^+ \vee \tau_{H_d}^-$, and keeping in mind Theorem 3.2 and 3.7 of [BLLN], we obtain results analogous to Theorems 4.2–4.4 with essentially the same proofs.

Theorem 4.5. *Let* $\langle X, d \rangle$ *be a metric space. Then the topology* $\tau_{W_d}^+ \vee \tau_{H_d}^-$ *is the weakest topology* τ *on* $CL(X)$ *such that for each* $F \in CL(X)$, $A \to e_d(F, A)$ *is* τ-*continuous on* $CL(X)$.

Note that the Hausdorff excess functional is, in this context, extended real valued.

Theorem 4.6. *Let* $\langle X, d \rangle$ *be a metric space. Then the Hausdorff metric topology* τ_{H_d} *on* $CL(X)$ *is the weakest topology* τ *on* $CL(X)$ *such that for each* $F \in CL(X)$, *both* $A \to D_d(F, A)$ *and* $A \to e_d(F, A)$ *are* τ-*continuous.*

In view of our characterization of the d-proximal topology in Theorem 3.8 in terms of excess functionals, we also have

Theorem 4.7. *Let* $\langle X, d \rangle$ *be a metric space. Then the Hausdorff metric topology* τ_{H_d} *on* $CL(X)$ *is the weakest topology* τ *on* $CL(X)$ *such that for each* $F \in CL(X)$, *both* $A \to e_d(A, F)$ *and* $A \to e_d(F, A)$ *are* τ-*continuous.*

There is a transparent proof of Theorem 4.7 that we note. Evidently, such excess functionals are Lipschitz continuous with respect to Hausdorff distance, and τ-continuity of the functionals with $F = A_0$ at $A = A_0$ gives $A_0 = H_d\text{-}\lim A_\lambda$ whenever $A_0 = \tau\text{-}\lim A_\lambda$. As noted earlier, $A \to e_d(A, B)$ for a fixed closed and bounded set B need not be continuous with respect to the Attouch-Wets topology.

Reasoning as in the proof of Theorem 4.4, we obtain

Theorem 4.8. *Let $\langle X, d \rangle$ be a metric space. Then the Hausdorff metric topology τ_{H_d} on $CL(X)$ is the weakest topology τ on $CL(X)$ such that for each metric ρ uniformly equivalent to d and for each closed subset F of X, $A \to e_\rho(F, A)$ is τ-continuous.*

5. More on hit-and-miss topologies as weak topologies

By Theorem 3.1 of [BLLN], the Vietoris topology on $CL(X)$ is generated by the family of distance functionals $\{\rho(x, \cdot): x \in X$ and ρ is compatible with the topology on $X\}$. What happens if we restrict our metrics to those that determine the same class of bounded sets? First, a definition.

Definition. Let $\langle X, d \rangle$ be a metric space. Then the *bounded Vietoris topology* τ_{bV} on $CL(X)$ associated with the metric d has as a subbase all sets of the form V^- where V is open and all sets of the form $(B^c)^+$ where B is closed and bounded.

There is no loss in generality in requiring that the open sets in the above definition be bounded to achieve symmetry in the definition.

Theorem 5.1. *Let $\langle X, d \rangle$ be a metric space, and let $\Sigma = \{\rho: \rho$ is a metric equivalent to d that determines the same bounded sets as $d\}$. Then τ_{bV} is the weak topology on $CL(X)$ determined by $\{\rho(x, \cdot): x \in X, \ \rho \in \Sigma\}$.*

Proof. Let τ_{weak} be the weak topology determined by the prescribed family of distance functionals. By Proposition 2.1 of [FLL], for each $\rho \in \Sigma$, $x \in X$, and $\alpha \geq 0$, the set $\{F \in CL(X): \rho(x, F) < \alpha\}$ is contained in τ_{bV}. Now suppose $A \in \{F \in CL(X): \rho(x, F) > \alpha\}$. Choosing $\beta > \alpha$ with $\rho(x, A) > \beta$, we have

$$A \in \{w \in X: \rho(x, w) > \beta\}^+ \subset \{F \in CL(X): \rho(x, F) > \alpha\}.$$

Together, these show $\tau_{\text{weak}} \subset \tau_{bV}$.

For the reverse inclusion, as in the proof of Theorem 3.7, it suffices to show that for each closed d-bounded set B and each $A \in (B^c)^+$, there exists $\rho \in \Sigma$, $y_0 \in X$, and $\delta > 0$ with

$$\{F \in CL(X): \rho(y_0, A) - \delta < \rho(y_0, F)\} \subset (B^c)^+.$$

Again, this is trivial if B is a singleton; so, we assume that B has at least two points. First, separate A and B by a Urysohn function φ, and construct an equivalent metric ρ_1 given by $\rho_1(x, y) = d(x, y) + |\varphi(x) - \varphi(y)|$. Evidently, bounded sets are not changed, and $D_{\rho_1}(A, B) \geq 1$. With $\alpha = D_{\rho_1}(A, B)/4 \operatorname{diam} B$, our desired metric ρ is given by

$$\rho(x, y) = \alpha \rho_1(x, y) + |\rho_1(x, B) - \rho_1(y, B)|.$$

Then $\rho \in \Sigma$. With $y_0 \in B$ arbitrary and $\delta = \alpha \rho_1(y_0, A)$, the proof of Theorem 3.7 yields

$$\{F \in CL(X): \rho(y_0, A) - \delta < \rho(y_0, F)\} \subset (B^c)^{++}_{\rho_1} \subset (B^c)^+. \quad \square$$

In view of Theorem 3.6, we have

Corollary 5.2. *Let $\langle X, d \rangle$ be a metric space, and let $\Sigma = \{\rho: \rho$ is a metric equivalent to d that determines the same bounded sets as $d\}$. Then the bounded Vietoris topology determined by the metric d is the weak topology determined by the family of gap functionals $\{D_\rho(B, \cdot): B \in CLB(X), \rho \in \Sigma\}$.*

By Theorem 4.8, the supremum of all Hausdorff metric topologies corresponding to compatible metrics for a metrizable space X is the weak topology determined by all functionals of the form $A \to e_d(F, A)$ where F ranges over the closed subsets of X and d ranges over the compatible metrics for the topology of X. This supremum topology, called the locally finite topology, admits an interesting hit-and-miss presentation, as described in [BHPV] (see also [NS]). For Ω a family of subsets of X, write $\Omega^- = \{F \in CL(X): \forall E \in \Omega, E \cap F \neq \varnothing\}$.

Definition. Let X be a metrizable space. Then the *locally finite topology* τ_{lf} on $CL(X)$ has as a subbase all sets of the form Ω^- where Ω is a locally finite family of open subsets of X, and all sets of the form V^+ where V is an open subset of X.

Theorem 5.3. *Let X be a metrizable space. Then the locally finite topology on $CL(X)$ is the weak topology determined by the family $\{e_d(F, \cdot): F \in CL(X)$ and d is a compatible metric for the topology of $X\}$.*

Proof. Apply Theorem 4.8 above and Theorem 2.1 of [BHPV]. $\square$

We now develop a bounded analog for the last result, using very different arguments from those presented in [BHPV]. The reader is invited to construct a proof of Theorem 2.1 of [BHPV] along these lines. First, a definition.

Definition. Let $\langle X, d \rangle$ be a metric space. The *bounded locally finite topology* τ_{blf} on $CL(X)$ determined by d has as a subbase all sets of the form Ω^- where Ω is a uniformly bounded locally finite family of open subsets of X, and all sets of the form $(B^c)^+$ where $B \in CLB(X)$.

Theorem 5.4. *Let $\langle X, d \rangle$ be a metric space. Then the bounded locally finite topology τ_{blf} determined by d is the weakest topology τ on $CL(X)$ such that $A \to e_\rho(B, A)$ is τ-continuous for each $B \in CLB(X)$ and for each compatible metric ρ determining the same bounded sets as the initial metric d. Thus, τ_{blf} is the supremum of the Attouch-Wets topologies corresponding to metrics that determine the same bounded subsets as d.*

Proof. The last assertion is immediate from Theorem 4.4, upon proving the first. As usual, denote the weak topology described above by τ_{weak}. We first show that $\tau_{weak} \subset \tau_{blf}$. To this end fix a closed and bounded set B and a metric ρ determining the same bounded sets as d. We will show that $A \to e_\rho(B, A)$ is τ_{blf}-continuous at a fixed set $A_0 \in CL(X)$. Lower semicontinuity holds if $e_\rho(B, A_0) = 0$. Otherwise, fix ε between 0 and $e_\rho(B, A_0)$, and choose $b \in B$

with $\rho(b, A_0) > e_\rho(B, A_0) - \varepsilon/2$. Then $B_0 = \{x: \rho(x, b) \leq \rho(b, A_0) - \varepsilon/2\}$ is a closed bounded set disjoint from A_0, and if $A \in (B_0^c)^+$, then $e_\rho(B, A) \geq \rho(b, A) > e_\rho(B, A_0) - \varepsilon$. Upper semicontinuity is harder. For each $b \in B$, choose $a_b \in A_0$ with $\rho(a_b, b) < \rho(b, A_0) + \varepsilon/3$. Now let $E = \{a_b: b \in B\}$. Evidently, E is bounded. By Zorn's lemma, there exists a maximal subset E' of E such that for each x and y in E' we have $\rho(x, y) \geq \varepsilon/3$. Clearly, the family of open balls $\Omega = \{S_{\varepsilon/10}[x]: x \in E'\}$ is uniformly bounded and locally finite; in fact, the family is discrete, i.e., each point in X has a neighborhood meeting at most one element of the family. We claim that if $A \in \Omega^-$, then $e_\rho(B, A) < e_\rho(B, A_0) + \varepsilon$. Fix $b \in B$. By the maximality of E', there exists $x \in E'$ such that $\rho(x, a_b) < \varepsilon/3$. Since A meets $S_{\varepsilon/10}[x]$, we have $\rho(b, A) < \rho(b, A_0) + \varepsilon/3 + \varepsilon/3 + \varepsilon/10$. Thus,

$$e_\rho(B, A) = \sup_{b \in B} \rho(b, A) < \sup_{b \in B} \rho(b, A_0) + \varepsilon = e_\rho(B, A_0) + \varepsilon.$$

It remains to show that $\tau_{\text{weak}} \supset \tau_{\text{blf}}$. By Theorem 5.1, each set of the form $(B^c)^+$ with B closed and bounded is in τ_{weak}, since distance functionals are excess functionals. Now let $\{V_i: i \in I\}$ be a bounded locally finite family of open sets, and suppose $A_0 \in \{V_i: i \in I\}^-$. For each $i \in I$, choose $x_i \in A_0 \cap V_i$. Although $i \to x_i$ need not be one-to-one, it is finite-to-one, by local finiteness. Now let $B = \{x_i: i \in I\}$. By local finiteness, we can choose for each $x \in B$ a number $\varepsilon_x > 0$ such that the family $\{S_{2\varepsilon_x}[x]: x \in B\}$ is discrete, and moreover, such that whenever $x \in V_i$ for some index i, then $S_{2\varepsilon_x}[x] \subset V_i$. Let $V = \{w \in X: \forall x \in B, d(w, x) > \varepsilon_x\}$. Then V is open so that $\{V\} \cup \{S_{2\varepsilon_x}[x]: x \in B\}$ is a locally finite open cover of X. Let $\{g_x: x \in B\} \cup \{g\}$ be a partition of unity subordinated to this cover [En, p. 374], where of course $g_x^{-1}((0, 1]) \subset S_{2\varepsilon_x}[x]$ for $x \in B$ and $g^{-1}((0, 1]) \subset V$. Then for each $w \in S_{\varepsilon_x}[x]$ we must have $g_x(w) = 1$ whereas if $w \notin S_{2\varepsilon_x}[x]$, we have $g_x(w) = 0$. Evidently, the metric ρ on X defined by

$$\rho(w, y) = d(w, y) + \sum_{x \in B} |g_x(w) - g_x(y)|$$

is equivalent to d and determines the same bounded sets, since the indexed sum on the right can be at most two. Clearly $A_0 \in \{A \in CL(X): e_\rho(B, A) < 1\}$ because $e_\rho(B, A_0) = 0$. Also, if $e_\rho(B, A) < 1$ holds for a closed set A, then for each $x \in B$ there exists $a \in A$ with $\rho(x, a) < 1$. This means that

$$1 - g_x(a) = |g_x(x) - g_x(a)| \leq \rho(x, a) < 1.$$

We conclude that $g_x(a) > 0$ so that A meets $S_{2\varepsilon_x}[x]$. Since $x \in B$ was arbitrary, we conclude that A meets V_i for each $i \in I$. Thus,

$$A_0 \in \{A \in CL(X): e_\rho(B, A) < 1\} \subset \{V_i: i \in I\}^-,$$

completing the proof that $\tau_{\text{weak}} \supset \tau_{\text{blf}}$. $\square$

6. SOME FINAL OBSERVATIONS

The reader may wonder why we have not considered compact sets as potential fixed left arguments in excess and gap functionals. The answer is simple, as may be quickly verified: no finer topology results using compact sets rather than singletons.

Although it may seem unnatural, we can mix families of generating functionals corresponding to different classes of metrics and/or different classes of sets. Here are some typical outcomes; the simple details are left to the reader.

Theorem 6.1. *Let* $\langle X, d \rangle$ *be a metric space. Then the weak topology on* $CL(X)$ *determined by the family of functionals* $\{e_d(F, \cdot): F \in CL(X)\} \cup \{D_d(B, \cdot): B \in CLB(X)\}$ *is* $\tau^+_{aw_d} \vee \tau^-_{H_d}$.

Theorem 6.2. *Let* $\langle X, d \rangle$ *be a metric space. Then the weak topology on* $CL(X)$ *determined by the family of functionals* $\{D_d(F, \cdot): F \in CL(X)\} \cup \{e_d(B, \cdot): B \in CLB(X)\}$ *is* $\tau^+_{H_d} \vee \tau^-_{aw_d}$.

Theorem 6.3. *Let* $\langle X, d \rangle$ *be a metric space. Then the weak topology on* $CL(X)$ *determined by the family of functionals* $\{e_d(F, \cdot): F \in CL(X)\} \cup \{\rho(x, \cdot): x \in X \text{ and } \rho \text{ equivalent to } d\}$ *is the supremum of the Hausdorff metric topology associated with* d *and the Vietoris topology.*

We conclude with a table showing all the weak topologies obtainable using distance functionals, gap functionals, and excess functionals, using a single metric, a uniform class of metrics, or all metrics. For bounded set arguments, it is understood that the metrics are expected to determine the same class of bounded sets.

Table 1a. Weak topologies on $CL(X)$ induced by families of distance functionals, gap functionals, and excess functionals.

$\rho \approx d$: the metrics ρ and d determine the same uniformity.

$\rho \underset{bd}{\approx} d$: the metrics ρ and d determine the same uniformity and the same bounded sets.

$\rho \underset{bd}{\sim} d$: the metrics ρ and d are equivalent and determine the same bounded sets.

	$\{\rho(x, \cdot): x \in X\}$	$\{D_\rho(F, \cdot): F \in CL(X)\}$	$\{e_\rho(F, \cdot): F \in CL(X)\}$
$\rho = d$	Wijsman topology	d-proximal topology	dual d-proximal topology
$\rho \approx d$	d-proximal topology	d-proximal topology	Hausdorff metric topology
all ρ	Vietoris topology	Vietoris topology	locally finite topology

Table 1b

	$\{\rho(x, \cdot): x \in X\}$	$\{D_\rho(B, \cdot): B \in CLB(X)\}$	$\{e_\rho(B, \cdot): B \in CLB(X)\}$
$\rho = d$	Wijsman topology	bounded d-proximal topology	bounded dual d-proximal topology
$\rho \underset{bd}{\approx} d$	bounded d-proximal topology	bounded d-proximal topology	Attouch-Wets topology
$\rho \underset{bd}{\sim} d$	bounded Vietoris topology	bounded locally finite topology	bounded locally finite topology

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