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ON KLEIN'S COMBINATION THEOREM. IV

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ABSTRACT. This paper contains an expansion of the combination theorems to cover the following problems. New rank 1 parabolic subgroups are produced, while, as in previous versions, all elliptic and parabolic elements are tracked. A proof is given that the combined group is analytically finite if and only if the original groups are; in the analytically finite case, we also give a formula for the hyperbolic area of the combined group (i.e., the hyperbolic area of the set of discontinuity on the 2-sphere modulo G) in terms of the hyperbolic areas of the original groups. There is also a new variation on the first combination theorem in which the common subgroup has finite index in one of the two groups.

This is the fourth formulation of the generalizations of Klein's combination theorem. The first formulation was given in [M1 and M2], where the amalgamated and conjugated subgroups were cyclic. The next formulation appeared in [M3], where we considered more general subgroups, but still required precisely invariant closed discs. In the third generation [M5], we no longer required the entire closed discs to be precisely invariant; we permitted the boundary of the disc to intersect translates of itself, but only at limit points; this permitted us to create rank 2 parabolic subgroups from doubly cusped rank 1 parabolic subgroups, but did not permit the creation of new rank 1 parabolic subgroups. In this version, we permit translates of the boundary of the disc to touch, but not cross, at a discrete set of ordinary points, and thus permit the production of new rank 1 parabolic subgroups.

The second version of the first combination theorem essentially says the following. We are given two discontinuous groups of Möbius transformations, G_1 and G_2, with a common subgroup J, where J is not equal to either G_1 or G_2. We are also given a simple closed curve W dividing the extended complex plane $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2, where B_m is precisely invariant under J in G_m (i.e., B_m is J-invariant, and if $g \in G_m - J$, then $g(B_m) \cap B_m = \varnothing$). Then $G = \langle G_1, G_2 \rangle$, the group generated by G_1 and G_2, is also discontinuous; G is the free product of G_1 and G_2, amalgamated across J; if we intelligently choose fundamental domains for G_1 and G_2, then their intersection will be a fundamental domain for G; every element of G that is not a conjugate of an element of either G_1 or G_2 is loxodromic (including hyperbolic).

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One can restate the basic hypothesis as follows: B_1 and B_2 are both J-invariant; the $(G_1 - J)$-translates of B_1 are disjoint discs in B_2; the $(G_2 - J)$-translates of B_2 are disjoint discs in B_1.

In general terms, the second version of the second combination theorem is as follows. We are given a single group G_0, with two subgroups J_1 and J_2, two closed discs B_1 and B_2, which have disjoint projections to $\Omega(G_0)/G_0$, where J_m preserves B_m, and we are given a Möbius transformation f mapping the outside of B_1 onto the inside of B_2, and conjugating J_1 onto J_2. The conclusions are that $G = \langle G_0, f \rangle$ is discontinuous; G is the HNN-extension of G_0 by f; if we intelligently choose a fundamental domain D for G_0, then $D - (B_1 \cup B_2)$ is a fundamental domain for G; every element of G that is not a conjugate of an element of G_0 is loxodromic.

The statements given in [M5] are somewhat more general; we permit the translates of the closed discs to have common boundary points, but we require that these common boundary points be limit points of the stabilizers of both discs. This entails a slight change in the conclusions; there may now be new parabolic elements in the final group; these commute with conjugates of parabolic elements of J. We also have the important conclusion that the final group is geometrically finite if and only if the original groups are.

In this paper, we weaken the hypotheses further and permit the translates of the closed discs to have common boundary points that are ordinary points of the stabilizers of both discs, but we require that these points of intersection also be ordinary points of our original group. We also add the conclusion that the final group is analytically finite if and only if the original groups are analytically finite, and we give a formula for the hyperbolic area.

The basic topological object in the use of combination theorems is a simple closed curve bounding two closed discs, and its translates under a Kleinian group. Our usual requirement is that the simple closed curve be almost disjoint from all its translates; that is, if the curve is W, and $g(W)$ is a translate, then $g(W)$ lies entirely in one of the closed discs bounded by W. Of course, the two curves might intersect; the main difficulties that we encounter in this paper are concerned with controlling these points of intersection.

We also formulate a version of the first combination theorem for the special case that J has finite index in G_2; this includes the case that G_2 stabilizes W, but contains an element interchanging B_1 and B_2. This case, which requires slightly different hypotheses and conclusions, is referred to as the variation on the first combination theorem (or first variation); it is treated in §III.

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0. Preliminaries

0.1. We will need the following notation. If A is a set, then $\text{Stab}(A) = \{g \in G \colon g(A) = A\}$ is the stabilizer of A.

We say that A is *precisely invariant* under the subgroup $H \subset G$ if $h(A) = A$ for all $h \in H$, and $g(A) \cap A = \varnothing$ for all $g \in G - H$.

We use the following notational convention throughout. In each section there is a given group G that contains all the other groups under consideration. This group G will usually be understood, rather than stated; for example, we refer to the stabilizer in G of a set A as $\mathrm{Stab}(A)$, while we refer to the stabilizer of A in the subgroup $H \subset G$ as $\mathrm{Stab}_H(A)$. Similarly, we refer to a set B as being a translate of A when there is some element of G mapping A to B; if we need to know that there is an element of $H \subset G$ mapping A to B, then we refer to B as being an H-translate of A.

0.2.1. A simple closed curve W divides the extended complex plane $\widehat{\mathbb{C}}$ into two open topological discs; another simple closed curve W' is *almost disjoint* from W if it is disjoint from one of these open discs.

Similarly, the simple closed curve W *weakly separates* two sets if they lie in distinct closed topological discs bounded by W.

A simple closed curve that is almost disjoint from all its translates is said to be *precisely embedded.*

If W is precisely embedded, then the set of points on W, which also lie on some translate of W can, in general, be quite complicated; there can also be points on W that are points of accumulation of translates of W.

0.2.2. If x is a fixed point of a parabolic element of G, then we say that x is *doubly cusped* if there are two disjoint open circular discs whose union is precisely invariant under $\mathrm{Stab}(x)$; each of these discs is called a *cusped region* at x; the union of the two discs is called a *doubly cusped region* at x.

If W is a curve, passing through the point x, then we say that W is *locally circular* at x if there is a neighborhood N of x so that $W \cap N$ is a circular arc.

0.2.3. Let W be a precisely embedded simple closed curve with geometrically finite stabilizer J. Assume that we are given a J-invariant set of points $\Theta \subset W \cap \Omega(J)$, called *rimpoints*, where Θ/J is a finite set, and W is locally circular at each point of Θ. We assume that for every $g \in G$, and every point $x \in (W \cap g(W))$, x and $g^{-1}(x)$ are either both points of $\Lambda(J)$, or they are both points of Θ. Assume further that every point of $\Omega(J)$ is either a point of $\Omega(G)$ or both a point of Θ and a doubly cusped rank 1 parabolic fixed point of G. Under these circumstances, we say that (W, Θ) is a (J, G)-*swirl*; if there is no danger of confusion, we will not specify the dependence on G and/or Θ, and simply say that W is a swirl.

The requirement that a swirl be locally circular at the rimpoints is in practice hardly a restriction; one could weaken this requirement significantly, but then the proofs would require more care.

0.2.4. Let B be a topological disc with geometrically finite stabilizer J. Suppose that $\mathrm{int}(B)$, the interior of B, is precisely invariant under J, and that $W = \partial B$ is a swirl with set of rimpoints Θ. Then we say that B, or (B, Θ), if we need to specify Θ, is a *simple disc.* Here again, if we need to specify the groups, we will also say that (B, Θ) is a (J, G)-simple disc.

If B is a simple disc, then, since $\mathrm{int}(B)$ is precisely invariant under J in G, ∂B weakly separates B from every $(G - J)$-translate of B.

A swirl W, or the simple disc B bounded by W, is *strong* if, for every parabolic element of G whose fixed point x lies on W, $\mathrm{Stab}(x)$ either has rank 2 or is doubly cusped. Note that if G is geometrically finite, then every swirl and every simple disc is necessarily strong.

Proposition 0.3. *Let B be a (J, G)-simple disc bounded by the swirl W. Then $\Omega(J) \cap B = \Omega(G) \cap B$.*

Proof. It follows from the definition of a swirl that the only other possibility is that there is a point $x \in \Omega(J)$, which is both a rimpoint and a doubly cusped parabolic fixed point of G. Let j be some nontrivial element of $\mathrm{Stab}(x)$. Since $j \notin J$, $j(B) \cap B \subset \Theta \cap \Lambda(J)$. It follows that B and $j(B)$ are distinct discs that are tangent at x; hence there are at most two of them. This contradicts the fact that $\mathrm{Stab}(x)$ has infinite order. □

0.4. The author thanks W. Abikoff and G. Swarup for conversations leading to the next simplification.

In the statement of the first combination theorem given in [M5], we required that $(\mathrm{int}(B_1), \mathrm{int}(B_2))$ be a proper interactive pair. That $(\mathrm{int}(B_1), \mathrm{int}(B_2))$ is an interactive pair is simply the statement that every element of $G_1 - J$ maps $\mathrm{int}(B_1)$ into $\mathrm{int}(B_2)$, and every element of $G_2 - J$ maps $\mathrm{int}(B_2)$ into $\mathrm{int}(B_1)$. The interactive pair is *proper* if there is either a point of $\mathrm{int}(B_2)$ that is not in a G_1-translate of $\mathrm{int}(B_1)$, or there is a point of $\mathrm{int}(B_1)$ that is not in a G_2-translate of $\mathrm{int}(B_2)$.

Proposition. *Let J be a subgroup of the Kleinian groups, G_1 and G_2, where J is a finitely generated quasifuchsian group of the first kind with limit set the simple closed curve W. Assume that W divides $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2, where $\mathrm{int}(B_m)$ is precisely invariant under J in G_m. Assume further that there is a $g_0 \in G_1$ so that $g_0(W) \neq W$. Then the complement of the union of the G_1-translates of B_1 in B_2 contains infinitely many points; in particular, $(\mathrm{int}(B_1), \mathrm{int}(B_2))$ is a proper interactive pair.*

Proof. Since J is of the first kind, and $\mathrm{int}(B_1)$ is precisely invariant under J in G_1, $\mathrm{int}(B_1)$ is a component of G_1. Since $g_0(W) \neq W$, $g_0(B_1)$ is a distinct component of G_1, and $g_0 J g_0^{-1} \neq J$. It then follows from [M4] that $J \cap g_0 J g_0^{-1}$ is a finitely generated subgroup of J of the second kind. Hence $J \cap g_0 J g_0^{-1}$ has infinite index in J. Every element of J, which is not in $g_0 J g_0^{-1}$, moves $g_0(B_1)$ to some other component of G_1; hence there are infinitely many distinct translates of B_1 in B_2.

It was shown by Abikoff [A] that since G_1 has infinitely many components, G_1 contains infinitely many distinct loxodromic elements stabilizing no component of G_1. The fixed points of these loxodromic elements must all lie in the complement of the union of the closures of the components of G_1. In particular, we have shown that there are infinitely many points in the complement of the union of the G_1-translates of B_1. □

0.5. There is a similar problem with the second combination theorem and interactive triples. Let G_0 be a Kleinian group with two distinguished subgroups, J_1 and J_2. Let X_1, X_2, and Y be disjoint nonempty sets. Let f be a Möbius transformation where $fJ_1f^{-1} = J_2$. The triple (Y, X_1, X_2) is *interactive* if (X_1, X_2) is precisely invariant under (J_1, J_2) in G_0 (that is, X_m is precisely invariant under J_m in G_0, and, for all $g \in G_0$, $g(X_1) \cap X_2 = \varnothing$); $f(Y \cup X_2) \subset X_2$; and $f^{-1}(Y \cup X_1) \subset X_1$.

Assume that we have an interactive triple, (Y, X_1, X_2). Let Y_0 be the complement in Y of the union of the G_0-translates of $X_1 \cup X_2$. The interactive triple is *proper* if $Y_0 \neq \varnothing$.

The basic setup for the second combination theorem is that we are given a Kleinian group G_0 with two geometrically finite subgroups, J_1 and J_2; we are given two closed topological discs, with distinct boundaries, B_1, and B_2, where B_m is a (J_m, G)-simple disc; and we are given a transformation, f, mapping the exterior of B_1 onto the interior of B_2, and conjugating J_1 onto J_2. We also assume that $(A, \operatorname{int}(B_1), \operatorname{int}(B_2))$ is an interactive triple, where A is the complement of $B_1 \cup B_2$. Let A_0 be the complement of the union of the G_0-translates of $(B_1) \cup (B_2)$.

Since J_m preserves B_m, and is geometrically finite, it is quasifuchsian.

Proposition 0.6. *If J_1 is a quasifuchsian group of the first kind, then A_0 contains infinitely many points; in particular, the interactive triple $(A, \operatorname{int}(B_1), \operatorname{int}(B_2))$ is proper.*

Proof. The requirement that $(A, \operatorname{int}(B_1), \operatorname{int}(B_2))$ be an interactive triple includes the facts that $\operatorname{int}(B_m)$ is precisely invariant under J_m in G_0, and that every G_0-translates of $\operatorname{int}(B_1)$ is disjoint from $\operatorname{int}(B_2)$; this implies that $\operatorname{int}(B_1)$ and $\operatorname{int}(B_2)$ are distinct regions in $\Omega(G_0)$; also, since J_1 and J_2 are of the first kind, they are components of G_0.

Since $A \neq \varnothing$, $\partial B_1 \neq \partial B_2$; it follows that $J_1 \neq J_2$. Assume for the sake of argument that there is a $j \in J_1 - J_2$. Then $j(B_2)$ and B_2 are distinct, and distinct from B_1. Since G_0 has at least three components, it has infinitely many components. Hence, exactly as in the proof of Proposition 0.4, there are infinitely many loxodromic elements of G_0 that stabilize no component; the fixed points of these loxodromic elements must lie in A_0. $\square$

0.7. An *analytically finite* Kleinian group G satisfies the conclusion of Ahlfors' finiteness theorem; that is, it is either of the first kind, or elementary, or Ω/G has finite hyperbolic area.

We will need the following conventions concerning area. If G is an analytically finite nonelementary Kleinian group of the second kind, then we define $\operatorname{area}(G)$ to be the hyperbolic area of Ω/G. If G is elementary, then it is automatically analytically finite, but the natural metric is now either Euclidean or spherical. If G has either one or two limit points, then the natural metric is Euclidean; in this case, we set $\operatorname{area}(G) = 0$. If G is finite, then the natural metric is spherical; in this case, we define $\operatorname{area}(G) = -4\pi/|G|$; this is the negative of the natural area. If G is of the first kind, then we also define $\operatorname{area}(G) = 0$. With these definitions, the area is, up to multiplication by -2π, the virtual Euler characteristic; that is, the area is essentially the negative Euler characteristic of Ω/G, up to multiplication by 2π, except that the special points (i.e., the projections of the elliptic fixed points in Ω) are counted as 0-cells with special weights.

0.8. We will also need some Euclidean metric considerations; $\operatorname{dia}(X)$ refers to the spherical diameter of the set X.

Proposition. *Let W be a (J, G)-swirl, and let $\{g_m(W)\}$ be a sequence of distinct translates of W. Then $\operatorname{dia}(g_m(W)) \to 0$.*

Proof. If $W \cap \Omega(J) = \varnothing$, then W is a block; in this case the desired result is given in [M5, p. 142]. If $W \cap \Omega(J) \neq \varnothing$, which we now assume, then G is of the second kind. Normalize G so that $\infty \in {}^\circ\Omega(G)$ (this is $\Omega(G)$ with the fixed points of elliptic elements removed). As in [M5, pp. 142 ff.], since

W is J-invariant, we can assume that a_m, the center of the isometric sphere for g_m, lies in the Ford region for J; call it E. Then we have the obvious bound: $\mathrm{dia}(g_m(W)) \leq r_m^2/\delta_m$, where r_m is the radius of the isometric sphere of g_m, and δ_m is the distance from a_m to W. Since $\sum r_m^6$ converges, there is nothing to prove unless the a_m accumulate at W. Suppose x is a limit point of the a_m on W. Since J is geometrically finite, $\overline{E}$ meets W only at a finite number of parabolic fixed points of J, and at points of $\Omega(J)$. Since W is a swirl, and $x \in \Lambda(G)$, if x is not a parabolic fixed point of J, then it is necessarily both a rimpoint and a doubly cusped parabolic fixed point of G.

Let λ_m be the distance from a_m to x.

If x is a parabolic fixed point of J, then a_m approaches x inside a cusp, so the distance from a_m to W is commensurate with λ_m. Hence

$$\mathrm{dia}(g_m(W)) \leq r_m^2/\delta_m \leq K r_m^2/\lambda_m .$$

Since x is not a point of approximation, the quantity on the right tends to zero.

If x is a rimpoint, and a parabolic fixed point of G but not of J, then, since W is circular near x, we can write W as the union of two open arcs, W_1 and W_2, where W_1 lies inside a doubly cusped region near x, and W_2 is disjoint from a smaller doubly cusped region. The a_m all lie outside the larger doubly cusped region, and accumulate at x. Since the larger doubly cusped region is precisely invariant under its stabilizer, any sequence of distinct translates of it has diameter tending to zero; hence $g_m(W_1) \to 0$. Since the a_m are bounded away from W_2, the argument above shows that $\mathrm{dia}(g_m(W_2)) \to 0$. Hence $\mathrm{dia}(g_m(W)) \to 0$. □

I. THE FIRST COMBINATION THEOREM

I.1. Our basic hypotheses for the first combination theorem are as follows. We are given two Kleinian groups, G_1 and G_2, with a common subgroup J, where J is geometrically finite and has index at least 2 in both G_1 and G_2. We are also given a J-invariant simple closed curve W, together with a J-invariant set of rimpoints Θ on W. W divides $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2; we assume that (B_m, Θ) is a (J, G_m)-simple disc. We also require that there be a $g_1 \in G_1 - J$ with $g_1(W) \neq W$. We still need a condition on certain of the rimpoints.

If $x \in \Theta$, then there need not be any $g \in G_m - J$ with $g(x) = x$; if there is such an element, then we say that x is a *true* G_m-rimpoint. A rimpoint that is both a true G_1-rimpoint and a true G_2-rimpoint is called a *double* rimpoint; the others are called *single* rimpoints.

Choose some fundamental set E for the action of J on $W \cap \Omega(J)$. Since J is geometrically finite, and Θ is a discrete subset of $\Omega(J)$, there are only finitely many rimpoints in E. Assume there is a double rimpoint $x = x_1$ in E. Then there is a $g_1 \in G_1 - J$ so that $x_2 = g_1(x_1)$ is again a rimpoint in E (note that g_1 might be an elliptic transformation with a fixed point at x_1). If x_2 is a single rimpoint, then there is nothing further to do; if x_2 is again a double rimpoint, then there is a $g_2 \in G_2 - J$ so that $g_2(x_2)$ is also a rimpoint in E. We continue in this manner until we either reach a single rimpoint, or we return to x_1. In the former case, we have a *chain* of rimpoints; in the latter case, we have a *cycle* of rimpoints.

A rimpoint on W that is J-conjugate to a rimpoint lying in a cycle of rimpoints is called *preparabolic*; the others are *ordinary* rimpoints.

We remark that a chain or cycle of rimpoints might have only one point in it.

In the case of rimpoints, it is clear that we could have started the chain at a single rimpoint x_1, continued through double rimpoints, $x_2, \dots x_{n-1}$, and finally arrived at a single rimpoint, x_n. From here on we assume that every chain of rimpoints starts and ends with single rimpoints; that is, we assume that every chain has maximal length.

For every preparabolic point x, we have constructed a transformation $g = g_n \circ \cdots \circ g_1$, with fixed point x, where the g_m are alternately in $G_1 - J$ and $G_2 - J$. This element is called a *cyclic stabilizer* at x. Since we permit rimpoints to be fixed points of elliptic elements in G_m, the cyclic stabilizer at x need not be unique.

I.2. We can also have cyclic stabilizers at parabolic fixed points of J; that is, a parabolic fixed point x of J is a *parabolic G_m-rimpoint* if there is an element $g \in G_m - J$ mapping x onto a point of B_m, necessarily a point on W. The *double parabolic rimpoints* are parabolic rimpoints for both G_1 and G_2. Since J is geometrically finite, there are only finitely many double parabolic rimpoints modulo J. As above, these come in chains and cycles; a point on a cycle of double parabolic rimpoints is a *double preparabolic* point. Also as above, there is an associated *cyclic stabilizer* at each double preparabolic point. Since such a cyclic stabilizer fixes a parabolic fixed point of J, it cannot be unique.

Condition (B) below requires that every cyclic stabilizer be parabolic. We remark that this hypothesis is not needed for the cyclic stabilizers at double preparabolic points; that is, we prove that G is discrete and that G is the free product of G_1 and G_2, amalgamated across J; this rules out any other possibility for such a cyclic stabilizer.

I.3. Let D_m be a fundamental set for G_m (that is, D_m is a fundamental set for the action of G_m on ${}^\circ\Omega(G_m)$) satisfying the following. D_m is maximal with respect to B_m (i.e., $D_m \cap B_m$ is a fundamental set for the action of J on B_m), and, in the complement of Θ, $D_1 \cap W = D_2 \cap W$. The sets D_1 and D_2 satisfying these conditions are called *compatible fundamental sets.*

In previous versions of this combination theorem, we set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$, and then showed that D is a fundamental set for G. In our case, this need not be true, we need to make modifications to account for the rimpoints.

Once we have chosen the fundamental domains D_1 and D_2, the chains and cycles of rimpoints are well defined.

Set $D' = (D_1 \cap B_2) \cup (D_2 \cap B_1)$. If $x_1, \dots, x_n$ is a cycle of double rimpoints, then condition (B) below asserts that each x_m is a parabolic fixed point in G; hence x_m is not in $\Omega(G)$. We delete all preparabolic rimpoints from D'.

If we have a chain of rimpoints containing an elliptic fixed point, then every point of this chain is a fixed point of an elliptic element of G; we delete all such points from D'.

If $x_1, \dots, x_n$ is a chain of rimpoints, where no x_i is an elliptic fixed point, then these points are all equivalent modulo G; hence we need only one of them in our fundamental set for G. We choose one single ordinary rimpoint in D'

from each chain of ordinary rimpoints, and delete all the others from D'. After these additions and deletions, we are left with the *modified set*, D.

I.4. The major conclusions of the first combination theorem are given below. The conclusions are numbered so as to agree with the numbering in [M5], although some of the formulations have been modified. Also, conclusions (xii) and (xiii) are new.

Statement of the first combination theorem

Theorem I (the first combination theorem). *Let G_1 and G_2 be Kleinian groups with a geometrically finite common subgroup J, where the index of J is at least two in both groups. Assume the following.*

(A) *There is a J-invariant simple closed curve W dividing $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2, and there is a set of rimpoints Θ given on W, so that (B_m, Θ) is a (J, G_m)-simple disc.*

(B) *Every cyclic stabilizer is parabolic.*

(C) *There is a $g_m \in G_m$ with $g_m(W) \neq W$.*

Let $G = \langle G_1, G_2\rangle$, let D_1 and D_2 be compatible fundamental sets for G_1 and G_2, respectively, and let D be the modified set for G. Then the following hold.

(i) $G = G_1 *_J G_2$, *(i.e., G is the free product of G_1 and G_2, with amalgamated subgroup J).*

(ii) *G is discrete.*

(iii) *Every element of G that is not a conjugate of an element of either G_1 or G_2, or a conjugate of a power of a cyclic stabilizer, is loxodromic.*

(iv) (W, Θ) *is a (J, G)-swirl; it is strong if and only if B_1 and B_2 are both strong simple discs.*

(vii) *The modified set D is a fundamental set for G.*

(viii) *Let S_m be the complement in B_{3-m} of the union of the G_m-translates of B_m. Then S_m is precisely invariant under G_m. Further, $\Omega(G)/G$ is the union of S_1/G_1 and S_2/G_2, where these two possibly disconnected surfaces (either or both of which might be empty) are joined along the projection of $W \cap \Omega(G)$. $W \cap \Omega(G)$ is the complement of the cyclic rimpoints in $W \cap \Omega(J)$. Two points of $W \cap \Omega(J)$ that are not rimpoints are G-equivalent if and only if they are J-equivalent.*

(ix) *Assume that G_1 and G_2 are both geometrically finite and that $W \cap \Omega(J)$ is smooth. Then there is a spanning disc Q for W (that is, Q is a properly embedded topological disc in hyperbolic 3-space $\mathbb{H}^3$ whose Euclidean boundary is W, where Q is precisely invariant under J). Further, $\mathbb{H}^3/G$ can be described as follows: Let B^3_m be the region in $\mathbb{H}^3$, bounded by the translates of Q, whose Euclidean boundary is B_m. Then $\mathbb{H}^3/G$ is the union of B^3_1/G_1 and B^3_2/G_2, where these two 3-orbifolds are joined along their common boundary, Q/J.*

(xi) *G is geometrically finite if and only if G_1 and G_2 are both geometrically finite.*

(xii) *G is analytically finite if and only if G_1 and G_2 are both analytically finite.*

(xiii) *If G is analytically finite, then*

$$\text{area}(G) = \text{area}(G_1) + \text{area}(G_2) - \text{area}(J).$$

Proof of the first combination theorem

Proof of (i). If J is of the second kind, then there are points of D_1 near W that do not lie in any G_1-translate of B_1; hence $(\text{int}(B_1), \text{int}(B_2))$ is a proper interactive pair. If J is of the first kind, then this result is the content of Proposition 0.4. The desired result now follows from [M5, Theorem VII.A.10].

Proof of (ii). It follows from conclusion (i) that every element of $G - J$ can be written in the form $g = g_n \circ \cdots \circ g_1$, where the g_m are alternately in $G_1 - J$ and $G_2 - J$; the g_m are not in general uniquely defined, but the *length*, n, is. The word $g_n \circ \cdots \circ g_1$ is called a (j, k)-form if $g_n \in G_j - J$, and $g_1 \in G_k - J$. It was shown in [M5, p. 138], that, since $(\text{int}(B_1), \text{int}(B_2))$ is an interactive pair, if $g = g_n \circ \cdots \circ g_1$ is a (j, k)-form then $g(\text{int}(B_k)) \in \text{int}(B_{3-j})$.

As in conclusion (viii), let S_m be the complement of the union of the G_m-translates of B_m in B_{3-m}.

Lemma I.1. *S_m is precisely invariant under G_m.*

Proof. It suffices to show that S_1 is precisely invariant under G_1. Every element of $g \in G$ is either an element of $J \subset G_1$, or an element of $G_1 - J$, or an element of $G_2 - J$, or g can be written as a (j, k)-form, $g = g_n \circ \cdots \circ g_1$, where $n > 1$.

Since S_1 is the complement of a G_1-invariant set, it is also G_1-invariant.

If $g \in G_2 - J$, then $g(S_1) \subset B_1$, which is disjoint from S_1.

If $g = g_n \circ \cdots \circ g_1$ is a $(2, 2)$-form, then, by the remark above, $g(S_1) \subset g(B_2) \subset B_1$, which is disjoint from S_1. If g is a $(2, 1)$-form, then $g_n \circ \cdots \circ g_2$ is a $(2, 2)$-form, and $g(S_1) = g_n \circ \cdots \circ g_2(g_1(S_1)) = g_n \circ \cdots \circ g_2(S_1)$; by the above, this is contained in B_1, which is disjoint from S_1.

If $g = g_n \circ \cdots \circ g_1$ is either a $(1, 1)$-form or a $(1, 2)$-form, then $g' = g_{n-1} \circ \cdots \circ g_1$ is either a $(2, 1)$-form or a $(2, 2)$-form. By the above, $g'(S_1) \subset B_1$. Then $g(S_1) = g_n \circ g'(S_1) \subset g_n(B_1)$, which is contained in the complement of S_1. □

Before returning to the proof of (ii), we make the following remark. Since there is some $g_0 \in G_1$ with $g_0(W) \neq W$, $g_0(B_1)$ is properly contained in B_2. Since B_1 is precisely invariant under J in G_1, for every $g \in G_1 - J$, $g(B_1)$ is properly contained in B_2.

We return to the proof of conclusion (ii). As a corollary of the above argument, we see that for any sequence $\{g_m\}$ of distinct elements of $G - J$, where each g_m has length at least two, $g_m(W)$ either lies inside a translate of B_1 in B_2, or lies inside a translate of B_2 in B_1. Since these are all topological discs, with disjoint interiors and nonempty complement, it is clear that $g_m(W)$ cannot converge to W. Since G_1 and G_2 are both discrete, for no sequence of distinct elements of either $G_1 - J$, or $G_2 - J$, can we have $g_m(W) \to W$.

Proof of (iii). Let g be an element of G, where g is not conjugate to an element of either G_1 or G_2. We assume that g has minimal length in its conjugacy class; in particular, we assume that $g = g_n \circ \cdots \circ g_1$, where $g_{2m} \in G_2 - J$, $g_{2m-1} \in G_1 - J$, and $n \geq 2$ is even. Since g has been written as a $(2, 1)$-form, $g(B_1) \subset B_1$; as remarked above, since $n \geq 2$, this inclusion is proper, from which it follows that g has infinite order. We also conclude that g has a fixed point in $g(B_1) \subset B_1$. Since g^{-1} is a $(1, 2)$-form, $g^{-1}(B_2) \subset B_2$; hence g also has a fixed point in $g^{-1}(B_2) \subset B_2$.

If g is parabolic, then its fixed point x lies on $W \cap g(W)$. Lying between W and $g(W)$, we also have $g_n(W)$, $g_n \circ g_{n-1}(W), \dots, g_n \circ \cdots \circ g_2(W)$; hence x also lies on all these translates of W. This can occur only if x is a translate of a preparabolic rimpoint and g is a corresponding cyclic stabilizer.

Proof of (iv). Since $\operatorname{int}(B_m)$ is precisely invariant under J in G_m, we already know that W is precisely embedded in G_m. Since S_m is precisely invariant under G_m in G, no translate of W crosses any G_m-translate of W. In particular, no translate of W crosses W; i.e., W is precisely embedded in G. It also follows, from the fact that S_m is precisely invariant under G_m that if some $g(W) \neq W$ touches W at a point x, then there is either a G_1-translate of W touching W at x, or there is a G_2-translate of W touching W at x; i.e., x and $g^{-1}(x)$ are either both points of Θ or they are both limit points of J.

We already know that J is geometrically finite, and that $J = \operatorname{Stab}_{G_1}(W) = \operatorname{Stab}_{G_2}(W)$. It follows from the above that $J = \operatorname{Stab}(W)$.

We also already know that Θ/J is a finite set, and that W is locally circular at the points of Θ. In order to show that W is a swirl, we still need to show that every point of $W \cap \Omega(J) - \Theta$ lies in $\Omega(G)$, that the ordinary rimpoints lie in $\Omega(G)$, and that the preparabolic rimpoints, all of which are parabolic fixed points of G, are doubly cusped; this is the content of the next two lemmas.

Lemma I.2. $(W \cap \Omega(J)) - \Theta \subset \Omega(G)$.

Proof. Let $x \in (W \cap \Omega(J)) - \Theta$. Then, since B_1 and B_2 are both simple discs, $x \in (\Omega(G_1) \cap \Omega(G_2))$. Since $x \in \Omega(G_m)$, and $x \notin \Theta$, there is a neighborhood N of x, so that no $(G_m - J)$-translate of B_m intersects N. It follows that no nontrivial translate of W intersects N, from which it follows that $x \in \Omega$. □

Lemma I.3. *The ordinary rimpoints are contained in* Ω, *and the preparabolic rimpoints are all doubly cusped parabolic fixed points of G.*

Proof. Let $x_0 \dots, x_n$ be a chain or cycle of rimpoints.

In the case that it is a chain, we can assume without loss of generality that x_0 and x_n are single rimpoints, and the others are double. In particular, we assume that there is no $g \in G_2 - J$ with $x_0 \in g(W) \cap W$, but there is a $g_1 \in G_1 - J$, with $x_1 = g_1(x_0) \in W$. Then there is a $g_2 \in G_2 - J$ with $x_2 = g_2 \circ g_1(x_0) \in W$, etc. Depending on whether n is even or odd, there is either no $g \in G_1 - J$ with $g(x_n) \in W$, or there is no $g \in G_2 - J$ with $g(x_n) \in W$; for the sake of definiteness, we assume the former.

In any case, since B_m is a simple disc for J in G_m, the points $x_0, \dots, x_n$ are all points of both $\Omega(G_1)$ and $\Omega(G_2)$. Hence, for a chain, x_0 has a neighborhood N_0 that meets no $(G_2 - J)$-translate of W, and, aside from $g_1^{-1}(W)$, meets no $(G_1 - J)$-translate of W. Similarly, for $i = 1, \dots, n-1$ in the case of a chain, and for all i in the case of a cycle, x_i has a neighborhood N_i that meets W, meets exactly one $(G_1 - J)$-translate of W, and meets exactly one $(G_2 - J)$-translate of W. Finally, again in the case of a chain, x_n has a neighborhood that meets no $(G_1 - J)$-translate of W and meets exactly one $(G_2 - J)$-translate of W. Let $h_i = (g_1 \circ \cdots \circ g_i)^{-1}$; let $N_i' = h_i(N_i)$; then $N = \bigcap N_i$ is a neighborhood of x_0.

In the case of a chain, we see exactly the $n+1$ translates of W: W, $h_1(W)$, $\dots, h_n(W)$, inside N; these all touch without crossing at x_0. Since there is no $(G_2 - J)$-translate of W in N_0, there is no G-translate of W in $N \cap B_2$. Similarly, there is no G-translate of W in N lying between any of the above translates of W meeting at x_0, and there is no G-translate of W in N lying on the other side of $h_n(W)$. We have shown that N is a neighborhood of x_0 meeting only finitely many G-translates of W; it follows that $x_0 \in \Omega(G)$.

In the case of a cycle, we see infinitely many translates of W inside N, but we pick out exactly the $n+1$ translates listed above. Then there is a parabolic transformation, the cyclic stabilizer, with fixed point at x_0, mapping W onto $h_n(W)$. We choose N sufficiently small so that these are both arcs of circles inside N, necessarily tangent. Inside N, aside from $W_1, \dots, W_{n-1}$, there are no G-translates of W lying between W and $h_n(W)$. It follows that there are no limit points of G in N lying between W and $h_n(W)$. Once we have a doubly cusped region containing no limit points, it follows from the Shimizu-Leutbecher lemma that there is a precisely invariant doubly cusped region contained inside it. □

The proof that (W, Θ) is strong if and only if (B_1, Θ) and (B_2, Θ) are both strong is included in the proof of (xi).

Proof of (vii). We already know [M5, p. 138] that $(D_1 \cap \operatorname{int}(B_2)) \cup (D_2 \cap \operatorname{int}(B_1))$ is precisely invariant under the identity in G. Since S_m is precisely invariant under G_m, we have the following picture of the action of G near W. All the translates of W lying in B_1 lie inside some G_2-translate of B_2, and all the translates of W lying in B_2 lie inside some G_1-translate of B_1. Hence there is a point $x \in W \cap g(W)$, for some $g \in G$, only if there is a $\hat{g}$ of length 1 with $x \in \hat{g}(W)$. We conclude that $D \cap W$ is precisely invariant under the identity in G.

Each point of $D_1 \cap B_2$ is G_1-equivalent to a unique point of $S_1 \cap \Omega(G_1)$. Since every point of $S_1 \cap \Omega(G_1)$ has a neighborhood that intersects no translate of W, $(D_1 \cap B_2) \subset \Omega$. Likewise, $(D_2 \cap B_1) \subset \Omega$.

We already know from Lemma I.2 that, except for the rimpoints, every point of $\Omega(J) \cap W$ is also in Ω. We also know from Lemma I.3 that the ordinary rimpoints are in Ω. We have shown that $D \subset \Omega$. Since we have exactly one point from each G-equivalence class of ordinary rimpoints in D, we have also shown that D is precisely invariant under the identity in G.

It remains only to show that every point of Ω is equivalent to some point of $\overline{D}$. We already know that every point of $(S_1 \cup S_2) \cap {}^{\circ}\Omega$ is equivalent to some point of D; hence the same is true for every translate of these sets. The only points which are not translates of these sets are the infinite points; these are the points that are separated from W by an infinite sequence of distinct translates of W. Since W is a swirl, by Proposition 0.8, any sequence of distinct translates of W have spherical diameter tending to zero. Hence every infinite point is a limit point of G.

Proof of (viii). The first statement is just Lemma I.1. If we choose the fundamental sets D_1 and D_2 to be constrained (that is, $\operatorname{int}(D_m)$ is a fundamental domain for G_m), then D is also constrained, from which the next statement follows. The statements about the G-equivalence of points of W have been proven above.

Before going on to conclusion (ix), we prove a lemma that will also be needed for the second combination theorem.

Lemma I.4. *Suppose W is a strong (J, G)-swirl, where $W \cap \Omega(J)$ is smooth. Then there is a spanning disc Q for W; that is, Q is a properly embedded disc in $\mathbb{H}^3$, where the Euclidean boundary of Q is W, and Q is precisely invariant under $J = \mathrm{Stab}(W)$.*

Proof. It is clear that there is some disc Q, properly embedded in $\mathbb{H}^3$, whose Euclidean boundary is W.

If $x \in W$ is a point of $\Omega(J)$, and x is not a rimpoint, then $x \in \Omega(G)$, and x lies on no nontrivial translate of W; hence we can find a Euclidean neighborhood N of x in 3-space, so that $Q \cap N$ is disjoint from every $(G-J)$-translate of Q.

If $x \in W$ is an ordinary rimpoint, then W is circular near x, and there are finitely many translates of W meeting W at x. Choose a neighborhood N of x, exactly as in the proof of Lemma I.3. Normalize G so that $x = \infty$. Then N is the outside of the disc of some radius ρ. There are exactly $n+1$ distinct translates of W meeting N, and they appear there as $n+1$ parallel lines. We now require that, in N, Q lies in the hyperbolic plane supported by W. We make this same requirement for every chain rimpoint. Then, since there are only finitely many rimpoints modulo J, we can choose the neighborhood N at each rimpoint so that all the translates of Q intersecting N appear, inside N, as parallel hyperbolic planes which are tangent at x. Having made this requirement, we see that, in $N \cap \mathbb{H}^3$, no translate of Q meets Q.

We next take up the case that x is a preparabolic rimpoint. Then, as above, we can find a neighborhood of x that meets only a finite number of translates of W modulo $\mathrm{Stab}(x)$, which we know is doubly cusped rank 1 parabolic. Again, we can choose N so that, inside N, all translates of W are circular. Normalizing so that $x = \infty$, the translates of W inside N appear as an infinite collection of parallel lines. Inside N, we require Q to be the collection of hyperbolic planes supported by these lines. With this requirement, we see that, in $N \cap \mathbb{H}^3$, no translate of Q meets Q.

As in [M5, p. 145] we define Q near the parabolic fixed points of elements of J by the same vertical extension as above. We observe that, for Q defined thus far, if $g(W) \neq W$, then $g(Q) \cap Q = \varnothing$. Using the fact that J is strong, it is now easy to extend Q to be a J-invariant properly embedded disc, with the following properties: the Euclidean boundary of Q is W; if $g \in G$, with $g(W) \neq W$, then the projection of $g(Q) \cap Q$ is compact in $\mathbb{H}^3/G$. Observe that once we have done this, the construction of the required disc follows from the argument given in [M5, p. 145 ff.].

Proof of (ix). By the above lemma, there is a precisely invariant spanning disc, Q. Since the Euclidean boundary of B_m^3 is S_m, which is precisely invariant under G_m, B_m^3 is precisely invariant under G_m. The translates of Q divide $\mathbb{H}^3$ into regions. By Proposition 0.8, the Euclidean diameter of any sequence of distinct translates of W tends to zero; the same proof yields that the Euclidean diameter of any sequence of distinct translates of Q tends to zero. Hence these translates of B_m^3 cover all of $\mathbb{H}^3$. The result now follows. □

Lemma I.5. *S_m contains infinitely many points.*

Proof. It suffices to show that S_1 contains infinitely many points. Since S_1 is the set of points inside B_2 disjoint from all $(G_1 - J)$-translates of B_1, this is exactly the statement of Proposition 0.4, if J is of the first kind. If J is of the second kind, then there are points of $\Omega(G) - \Theta$ on W; nearby points in B_2 must lie in S_1. □

Proof of (xi). We start with the assumption that G_1 and G_2 are both geometrically finite. It follows from the fact that S_m is precisely invariant under G_m that if P is a doubly cusped rank 1 parabolic subgroup of G_m, where the fixed point of P does not lie on W, or on any G_m-translate of W, then P is doubly cusped. If P is a doubly cusped rank 1 parabolic subgroup of G_m, whose fixed point z lies on W, then either z is also the fixed point of a cyclic stabilizer, in which case $\mathrm{Stab}_G(z)$ has rank 2, or not. If not, then, since z lies on W, and G_1 and G_2 are both geometrically finite, z is doubly cusped in both G_1 and G_2. There are also only finitely many translates of W touching W at z, so there is a farthest one in B_1, call it W_1, and there is a farthest one in B_2, call it W_2 (these correspond to the ends of the chain of parabolic rimpoints). Write $W_m = g_m(W)$; since G_1 and G_2 are both geometrically finite, $g_m^{-1}(z)$ is doubly cusped in G_{3-m}, from which it easily follows that z is doubly cusped.

The only other parabolic subgroups of G are the conjugates of cyclic stabilizers; we saw in Lemma I.3 that these are doubly cusped.

We note that the argument above also shows that if B_1 and B_2 are both strong simple discs, then W is a strong swirl. If say B_1 is not strong, then there is a parabolic fixed point $z \in W$, where z is not doubly cusped in G_1. Since B_1 contains a cusped region for every parabolic element of $J \subset G_1$, there can be no $(G_1 - J)$-translate of W at z. It follows that z is not the fixed point of a cyclic stabilizer; hence $\mathrm{Stab}(z)$ has rank 1 in G. Since z is not doubly cusped in G_1, it is surely not doubly cusped in G. We have shown that W is a strong swirl if and only if B_1 and B_2 are both strong simple discs; this concludes the proof of (iv).

We continue with the proof of (xi). Since G_m is geometrically finite, it is analytically finite; hence S_m, which is precisely invariant under G_m, projects to a surface of finite type, with its boundary arcs on the projection of W. It follows that every point in $\overline{S}_m$ either lies on a translate of W, or is a point of $\Omega(G_m) \cap S_m = \Omega(G) \cap S_m$, or is a limit point of G_m. Every point on W is either a limit point of J, or a rimpoint, or a point of Ω. We have shown that every point in $\overline{S}_m$ is either a point of Ω, or a rank 2 parabolic fixed point, or a doubly cusped rank 1 parabolic fixed point, or a point of approximation.

Let x be a limit point of G, where x is not a parabolic fixed point, and x is not a translate of any point in the closure of either S_1 or S_2; in particular, x does not lie on W or on any translate of W. This means that there is a sequence of translates of W, call it $\{W_j\}$, where $W_1 = W$, so that each W_j weakly separates x from W_{j-1}. We remark further that since $\mathrm{dia}(W_j) \to 0$, and x does not lie on any one of the W_j, we actually have that x lies in the open disc bounded by W_j. Write $W_j = g_j(W)$, and set $h_j = g_j^{-1}$. Note that $h_j(x)$ lies in either $\mathrm{int}(B_1)$ or $\mathrm{int}(B_2)$, and W weakly separates $h_j(x)$ from $h_j(W)$. For each j, find an element $k_j \in J$, with $k_j \circ h_j(x) \in E$, a constrained fundamental set for J.

We first take up the case that $k_j \circ h_j(x)$ is bounded away from W. Then it is surely bounded away from $k_j \circ h_j(W)$. We choose a subsequence, which we call by the same name, so that $k_j \circ h_j(z)$ converges to some point y, uniformly on compact subsets of the complement of some limit point x'. Since $k_j \circ h_j(W)$ is uniformly bounded away from $k_j \circ h_j(x)$, we must have that $x = x'$, and $k_j \circ h_j(z) \to y$ for all $z \neq x$. It follows that x is a point of approximation.

Since J is geometrically infinite, E comes close to W only near parabolic fixed points of elements of J, and near points of $\Omega(J)$. If y is a parabolic fixed point of J, then y is either doubly cusped, in which case, $k_j \circ h_j(x)$ is bounded away from y, or $\mathrm{Stab}(y)$ has rank 2.

The only points of $\Omega(J)$ that are not in $\Omega(G)$ are the preparabolic rimpoints, and these are all doubly cusped rank 1 parabolic fixed points of G. Assume that $k_j \circ h_j(x)$ approaches the parabolic fixed point y where y is either a rimpoint on W, or a parabolic fixed point of J that has rank 2 in G. We can assume that $k_j \circ h_j(x)$ approaches y from inside $E \cap B_1$; the argument in the case that $k_j \circ h_j(x) \in B_2$ is essentially the same. Then $k_j \circ h_j(W)$ lies in B_2. If $k_j \circ h_j(W)$ is bounded away from y, then we can use the argument above to conclude that x is a point of approximation. If $k_j \circ h_j(W)$ also approaches y, then since $k_j \circ h_j(x)$ does not approach tangentially, we can find $p_j \in \mathrm{Stab}(y)$ so that $p_j \circ k_j \circ h_j(x)$ still lies in B_1 and is bounded away from W, while $p_j \circ k_j \circ h_j(W)$ is still in B_2; in fact it is closer to y. We have shown in this case as well that x is a point of approximation. This concludes the proof that G is geometrically finite.

For the converse, assume that G is geometrically finite, but G_1 is not. Let x be a parabolic fixed point of G_1. There is nothing to prove if $\mathrm{Stab}_G(x)$ has rank 2 in G_1. If $\mathrm{Stab}_G(x)$ has rank 1, then it is doubly cusped in G, hence doubly cusped in G_1. The only other possibility is that $\mathrm{Stab}_G(x)$ has rank 2, and $\mathrm{Stab}_{G_1}(x)$ has rank 1. This can occur only if x lies on some translate of W; we can assume that x lies on W itself. We showed in Lemma I.3 that if x is a parabolic rimpoint, then $\mathrm{Stab}_G(x)$ has rank 1. Hence x is a parabolic fixed point both of an element of J and of a cyclic stabilizer. In particular, there is a $g_1 \in G_1 - J$ so that $g_1(B_1)$ touches W at x. We now have two disjoint precisely invariant topological discs at x. Since (B_1, Θ) is a (J, G_1)-simple disc, x is doubly cusped in G_1.

If x is a limit point of G_1, then x lies in the closure of S_1, which is precisely invariant under G_1. If x is a point of approximation for G, then there is a sequence $\{g_i\}$ of distinct elements of G, so that for all $z \neq x$, $g_i(z)$ and $g_i(x)$ are bounded away from each other. By Lemma I.5, there are infinitely many points $z \in S_2$ different from x. Note that $g_i(x)$ and $g_i(z)$ are both separated from W by the same set of translates of W. If the number of such translates goes to infinity with i (i.e., if the length, $|g_i| \to \infty$ as $i \to \infty$), then their spherical diameter tends to zero, so the spherical distance between $g_i(x)$ and $g_i(z)$ tends to zero. We have shown that the length $|g_i|$ is bounded. The same argument shows that for i sufficiently large, the set of translates of W lying between W and $g_i(x)$ is independent of i. It follows that for i sufficiently large, $g_i(x)$ and $g_i(z)$ lie in some fixed $g(S_1)$. Then there is a sequence $\{h_i\}$ of elements of $G_1 - J$ so that $g_i = g \circ h_i$. Since the spherical distance between $g_i(x)$ and $g_i(z)$ is bounded away from zero, so is the spherical distance between $h_i(x)$ and $h_i(z)$. Since this is true for every $z \in S_1$ other than x, x is a point of approximation for G_1.

Proof of (xii). Assume first that G_1 and G_2 are analytically finite. By conclusion (viii), we can construct Ω/G by deleting the projection of $\mathrm{int}(B_1)$ from $\Omega(G_1)/G_1$, and deleting the projection of $\mathrm{int}(B_2)$ from $\Omega(G_2)/G_2$, and then joining together the remaining possibly disconnected surfaces, call them X_1 and X_2, along $((W \cap \Omega(G)) - \Theta)/J \cup \Theta'/G$, where Θ' is the set of ordinary rimpoints. If X_1 and X_2 are both empty, then G is of the first kind, hence analytically finite. If they are not both empty, then, since J is geometrically finite, and there are only finitely many rimpoints on W modulo J, $(W \cap \Omega(G))/J$ consists of a finite number of arcs. Since G_m is analytically finite, the boundary of X_m consists of these finitely many arcs, together with the finitely many parabolic punctures on X_m; also, there are only finitely many special points on X_m.

Gluing X_1 to X_2, we obtain a new possibly disconnected surface $X = \Omega/G$, whose only boundary points are the finitely many parabolic punctures of X_1, the finitely many parabolic punctures of X_2, and the finitely many parabolic punctures coming from the preparabolic rimpoints on W. The only special points on X are the finitely many special points of X_1, the finitely many special points of X_2, and perhaps the projections of the ordinary rimpoints, of which again there are only finitely many. We have shown that G is analytically finite.

For the converse, assume that G is analytically finite, and construct X_1 and X_2 as above. Then $\Omega(G_m)/G_m$ is X_m to which we adjoin $\mathrm{int}(B_m)/J$. Since J is geometrically finite, X_m has finitely many boundary arcs, and $\mathrm{int}(B_m)/J$ has finitely generated fundamental group (as an orbifold). Since G is analytically finite, X_m is either empty, or has finitely generated fundamental group (as an orbifold). Joining these two orbifolds along their common boundaries leaves us with an orbifold with finitely generated fundamental group, where every boundary component is a parabolic puncture; from which is follows that either G_m is elementary, or $\Omega(G_m)/G_m$ has finite hyperbolic area.

Proof of (xiii). Since G is analytically finite, we can find a possibly disconnected compact surface $\widehat{S}$, where $S = \Omega/G$ is conformally embedded in $\widehat{S}$, and $\widehat{S} - S$ consists of a finite number of points. Except for a factor of -2π, the area of G is equal to the Euler characteristic of $\widehat{S}$, where the special points are given special weight; that is, we count a special point of order ν as having weight $1/\nu$, as opposed to an ordinary point, which has unit weight; we also regard parabolic punctures (i.e., the points of $\widehat{S} - S$) as being special points of order ∞, where $1/\infty = 0$.

We find a cell decomposition of $\Omega(G_1)/G_1$, where the special points and projections of the rimpoints are all 0-cells, and $W \cap \Omega(G_1)$ projects onto a set of 1-cells. We likewise find a cell decomposition of $\Omega(G_2)/G_2$, where the special points are all 0-cells, and where $W \cap \Omega(G_2) = W \cap \Omega(G_1)$ has the same invariant cell decomposition.

Note that $B_m \cap \Omega(J) = (B_m \cap \Omega) \cup \Theta$. Set $\Omega_- = \Omega - \Theta$. Then, by using precise invariance, we can write

$$\begin{aligned} \mathrm{area}(G_m) = {} & \mathrm{area}(S_m/G_m) + \mathrm{area}(\mathrm{int}(B_m)/J) \\ & + \mathrm{area}(W \cap \Omega_-/J) + \mathrm{area}(\Theta/G_m). \end{aligned}$$

We also have

$$\begin{aligned}\operatorname{area}(J) = {} & \operatorname{area}(\operatorname{int}(B_1)/J) + \operatorname{area}(\operatorname{int}(B_2)/J) \\ & + \operatorname{area}(W \cap \Omega_-/J) + \operatorname{area}(\Theta/J),\end{aligned}$$

and

$$\begin{aligned}\operatorname{area}(G) = {} & \operatorname{area}(S_1/G_1) + \operatorname{area}(S_2/G_2) \\ & + \operatorname{area}(W \cap \Omega_-/J) + \operatorname{area}(\Theta/G).\end{aligned}$$

The above computation ignores the chains and cycles of parabolic fixed points of J, for these points all have weight zero in all the above computations.

Combining the above, we see that in order to prove our assertion; i.e., that

$$\operatorname{area}(G_1) + \operatorname{area}(G_2) = \operatorname{area}(G) + \operatorname{area}(J),$$

it suffices to show that

$$(*) \qquad \operatorname{area}(\Theta/G_1) + \operatorname{area}(\Theta/G_2) = \operatorname{area}(\Theta/G) + \operatorname{area}(\Theta/J).$$

We prove $(*)$ for each chain and cycle separately.

We need a slight change of notation for the computation of the contributions of the rimpoints to the different areas. We write $\{x_1, \dots, x_n\}$ as a chain or cycle of rimpoints, where the points $x_1, \dots, x_n$ are all distinct.

If $\{x_1, \dots, x_n\}$ is a chain of rimpoints, then these points project to one 0-cell on Θ/G, and to n distinct 0-cells on Θ/J. Assuming that none of the x_j are elliptic fixed points, they come in pairs; we can assume without loss of generality that x_1 and x_2 are paired in G_1, x_2 and x_3 are paired in G_2, etc. Hence, if n is even, these points project to $n/2$ distinct 0-cells in Θ/G_1, and $(n/2)+1$ distinct 0-cells in Θ/G_2; if n is odd, there are $(n+1)/2$ 0-cells in both Θ/G_1 and Θ/G_2. We have shown that this chain of rimpoints satisfies $(*)$.

Continuing with the case that we have a chain of rimpoints, now assume that one of the points is an elliptic fixed point; since W is precisely embedded, the stabilizer of this point necessarily has order 2. Note that a chain contains at most one elliptic fixed point. Then the contribution to $\operatorname{area}(\Theta/G)$ is $1/2$, and the contribution to $\operatorname{area}(\Theta/J)$ is n. We compute the contributions to $\operatorname{area}(\Theta/G_1)+$ $\operatorname{area}(\Theta/G_2)$ as follows. We can assume that x_1 is the elliptic fixed point, and that the elliptic element lies in $G_1 - J$. If n is even, then the contribution to $\operatorname{area}(\Theta/G_1)$ is $(n+1)/2$, and the contribution to $\operatorname{area}(\Theta/G_2)$ is $n/2$; if n is odd, then the contributions to $\operatorname{area}(\Theta/G_1)$ and $\operatorname{area}(\Theta/G_2)$ are $n/2$ and $(n+1)/2$, respectively. In either case, we have shown that $(*)$ holds.

We next take up the case of a cycle $\{x_1, \dots, x_n\}$ of rimpoints, none of which are elliptic fixed points. Then the contributions to $\operatorname{area}(\Theta/G)$ and $\operatorname{area}(\Theta/J)$ are 0 and n, respectively. One easily sees that n is necessarily even, and that the cycle contributes $n/2$ to the areas of both Θ/G_1 and Θ/G_2.

Finally, we take up the case of a cycle, where there is an elliptic fixed point in the cycle. We can assume that x_1 is a fixed point of some elliptic $g_1 \in G_1 - J$. Since x_1 is a doubly cusped rank one parabolic fixed point in G, the order of g_1 is necessarily 2. Hence g_1 is the unique element of $G_1 - J$ with $g_1(x_1) \in W$.

Since x_1 lies in a cycle of rimpoints, there is a $g_2 \in G_2 - J$ with $x_2 = g_2(x_1) \in W$. If $x_2 = x_1$, then, as above, g_2 has order 2, and the cycle is complete. In this case, it is easy to verify $(*)$.

If $x_2 \neq x_1$, then we continue; note that, in this case, g_2 is the unique element of $G_2 - J$ with $g_2(x_1) \in W$. There is a $g_3 \in G_1 - J$ with $x_3 = g_3(x_2) \in W$; etc. Since we have a cycle of rimpoints, there is some first $x_m = x_1$. Then g_{m-1} cannot be equal to g_1, so we must have $g_{m-1} = g_2^{-1}$. Continuing backwards, we see that the cycle can be regarded as a chain, with x_1 at one end, and some x_n at the other, where x_n is also a fixed point of an elliptic element of either $G_1 - J$ or $G_2 - J$. In either case, we see that, as above, the contributions of this cycle to $\text{area}(\Theta/G)$ and $\text{area}(\Theta/J)$ are 0 and n, respectively. Since the elliptic elements with fixed points at x_1 and x_n both have order 2, one easily sees that the cycle contributes $n/2$ to the areas of both Θ/G_1 and Θ/G_2. □

II. The second combination theorem

II.1. The general setup for the second combination theorem is as follows. We are given a Kleinian group G_0 with two distinguished subgroups, J_1 and J_2; we are given two closed topological discs, B_1 and B_2, with boundary curves W_1 and W_2, respectively; and we are given a set of rimpoints Θ_m on W_m so that (B_m, Θ_m) is a (J_m, G_0)-simple disc. We further assume that for every $g \in G_0$, every point of $g(B_1) \cap B_2$ is either both a limit point of J_2 and a g-image of a limit point of J_1, or a rimpoint of B_2 and a g-image of a rimpoint of B_1; in particular, the $(G_0 - J_1)$-translates of $\text{int}(B_1)$ and the $(G_0 - J_2)$-translates of $\text{int}(B_2)$ are all disjoint (i.e., $(\text{int}(B_1), \text{int}(B_2))$ is precisely invariant under (J_1, J_2) in G_0). We also assume that we are given a transformation f, where f maps the exterior of B_1 onto the interior of B_2, $f(\Theta_1) = \Theta_2$, and f conjugates J_1 onto J_2. Given these conditions, we say that the pair, (B_1, B_2), is *jointly f-simple.*

II.2. Let $z \in \Theta_1$. We say that z is *occupied* if either there is a $g \in G_0 - J_1$ so that $z \in g(B_1)$, or there is a $g \in G_0$, not necessarily nontrivial, so that z also lies in $g(B_2)$. Similarly, we say that $z \in \Theta_2$ is *occupied* if either there is a $g \in G_0 - J_2$ so that $z \in g(B_2)$, or there is a $g \in G_0$ so that $z \in g(B_1)$. If both $z \in \Theta_1$ and $f(z) \in \Theta_2$ are occupied, then we say that z and $f(z)$ are *double* rimpoints. A rimpoint that is not a double rimpoint is a *single* rimpoint.

Choose a fundamental set E_1 for the action of J_1 on W_1, and set $E_2 = f(E_1)$; since f conjugates J_1 onto J_2, E_2 is a fundamental set for the action of J_2 on W_2. As in §I, the rimpoints in E_1 and E_2 fall into chains and cycles as follows. Start with a double rimpoint, $z_1 \in E_1$; then there is a $g_1 \in G_0 - J_1$, so that z_1 also lies in $g_1^{-1}(E_1 \cup E_2)$; set $z_2 = g_1(z_1)$. If $z_2 \in E_1$, and $z_3 = f(z_2)$ is not a double rimpoint, then we have reached the end of the chain; we have likewise reached the end of the chain if $z_2 \in E_2$ and $z_3 = f^{-1}(z_2)$ is not a double rimpoint. If z_3 is a double rimpoint, then there is a $g_2 \in G_0$ so that z_3 also lies in $g_2^{-1}(E_1 \cup E_2)$. Set $z_4 = g_2(z_3)$, and continue. Since there are only finitely many rimpoints in $E_1 \cup E_2$, this process either ends after finitely many steps, in which case these points lie in a *chain* of rimpoints, or it is periodic, in which case they lie in a *cycle* of rimpoints.

The rimpoints lying in chains are called *ordinary* rimpoints; those lying in cycles are called *preparabolic* rimpoints.

Each chain of rimpoints has single rimpoints at its ends; all others are double rimpoints. We also have a *cyclic stabilizer* for each preparabolic rimpoint z; this is the first element of the form: $g = f^{\varepsilon_n} \circ g_n \circ \cdots \circ f^{\varepsilon_1} \circ g_1$, as above, where each ε_i is either 0 or ± 1, and $g(z) = z$. As in §I, we permit fixed points of elliptic elements of G_0 to be rimpoints. If there are such points in a cycle, then the cyclic stabilizer is not unique.

If $z \in W_1$ is a rimpoint, then so is $f(z) \in W_2$; it follows that every chain of rimpoints has at least two elements. However, a cycle of rimpoints might only have one element; in particular, f itself might be a cyclic stabilizer.

II.3. We can also have cyclic stabilizers for parabolic fixed points of J_1 and/or J_2; that is, a parabolic fixed point x of B_1 is a *parabolic rimpoint* if there is an element $g \in G_0 - J_1$ mapping x onto a parabolic fixed point on either W_1 or W_2. Since f maps parabolic fixed points of J_1 onto parabolic fixed points of J_2, these also fall into chains and cycles; for each such point x in a cycle of parabolic rimpoints, there are infinitely many *cyclic stabilizers* with fixed point x.

II.4. As in the first combination theorem, we have the additional assumption that every cyclic stabilizer with fixed point in Θ_1 or Θ_2 is parabolic.

We remark here, as in §I, that the cyclic stabilizers with fixed points at parabolic fixed points of either J_1 or J_2 are automatically parabolic.

II.5. Let D_0 be a fundamental set for G_0 satisfying the following conditions. For $m = 1, 2$, D_0 is maximal with respect to B_m; that is, $D_0 \cap B_m$ is a fundamental set for the action of J_m on B_m. We also require that $f(D_0 \cap W_1) = D_0 \cap W_2$. We call D_0 satisfying these conditions a *coordinated fundamental set* for G_0.

Let D' be the intersection of D_0 with the complement of $\operatorname{int}(B_1) \cup B_2$. We need to delete certain of the rimpoints from D'.

Once we have chosen D_0, the chains and cycles of rimpoints are well defined.

If $x_1, \dots, x_n$ is a cycle of double rimpoints, then each x_m is a parabolic fixed point in G; hence x_m is not in Ω. We delete all preparabolic rimpoints from D'.

We similarly delete all ordinary rimpoints from D', if they are part of a chain of rimpoints where one of the points of the chain is an elliptic fixed point of G_0.

If $x_1, \dots, x_n$ is a chain of rimpoints, where no x_m is an elliptic fixed point, then they are all equivalent; hence we need only one of them in D. We choose one of the single rimpoints lying in D', and delete all the others.

We define the *adjusted set* D to be D' with the above rimpoints deleted.

II.6. The major conclusions of the second combination theorem are given below. The conclusions are numbered so as to agree with the numbering in [M5], although some of the formulations have been modified. Also, conclusions (xii) and (xiii) are new.

Statement of the second combination theorem

Theorem II (the second combination theorem). *Let J_1 and J_2 be geometrically finite subgroups of the Kleinian group, G_0. Assume the following.*

(A) *For $m = 1, 2$, there is a J_m-invariant closed topological disc B_m, with boundary loop W_m; there is a set of rimpoints Θ_m given on W_m; and there is a Möbius transformation, f, mapping the exterior of B_1 onto the interior of B_2, so that (B_1, B_2) is jointly f-simple (i.e., (B_m, Θ_m) is a (J_m, G_0)-simple disc;*

if there is an $x \in B_1 \cap g(B_2)$, *for some* $g \in G_0$, *then either* $x \in \Lambda(J_1) \cap g(\Lambda(J_2))$, *or* $x \in \mathbf{\Theta}_1 \cap g(\mathbf{\Theta}_2)$; f *conjugates* J_1 *onto* J_2; *and* $f(\mathbf{\Theta}_1) = \mathbf{\Theta}_2)$.

(B) *Every cyclic stabilizer is parabolic.*

(C) $A = \widehat{\mathbb{C}} - (B_1 \cup B_2) \neq \varnothing$.

Let D_0 *be a coordinated fundamental set for* G_0; *let* D *be the corresponding adjusted set; let* A_0 *be the complement of the union of the* G_0*-translates of* $(B_1 \cup B_2)$; *and let* $G = \langle G_0, f \rangle$. *Then*

(i) $G = G_0 *_f$ (*i.e.,* G *is the HNN-extension of* G_0 *by the element* f *conjugating the subgroup* J_1 *onto the subgroup* J_2).

(ii) G *is discrete.*

(iii) *Every element of* G *that is not a conjugate of an element of* G_0, *and is not a conjugate of a cyclic stabilizer, is loxodromic.*

(iv) W_1 *is precisely embedded, and* $(W_1, \mathbf{\Theta}_1)$ *is a* (J_1, G)*-swirl; it is strong if and only if* B_1 *and* B_2 *are both strong simple discs.*

(viii) D *is a fundamental set for* G.

(ix) A_0 *is precisely invariant under* G_0. *Let* $A_0^* = \overline{A}_0 \cap \Omega(G_0)$; *then* $\Omega/G = A_0^*/G_0$, *where the two possibly disconnected and possibly empty boundaries,* $(W_1 \cap \Omega(G))/J_1$ *and* $(W_2 \cap \Omega(G))/J_2$ *are identified by* f.

(x) G *is geometrically finite if and only if* G_0 *is geometrically finite.*

(xi) *Assume that* G_0 *is geometrically finite, and that* $W_1 \cap \Omega(J_1)$ *is smooth. Then there is a spanning disc* Q_m *for* W_m, *where* (Q_1, Q_2) *is precisely invariant under* (J_1, J_2), *and* $f(Q_1) = Q_2$. *Further,* $\mathbb{H}^3/G$ *can be described as follows. Let* A_0^3 *be the region in* $\mathbb{H}^3$, *bounded by the translates of* $Q_1 \cup Q_2$, *whose Euclidean boundary is* A_0. *Then* $\mathbb{H}^3/G$ *is* A_0^3/G_0, *where the two boundaries,* Q_1/J_1 *and* Q_2/J_2, *are identified by* f.

(xii) G *is analytically finite if and only if* G_0 *is analytically finite.*

(xiii) *If* G *is analytically finite, then*

$$\operatorname{area}(G) = \operatorname{area}(G_0) - \operatorname{area}(J_1) = \operatorname{area}(G_0) - \operatorname{area}(J_2).$$

Proof of the second combination theorem

The proof closely follows that of §I; the main differences lie in the combinatorial group theory.

Proof of (i). By hypothesis (A), ($\operatorname{int}(B_1)$, $\operatorname{int}(B_2)$) is precisely invariant under (J_1, J_2) in G_0; also $f(A \cup \operatorname{int}(B_2)) \subset \operatorname{int}(B_2)$, and $f^{-1}(A \cup \operatorname{int}(B_1) \subset \operatorname{int}(B_1)$. By hypothesis (C), $A \neq \varnothing$. Hence $(A, \operatorname{int}(B_1), \operatorname{int}(B_2))$ is an interactive triple.

If J_1 is of the second kind, then A_0 is obviously infinite. If J_1 is of the first kind, then it follows from Proposition 0.6 that this triple is proper; it then follows from [M5, p. 160] that G is the HNN-extension of G_0.

It shown in [M5, p. 160] that A_0 is precisely invariant under G_0 in G.

Proof of (ii). Since A_0 is precisely invariant under G_0, every translate of W_1 lying in A is weakly separated from W_1 by a G_0-translate of either W_1 or W_2. Similarly, every translate of W_1 lying between W_1 and $f^{-1}(W_1)$ is weakly separated from W_1 by the f^{-1} image of a G_0-translate of either W_1 or W_2. It follows that if we had a sequence of distinct elements $\{g_n\}$ of G, with $g_n \to 1$, then, since $g_n(W_1) \to W_1$, we must have a sequence h_n of elements of either G_0 or $f^{-1}G_0 f$, with $h_n(W_1) \to W_1$. Since G_0 is discrete, this cannot happen.

Before going on to the proof of conclusion (iii), we recall the following; see [M5, p. 159].

Lemma II.1. *Let* $(A,\ \mathrm{int}(B_1),\ \mathrm{int}(B_2))$ *be a proper interactive triple, and let*

$$g = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1} \circ g_1$$

be a normal form (that is, $g_m \in G_0$ *, and only* g_1 *can be the identity;* α_m *is an integer, and only* α_n *can be zero; if* $\alpha_m > 0$ *and* $g_{m+1} \in J_2$ *, then* $\alpha_{m+1} > 0$ *; if* $\alpha_m < 0$ *and* $g_{m+1} \in J_1$ *, then* $\alpha_{m+1} < 0$ *). Then the following conclusions hold.*
If $\alpha_n > 0$ *and* $\alpha_1 > 0$*, then* $g(A \cup \mathrm{int}(B_2)) \subset \mathrm{int}(B_2)$ *;*
if $\alpha_n > 0$ *and* $\alpha_1 < 0$*, then* $g(A \cup \mathrm{int}(B_1)) \subset \mathrm{int}(B_2)$ *;*
if $\alpha_n < 0$ *and* $\alpha_1 > 0$*, then* $g(A \cup \mathrm{int}(B_2)) \subset \mathrm{int}(B_1)$ *;*
if $\alpha_n < 0$ *and* $\alpha_1 < 0$*, then* $g(A \cup \mathrm{int}(B_1)) \subset \mathrm{int}(B_1)$ *.*

Before going on with our proof, we make the following observation. Every element of $G - G_0$ can be written either in one of the four normal forms listed above, or in the form $g_1 \circ g_2$, where g_1 is a nontrivial element of G_0 , and g_2 is one of the above normal forms.

Lemma II.2. A_0 *contains infinitely many points.*

Proof. If J_1 is of the first kind, then so is J_2 ; in this case the result is immediate from Lemma 0.6.

If J_1 is of the second kind, then, since $A \neq \varnothing$, $W_1 \neq W_2$. Since the limit points of J_m , and the points of Θ_m , are nowhere dense on W_m , there must be a point $x \in W_1$, where $x \notin W_2$, $x \notin \Theta_1$, and $x \in \Omega(G_0)$. Such a point lies on no translate of W_2 , and lies on no translate of W_1 other than W_1 itself. Also, since $x \in \Omega(G_0)$, there is a neighborhood N of x that meets no G_0-translate of either W_1 or W_2 other than W_1 itself. Since B_1 and B_2 are simple discs, N meets no translate of either B_1 or B_2 other than B_1 itself. Hence there are infinitely many distinct points of A_0 in N . □

Proof of (iii). Let g an element of G , where g is not conjugate to any element of G_0 ; write g in normal form as above, and assume that this normal form has minimal length among all its conjugates; in particular, $g_1 \neq 1$, and $\alpha_n \neq 0$. We assume that $\alpha_n > 0$; the case that $\alpha_n < 0$ is treated analogously. There are now two cases to consider according as $\alpha_1 > 0$ or $\alpha_1 < 0$.

If $\alpha_1 > 0$, then, by Lemma II.1, $g(\mathrm{int}(B_2)) \subset \mathrm{int}(B_2)$; hence g has a fixed point in $g(B_2)$. Note that since $A_0 \neq \varnothing$, this inclusion is proper; in particular, g has infinite order.

If g is parabolic, then the fixed point of g lies on both W_2 and $g(W_2)$. Of necessity, the fixed point of g also lies on every translate of either W_1 or W_2 lying between these two. It is now easy to see that g is a conjugate of a power of a cyclic stabilizer.

If $\alpha_1 < 0$, then if g_1 were in J_2 , we could reduce the length $(= \sum |\alpha_m|)$ by using the relation, $f^{-1} J_2 = J_1 f^{-1}$, and conjugating by f ; hence $g_1 \notin J_2$. Then $g_1(B_2) \subset A$; it then follows from Lemma II.1 that $g(\mathrm{int}(B_2)) \subset \mathrm{int}(B_2)$. This inclusion is also proper. Hence, as above, g can be parabolic only if it is a conjugate of a power of a cyclic stabilizer.

Proof of (iv). We already know that W_1 and W_2 are precisely embedded in G_0 . Every element of $G - G_0$ can be written in the form $g = g_1 \circ g_2$, where

$g_1 \in G_0$, and g_2 is one of the normal forms listed in Lemma II.2. If g_1 is trivial, then it follows from Lemma II.1 that $g(W_1) \in (B_1 \cup B_2)$. If $g_1 \neq 1$, then $g(W_1)$ is contained in some G_0-translate of $(B_1 \cup B_2)$. In any case, $g(W_1)$ does not cross W_1; hence W_1 is precisely embedded.

We have also shown that every translate of W_1 either lies inside B_1, or inside B_2, or inside a G_0-translate of $B_1 \cup B_2$. Looking on the side of W_1 towards A, we see that a G-translate of W_1 can touch W_1 only at a point where a G_0-translate of either W_1 or W_2 touches it (the G_0-translate of W_2 might be the identity). The exact same argument shows that a G-translate of W_2, lying on the same side of W_2 as A, can touch W_2 only at a point where there is a G_0-translate of either W_1 or W_2 touching it. Observe that f^{-1} maps W_2 to W_1, and maps the side of W_2 facing A onto the side of W_1 facing away from A.

Now suppose there is a $g \in G$ and there is a point $x \in W_1 \cap g(W_1)$. Then there is a $g_0 \in G_0$ with either $x \in W_1 \cap g_0(W_1)$, or $x \in W_1 \cap g_0(W_2)$, or $f(x) \in W_2 \cap g_0(W_1)$, or $f(x) \in W_2 \cap g_0(W_2)$. In any case, such a point is either both a rimpoint on W_1, and the G_0-image of a rimpoint on either W_1 or W_2, or both a point of $\Lambda(J_1)$ and the image of a point of either $\Lambda(J_1)$ or $\Lambda(J_2)$.

We already know that $J_1 = \operatorname{Stab}(W_1)$ is geometrically finite, and that W_1 is circular near every point of Θ_1. It remains to show that the points of $\Omega(J) \cap W_1$ that are not rimpoints are in $\Omega(G)$; that the ordinary rimpoints of Θ_1 are also points of $\Omega(G)$; and that the preparabolic rimpoints are doubly cusped parabolic fixed points of elements of G.

If $x \in (W_1 \cap \Omega(J))$ is not a rimpoint, then, since B_1 is a (J_1, G_0)-simple disc, $x \in \Omega(G_0)$, and there is a neighborhood N_1 of x whose intersection with A meets no G_0-translate of W_1 or W_2. Similarly, $f(x) \in W_2$ is a point of $\Omega(G_0)$, and it has a neighborhood N_2 whose intersection with A meets no G_0-translate of either W_1 or W_2. Then $N = N_1 \cap f^{-1}(N_2)$ is a neighborhood of x which meets no G-translate of W_1 other than W_1 itself. Hence $x \in \Omega(G)$.

If x is an ordinary rimpoint on W_1, then x is one point in a chain of rimpoints, $x_1, \dots, x_n$, where x_1 and x_n are single rimpoints; the others are double. For each $i = 2, \dots, n$, there is a transformation $g_i \in G$, with $g_i(x_i) = x_{i-1}$; each g_i is either an element of G_0, or the transformation f, or the transformation f^{-1}. We set $g_1 = 1$. Then $h_i = g_1 \circ \cdots \circ g_i$ maps x_i to x_1.

We assume, for the sake of argument, that $x_i \in W_1$. Since x_1 is a single rimpoint, either it has a neighborhood N_1 whose intersection with A meets no G_0-translate of either W_1 or W_2, or $f(x_1)$ has a neighborhood N_1' whose intersection with A meets no G_0-translate of either W_1 or W_2. In the latter case, set $N_1 = f^{-1}(N_1')$. In any case, N_1 meets no G-translate of either W_1 or W_2 on one of the two sides of W_1.

Each of $x_2, \dots, x_{n-1}$ is a double rimpoint. Assume for the sake of argument that x_i lies on W_1. Then there is exactly one G_0-translate of either W_1 or W_2 lying inside A and touching W_1 at x_i; call it $W_i^!$. Since $x_i \in \Omega(G_0)$, it has a neighborhood N_i so that, in the region between W_1 and $W_i^!$, N_i meets no G_0-translate of either W_1 or W_2; we can choose N_i so that, inside N_i, these two translates of W_1 and/or W_2 appear as circular arcs. It follows that, in that same region, N_i meets no other G-translate of either W_1 or W_2.

We do not know whether x_n lies on W_1 or W_2; however, exactly as in the case of x_1, x_n has a neighborhood N_n which meets no translate of either W_1 or W_2 on one side of the curve on which it lies. One easily sees that $N = \bigcap h_n(N_n)$, is a neighborhood of x_1 which meets the n G-translates of W_1 defined by the chain, and no others. It follows that $x_1 \in \Omega(G)$; hence, since the x_i are all G-equivalent, $x_i \in \Omega(G)$.

Of course if x is a preparabolic rimpoint, then $x \notin \Omega(G)$, and there is a parabolic element of G with its fixed point at x.

Now suppose $x = x_1$ is a preparabolic rimpoint. Exactly as above, we can assume that $x_1 \in W_1$, and we can find $x_2, \dots, x_n$, where these are now all double rimpoints, and we can find g_i, as above, with $g_i(x_i) = x_{i-1}$, $i = 2, \dots, n$; we set $g_1 = 1$, and we set $h_i = g_1 \circ \cdots \circ g_i$. The point x_n lies on either W_1 or W_2, call it W'.

We choose the neighborhoods N_i exactly as above, except that, since x_1 and x_n are double rimpoints, N_1 and N_n meet exactly two G_0-translates of W_1 and/or W_2. We again set $N = \bigcap h_i(N_i)$. We observe that the cyclic stabilizer at x_1 maps W_1 onto $h_n(W')$, and that, inside N, these two translates of W_1 appear as circular arcs, and, aside from the $n-1$ translates going through x_1, there are no G-translates of W_1 inside N. It now easily follows that x_1 is doubly cusped. This completes the proof that W_1 is a (J_1, G)-swirl.

We delay the proof that (W_1, Θ_1) is strong if and only if (B_1, Θ_1) and (B_2, Θ_2) are both strong until after the proof of (viii).

Proof of (viii). Since A_0 is precisely invariant under G_0, $D \cap A_0$ is precisely invariant under the identity. We saw above that a translate of W_1 intersects W_1 only in limit points of J_1 and rimpoints. We have chosen the intersection of D with $W_1 \cup W_2$ so as to account for the identifications of the rimpoints. Hence D is precisely invariant under the identity.

It is immediate that if z is a point of D in the interior of A_0, then $z \in \Omega$. Since any sequence of G_0-translates of either W_1 or W_2 has spherical diameter tending to zero, every point of ∂A_0 that is not on any translate of either W_1 or W_2 is a limit point of G_0; hence such a point cannot be in D. The only possibility left is that z in D also lies on W_1. We saw above that if z is either not a rimpoint, or a chain rimpoint, then it lies in Ω. We have shown that $D \subset \Omega$.

Since W_1 is a swirl, we can apply Proposition 0.8 to conclude that every point of $\widehat{\mathbb{C}}$ either lies in a translate of $\overline{A}_0$, or is a limit point of G. It follows that every point of Ω is equivalent to some point of D. This concludes the proof that D is a fundamental set for G.

We now conclude the proof of (iv). We still need to show that W_1 is strong if and only if B_1 and B_2 are both strong. If B_1 and B_2 are both strong, then every parabolic fixed point of either J_1 or J_2 is doubly cusped in G_0. It easily follows that every point in a chain of such parabolic fixed points is doubly cusped in G, for there are only finitely many G-translates of W_1 at such a point, and each of the extreme translates has a cusped region that intersects no translate of W_1. Hence the chains of parabolic fixed points of J_1 and J_2 are doubly cusped. Of course, the cycles of parabolic fixed points on W_1 all have rank 2 stabilizers.

For the converse, we assume that W_1 is a strong (J_1, G)-swirl. We need only consider a parabolic fixed point, $x \in W_1$, where $\mathrm{Stab}_G(x)$ has rank 2, and $\mathrm{Stab}_{G_0}(x)$ has rank 1. We choose an element $p \in \mathrm{Stab}_G(x)$, so that p, together with $\mathrm{Stab}_{G_0}(x)$ generates $\mathrm{Stab}_G(x)$. As above, there are finitely many translates of W_1 between W_1 and $p(W_1)$; write these as $g_1(W_1), \dots, g_n(W_1)$. For each $m = 1, \dots, n$, there is a G_0-translate of either W_1 or W_2 touching W_1 at $g_m^{-1}(x)$; there is likewise a G_0-translate of either W_1 or W_2 touching W_2 at $f \circ g_m^{-1}(x)$. Since B_1 and B_2 contain cusped regions at each of these points, each of these points is doubly cusped in G_0.

Proof of (ix). We have already shown that A_0 is precisely invariant under G_0. The other statement now follows almost immediately from conclusion (viii).

Proof of (x). We first assume that G_0 is geometrically finite. If P is a cusped rank 1 parabolic subgroup of G_0, where the fixed point of P does not lie on any G_0-translate of either W_1 or W_2, then, since A_0 is precisely invariant under G_0, P is doubly cusped. Since G_0 is geometrically finite, B_1 and B_2 are both strong simple discs; hence, by conclusion (iv), W_1 is a strong (J_1, G)-rimblock. It follows that every parabolic fixed point on W_1 either has rank 2 or is doubly cusped.

Let x be a limit point of G that is not a parabolic fixed point. Every point of $\Omega(G_0) \cap \overline{A}_0$ is G_0-equivalent to either a point of D or to a point of either W_1 or W_2; we saw above that these points are all either in Ω or parabolic fixed points of G. Hence x does not lie in any translate of $\Omega(G_0) \cap \overline{A}_0$.

If x is a point of approximation for G_0, then of course it is a point of approximation for G. We have shown that every point $\overline{A}_0$ is either a point of Ω, or a doubly cusped parabolic fixed point, or a rank 2 parabolic fixed point, or a point of approximation.

We next assume that x does not lie in any translate of $\overline{A}_0$. Then there is a sequence of translates of W_1, call it $\{V_j\}$, with $V_1 = W_1$, so that each V_j weakly separates x from V_{j-1}. It follows from Proposition 0.8 that the spherical diameter of the V_j tends to zero; hence $V_j \to x$. We also note that x does not lie on any one of the V_j. There are now two possibilities: we either write V_j as $g_j(W_1)$, where $h_j(x) = g_j^{-1}(x)$ lies in $\mathrm{int}(B_1)$, and W_1 weakly separates $h_j(x)$ from $h_j(W_1)$, or we write V_j as $g_j(W_2)$, where $h_j(x) = g_j^{-1}(x)$ lies in $\mathrm{int}(B_2)$, and W_2 weakly separates $h_j(x)$ from $h_j(W_1)$. The two cases are essentially equivalent; we assume without loss of generality that we are in the first case. For each j, find an element $k_j \in J_1$, with $k_j \circ h_j(x) \in E_1$, a constrained fundamental set for J_1. Note that if $k_j \circ h_j(x)$ is bounded away from W_1, then it is surely bounded away from $k_j \circ h_j(W_1)$; i.e., the spherical distance between $k_j \circ h_j(x)$ and $k_j \circ h_j(W_1)$ is bounded from below, from which it follows that x is a point of approximation. We now assume that $k_j \circ h_j(x)$ approaches W_1.

Since J_1 is a geometrically finite quasifuchsian group, we can assume that we have chosen the fundamental domain E_1 so that its Euclidean boundary intersects W_1 only at parabolic fixed points of elements of J_1, and at points of $\Omega(J_1)$. We can also assume that near the parabolic fixed points of J_1, E_1 lies inside a doubly cusped region. We first take up the case that $k_j \circ h_j(x)$ approaches the parabolic fixed point z_0 of J_1. There are now two possibilities: z_0, as a parabolic fixed point of G, either has rank 1 or has rank 2.

We first take up the case that z_0 has rank 1. Since G_0 is geometrically finite, every rank 1 parabolic fixed point of G_0 is doubly cusped; hence B_1 and B_2 are strong simple discs. It follows from conclusion (iv) that W_1 is a strong (J_1, G)-swirl; in particular, every rank 1 parabolic fixed point of J_1, which also has rank 1 in G, is doubly cusped. A doubly cusped region near a parabolic fixed point can contain no limit points, hence there are no limit points of G inside E_1 near any rank 1 parabolic fixed point of J_1. In particular, since x is a limit point of G, $k_j \circ h_j(x)$ cannot approach z_0 from inside E_1.

If z_0 is a rank 2 parabolic fixed point, then normalize G so that $z_0 = \infty$, and so that the parabolic stabilizer of $\mathrm{Stab}_{J_1}(z_0)$ is generated by $g_1(z) = z + 1$. Then there is an additional generator of $\mathrm{Stab}(z_0)$ of the form $g_2(z) = z + \tau$, where $\Im(\tau) > 0$. We can also assume that, for $\Im(z)$ sufficiently large, $\overline{E}_1$ is the strip $\{|\Re(z)| \le 1/2\}$. Since $k_j \circ h_j(x) \to \infty$ inside E_1, we can also assume that $\Im(k_j \circ h_j(x)) \to +\infty$. Then we can find integers $\alpha_j < 0$ so that $g_2^{\alpha_j} \circ k_j \circ h_j(x)$ is bounded, and bounded away from W_1, while $g_2^{\alpha_j} \circ k_j \circ h_j(W_1)$, which has smaller imaginary part that $k_j \circ h_j(W_1)$, is still separated from $g_2^{\alpha_j} \circ k_j \circ h_j(x)$ by W_1. We have shown that x is a point of approximation in this case.

We next take up the case that $k_j \circ h_j(x) \to y_0 \in \Omega(J_1) \cap W_1$. Since each $k_j \circ h_j(x)$ is a limit point of G, these points cannot accumulate at a point of $\Omega(G)$; hence we need only look at points on the boundary of E_1 that are limit points of G.

The only points of $\Omega(J_1)$ that are not in $\Omega(G)$ are the preparabolic rimpoints. If $k_j \circ h_j(x)$ approaches the preparabolic rimpoint y on W_1 from inside E_1, then, exactly as above, we can find $p_j \in \mathrm{Stab}(y)$ so that $p_j \circ k_j \circ h_j(x)$ is bounded away from W_1, while W_1 still weakly separates $p_j \circ k_j \circ h_j(x)$ from $p_j \circ k_j \circ h_j(W_1)$. We have shown in this case as well that x is a point of approximation. This completes that proof that if G_0 is geometrically finite, then G is geometrically finite.

For the converse, assume that G is geometrically finite. Let x be a parabolic fixed point of G_0. There is nothing to prove if $\mathrm{Stab}_{G_0}(x)$ has rank 2; if $\mathrm{Stab}_G(x)$ has rank 1, then x is doubly cusped in G, so it is necessarily doubly cusped in G_0. The only other possibility is that $\mathrm{Stab}_G(x)$ has rank 2, while $\mathrm{Stab}_{G_0}(x)$ has rank 1. This can occur only if x is a translate of the fixed point of a cyclic stabilizer. We can assume that x lies on W_1. Then, as in [M5, p. 167], there is one cusped region for x in B_1. There is also a $g \in G_0$ with $g^{-1}(x)$ lying on either W_1 or W_2; then there is a second cusped region for x in either $g(B_1)$ or $g(B_2)$. We have shown that every parabolic fixed point of G_0 either has rank 2 or is doubly cusped.

Let x be a limit point of G_0 that is not a parabolic fixed point; since A_0 is G_0-invariant, $x \in \overline{A}_0$. Then, since x is a point of approximation, there is a sequence $\{g_j\}$ of distinct elements of G, and there are limit points $x_0 \neq y_0$, so that $g_i(x) \to x_0$, and $g_j(z) \to y_0$, for all $z \neq x$. Note that for $z \in \overline{A}_0$, $g_j(x)$ and $g_j(z)$ are both weakly separated from W_1 by the same set of translates of W_1. If there were infinitely many distinct such translates, then their spherical diameter would tend to zero, so the spherical distance between $g_j(x)$ and $g_j(z)$ would tend to zero. It follows that for j sufficiently large, there is a single element, $g \in G$ so that $g_j(\overline{A}_0) = g(\overline{A}_0)$. Then we can write $g_j = g \circ h_j$, where $h_j \in G_0$. Since the spherical distance between $g_j(x)$ and $g_j(z)$ is bounded from below, so is the spherical distance between $h_j(x)$ and $h_j(z)$. It follows that x is a point of approximation for G_0.

Proof of (xi). By Lemma I.5, there is a spanning disc Q_1 for W_1, where Q_1 is precisely invariant under J_1. It easily follows that A_0^3, the region in $\mathbb{H}^3$ cut out by the translates of Q_1 and having A_0 as its Euclidean boundary, is precisely invariant under G_0. Since $f(A_0^3)$ is the corresponding region on the other side of Q_1, the desired result now follows from Proposition 0.8.

Proof of (xii). This statement follows almost at once from conclusion (ix) together with the facts that J_1 and J_2 are both geometrically finite, and that B_1 and B_2 are both simple discs.

Proof of (xiii). As in the proof of the first combination theorem, we conformally embed $\Omega(G)/G$ in a Riemann surface $\overline{S}$ so that the complement of the image of $\Omega(G)/G$ is a finite number of points. Then, except that some of the vertices have special weights, we can regard $\operatorname{area}(G)$ as being the Euler characteristic of $\overline{S}$ multiplied by -2π.

Let $\Theta = \Theta_1 \cup \Theta_2$, and let $\Omega_- = \Omega - \Theta$.

We find a cell decomposition of $\overline{S}$, where the special points, points of $\overline{S}-S$, and points in the projection of Θ, are all 0-cells; we also require that $W_m \cap \Omega(G_0)$ projects onto some 1-cells, and that the projection of f maps the 1-cells of the projection of W_1 onto the 1-cells in the projection of W_2.

Then we can write

$$\begin{aligned}\operatorname{area}(G_0) = {} & \operatorname{area}(A_0/G_0) + \operatorname{area}(\operatorname{int}(B_1)/J_1) + \operatorname{area}(\operatorname{int}(B_2)/J_2) \\ & + \operatorname{area}((W_1 \cap \Omega_-)/J_1) + \operatorname{area}((W_2 \cap \Omega_-)/J_2) \\ & + \operatorname{area}(((W_1 \cup W_2) \cap \Theta)/G_0),\end{aligned}$$

and

$$\begin{aligned}\operatorname{area}(J_1) = {} & \operatorname{area}(\operatorname{int}(B_1)/J_1) + \operatorname{area}(\operatorname{int}(B_2)/J_2) \\ & + \operatorname{area}((W_1 \cap \Omega_-)/J_1) + \operatorname{area}((W_1 \cap \Theta)/J_1).\end{aligned}$$

We can also write

$$\operatorname{area}(G) = \operatorname{area}(A_0/G_0) + \operatorname{area}((W_1 \cap \Omega_-)/J_1) + \operatorname{area}((W_1 \cap \Theta)/G).$$

We see from above that, in order to prove our assertion; i.e., that

$$\operatorname{area}(G) = \operatorname{area}(G_0) - \operatorname{area}(J_1),$$

it suffices to show that

$$(*) \qquad \operatorname{area}(((W_1 \cup W_2) \cap \Theta)/G_0) = \operatorname{area}((W_1 \cap \Theta)/G) + \operatorname{area}((W_1 \cap \Theta)/J_1).$$

Let $x_1, \dots, x_n$ be a chain of ordinary rimpoints in $\Omega(G)$. Since this is a chain, and not a cycle, at most one of these points is an elliptic fixed point in G_0; since W_1 is precisely embedded, the order of the elliptic fixed point is at most two. There are exactly n G-translates of W_1 passing through x_1, and there is a neighborhood N of x_1 that intersects no other translate of W_1. Hence, near this point, the projection of W_1 to $\Omega(G)/G$ appears as n distinct tangent circles. Since the preimages of these circular arcs are disjoint, we can locally deform them into n parallel arcs. That is, we can deform W_1 and W_2 slightly near every point of the chain, and also near the f or f^{-1} image of the single rimpoints at the ends of the chain, to obtain new simple closed curves, W_1' and W_2'. We also have new sets of rimpoints, Θ_1' and Θ_2', which are the old rimpoints, with the points of this chain, and all their J_1 and J_2 translates, deleted. It is easy to see that we can make this deformation so that (W_m', Θ_m') is still a (J_m, G_0)-swirl; $f(W_1') = W_2'$; and (W_1', W_2') is still jointly f-simple; i.e., the hypotheses of our combination theorem still hold with W_m replaced by W_m'. Since this deformation leaves unchanged $\text{area}(G)$, $\text{area}(G_0)$, and $\text{area}(J_1)$, we need prove $(*)$ only for cycles of preparabolic rimpoints.

We first assume that we are given a cycle of rimpoints, none of which is an elliptic fixed point of G_0. Consider the n distinct points $\{x_1, \dots, x_n\}$ in the cycle lying on W_1. There are also n distinct points in the cycle lying on W_2; these are $\{f(x_1), \dots, f(x_n)\}$. Some of these points on W_2 might also lie on W_1; in fact, some of the $f(x_i)$ might be equal to some of the x_j. These rimpoints are all parabolic fixed points in G, hence they contribute zero to the area of $(W_1 \cap \Theta)/G$; we have to show that the contributions to the areas of $(W_1 \cap \Theta)/G_0$ and $(W_1 \cap \Theta)/J_1$ are equal. Each of these $2n$ points lies on either W_1 or W_2, and has a unique G_0-equivalent point, also in the cycle, and also lying either on W_1 or W_2. Hence the contribution to the area of $(W_1 \cap \Theta)/G_0 + (W_2 \cap \Theta)/G_0$ is exactly n. Of course the n points of the cycle on W_1 all project to distinct points of $\Omega(J_1)/J_1$; hence their contribution to the area of $(W_1 \cap \Omega(J_1))/J_1$ is also n.

We next take up the case that one of the points of our cycle, say x_1, is a fixed point of an elliptic element of G_0, necessarily of order 2. Write $x_2 = f(x_1)$, and continue with the cycle. Note that in order for the cycle to come back to x_1, it must reach some $x_q \neq x_1$, which is also a fixed point of a half-turn in G_0, and then return back to x_1, where all the points of the cycle between x_1 and x_q are not elliptic fixed points. The computations are now essentially the same as above. The contribution of the cycle to the area of $(W_1 \cap \Theta)/G$ is zero; the contribution to the area of $(W_1 \cap \Theta)/J_1$ is n, where n is the number of distinct points of the cycle on W_1. Except for x_1 and x_q, each of which is paired with itself, the other $2n-2$ points of the cycle on W_1 and W_2 are each paired by G_0 with exactly one other point (this shows that $q = n$). Hence the contribution to the area of $((W_1 \cup W_2) \cap \Theta)/G_0$ is $(n-1) + 1/2(2) = n$.

III. The first variation

In this variation on the first combination theorem, we have the same hypotheses for G_1, but somewhat different hypotheses for G_2. For $g \in G_2$, we permit significantly larger sets of the form $g(W) \cap W$, although we still require that $\text{int}(B_2)$ be precisely invariant under J in G_2 (in particular, W is precisely embedded), and we have the additional requirement that J have finite index in G_2.

Our basic hypotheses for the first variation are essentially as follows. We are given two Kleinian groups, G_1 and G_2, with a common subgroup J, where J is geometrically finite and has index at least 2 in both G_1 and G_2. We also assume that we are given a J-invariant simple closed curve W, together with a J-invariant set of rimpoints $\Theta \subset W$. W divides $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2; we assume that $\text{int}(B_m)$ is precisely invariant under J in G_m, and we assume that (B_1, Θ) is a (J, G_1)-simple disc. We also assume that J has finite index in G_2, and that for every $g \in G_2$, and for every $x \in \Theta$, either $g(x) \notin W$, or $g(x) \in \mathbf{\Theta}$. We require that there be a $g_1 \in G_1 - J$ with $g_1(W) \neq W$, and we require that every cyclic stabilizer be parabolic.

Note that since J has finite index in G_2, $\Lambda(G_2) = \Lambda(J) \subset W$.

As in the first combination theorem, the conditions above imply that if there is a $g \in G_m - J$, and there is a point $x \in W$, with $g(x) \in W$, then, since $g(B_m) \subset B_{3-m}$, x can be a parabolic fixed point of G_m only if it is a parabolic fixed point of J. That is, $\Omega(G_m) \cap W = \Omega(J) \cap W$.

III.1. Observe that our hypotheses includes the following two possibilities. For both examples, W consists of the two rays emanating from the origin, $\{\arg(z) = \pm\pi/n\}$. In the first example, J is trivial; G_1 is a Fuchsian group of the second kind, acting on the upper half-plane, where $\{\pi/n < \arg(z) < (2n-1)\pi/n\}$ is precisely invariant under the identity (but its closure need not be); and $G_2 = \langle z \to e^{2\pi i/n} z\rangle$. In the second example, J is hyperbolic cyclic, generated by $z \to \lambda z$, $\lambda > 1$; G_1 is Fuchsian of the second kind, where $\{\pi/n < \arg(z) < (2n-1)\pi/n\}$ is precisely invariant under J (but its closure need not be); and G_2 is generated by $g(z) = e^{2\pi i/n}\lambda^{1/n} z$ (see [M5, p. 193 ff.]).

Another use of this theorem is as follows. G_1 is a Fuchsian group, acting on the upper half-plane, where G_1 has an extension; that is, there is a Möbius transformation f, interchanging the upper and lower half-planes, where $fG_1f^{-1} = G_1$, and $f^2 \in G_1$. In this case, W is the limit circle of G_1; $G_2 = \langle f\rangle$; and $J = \langle f^2\rangle$.

III.2. We define *single* and *double* rimpoints exactly as in §I; that is, a rimpoint x is a double rimpoint if there is both a $g_1 \in G_1 - J$ with $g_1(x) \in W$, and there is a $g_2 \in G_2 - J$ with $g_2(x) \in W$. We similarly define *chains* and *cycles* of rimpoints; the points in a chain of rimpoints are called *ordinary*; the points in a cycle of rimpoints are called *preparabolic.* We also define the *cyclic stabilizer* at a preparabolic rimpoint. As in §I, each chain starts and ends with a single rimpoint, while each rimpoint in a cycle is a double rimpoint.

We also define parabolic chains and cycles, and cyclic stabilizers for these points, exactly as in §I.

III.3. Let D_m be a fundamental set for G_m that is maximal with respect to B_m; in particular, each rimpoint in $\mathcal{R}\Omega(G_m)$ is G_m-equivalent to a unique rimpoint in D_m. We also require that $D_2 \cap W \subset D_1$. D_1 and D_2 satisfying these conditions are called *compatible* fundamental sets.

Set $D' = (D_1 \cap \operatorname{int}(B_2)) \cup (D_2 \cap B_1)$. The *modified set* D is obtained from D' by deleting all the preparabolic rimpoints, and deleting all the ordinary rimpoints, but including exactly one ordinary rimpoint from each chain that includes no elliptic fixed points.

III.4. We note that the action of G_2 on W induces an equivalence relation on $\Omega(J) \cap W$; since J has finite index in G_2, there are only finitely many points in each equivalence class that are distinct modulo J. We define W/G_2, or, more precisely, $(W \cap \Omega(G_2))/G_2$, to be $W \cap \Omega(J)$ factored by this equivalence relation.

III.5. The conclusions of the theorem below are numbered so as to agree with the numbering in §I.

Statement of the first variation

Theorem III (The first variation). *Let G_1 and G_2 be Kleinian groups with a geometrically finite common subgroup J, where $[G_m : J] \geq 2$ and $[G_2 : J] < \infty$. Assume the following.*

(A) *There is a J-invariant simple closed curve W dividing $\widehat{\mathbb{C}}$ into two closed topological discs, B_1 and B_2, where $\operatorname{int}(B_2)$ is precisely invariant under J in G_2, and there is a set of rimpoints Θ given on W so that (B_1, Θ) is a (J, G_1)-simple disc. Assume further that for every $g \in G_2$, and for every $x \in \Theta$, either $g(x) \notin W$ or $g(x) \in \Theta$.*

(B) *Every cyclic stabilizer is parabolic.*

(C) *There is a $g_1 \in G_1$ with $g_1(W) \neq W$.*

Let $G = \langle G_1, G_2 \rangle$; let D_1 and D_2 be integrated fundamental sets for G_1 and G_2, respectively, and let D be the modified set for G. Then the following hold.

(i) $G = G_1 *_J G_2$.

(ii) *G is discrete.*

(iii) *Every element of G that is not a conjugate of an element of either G_1 or G_2, or a conjugate of a power of a cyclic stabilizer, is loxodromic.*

(iv) *W is a precisely embedded simple closed curve.*

(vii) *The modified set D is a fundamental set for G.*

(viii) *Let S_m be the complement in B_{3-m} of the union of the G_m-translates of B_m. Then S_m is precisely invariant under G_m in G. Further, $\Omega(G)/G$ is the union of S_1/G_1 and S_2/G_2, where these are joined along their common boundary, $(W \cap \Omega(G))/G_2$. $W \cap \Omega(G)$ is the complement of the cyclic rimpoints in $W \cap \Omega(J)$.*

(ix) *Assume that $W \cap \Omega(J)$ is smooth, and that G_1 is geometrically finite. Then there is a spanning disc Q for W, where Q is precisely invariant under* $\operatorname{Stab}(W)$. *Further, $\mathbb{H}^3/G$ can be described as follows: Let B_m^3 be the region in $\mathbb{H}^3$ bounded by the translates of Q, whose Euclidean boundary is B_m. Then $\mathbb{H}^3/G$ is the union of B_1^3 and B_2^3, where these two are identified across their common boundary, Q/J.*

(xi) *G is geometrically finite if and only if G_1 is geometrically finite.*

(xii) *G is analytically finite if and only if G_1 is analytically finite.*

(xiii) *If G is analytically finite, then*

$$\operatorname{area}(G) = \operatorname{area}(G_1) + \operatorname{area}(G_2) - \operatorname{area}(J).$$

The proof of this theorem is essentially the same as the proof of Theorem I; we do not give a formal proof, but rather a discussion of how to modify the proof of Theorem I so as to be applicable here.

Before going on, we remark that since G_2 is a finite extension of the geometrically finite group, J, it is also geometrically finite; hence also analytically finite.

Proof of the first variation

The proof given in §I that $(\operatorname{int}(B_1), \operatorname{int}(B_2))$ is a proper interactive pair is valid here as well; since the proofs of conclusions (i), (ii), and (iii) in §I depend only on this fact, they are also valid here.

The proof of conclusion (iv) is the same as that in §I; that is, we first prove that S_m is precisely invariant under G_m. Here, however, we may have an element of G_2 mapping a nontrivial arc of $W \cap \Omega(J)$ onto another such arc, in which case W is not a swirl.

Set $\widehat{W} = \bigcup_{g \in G_2} g(W)$, and let $\widehat{\Theta} = \bigcup_{g \in G_2} g(\Theta)$. Then $\widehat{W}$ is a finite union of translates of W; hence it divides $\widehat{\mathbb{C}}$ into a finite number of regions. Since S_m is precisely invariant under G_m, and W is precisely embedded, it follows that every nontrivial translate of $\widehat{W}$ is weakly separated from $\widehat{W}$ by either a $(G_1 - J)$-translate of W, or by a translate of W of the form $g_2 \circ g_1$, where $g_m \in G_m - J$. It follows from this that a translate of $\widehat{W}$ can touch $\widehat{W}$ only at a limit point of $G_2 =$ Stab$(\widehat{W})$, or at a point of $\widehat{\Theta}$.

Condition (A) tells us that we can form chains and cycles of rimpoints on W exactly as in §I; we can define the ordinary and preparabolic rimpoints accordingly. We also prove, exactly as in §I that the ordinary rimpoints are points of $\Omega(G)$, and, using condition (B), that the preparabolic rimpoints are doubly cusped parabolic fixed points of G.

It follows from the above that if $\{g_m(\widehat{W})\}$ is a sequence of distinct translates of $\widehat{W}$, then, for every $g_2 \in G_2$, $\{g_m \circ g_2(W)\}$ is a sequence of translates of W satisfying the hypotheses of Proposition 0.8 (that is, they are almost disjoint, two of them intersect only at common limit points or at common rimpoints, and, except for the preparabolic rimpoints, $\widehat{W} \cap \Omega(J) \subset \Omega(G)$). It follows that the spherical diameter of $g_m(\widehat{W}) \to 0$.

For the proof of (vii), as in §I, we observe that since S_m is precisely invariant under G_m, $D_m \cap S_m$ is precisely invariant under the identity. It follows that D is precisely invariant under the identity in G.

Since S_1 is precisely invariant under G_1, a sequence of translates of W can accumulate to a point of $W \cap \Omega(J)$ from inside B_2 only at a rimpoint. Similarly, since J has finite index in G_2, if there is a sequence of distinct translates of W approaching W from inside B_1, then there is a set of the form $g_2 \circ g_1(B_1)$, $g_m \in G_m - J$, and there is a subsequence of these translates of W all of which lie in this set. It follows that a sequence of distinct translates of W can approach W only at a limit point of J, or at a rimpoint. Since the ordinary rimpoints are all in Ω, $D \subset \Omega$.

We saw above that any sequence of distinct translates of W has spherical diameter tending to zero; hence, as in §I, $\Omega(G) \subset \overline{S}_1 \cup \overline{S}_2$. It follows that D is a fundamental set for G.

For conclusion (viii), we already know that S_m is precisely invariant under G_m in G. We also know from conclusion (vii) that every point of $\Omega(G)$ is equivalent to either a point of S_1, or a point of S_2, or a point of W.

The only difficulty in proving conclusion (ix) is the construction of the spanning disc. We remark that since J has finite index in G_2, and W is precisely embedded under J in G_2, there is a spanning disc Q_2 for W in G_2. We start with Q_2, and make the necessary modifications exactly as in §I.

We start the proof of the conclusion (xi) with the observation that the proof of Lemma I.5 shows that S_1 contains infinitely many points; of course, S_2 might be empty. The remainder of the proof of (xi) is essentially the same as that given in §I.

For conclusion (xii), we need only remark that G_2 is automatically analytically finite, and that S_1/G_1 and S_2/G_2 are glued together along their common boundary, which, in this case, is $(W \cap \Omega(G))/G_2$.

For the area computation, we choose a cell decomposition for $\Omega(G)/G$ as in §I, where the cell decomposition of the projection of W is invariant under the action of G_2. We again let $\Omega_- = \Omega - \Theta$.

Then, as in §I, we write

$$\begin{aligned}
\operatorname{area}(G_1) &= \operatorname{area}(S_1/G_1) + \operatorname{area}(\operatorname{int}(B_1)/J) + \operatorname{area}((W \cap \Omega_-)/J) + \operatorname{area}(\Theta/G_1);\\
\operatorname{area}(G_2) &= \operatorname{area}(S_2/G_2) + \operatorname{area}(\operatorname{int}(B_2)/J) + \operatorname{area}((W \cap \Omega_-)/G_2) + \operatorname{area}(\Theta/G_2);\\
\operatorname{area}(J) &= \operatorname{area}(\operatorname{int}(B_1)/J) + \operatorname{area}(\operatorname{int}(B_2)/J) + \operatorname{area}(\Omega_-/J) + \operatorname{area}(\Theta/J);\\
\operatorname{area}(G) &= \operatorname{area}(S_1/G_1) + \operatorname{area}(S_2/G_2) + \operatorname{area}(\Omega_-/G) + \operatorname{area}(\Theta/G).
\end{aligned}$$

Since

$$(W \cap \Omega_-)/G_1 = (W \cap \Omega_-)/J,$$

and

$$(W \cap \Omega_-)/G_2 = (W \cap \Omega_-)/G,$$

we need only show that

$$\Theta/G_1 + \Theta/G_2 = \Theta/G + \Theta/J.$$

This follows from the relevant area calculations given in §I.

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