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$W^{2,p}$-SOLVABILITY OF THE DIRICHLET PROBLEM FOR NONDIVERGENCE ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS

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ABSTRACT. We prove a well-posedness result in the class $W^{2,p} \cap W_0^{1,p}$ for the Dirichlet problem

$$\begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume the coefficients of the elliptic nondivergence form equation that we study are in $\mathrm{VMO} \cap L^\infty$.

1. INTRODUCTION

We consider the Dirichlet problem

$$(*) \qquad \begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded open subset Ω of $\mathbb{R}^n$.

Here we assume L to be a linear elliptic operator in Ω in *nondivergence* form whose possibly discontinuous coefficients are taken in the space VMO (for a precise definition see §2 below). The space VMO, introduced by Sarason [S], is the subspace of the functions in the John-Nirenberg space BMO whose BMO norm over a ball vanishes as the radius of the ball tends to zero. This property implies a number of good features of VMO functions not shared by general BMO functions; in particular they can be approximated by smooth functions.

It is easy to check that bounded uniformly continuous functions (BUC) are in VMO as well as functions of the Sobolev spaces $W^{1,n}$ and $W^{\vartheta, n/\vartheta}$ ($\vartheta \in]0, 1[$) (see [CFL, §2]).

Our main result in this paper is the well-posedness of problem $(*)$ in the class $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p \in]1, +\infty[$. The result has been known for a long time in the BUC case. (See [K, Gr, GT].) Furthermore there is a classical result by Miranda [M1] in the case of $W^{1,n}$ coefficients and $p = 2$. That the result should be true for $p = 2$ in the $W^{\vartheta, n/\vartheta}$ ($\vartheta \in]0, 1[$) case we heard to be an old conjecture of the same author. Regarding this, we wish to mention the interesting paper by Canfora and Zecca [CZ] which deals with the special case $n = 3$, $\vartheta = 3/4$. (For related results see [Cn, Z1, Z2].)

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The largely standard techniques of our proof consist in obtaining suitable interior and boundary estimates for the solution of problem $(*)$ and then, from these, an a priori estimate for the solutions of $(*)$. By using the above mentioned good behaviour of mollifiers in VMO we get the existence of the solution of $(*)$.

In realizing this program the technical difficulties arise in obtaining the interior and boundary estimates and in proving the uniqueness of the solution to problem $(*)$.

Both the interior and boundary estimates are consequences of explicit representation formulas for the solution of problem $(*)$ and the boundedness in L^p of some integral operators appearing in those formulas. These operators appear to be *new* (in the sense that we were unable to explicitly find their boundedness properties studied in the literature), and are studied using very classical techniques to reduce them to simpler operators. In particular, for the interior estimates, which we studied in detail in our previous work [CFL], we used a spherical harmonics development to reduce the singular integral operators appearing in the representation formula to a series of Calderon-Zygmund singular integrals and to a series of singular commutators like those considered by Coifman, Rochberg, and Weiss in [CRW]. In the study of this last operator, the VMO assumption on the coefficients is of the greatest relevance.

The boundary estimates are similar. Indeed, the representation formula obtained using the half space Green function involves the same integral operators of the interior case and two more, *less singular*, operators somewhat resembling Hardy's operator.

Finally, for the uniqueness, the VMO assumption again played a crucial role, assuring that some operators in L^p are contractions on this space.

Some results close to ours have been obtained recently by Caffarelli in his deep paper [Cf]. These results, although of local character as stated in [Cf], could probably be extended to obtain another proof of our Theorems 4.3 and 4.4. However, because of the essential use of the Pucci-Alexandroff maximum principle, Caffarelli's proof requires the assumption $p \geq n$. We take this opportunity to thank Luis Caffarelli for discussing with us some aspects of his work.

Also we mention here that, because of the technique we used, it appears possible to extend our results to higher order elliptic and parabolic equations. The development of this project will be our aim in the near future.

2. Some preliminary facts from real analysis

We recall the definitions and some useful properties of the spaces BMO and VMO. The proofs of these by now well-known facts may be found in [S] or in some general reference texts, e.g., [G].

We say that a locally integrable function f in $\mathbb{R}^n$ is in the space BMO if

$$\sup_B \fint_B |f(x) - f_B|\, dx = \|f\|_* < +\infty,$$

where B ranges in the class of the balls in $\mathbb{R}^n$. Here f_B is the average $\fint_B f(x)\, dx$.

For $f \in \mathrm{BMO}$ and $r > 0$ we set

$$\sup_{\rho \le r} \fint_B |f(x) - f_B|\, dx = \eta(r), \tag{2.1}$$

where B ranges in the class of the balls with radius ρ less than or equal to r. We will say that a function $f \in \mathrm{BMO}$ is in the space VMO if $\eta(r)$ in (2.1) vanishes as r tends to zero. We will refer to $\eta(r)$ as the VMO modulus of f. We have

Theorem 2.1. *For $f \in \mathrm{BMO}$ the following conditions are equivalent*

(i) *f is in* VMO;

(ii) *f is in the* BMO *closure of the set of the uniformly continuous functions which belong to* BMO;

(iii) $\lim_{y \to 0} \|f(x-y) - f(x)\|_* = 0$.

We explicitly observe that if $f \in \mathrm{VMO}$ with VMO modulus η there exists a constant $c = c(n)$ such that

$$\|f(x-y) - f(x)\|_* \le c\eta(r), \qquad |y| < r,$$

(see [G, pp. 250–251]); so that the usual mollifiers converge to f in the BMO norm. More precisely, given $f \in \mathrm{VMO}$ with VMO modulus $\eta(r)$, we can find a sequence of C^∞ functions $\{f_h\}$ converging to f in BMO as $h \to 0$ with VMO moduli η_h such that $\eta_h(r) \le \eta(r)$.

We start by recalling the definition and some useful properties of singular integrals

Definition 2.2. Let $k \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We say that $k(x)$ is a *Calderon-Zygmund kernel* (*C-Z kernel*) if

(i) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

(ii) k is homogeneous of degree $-n$;

(iii) $\int_\Sigma k(x)\, d\sigma = 0$, where $\Sigma = \{x \in \mathbb{R}^n \colon |x| = 1\}$.

Theorem 2.3. *Let Ω be an open subset of $\mathbb{R}^n$. Let $k \colon \Omega \times \{\mathbb{R}^n \setminus \{0\}\} \to \mathbb{R}$ be a function satisfying*

(i) *$k(x, \cdot)$ is a C-Z kernel for a.a. $x \in \Omega$,*

(ii) $\max_{|j| \le 2n} \|(\partial^j / \partial z^j) k(x, z)\|_{L^\infty(\Omega \times \Sigma)} = M < +\infty$.

For $f \in L^p(\Omega)$, $(1 < p < +\infty)$, $\varphi \in L^\infty(\mathbb{R}^n)$, and $x \in \Omega$, set

$$K_\varepsilon f(x) = \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} k(x, x-y) f(y)\, dy,$$

$$C_\varepsilon[\varphi, f](x) = \varphi(x) K_\varepsilon f(x) - K_\varepsilon(\varphi f)(x)$$

$$= \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} k(x, x-y)[\varphi(x) - \varphi(y)] f(y)\, dy.$$

Then, for any $f \in L^p(\Omega)$ there exist Kf, $C[\varphi, f] \in L^p(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \|K_\varepsilon f - Kf\|_{L^p(\Omega)} = \lim_{\varepsilon \to 0} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{L^p(\Omega)} = 0.$$

Moreover, there exists a constant $c = c(n, p, M)$ such that

$$\|Kf\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}, \qquad \|C[\varphi, f]\|_{L^p(\Omega)} \le c\|\varphi\|_* \|f\|_{L^p(\Omega)}.$$

Theorem 2.4. *Let k and c be as in Theorem 2.3. Also let $a \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n)$, and let η be the VMO modulus of a. Then, for any $\varepsilon > 0$, there exists a positive $\rho_0 = \rho_0(\varepsilon, \eta)$ such that*

$$\|C[a, f]\|_{L^p(\Omega_r)} \leq c\varepsilon \|f\|_{L^p(\Omega_r)} \quad \forall f \in L^p(\Omega_r),$$

for any ball B_r, $r \in]0, \rho_0[$, and $B_r \cap \Omega = \Omega_r \neq \varnothing$.

The proof of the first theorem follows closely a classical argument based on the expansion of the kernel into spherical harmonics (see, e.g., [Cl]). The proof of Theorem 2.4 is a straightforward consequence of Theorem 2.3 and property (ii) in Theorem 2.1. Both proofs are given in detail in [CFL].

For further developments we need to study the boundedness in L^p of some other integral operators. The techniques we employ in the proofs here are also quite standard. Because we are unable to give a precise reference we give the complete arguments.

Let $\mathbb{R}^n_+ = \{x = (x', x_n) \colon x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ and for $x \in \mathbb{R}^n$ let $\tilde{x} = (x', -x_n)$.

Theorem 2.5. *Let $f \in L^p(\mathbb{R}^n_+)$, $1 < p < +\infty$. For $x \in \mathbb{R}^n_+$ set*

$$\widetilde{K} f(x) = \int_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x} - y|^n}\, dy.$$

Then there exists a constant $c = c(n, p)$ such that $\|\widetilde{K} f\|_p \leq c\|f\|_p$, where the norms are taken in $L^p(\mathbb{R}^n_+)$.

Proof. For $x \in \mathbb{R}^n_+$ let

$$\begin{aligned} I(x_n) &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^n_+} \frac{|f(y)|}{(|x' - y'|^2 + (x_n + y_n)^2)^{n/2}}\, dy \right)^p dx' \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_0^{+\infty} \left(\int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{(|x' - y'|^2 + (x_n + y_n)^2)^{n/2}}\, dy' \right) dy_n \right)^p dx'. \end{aligned}$$

Using the Minkowsky and Young inequalities we obtain

$$\begin{aligned} I(x_n) &\leq \left[\int_0^{+\infty} \left(\int_{\mathbb{R}^{n-1}} |f(y', y_n)|^p\, dy' \right)^{1/p} \left(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|^2 + (x_n + y_n)^2)^{n/2}} \right) dy_n \right]^p \\ &= \left(\int_0^{+\infty} \frac{\varphi(y_n)}{x_n + y_n}\, dy_n \right)^p \left(\int_{\mathbb{R}^{n-1}} \frac{dt}{(|t|^2 + 1)^{n/2}} \right)^p, \end{aligned}$$

where we set

$$\varphi(y_n) = \left(\int_{\mathbb{R}^{n-1}} |f(y', y_n)|^p\, dy' \right)^{1/p}.$$

Integrating in $]0, +\infty[$ we get

$$\|\widetilde{K} f\|_p^p \leq c(n, p) \int_0^{+\infty} \left(\int_0^{+\infty} \frac{\varphi(\lambda x_n)}{1 + \lambda}\, d\lambda \right)^p dx_n.$$

Again, by Minkowsky, we obtain

$$\begin{aligned} \|\widetilde{K} f\|_p^p &\leq c(n, p) \left(\int_0^{+\infty} \left(\int_0^{+\infty} \left(\frac{\varphi(\lambda x_n)}{1 + \lambda} \right)^p dx_n \right)^{1/p} d\lambda \right)^p \\ &= c(n, p) \left(\int_0^{+\infty} \frac{1}{1 + \lambda} \frac{1}{\lambda^{1/p}}\, d\lambda \right)^p \|f\|_p^p. \end{aligned}$$

Theorem 2.6. *Let* $f \in L^p(\mathbb{R}^n_+)$, $1 < p < +\infty$, $a \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n)$. *For* $x \in \mathbb{R}^n_+$ *set*

$$\widetilde{C}[a, f](x) = \int_{\mathbb{R}^n_+} \frac{a(x) - a(y)}{|\tilde{x} - y|^n} f(y)\, dy.$$

Then there exists $c = c(n, p)$ *such that* $\|\widetilde{C}[a, f]\|_p \le c(n, p)\|a\|_* \|f\|_p$.

Proof. Let Q be a cube with sides parallel to the coordinate axes contained in $\mathbb{R}^n_+$. Given $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ we set

$$M_+\varphi(x) = \sup_{Q \ni x} \fint_Q |\varphi(y)|\, dy, \qquad \varphi^\#_+(x) = \sup_{Q \ni x} \fint_Q |\varphi(y) - \varphi_Q|\, dy.$$

It is easy to adapt the usual proofs given in $\mathbb{R}^n$ (see, e.g., [GR]) to prove that, for $\varphi \in L^p(\mathbb{R}^n_+)$, $1 < p < +\infty$,

$$\|M_+\varphi\|_p \le c(n, p)\|\varphi\|_p, \qquad \|\varphi\|_p \le c(n, p)\|\varphi^\#_+\|_p.$$

Following an idea of Strömberg (see [T, pp. 417–418]) we will get the conclusion, using Theorem 2.5, as soon as we prove the following pointwise inequality

$$\widetilde{C}[a, f]^\#_+(x) \le c(n, p)\|a\|_* \{(M_+(|\widetilde{K} f|^r)(x))^{1/r} + (M_+(|f|^r)(x))^{1/r}\},$$

where $1 < r < p$. For $Q \subseteq \mathbb{R}^n_+$ denote by ℓ_Q the side of Q and by y_Q its center. Also denote by $\imath Q$ the cube centered at y_Q with side $2^{\imath - 1}\ell_Q$, $\imath \in \mathbb{N}$. We write for any Q containing x

$$\begin{aligned}\widetilde{C}[a, f](x) &= (a(x) - a_Q)\widetilde{K} f(x) + \widetilde{K}((a_Q - a(\cdot))f(\cdot)\chi_{2Q})(x) \\ &\quad + \widetilde{K}((a_Q - a(\cdot))f(\cdot)\chi_{\mathbb{R}^n_+ \setminus 2Q})(x) \\ &\equiv I(x) + J(x) + L(x),\end{aligned}$$

where χ_A is the characteristic function of the set A.

We have

$$\begin{aligned}\fint_Q |I(y) - I_Q|\, dy &\le 2\left(\fint_Q |a(y) - a_Q|^{r'}\, dy\right)^{1/r'} \left(\fint_Q |\widetilde{K} f(y)|^r\, dy\right)^{1/r} \\ &\le c(n, r)\|a\|_* (M_+(|\widetilde{K} f|^r)(x))^{1/r} \qquad \left(\frac{1}{r} + \frac{1}{r'} = 1\right),\end{aligned}$$

where we used the John-Nirenberg lemma. Fix q, $1 < q < r$. Using Theorem 2.5 we have

$$\begin{aligned}\fint_Q |J(y)|\, dy &\le c(n, q)\left(\frac{1}{|Q|}\int_{2Q \cap \mathbb{R}^n_+} |a - a_Q|^q |f|^q\, dy\right)^{1/q} \\ &\le c(n, q, r)\|a\|_* \left(\frac{2^n}{|2Q|}\int_{2Q \cap \mathbb{R}^n_+} |f|^r\, dy\right)^{1/r} \\ &= c(n, q, r)\|a\|_* \left(\frac{1}{|Q'|}\int_{Q'} |f|^r\, dy\right)^{1/r} \\ &\le c(n, q, r)\|a\|_* (M_+(|f|^r)(x))^{1/r},\end{aligned}$$

where we set Q' for the cube contained in $\mathbb{R}^n_+$ containing $2Q \cap \mathbb{R}^n_+$ and having the same measure as $2Q$.

Finally we estimate $L_+^\#$. We have

$$\fint_Q |L(y) - L_Q|\, dy \le 2 \fint_Q |L(y) - L(y_Q)|\, dy$$

and

$$\begin{aligned}|L(y) - L(y_Q)| &\le c(n) \int_{\mathbb{R}_+^n \setminus 2Q} \frac{|\tilde{y} - \tilde{y}_Q|}{|\tilde{y}_Q - z|^{n+1}} |f(z)|\, |a(z) - a_Q|\, dz \\ &\le c(n) \ell_Q \left(\int_{\mathbb{R}_+^n} \frac{|f(z)|^r}{|\tilde{y}_Q - z|^{n+1}}\, dz \right)^{1/r} \left(\int_{\mathbb{R}_+^n} \frac{|a(z) - a_Q|^{r'}}{|\tilde{y}_Q - z|^{n+1}}\, dz \right)^{1/r'}.\end{aligned}$$

We now set $\widetilde{Q}$ equal to the cube in $\mathbb{R}_-^n$ symmetrical to Q and let k be the least integer such that $k\widetilde{Q} \cap \mathbb{R}_+^n \ne \varnothing$. Then we have

$$\begin{aligned}\int_{\mathbb{R}_+^n} \frac{|f(z)|^r}{|\tilde{y}_Q - z|^{n+1}}\, dz &= \sum_{i=k}^{+\infty} \int_{\mathbb{R}_+^n \cap (i\widetilde{Q} \setminus (i-1)\widetilde{Q})} \frac{|f(z)|^r}{|\tilde{y}_Q - z|^{n+1}}\, dz \\ &\le \frac{c(n)}{\ell_Q} \sum_{i=k}^{+\infty} \frac{1}{2^i |Q_i'|} \int_{Q_i'} |f(z)|^r\, dz \le \frac{c(n)}{\ell_Q} M_+(|f|^r)(x),\end{aligned}$$

where Q_i' is the cube contained in $\mathbb{R}_+^n$ containing $i\widetilde{Q} \cap \mathbb{R}_+^n$ and having the same measure as $i\widetilde{Q}$. Obviously $Q \subseteq Q_i'$.

In the same way we obtain

$$\int_{\mathbb{R}_+^n} \frac{|a - a_Q|^{r'}}{|\tilde{y}_Q - z|^{n+1}}\, dz \le \frac{c(n)}{\ell_Q} \sum_{i=1}^{+\infty} \frac{1}{2^i |iQ|} \int_{iQ} |a(z) - a_Q|^{r'}\, dz.$$

Recalling that $|a_{iQ} - a_Q| \le c(n) i \|a\|_*$ we obtain

$$\int_{\mathbb{R}_+^n} \frac{|a(z) - a_Q|^{r'}}{|\tilde{y}_Q - z|^{n+1}}\, dz \le \frac{c(n, r)}{\ell_Q} \|a\|_*^{r'},$$

and then

$$\fint_Q |L(y) - L_Q|\, dy \le c(n, p) \|a\|_* (M_+(|f|^r)(x))^{1/r}.$$

In the following we will set

$$B_r^+ = \{(x', x_n) \in \mathbb{R}^n : |x| < r,\ x_n > 0\}.$$

The following is an easy consequence of Theorems 2.1 and 2.6 (for a similar result see Theorem 2.13 in [CFL]).

Theorem 2.7. *Let* $1 < p < +\infty$ *and* $c = c(n, p)$ *as in Theorem* 2.6. *Let* $a \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n)$ *and* η *its* VMO *modulus. Then for any* $\varepsilon > 0$ *there exists a positive* $\rho_0 = \rho_0(\eta, \varepsilon)$ *such that for any* $r \in]0, \rho_0[$ *we have*

$$\|\widetilde{C}[a, f]\|_{L^p(B_r^+)} \le c\varepsilon \|f\|_{L^p(B_r^+)}, \quad \forall f \in L^p(B_r^+).$$

3. An a priori estimate in a special case

Define $W_{\gamma_0}^{2,p}(B_\sigma^+)$ to be the closure in $W^{2,p}$ of the subspace

$$C_{\gamma_0} = \{u : u \text{ is the restriction to } B_\sigma^+ \text{ of a function belonging to } C_0^\infty(B_\sigma),\ u(x', 0) = 0\}.$$

We will make the following assumption, and we will refer to it as assumption (H).

$$\left.\begin{array}{ll}(3.1) & \text{Let } n \geq 3, \quad b_{ij} \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n), \quad i, j = 1, \dots, n, \\ & b_{ij} = b_{ji}, \quad i, j = 1, \dots, n \quad \text{a.e. in } B_\sigma^+, \\ (3.2) & \exists \mu > 0 \colon \mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^n b_{ij}\xi_i\xi_j \leq \mu|\xi|^2 \quad \text{a.e. in} B_\sigma^+, \ \forall \xi \in \mathbb{R}^n.\end{array}\right\} \text{(H)}$$

Also set

$$\widetilde{L} = \sum_{i,j=1}^n b_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

and

$$\Gamma(x, t) = \frac{1}{(n-2)\omega_n (\det b_{ij})^{1/2}} \left(\sum_{i,j=1}^n B_{ij}(x) t_i t_j \right)^{(2-n)/2},$$

$$\Gamma_i(x, t) = \frac{\partial}{\partial t_i}\Gamma(x, t), \qquad \Gamma_{ij}(x, t) = \frac{\partial^2}{\partial t_i \partial t_j}\Gamma(x, t),$$

for a.a. $x \in B_\sigma^+$ and $\forall t \in \mathbb{R}^n \backslash \{0\}$, where the B_{ij} are the entries of the inverse of the matrix $(b_{ij})_{i,j=1,\dots,n}$.

We need some more notation:

$$b(x) = (b_{in}(x))_{i=1,\dots,n}, \qquad T(x; y) = x - \frac{2x_n}{b_{nn}(y)} b(y).$$

Finally set $T(x) = T(x; x)$ and $B(y) = T(e_n; y)$ where, as usual, $e_n = (0, 0, \dots, 0, 1)$. We have

Lemma 3.1. *There exists a positive constant $c = c(n, \mu)$ such that*

$$|\tilde{x} - y| \leq c|T(x) - y| \quad \forall y \in \mathbb{R}^n_+ \text{ and a.a. } x \in B_\sigma^+.$$

Here $\tilde{x} = (x', -x_n)$.

Proof. Clearly $|T(x) - y| \geq x_n + y_n \geq x_n$. Hence

$$\frac{|T(x) - \tilde{x}|}{|T(x) - y|} \leq \frac{1}{x_n} \left| x - \frac{2x_n b(x)}{b_{nn}(x)} - \tilde{x} \right| = 2 \left| e_n - \frac{b(x)}{b_{nn}(x)} \right| \leq c(n, \mu);$$

then

$$|\tilde{x} - y| \leq |T(x) - \tilde{x}| + |T(x) - y| \leq (1 + c(n, \mu))|T(x) - y|.$$

In the following we will call $\widetilde{B}_\sigma$ the subset of B_σ^+ where (3.1) and (3.2) hold.

Theorem 3.2. *Assume* (H) *and let $u \in W^{2,p}_{\gamma_0}(B_\sigma^+)$. Then*

(3.3)

$$u_{x_i x_j}(x) = \mathrm{P.V.} \int_{B_\sigma^+} \Gamma_{ij}(x, x - y) \left\{ \sum_{h,k=1}^n (b_{hk}(x) - b_{hk}(y)) u_{x_h x_k}(y) + \widetilde{L}u(y) \right\} dy$$
$$+ \widetilde{L}u(x) \int_{|t|=1} \Gamma_i(x, t) t_j \, d\sigma_t + I_{ij}(x),$$

where: *for* $i, j = 1, \dots, n-1$

$$I_{ij}(x) = \int_{B_\sigma^+} \Gamma_{ij}(x, T(x) - y) \left\{ \sum_{h,k=1}^{n} (b_{hk}(x) - b_{hk}(y)) u_{x_h x_k}(y) + \widetilde{L}u(y) \right\} dy,$$

for $i = 1, \dots, n$

$$I_{in}(x) = I_{ni}(x) = \int_{B_\sigma^+} \left(\sum_{j=1}^{n} \Gamma_{ij}(x, T(x) - y) B_j(x) \right) \{\cdots\} \, dy,$$

and

$$I_{nn}(x) = \int_{B_\sigma^+} \left(\sum_{i,j=1}^{n} \Gamma_{ij}(x, T(x) - y) B_i(x) B_j(x) \right) \{\cdots\} \, dy;$$

in the formulas above $B_i(x)$ *is the* ith *component of the vector* $B(x)$ *and in the curly brackets there is always the same expression as in the first case.*

Proof. Let $x_0 \in \widetilde{B}_\sigma$ and $u \in C_{\gamma_0}$. Setting

$$\widetilde{L}_0 u(x) = \sum_{i,j=1}^{n} b_{ij}(x_0) u_{x_i x_j}(x)$$

and using the half space Green function for $\widetilde{L}_0$ we obtain

$$\begin{aligned} u(x) &= \int_{B_\sigma^+} \{\Gamma(x_0, x - y) - \Gamma(x_0, T(x; x_0) - y)\} \widetilde{L}_0 u(y) \, dy \\ &= \int_{B_\sigma^+} \Gamma(x_0, x - y) \widetilde{L}_0 u(y) \, dy - \int_{B_\sigma^+} \Gamma(x_0, T(x; x_0) - y) \widetilde{L}_0 u(y) \, dy \qquad (3.4) \\ &\equiv I'(x) - I''(x). \end{aligned}$$

Differentiating I' twice in (3.4) and writing $\widetilde{L}_0 u = (\widetilde{L}_0 - \widetilde{L}) u + \widetilde{L} u$ we obtain by classical results (see [M2])

$$\begin{aligned} I'_{x_i x_j}(x) = \text{P.V.} \int_{B_\sigma^+} \Gamma_{ij}(x_0, x - y) \left\{ \sum_{h,k=1}^{n} (b_{hk}(x_0) - b_{hk}(y)) u_{x_h x_k}(y) + \widetilde{L}u(y) \right\} dy \\ + \widetilde{L}u(x) \int_{|t|=1} \Gamma_i(x_0, t) t_j \, d\sigma_t, \quad \forall x \in B_\sigma^+. \end{aligned}$$

As for I'' differentiation is easier because it is possible to differentiate inside the integral. Then, for $u \in C_{\gamma_0}$, (3.3) is immediately obtained by setting $x = x_0$ in the formula giving the second derivatives of (3.4). A density argument, using Theorems 2.3, 2.5, 2.6 and Lemma 3.1, gives the conclusion for $u \in W_{\gamma_0}^{2,p}$.

Theorem 3.2. *Assume* (H). *Let* $q, p \in]0, +\infty[$, $q \le p$. *Set* $\tilde{\eta} = (\sum_{i,j=1}^{n} \tilde{\eta}_{ij})^{1/2}$, *where* $\tilde{\eta}_{ij}$ *is the* VMO *modulus of* b_{ij}, *and*

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha}{\partial t^\alpha} \Gamma_{ij}(x, t) \right\|_{L^\infty(B_\sigma^+ \times \Sigma)}.$$

Then there exists a positive number $\rho_0 = \rho_0(n, q, p, M, \mu, \tilde{\eta})$, $\rho_0 < \sigma$, *such that for any* $r \in]0, \rho_0[$ *and any* $u \in W_{\gamma_0}^{2,q}(B_r^+)$ *with* $\widetilde{L}u \in L^p(B_r^+)$ *we have*

$u \in W^{2,p}(B_r^+)$. *Furthermore there exists a constant* $c = c(n, p, M, \mu, \tilde{\eta})$ *such that*

$$\|u_{x_i x_j}\|_{L^p(B_r^+)} \leq c\|\widetilde{L}u\|_{L^p(B_r^+)}. \tag{3.5}$$

Proof. Set for $i, j, h, k = 1, \ldots, n$,

$$S_{ijhk}(f)(x) = \mathrm{P.V.}\int_{B_r^+} \Gamma_{ij}(x, x-y)(b_{hk}(x) - b_{hk}(y))f(y)\, dy,$$

and for $i, j = 1, \ldots, n-1$, $h, k = 1, \ldots, n$

$$\widetilde{S}_{ijhk}(f) = \int_{B_r^+} \Gamma_{ij}(x, T(x) - y)(b_{hk}(x) - b_{hk}(y))f(y)\, dy,$$

for $i = 1, \ldots, n-1$, $h, k = 1, \ldots, n$

$$\widetilde{S}_{inhk}(f) = \int_{B_r^+} \left(\sum_{j=1}^{n} \Gamma_{ij}(x, T(x) - y)B_j(x)\right)(b_{hk}(x) - b_{hk}(y))f(y)\, dy,$$

and finally for $h, k = 1, \ldots, n$

$$\widetilde{S}_{nnhk}(f) = \int_{B_r^+} \left(\sum_{i,j=1}^{n} \Gamma_{ij}(x, T(x) - y)B_i(x)B_j(x)\right)(b_{hk}(x) - b_{hk}(y))f(y)\, dy,$$

where $r \in]0, \sigma]$ and $f \in L^\nu(B_r^+)$.

Recalling Lemma 3.1 and Theorems 2.4 and 2.7 we can fix ρ_0 so small that $\sum_{ijhk} \|S_{ijhk} + \widetilde{S}_{ijhk}\| < 1$, where the norm of operators $S_{ijhk} + \widetilde{S}_{ijhk}$ is the norm in the space of linear operators from $L^\nu(B_r^+)$ in itself if $r \in]0, \rho_0[$ and $\nu \in [q, p]$.

Consider $u \in W_{\gamma_0}^{2,p}(B_r^+)$ with $\widetilde{L}u \in L^p(B_r^+)$, $r \in]0, \rho_0[$, and set

$$\begin{aligned} h_{ij}(x) = {} & \mathrm{P.V.}\int_{B_r^+} \Gamma_{ij}(x, x-y)\widetilde{L}u(y)\, dy \\ & + \widetilde{L}u(x)\int_{|t|=1} \Gamma_i(x, t)t_j\, d\sigma_t + \widetilde{I}_{ij}(x), \end{aligned}$$

where

$$\widetilde{I}_{ij} = \begin{cases} \displaystyle\int_{B_r^+} \Gamma_{ij}(x, T(x) - y)\widetilde{L}u(y)\, dy, & \text{for } i, j = 1, \ldots, n-1, \\ \displaystyle\int_{B^+} \left(\sum_{\ell=1}^{n} \Gamma_{i\ell}(x, T(x) - y)B_\ell(x)\right)\widetilde{L}u(y)\, dy, & \\ & \text{for } i = 1, \ldots, n-1,\ j = n, \\ \displaystyle\int_{B_r^+} \left(\sum_{\ell,m=1}^{n} \Gamma_{ij}(x, T(x) - y)B_\ell(x)B_m(x)\right)\widetilde{L}u(y)\, dy, & \text{for } i = j = n. \end{cases}$$

Clearly $h_{ij} \in L^p(B_r^+)$.

Consider $w \in [L^p(B_r^+)]^{n^2}$ and define $Tw\colon [L^p(B_r^+)]^{n^2} \to [L^p(B_r^+)]^{n^2}$ by setting

$$Tw = ((Tw)_{ij})_{i,j=1,\dots,n}$$
$$= \left(\sum_{h,k=1}^{n} (S_{ijhk} + \widetilde{S}_{ijhk})(w_{ij}) + h_{ij} \right)_{i,j=1,\dots,n}.$$

The operator T is a contraction in $[L^p(B_r^+)]^{n^2}$ and then has a unique fixed point $\tilde{w}$. Since, by (3.3), $(u_{x_i x_j})_{i,j=1,\dots,n}$ is also a fixed point in $[L^q(B_r^+)]^{n^2}$ the uniqueness of fixed point implies

$$u_{x_i x_j} = \tilde{w}_{ij} \in L^p(B_r^+) \quad \forall i, j = 1, \dots, n.$$

Then (3.5) is an easy consequence of formula (3.3), Theorems 2.3, 2.4, 2.5, 2.7, and Lemma 3.1.

4. The Dirichlet problem

In this section we make the following assumptions, and we will refer to them collectively as assumption (A).

$$\left.\begin{array}{l}
\text{Let } \Omega \text{ an open bounded subset of } \mathbb{R}^n,\ n \ge 3, \text{ with } \partial\Omega \in C^{1,1}, \\
\qquad L = \displaystyle\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \\
\text{where} \\
(4.1) \qquad a_{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n) \quad \forall i, j = 1, \dots, n, \\
(4.2) \qquad a_{ij}(x) = a_{ji}(x) \quad \forall i, j = 1, \dots, n, \text{ a.e. in } \Omega, \\
(4.3) \qquad \exists \lambda > 0\colon \lambda^{-1}|\xi|^2 \le \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n.
\end{array}\right\} \quad (A)$$

Furthermore call $\eta_{ij}(r)$ the VMO modulus of a_{ij} $(i, j = 1, \dots, n)$ and set $\eta(r) = (\sum_{i,j=1}^n \eta_{ij}^2(r))^{1/2}$. Finally let Γ, Γ_i, $\Gamma_{ij}(x, t)$ have the same meaning as in the previous section with a_{ij} replacing b_{ij} at any occurrence and set

$$\max_{i,j=1,\dots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x, t)}{\partial t^\alpha} \right\|_{L^\infty(\Omega \times \Sigma)} = M.$$

Theorem 4.1. *Assume* (A). *Let* $q, p \in]1, +\infty[$, $q \le p$, $f \in L^p(\Omega)$, $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, *and* $Lu = f$ *a.e. in* Ω. *Then* $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. *Moreover given* $\Omega' \subset\subset \Omega$, Ω' *open, there exists a constant*

$$c = c(n, p, M, \mathrm{dist}(\Omega', \partial\Omega), \lambda, \eta)$$

such that

$$\|u\|_{W^{2,p}(\Omega')} \le c\{\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\}. \tag{4.4}$$

Proof. The proof, via a covering argument, follows closely the lines of Theorem 3.2 above. A detailed exposition may be found in [CFL, Theorem 4.2].

Theorem 4.2. *Assume* (A). *Let* $q, p \in]1, +\infty[$, $q \leq p$, $f \in L^p(\Omega)$, $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, *and* $Lu = f$ *a.e. in* Ω. *Then* $u \in W^{2,p}(\Omega)$ *and there exists a constant* $c = c(n, p, M, \partial\Omega, \lambda, \eta)$ *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\{\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\}. \tag{4.5}$$

Proof. By a *covering and flattening* argument the conclusion follows in a standard way from Theorems 3.2 and 4.1.

We wish only to make some comments on the effect of the stretching on our class of coefficients. More precisely, the assumption on the boundary implies the existence for all $x_0 \in \partial\Omega$ of a neighborhood $U(x_0)$ and of a $C^{1,1}$-diffeomorphism G which maps $U(x_0) \cap \Omega$ onto B_r^+ (same notation as in the previous section). In the transformed equation the principal part coefficients are

$$b_{ij}(y) = \sum_{h,k=1}^{n} a_{hk}(G^{-1}(y)) \frac{\partial}{\partial x_h} G_i(G^{-1}(y)) \frac{\partial}{\partial x_k} G_j(G^{-1}(y)),$$

$y \in B_r^+$, $i, j = 1, \dots, n$, and G^{-1} is the inverse of G.

We observe that G may be extended to a diffeomorphism of $\mathbb{R}^n$ onto itself with preservation of the norm; we now have that the $b_{ij}(y)$ are defined and bounded in $\mathbb{R}^n$.

It is easy (changing variables) to check that $a_{ij}(G^{-1}(y))$ is in VMO and its VMO modulus is comparable with the VMO modulus of $a_{ij}(x)$ through the $C^{1,1}$-norm of G. Then the b_{ij} are in VMO because of the boundedness and uniform continuity of $\partial G_i(G^{-1}(y))/\partial x_h$, $i, h = 1, \dots, n$, and their VMO moduli are easily estimated in terms of η, the $C^{1,1}$-norm of G and the continuity moduli of the derivatives of G.

Theorem 4.3 (Uniqueness). *Assume* (A). *Then the solution of the Dirichlet problem*

$$\begin{cases} Lu = 0 \quad \text{a.e. in } \Omega, \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \end{cases} \qquad (1 < p < +\infty)$$

is zero in Ω.

Proof. The function 0 belongs to $L^n(\Omega)$. By Theorem 4.2 it follows that $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$; hence, recalling the Pucci-Alexandroff maximum principle, the conclusion follows.

Theorem 4.4 (Existence). *Assume* (A). *Let* $f \in L^p(\Omega)$, $p \in]1, +\infty[$. *Then the Dirichlet problem*

$$\begin{cases} Lu = f \quad \text{a.e. in } \Omega, \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \end{cases}$$

has a (*unique*) *solution* u. *Furthermore there exists a positive constant* $c = c(n, p, M, \partial\Omega, \lambda, \eta)$ *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)}. \tag{4.6}$$

Proof. First we prove (4.6). The existence result will then follow in a standard way, approximating the equation with a similar one with smooth coefficients. In fact it was observed in Theorem 2.1 that the η function of the smoothed coefficients is dominated by the η of the original.

To prove (4.6) we argue by contradiction. If (4.6) is not true, there exists a sequence of operators

$$\left\{L^{(m)}=\sum_{i,j=1}^{n}a_{ij}^{(m)}(x)\frac{\partial^2}{\partial x_i\partial x_j}\right\},$$

verifying assumption (A) with the VMO moduli and the L^∞-norms of $a_{ij}^{(m)}$, $m\in\mathbb{N}$, uniformly bounded by those of a_{ij}, and a sequence of functions $\{u^{(m)}\}$, $u^{(m)}\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, satisfying

$$\|u^{(m)}\|_{W^{2,p}(\Omega)}=1,\qquad \lim_m\|L^{(m)}u^{(m)}\|_{L^p(\Omega)}=0.$$

Fix any ball $B\subseteq\mathbb{R}^n$. By Theorem 2.1(iii) and the remarks following that theorem, the sequence $\{a_{ij}^{(m)}-(a_{ij}^{(m)})_B\}$ is compact in $L^1(B)$ by a well-known compactness result. Then it is possible to find a subsequence of $\{a_{ij}^{(m)}\}$ converging a.e. in B. By considering an increasing sequence of balls with union $\mathbb{R}^n$ it is possible to find a subsequence, which we still call $\{a_{ij}^{(m)}\}$, converging a.e. in $\mathbb{R}^n$ to a function α_{ij}. Clearly the functions α_{ij} verify assumption (A). Set

$$L^{(\alpha)}=\sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}.$$

Furthermore there exists a subsequence of $\{u^{(m)}\}$, which we relabel as $\{u^{(m)}\}$, converging weakly to a function $u^{(\alpha)}\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and then $\|u^{(m)}\|_{L^p(\Omega)}$ converges to $\|u^{(\alpha)}\|_{L^p(\Omega)}$.

Since for $\varphi\in L^{p'}(\Omega)$, $p'=p/(p-1)$, we have

$$\int_\Omega|(L^{(m)}u^{(m)}-L^{(\alpha)}u^{(\alpha)})\varphi|\,dx \le\sum_{i,j=1}^{n}\left\{\|(a_{ij}^{(m)}-\alpha_{ij})\varphi\|_{L^{p'}(\Omega)}+\int_\Omega|(u_{x_ix_j}^{(m)}-u_{x_ix_j}^{(\alpha)})a_{ij}^{(\alpha)}\varphi|\,dx\right\},$$

$\{L^{(m)}u^{(m)}\}$ converges weakly in $L^p(\Omega)$ to $L^{(\alpha)}u^{(\alpha)}$. Hence $L^{(\alpha)}u^{(\alpha)}=0$ a.e. in Ω and by Theorem 4.3 $u^{(\alpha)}=0$. Thus $\|u^{(m)}\|_{L^p(\Omega)}$ converges to zero, which, on account of (4.5), contradicts $\|u^{(m)}\|_{W^{2,p}(\Omega)}=1$.

5. Concluding remarks

After this work was completed we noticed that our proofs could be modified in order to replace the VMO assumption by the smallness of the BMO norm (depending on p). Also let us observe that given a function f with weak-L^n derivatives, its BMO norm is bounded by the weak-L^n norm of the gradient.

These remarks suggest that the sharp result of Alvino and Trombetti [AT], dealing with an existence and uniqueness result for $p=2$, could be extended to cover some neighborhood of $p=2$.

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REFERENCES

[AT] A. Alvino and G. Trombetti, *Second order elliptic equations whose coefficients have their first derivatives weakly-L^n*, Ann. Mat. Pura Appl. (4) **138** (1984), 331–340.

[Cf] L. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. of Math. **130** (1989), 189–213.

[Cl] A. P. Calderon, *Integrales singulares y sus aplicaciones a ecuaciones diferenciales hyperbolicas*, Cursos y Seminarios de Matematicas 3, Univ. de Buenos Aires, 1960.

[Cn] A. Canfora, *Extension of an integral inequality of C. Miranda and applications to the elliptic equations with discontinuous coefficients*, Czechoslovak Math. J. **39** (1989), 385–421.

[CZ] A. Canfora and P. Zecca, *Su di una equazione ellittica in $\mathbb{R}^n$ a coefficienti discontinui e non derivabili*, Ann. Mat. Pura Appl. (4) **141** (1985), 369–384.

[CFL] F. Chiarenza, M. Frasca, and P. Longo, *Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. (to appear).

[CRW] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976).

[GR] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Stud., vol. 116, North-Holland, Amsterdam, 1985.

[G] J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.

[Gr] D. Greco, *Nuove formole integrali di maggiorazione per le soluzioni di un'equazione lineare di tipo ellittico ed applicazioni alla teoria del potenziale*, Ricerche Mat. **5** (1956), 126–149.

[K] A. I. Koshelev, *On boundedness in L^p of solutions of elliptic differential equations*, Mat. Sb. **38** (1956), 359–372.

[M1] C. Miranda, *Sulle equazioni ellittiche del secondo ordine a coefficienti discontinui*, Ann. Mat. Pura Appl. **63** (1963).

[M2] ——, *Partial differential equations of elliptic type*, Springer-Verlag, Berlin, 1970.

[S] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405.

[T] A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, New York, 1986.

[Z1] P. Zecca, *Teoremi di esistenza e di unicita' di soluzioni periodiche per una equazione ellittica a coefficienti discontinui e non derivabili*, Ricerche Mat. **35** (1986), 61–74.

[Z2] ——, *Un nuovo teorema di esistenza per un'equazione ellittica a coefficienti discontinui e non derivabili*, Ricerche Mat. **35** (1986), 215–229.

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