SUPER EFFICIENCY IN VECTOR OPTIMIZATION

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ABSTRACT. We introduce a new concept of efficiency in vector optimization. This concept, super efficiency, is shown to have many desirable properties. In particular, we show that in reasonable settings the super efficient points of a set are norm-dense in the efficient frontier. We also provide a Chebyshev characterization of super efficient points for nonconvex sets and a scalarization theory when the underlying set is convex.

0. INTRODUCTION

Decision-making problems appearing in economics, management science and operations research require frequently that decision making be based on optimizing several criteria. Vector optimization has provided an organized constructive approach to these problems. Throughout this note, we consider minimization problems. *Efficient decisions* are those decisions not minorized by any others.

As observed by Kuhn and Tucker and later Geoffrion, a subset of an efficient decision set may not be satisfactorily characterized by a scalar minimization problem, so the concept of *proper efficiency* was introduced by Kuhn-Tucker, Geoffrion, and modified and formulated in a more general framework by Borwein, Benson, Henig, and Hartley among many other authors [Kuhn 1], [Benson 1], [Borwein 1, 2, 4, 6], [Geoffrion 1], [Henig 1], [Hartley 1] and the references therein. The motivation for introducing proper efficiency is that it enables one to eliminate certain anomalous efficient decisions and to prove the existence of equivalent scalar problems whose solutions produce at least most of the efficient decisions, namely the proper ones. It has been amply demonstrated that proper efficiency is a natural concept in vector optimization.

In this note, we introduce a new kind of proper efficiency, namely *super efficiency*. Super efficiency refines the notions of efficiency and other kinds of proper efficiency, and provides a concise (and equivalent) scalar characterization and duality results when the underlying decision problem is convex. We also

study density results for super efficiency and a Chebyshev scalar characterization of super efficiency.

1. PRELIMINARIES

For the sake of simplicity, we make the following assumptions (unless specifically stated otherwise). Throughout the paper, X will always be an ordered real normed linear space and a subset C of X is always assumed to be nonempty. The partial ordering cone S of X is always assumed to be closed, convex ($S + S \subset S$) and pointed, that is $S \cap -S = \{0\}$.

We say that the ordering cone S of X is *generating* if $X = S - S$. We associate a *dual cone* with S, denoted by S^+, in X^* (the norm dual of X),

$$S^+ := \{\phi \in X^* | \phi(s) \geq 0, \ \forall s \in S\}.$$

Then S^+ is a convex cone which is closed in $\sigma(X^*, X)$, the weak-star topology, and S^+ is pointed if S is generating.

Recall that an ordering cone is *normal* if $(B-S) \cap (S-B)$ is bounded, and so is a basic neighborhood. Here B is the closed unit ball of the space S. *Order intervals* are defined by

$$[a, b]_s := \{x | a \leq_S x \leq_S b\}.$$

Also S^+ is a *strict b-cone* if for some $\varepsilon > 0$,

$$\phi \in X^*, \qquad \|\phi\| \leq \varepsilon \Rightarrow \phi = \lambda_1 - \lambda_2, \qquad \lambda_i \in S^+ \text{ and } \|\lambda_i\| \leq 1, \ i = 1, 2.$$

In a Banach space, S is normal if and only if S^+ is generating and if and only if S^+ is a strict b-cone [Peressini 1].

Recall that a *base* of a cone S is a convex subset Θ of S such that

$$S = \bigcup\{\lambda\theta | \lambda \geq 0 \text{ and } \theta \in \Theta\} \quad \text{and} \quad 0 \notin \operatorname{cl}(\Theta).$$

Of course, S is pointed whenever S has a base.

We say the ordering cone S has the *Daniell property* if the infimum of each monotone decreasing net in S exists and is also the topological limit of the net. In [Borwein 3] it is proved that S is Daniell if
  (i) S has weakly compact intervals; or
  (ii) if S has a closed bounded complete base.

A *vector lattice* is an ordered vector space in which $\max\{a, b\}$ always exists. A *normed lattice* is a normed vector lattice in which the norm is *absolute*: $\|x\| = \| |x| \|$ and *monotone*: $0 \leq_S x \leq_S y$ implies that $\|x\| \leq \|y\|$. A *Banach lattice* is a Riesz lattice over a Banach space.

Recall also that the *recession cone* of a set C, denoted by $O^+(C)$, is the set of all x such that

$$c + \lambda x \in C \quad \text{for all } \lambda > 0 \text{ and all } c \in C.$$

The *generalized (weak) recession cone* of a set C, denoted by $R(C)$ $[R_\sigma(C)]$, is the potentially larger set defined by

$$R(C) := \{\text{norm limits of } \lambda_n c_n | \lambda_n \to 0 \text{ and } c_n \in C\}$$
$$[R_\sigma(C) := \{\text{weak limits of } \lambda_n c_n | \lambda_n \to 0 \text{ and } c_n \in C\}].$$

The generalized (weak) recession cone of C is the recession cone of C when C is closed and convex. Note that when C is bounded, $R_\sigma(C) = \{0\}$.

We shall frequently use the following notations:

(1) cone(A) denotes the cone generated by the set A, i.e.

$$\text{cone}(A) := \bigcup \{tA | t \geq 0\};$$

while cl[cone(A)] denotes the closure of cone(A).

(2) S^{+i} denotes the set of all strictly positive linear functionals in S^+, that is

$$S^{+i} := \{f \in X^* | f(s) > 0, \text{ for all } s \text{ in } S \text{ and } s \neq 0\}.$$

It follows directly from the Hahn-Banach theorem that S^{+i} is nonempty exactly when S has a base.

(3) We use int(C) to denote the interior of the set C and C^0 to denote the *polar* of C,

$$C^0 := \{f \in X^* | f(c) \leq 1, \text{ for all } c \text{ in } C\}.$$

(4) We denote the set of *efficient points* of C with respect to the partial ordering cone S by $E(C, S)$. Thus, $x \in E(C, S)$ if $(C - x) \cap -S = \{0\}$.

(5) $PE(C, S)$ denotes the set of all *proper efficient* points of C in the sense of Borwein [Borwein 1]. That is, $x \in PE(C, S)$ if

$$\text{cl}[\text{cone}(C - x)] \cap -S = \{0\}.$$

(6) Let S be an ordering cone in a normed linear space X with a base Θ, we can associate S with another convex cone $S_\varepsilon(\Theta)$ defined by

$$S_\varepsilon(\Theta) := \text{cl}[\text{cone}(\Theta + \varepsilon B)].$$

Here B is the closed unit ball of X. We say $x \in C$ is *Henig efficient*, denoted by $x \in HE(C, S)$, [more properly, $x \in HE(C, \Theta)$] if

$$\text{cl}[\text{cone}(C - x)] \cap -S_\varepsilon(\Theta) = \{0\}.$$

$S_\varepsilon(\Theta)$, called the *Henig Dilating Cone*, plays an important role in our development. Some properties of $S_\varepsilon(\Theta)$ are summarized in the following theorem.

Theorem 1.1. *Let X be a normed linear space, $S \subset X$ a closed and convex cone with a closed base Θ. Let $\delta := \inf\{\|\theta\| \,|\, \theta \in \Theta\}$, then $\delta > 0$. Define*

$$S_\varepsilon := S_\varepsilon(\Theta) := \text{cl}[\text{cone}(\Theta + \varepsilon B)].$$

Then for any $\beta > 0$, $S_\varepsilon \subset S_{\varepsilon+\beta}$. For $0 \leq \varepsilon < \delta$ we have

(1) $S_\varepsilon(\Theta) = \text{cone}[\text{cl}(\Theta + \varepsilon B)]$;

(2) $S_\varepsilon(\Theta)$ *is pointed*;

(3) $\bigcap\{S_\varepsilon(\Theta) | \varepsilon > 0\} = S$;

(4) *S is normal if and only if $S_\varepsilon(\Theta)$ is normal*;

(5) *If, in addition, X is complete and Θ is bounded then $S_\varepsilon(\Theta)$ is Daniell and normal*;

(6) *If Θ is weakly compact and X is reflexive then $S_\varepsilon(\Theta)$ is locally weakly compact*;

(7) *If X is reflexive and S is normal, then $S_\varepsilon(\Theta)$ has weakly compact intervals and so is Daniell.*

Proof. (1) Let

$$\Theta_\varepsilon := \text{cl}(\Theta + \varepsilon B).$$

Since $S_\varepsilon(\Theta)$ is closed, $\mathrm{cl}[\mathrm{cone}(\Theta_\varepsilon)] = S_\varepsilon(\Theta)$. Let d be in $\mathrm{cl}[\mathrm{cone}(\Theta_\varepsilon)]$. Then since $0 \notin \Theta_\varepsilon$ for $\varepsilon < \delta$,

$$(1.1) \qquad\qquad \mathrm{cl}[\mathrm{cone}(\Theta_\varepsilon)] = \mathrm{cone}(\Theta_\varepsilon) \cup R(\Theta_\varepsilon)$$

where $R(\Theta_\varepsilon) := \{\lim t_\alpha \theta_\alpha^\varepsilon | t_\alpha \to 0,\ t_\alpha \geq 0,\ \theta_\alpha^\varepsilon \in \Theta_\varepsilon\}$. Indeed, x in $\mathrm{cl}[\mathrm{cone}(\Theta_\varepsilon)]$ implies that x is either in $\mathrm{cone}(\Theta_\varepsilon)$ or

$$x = \lim t_\alpha \theta_\alpha^\varepsilon \quad \text{for } t_\alpha \geq 0 \text{ and } \theta_\alpha^\varepsilon \in \Theta_\varepsilon.$$

Without loss of generality (taking subnets if necessary), we may assume that t_α converges. If t_α tends to infinity, then t_α^{-1} tends to zero, which implies that $\theta_\alpha^\varepsilon$ tends to zero. This is impossible as 0 is not in the closure of Θ. Hence, t_α converges to t for some $t < \infty$. If $t = 0$ then x is in $R(\Theta_\varepsilon)$ by definition. If $t \neq 0$ then as $\Theta_\varepsilon := \mathrm{cl}(\Theta + \varepsilon B)$,

$$\theta_\alpha^\varepsilon \to t^{-1} x \in \Theta_\varepsilon.$$

Hence, $x \in \mathrm{cone}(\Theta_\varepsilon)$. Therefore, (1.1) is verified. Note that it is easy to check that

$$R(\Theta_\varepsilon) = R(\Theta) \subset S \subset \mathrm{cone}(\Theta_\varepsilon)$$

because Θ is a base for S and S is closed. Whence $\mathrm{cone}(\Theta_\varepsilon) = S_\varepsilon(\Theta)$.

(2) By (1), $S_\varepsilon(\Theta) = \mathrm{cone}(\Theta_\varepsilon)$ has a closed base Θ_ε, hence $S_\varepsilon(\Theta)$ is pointed.

(3) It is clear that $\bigcap_{\varepsilon > 0} S_\varepsilon \supset S$. Let d be in $\bigcap_{\varepsilon > 0} S_\varepsilon$. We may assume that

$$d = \lim t_n(\theta_n + (1/n)b_n)$$

with $\theta_n \in \Theta$, $b_n \in B$ and $t_n > 0$. Then

$$\lim \|t_n^{-1} d - \Theta_n\| = \lim(1/n)\|b_n\| = 0.$$

Since $0 \in \Theta$, t_n is bounded. Assume that $t_{n'} \to \mu < \infty$. Note that since Θ is closed, we have $d = \mu\theta \in S$ for some $\theta \in \Theta$. Hence d is in S.

(4) By [Peressini 1, p. 64; Borwein 6] the normality of S is equivalent to the uniform boundedness of order intervals. Suppose that S is normal, then order intervals of S are uniformly norm-bounded. Let $0 \leq_{S_\varepsilon} x \leq_{S_\varepsilon} a$ be given. Since

$$x \in S_\varepsilon = \mathrm{cone}(\Theta_\varepsilon) = \mathrm{cone}[\mathrm{cl}(\Theta + \varepsilon B)] \subset \mathrm{cone}(\Theta + \varepsilon' B)$$

for any $\delta > \varepsilon' > \varepsilon$, we may assume that

$$x = t_1(\theta_1 + \varepsilon' b_1); \qquad a - x = t_2(\theta_2 + \varepsilon' b_2)$$

with $\theta_i \in \Theta$, $b_i \in B$, $t_i > 0$ $(i = 1, 2)$. Thus

$$(1.2) \qquad\qquad a = (t_1 + t_2)(\theta + \varepsilon' b)$$

where $\theta = (t_1\theta_1 + t_2\theta_2)/(t_1 + t_2)$ and $b = (t_1 b_1 + t_2 b_2)/(t_1 + t_2)$. Define $s_i := t_i\theta_i$ for $i = 1, 2$. Then

$$s_i \leq_S s_1 + s_2 = (t_1 + t_2)\theta.$$

By the normality of S, there exists a $k > 0$, independent of x, such that

$$\|s_i\| \leq k\|(t_1 + t_2)\theta\|$$

and so

$$\begin{aligned}
\|x\| &\leq k\|(t_1 + t_2)\theta\| + \varepsilon'(t_1 + t_2) \\
&\leq k\|(t_1 + t_2)\theta + (t_1 + t_2)\varepsilon' b\| + k\|(t_1 + t_2)\varepsilon' b\| + \varepsilon'(t_1 + t_2) \\
&\leq k\|a\| + k(t_1 + t_2)\varepsilon' + \varepsilon'(t_1 + t_2) \leq k\|a\| + \varepsilon'(1 + k)(t_1 + t_2).
\end{aligned}$$

Since $\Theta_{\varepsilon'}$ is a base, the coefficient $t_1 + t_2$ in (1.2) is uniformly bounded, say by $N\|a\|$ independent of x. Thus

$$M(a) := k\|a\| + \varepsilon'(1+k)N\|a\|$$

is a constant independent of the choice of x. Hence, $[0, a]_{S_\varepsilon}$ is uniformly bounded in norm. Therefore S_ε is normal. Conversely, if S_ε is normal for some $\varepsilon > 0$, then as $S_\varepsilon \supset S$, S is normal.

(5) If Θ is bounded, then $S_\varepsilon = \text{cone}[\text{cl}(\Theta + \varepsilon B)]$ has a closed and bounded base. Since X is complete, S_ε is Daniell [Borwein 3]; S_ε is also normal [Borwein 3].

(6) When Θ is weakly compact and X is reflexive, $\Theta + \varepsilon B$ is weakly compact. Hence S_ε is locally weakly compact by a theorem of Klee [Klee 1].

(7) By (4), S being normal implies that S_ε is normal for $\varepsilon < \delta$. Thus order intervals of S_ε are norm-bounded. They are also closed (because S_ε is closed) and convex. Hence, intervals are weakly compact as X is reflexive. $\square$

2. Super efficiency

In this section, we introduce a new efficiency concept, namely *super efficiency*. After defining the concept in a general setting we provide equivalent formulations of super efficiency for several important special cases. Existence results are then presented.

Definition 2.1. Let X be a real normed linear space. We say that x is a *super efficient point* of a nonempty subset C of X with respect to the ordering cone S, written $x \in SE(C, S)$, if there is a real number $M > 0$ such that

$$(2.1) \qquad K \cap (B - S) \subset MB$$

where $K := \text{cl}[\text{cone}(C - x)]$ and B is the closed unit ball of X.

Since X is a normed linear space, (2.1) is unchanged if K is replaced by $\text{cone}(C - x)$. Thus, (2.1) can be expressed explicitly in terms of the norm on X:

for any c in C and $t > 0$, whenever $t(c - x) \leq_S b$ for some $b \in B$,
$$\|t(c - x)\| \leq M.$$

This is equivalent to saying that for each c in C, if y is in X and $c - x \leq_S y$, then

$$(2.2) \qquad \|c - x\| \leq M\|y\|$$

with a uniform constant M (depending only on x, not on y or c).

Note that when $y = 0$, (2.2) becomes

$$c - x \leq_S 0 \Rightarrow \|c - x\| = 0, \quad \text{for all } c \in C,$$

which implies that super efficient points are efficient, i.e.

$$SE(C, S) \subset E(C, S).$$

The relationships between super efficiency and other efficiencies will be explored in detail in the following sections.

Super efficiency has a very simple description in a normed lattice. In this setting, $x \in SE(C, S)$ is equivalent to the existence of some M with

$$(2.3) \qquad \|c - x\| \le M \|(c - x)^+\|$$

for each $c \in C$, where $(c - x)^+$ is the positive part of $c - x$, see [Jameson 1].

To see this, we put $y := (c - x)^+$ in (2.2) and have (2.3) immediately. On the other hand, if $c - x \le_S y$ for y in X, then $(c - x)^+ \le_S y^+$. Hence

$$\|c - x\| \le M \|(c - x)^+\| \le M \|(y)^+\| \quad (\text{as } \| \cdot \| \text{ is monotone})$$
$$\le M \| |y| \| = M \|y\| \quad (\text{as } \| \cdot \| \text{ is absolute})$$

which is exactly (2.3).

When the set C is convex, our definition of super efficiency has a concise dual form. Indeed, we prove in the following proposition that (2.1) is equivalent to

$$(2.4) \qquad X^* = K^+ - S^+ = (C - x)^+ - S^+.$$

Proposition 2.2. *Let X be a Banach space, $C \subset X$ a convex subset and S a closed cone (S as always is convex). Then (2.1) is equivalent to (2.4).*

Proof. When C is convex, $K := \text{cl}[\text{cone}(C - x)]$ is also convex. Hence

$$H(x) := \begin{cases} x + S, & x \in K, \\ \varnothing, & x \notin K, \end{cases}$$

is a closed convex process. It follows from direct calculations that for any y in X, $H^{-1}(y) = K \cap (y - S)$ and the *adjoint process* [Borwein 5] is

$$H^*(y^*) = \begin{cases} y^* - K^+, & y^* \in S^+, \\ \varnothing, & y^* \notin S^+. \end{cases}$$

By definition, x is in $SE(C, S)$ if and only if H^{-1} is bounded which is equivalent, by the dual version of Corollary 2.1 in [Borwein 5] (as H is closed), to H^* being open at 0, which in turn is equivalent to H^* being surjective:

$$H^*(K^+) = K^+ - S^+ = X^*.$$

This is (2.4). □

The following corollary gives a characterization of super efficiency by scalar optimization.

Corollary 2.3. *Let S be a normal cone in a normed space X. For any convex set C of X, x_0 is in $SE(C, S)$ if and only if for any ψ in S^+ (or in X^*), one can find ϕ in S^+ such that $\phi \ge_{S^+} \psi$ and*

$$(2.5) \qquad \inf_{x \in C} \phi(x) = \phi(x_0).$$

Proof. Note that S is normal in a normed space if and only if S^+ is generating [Peressini 1, pp. 72 and 76] while (2.5) is equivalent to $S^+ \subset (C - x_0)^+ - S^+$. (Moreover, (2.5) implies that $X^* = S^+ - S^+$.) □

Now we present our first existence result for super efficiency.

Proposition 2.4. *Let X be a normed space, $S \subset X$ an ordering cone with a bounded base and C a weakly compact subset of X. Then $SE(C, S) \neq \varnothing$.*

Proof. Since S has a bounded base Θ,

$$\text{norm-int}(S^+) \neq \varnothing.$$

[Jameson 1, p. 122]. Let ψ be in $\text{int}(S^+)$, then $\psi + \varepsilon B^* \subset S^+$ for some $\varepsilon > 0$, where B^* is the closed unit ball in X^*. Thus for any $\lambda \in X^*$, we can always find $n > 0$ so that $\lambda \leq_{S^+} n\psi$. Moreover, since C is weakly compact, there is x_0 in C such that

$$\min_{x \in C} \psi(x) = \psi(x_0).$$

In other words, for all c in C we always have $\psi(c - x_0) \geq 0$. Note also that since S has a bounded base, it is normal [Borwein 3]. The fact that x_0 is in $SE(C, S)$ now follows from Corollary 2.3. □

The following proposition and its corollary describes super efficiency in terms of the Henig dilating cone S_ε.

Proposition 2.5. *Let X be a normed linear space, $S \subset X$ a closed ordering cone, $C \subset X$ a nonempty subset and $x_0 \in C$. Define $K := \text{cl}[\text{cone}(C - x_0)]$ as before. Suppose that S has a bounded base Θ and let $\delta := \inf\{\|\theta\| \,|\, \theta \in \Theta\}$. Suppose also that for some $0 < \varepsilon < \delta$,*

$$(2.6) \qquad\qquad K \cap -S_\varepsilon = \{0\}.$$

Then $x_0 \in SE(C, S)$.

Proof. Suppose that x_0 is not in $SE(C, S)$. Then we can select a sequence $\{y_n\}$ in $K \cap (B - S)$ with $\|y_n\|$ approaching to infinity, where B is again the closed unit ball of X. Since y_n is in $B - S$ and S has a base Θ, y_n can be written as

$$y_n = b_n - \lambda_n \theta_n \quad \text{and} \quad y_n \in K$$

with $\theta_n \in \Theta$, $\|b_n\| \leq 1$, and $\lambda_n > 0$, $n = 1, 2, \ldots$.

Since Θ is bounded, we may assume that $\|\theta_n\| \leq M$, $n = 1, 2, \ldots$. Now

$$\|y_n\| - 1 \leq \|y_n - b_n\| \leq \lambda_n M.$$

Hence λ_n tends to infinity. Therefore,

$$0 \neq \lambda_n^{-1} y_n = \lambda_n^{-1} b_n - \theta_n \in K \cap (\varepsilon B - \Theta) \subset K \cap -S_\varepsilon$$

for n sufficiently large, which contradicts (2.6). □

Corollary 2.6. *Let X be a normed space and suppose the ordering cone S have a closed base. Let $\delta := \inf\{\|\theta\| \,|\, \theta \in \Theta\}$ and let $S_\varepsilon := S_\varepsilon(\Theta)$. Then for any set C in X the following are equivalent:*
   (1) $x_0 \in E(C, S_\varepsilon)$ for some $0 < \varepsilon < \delta$;
   (2) $x_0 \in PE(C, S_\varepsilon)$ for some $0 < \varepsilon < \delta$.
If, in addition, Θ is a bounded base, then the above are equivalent to
   (3) $x_0 \in SE(C, S)$.
Proof. From the definitions

$$PE(C, S_\varepsilon) \subset E(C, S_\varepsilon)$$

is clear for any $\varepsilon > 0$. On the other hand, x_0 is in $E(C, S_\varepsilon)$ if and only if

$$\text{cone}(C - x_0) \cap -S_\varepsilon = \{0\}.$$

By Theorem 1.1(1), for $0 < \varepsilon < \delta$, $\text{cl}(\varepsilon B - \Theta)$ is a base and $-S_\varepsilon = \text{cone}[\text{cl}(\varepsilon B - \Theta)]$. Thus, for any $0 < \varepsilon' < \varepsilon$, $\text{cone}(C - x_0) \cap (\varepsilon' B - \Theta) = \varnothing$. This implies that

$$\text{cone}(C - x_0) \cap [(2^{-1}\varepsilon' B - \Theta) + 2^{-1}\varepsilon' \, \text{int}(B)] = \varnothing.$$

Noting that the second set is open, we have,

$$\text{cl}[\text{cone}(C - x_0)] \cap \text{cl}(2^{-1}\varepsilon' B - \Theta) = \varnothing.$$

Therefore, by Theorem 1.1(1) again,

$$\text{cl}[\text{cone}(C - x_0)] \cap \text{cone}[\text{cl}(2^{-1}\varepsilon' B - \Theta)] = \text{cl}[\text{cone}(C - x_0)] \cap -S_{\varepsilon''} = \{0\}$$

for $\varepsilon'' := 2^{-1}\varepsilon'$. This proves that x_0 is in $PE(C, S_{\varepsilon''})$. Hence (1) implies (2).

Now let us assume that S has a bounded base. Then (2) $\Rightarrow$ (3) follows from Proposition 2.5. Conversely, suppose x_0 is in $SE(C, S)$ while $K \cap -S_\varepsilon = \{0\}$ fails for all $\varepsilon > 0$. Then for $n = 1, 2, \ldots$,

$$K \cap (1/nB - \Theta) \neq \varnothing.$$

Take k_n in $K \cap ((1/n)B - \Theta)$. Note that for n sufficiently large,

$$\|k_n\| \geq \delta - 1/n > 0.$$

Now nk_n is in $K \cap (B - S)$ with $\|nk_n\| \to \infty$. This contradicts the assumption that x_0 is in $SE(C, S)$. $\square$

Our next theorem says that every bounded closed set in a Banach space has super efficient points provided the ordering cone has a bounded base.

Theorem 2.7. *Let X be a Banach space and let S have a closed bounded base Θ. Then every bounded closed set C possesses super efficient points.*

Proof. Let $\delta := \inf\{\|\theta\| \mid \theta \in \Theta\}$. Let $S_\varepsilon(\Theta)$ be the Henig cone for $0 < \varepsilon < \delta$. Then S_ε has a closed bounded base and $\text{int}(S_\varepsilon)$ is nonempty. Thus S_ε is Daniell [by Theorem 1.1(5)] and C is bounded below in S_ε. By Theorem 2.1 in [Borwein 4], $E(C, S_\varepsilon)$ is nonempty. Hence, $SE(C, S)$ is nonempty by Corollary 2.6. $\square$

Example 2.8. (a) Let $X = l_1(\mathbb{N})$ and $S = l_{1+}(\mathbb{N})$ (the natural ordering cone). Then S has a bounded base. Let C be the closed unit ball of X, i.e.,

$$C = \{x \in l_1(\mathbb{N}) \mid \|x\|_1 \leq 1\}.$$

Let $\phi = (1, 1, \ldots)$ be in $l_\infty(\mathbb{N}) = [l_1(\mathbb{N})]^*$. Then

$$\inf\{\phi(x) \mid x \in C\} = -1.$$

Note that ϕ attains its minima on $T := \{x \in C \mid \|x\| = 1 \text{ and } x \leq_s 0\}$. Note also that for any λ in $l_\infty(\mathbb{N})$ we can always find an integer $n > 0$ so that $\lambda \leq_{S^+} n\phi$. Thus $SE(C, S)$ contains T by Corollary 2.3. On the other hand, if x is in $C \backslash T$, then x is not even efficient. Thus, $SE(C, S) = T$.

   (b) If we let

$$C_1 = \left\{ x \in l_1(\mathbb{N}) \mid \sum x_n / 2^n = 0 \right\}.$$

Then $SE(C_1, S) = \varnothing$ while $PE(C_1, S) = C_1$. Indeed,

$$C_1 - x = C_1 = \mathrm{cl}[\mathrm{cone}(C_1 - x)] =: K$$

for all x in C_1 (since C_1 is a closed subspace) and $K \cap -S = \{0\}$. Hence $PE(C_1, S) = C_1$. But, since $(1/2^n) \notin \mathrm{int}(S^+)$,

$$C_1^+ - S^+ = R(2^{-n}) - S^+ \neq l_\infty(\mathbb{N}),$$

where, $(C_1)^+ = R(2^{-n})$ is the set of all scalar multiples of (2^{-n}) as C_1 is the kernel of (2^{-n}). In particular, $(1, 1, 1, \dots)$ is not in $(C_1)^+ - S^+$. Therefore, $SE(C_1, S) = \varnothing$ by Proposition 2.2. $\square$

Note that C is bounded while C_1 is not. So the example shows that the boundedness of C is needed in Theorem 2.7. The following proposition gives a scalar characterization of super efficient points in an important case.

Proposition 2.9. *Let X be a normed space. If S has a closed bounded base Θ and C is convex then x is in $SE(C, S)$ if and only if there is ϕ in the norm-interior of S^+ such that $\phi(C - x) \geq 0$.*

Proof. If x is in $SE(C, S)$ then

$$(C - x)^+ - S^+ = X^*$$

(by Proposition 2.2). Since Θ has a bounded base, $\mathrm{int}(S^+) \neq \varnothing$ [Jameson 1]. Therefore,

$$(C - x)^+ - \mathrm{int}(S^+) = X^*,$$

since $(C - x)^+ - \mathrm{int}(S^+)$ is a convex set with nonempty interior whose closure is X^*. (It then must be equal to X^* otherwise, it is contained in a closed half-space.) This implies that

$$0 \in (C - x)^+ - \mathrm{int}(S^+).$$

So there is ϕ in X^* such that

$$\phi \in (C - x)^+ \cap \mathrm{int}(S^+).$$

Conversely,

$$\phi \in (C - x)^+ \cap \mathrm{int}(S^+)$$

implies that

$$0 \in (C - x)^+ - \mathrm{int}(S^+) \subset \mathrm{int}[(C - x)^+ - S^+],$$

which gives $X^* = K^+ - S^+$ and hence x is in $SE(C, S)$. $\square$

3. SUPER EFFICIENCY VIA OTHER EFFICIENCIES

In this section, we shall consider in detail the relationships between super efficiency and other efficiencies such as Borwein efficiency, Henig efficiency, and Hartley efficiency. By exploring such relationships, we show that super efficiency is indeed a central kind of proper efficiency.

Proposition 3.1. *Let X be a normed space, S an ordering cone and C a subset of X. Then,*

(3.1) $$SE(C, S) \subset PE(C, S).$$

Proof. Let x be in $SE(C, S)$. For each d in $\mathrm{cl}[\mathrm{cone}(C-x)] \cap -S$, we assume that $d = \lim d_n$ with d_n in $\mathrm{cone}(C - x)$. Let $d_n = t_n(c_n - x)$ with c_n in C, $t_n > 0$. For all n, we have

$$d_n \in -S + \|d_n - d\| B.$$

Thus for some b_n in B,

$$c_n - x \leq_S t_n^{-1} \|d_n - d\| b_n.$$

As x is in $SE(C, S)$, there exists $M > 0$ with $\|c_n - x\| \leq M t_n^{-1} \|d_n - d\|$, or

$$\|d_n\| \leq M \|d_n - d\|,$$

which implies that $\|d\| \leq 0$. Therefore

$$\mathrm{cl}[\mathrm{cone}(C - x)] \cap -S = \{0\}. \quad \square$$

Our next proposition indicates the important fact that in a finite dimensional space, super efficiency is the same as proper efficiency.

Proposition 3.2. *Let X be a normed linear space with S a closed and convex ordering cone. Let C be a closed subset of X and lie in a finite dimensional subspace of X. Then*

$$(3.2) \qquad\qquad SE(C, S) = PE(C, S).$$

Proof. It suffices to prove that

$$PE(C, S) \subset SE(C, S).$$

Let x be in $PE(C, S)$ which is contained in $E(C, S)$. If x is not in $SE(C, S)$, then for $n = 1, 2, \ldots$ we can select c_n in C, h_n in X with $c_n \leq_s x + h_n$ and

$$(3.3) \qquad\qquad \|c_n - x\| > 2^n \|h_n\|.$$

If $\|h_n\| = 0$, then $c_n \leq_S x$. This violates the fact that x is in $E(C, S)$. So $\|h_n\| \neq 0$. Let $d_n := (c_n - x)/\|c_n - x\|$, then we have

$$(3.4) \qquad\qquad d_n \leq_S h_n/\|c_n - x\|.$$

As $\{d_n\}$ is in a finite dimensional space and is bounded, we may assume (by extracting a subsequence if necessary) that $d_n \to d_0$. Then

$$d_0 \in \mathrm{cl}[\mathrm{cone}(C - x)] = K,$$

and $d_0 = \lim d_n \leq_S \lim h_n/\|c_n - x\| = 0$ by (3.4) and (3.3). Since $\|d_n\| = 1$ for all n, $d_0 \neq 0$. This contradicts our assumption that $x \in PE(C, S)$. $\quad \square$

The next two propositions show that every super efficient point is a Henig efficient point; when the base of the cone is bounded, the two efficiencies are the same.

Proposition 3.3. *Let X be a normed linear space with an ordering cone S. If S has a closed base Θ, then*

$$(3.5) \qquad\qquad SE(C, S) \subset HE(C, \Theta).$$

Proof. Let $\delta := \inf\{\|\theta\| \,|\, \theta \in \Theta\}$. Then $\delta > 0$. Suppose x is in $SE(C, S)$, then there exists a constant $M > 0$ such that for each c in C,

$$c - x \leq_s y \Rightarrow \|c - x\| \leq M \|y\| \quad \text{for all } y \text{ in } X.$$

Let ε be chosen so that $\varepsilon < \delta/(1 + M)$. Suppose there is z such that

$$z \in \text{cone}(C - x) \cap (\varepsilon B - \Theta),$$

it then follows that

$$z = \lambda(c - x) = \varepsilon b - \theta \leq_S \varepsilon b$$

for b in B and θ in Θ and $\lambda > 0$. Thus, $\lambda^{-1}\|z\| \leq \lambda^{-1} M \varepsilon$ because x is in $SE(C, S)$. Hence,

$$\lambda^{-1}(\delta - \varepsilon) \leq \lambda^{-1}\|\theta - \varepsilon b\| = \lambda^{-1}\|z\| \leq \lambda^{-1} M \varepsilon,$$

and $\delta/(M + 1) \leq \varepsilon$. This is a contradiction. Therefore,

$$\text{cone}(C - x) \cap (\varepsilon B - \Theta) = \varnothing.$$

This implies that

$$\text{cone}(C - x) \cap \text{cone}(\varepsilon B - \Theta) = \{0\}.$$

Thus, as $\text{cone}[\text{cl}(\varepsilon' B - \Theta)] \subset \text{cone}(\varepsilon B - \Theta)$ for some $0 < \varepsilon' < \varepsilon$, one has

$$\text{cone}(C - x) \cap -S_{\varepsilon'} = \{0\}.$$

Thus $x \in E(C, S_{\varepsilon'}) \subset HE(C, \Theta)$. □

Proposition 3.4. *If X is a normed linear space and $S \subset X$ is an ordering cone with a norm-bounded base Θ, then*

$$HE(C, \Theta) = SE(C, S).$$

Proof. If $x \in HE(C, \Theta)$, then $x \in E(C, S_\varepsilon)$ for small $\varepsilon > 0$. Thus

$$(3.6) \qquad\qquad \text{cone}(C - x) \cap (\varepsilon B - \Theta) = \varnothing.$$

Take any y in X with $c - x \leq_S y$, then we may assume that

$$c - x = y - \lambda \theta$$

for some θ in Θ and $\lambda \geq 0$. If $\lambda = 0$, then $\|c - x\| = \|y\|$. If $\lambda > 0$,

$$\lambda^{-1}(c - x) = (\lambda^{-1} y - \theta).$$

Now by (3.6) $\|\lambda^{-1} y\| \geq \varepsilon$, or

$$(3.7) \qquad\qquad \lambda/\|y\| \leq \varepsilon^{-1}.$$

If we define $m := \sup\{\|\theta\| \mid \theta \in \Theta\}$ (m is finite as Θ is bounded) then

$$\begin{aligned}
\|c - x\| &\leq \|y\| + \lambda \sup\|\theta\| \\
&\leq \|y\| + \lambda m = \|y\|(1 + \lambda m/\|y\|) \\
&\leq \|y\|(1 + \varepsilon^{-1} m) = M\|y\|, \qquad \text{[by (3.7)]}
\end{aligned}$$

where $M := 1 + \varepsilon^{-1} m$. Therefore we have $\|c - x\| \leq M\|y\|$ whenever $c - x \leq_S y$. In other words, x is in $SE(C, S)$. □

Let us recall Hartley's notion of proper efficiency which is extended in [Borwein 2]: a point x of a set C in a normed space X with an ordering cone S is a *Hartley efficient point*, denoted by $x \in HPE(C, S)$, if there exists a constant $M > 0$ such that whenever there is $\phi \in S^+$ with $\phi(c - x) < 0$ for some c in C one can find $\psi \in S^+$ with

$$\phi(c - x)/\|\phi\| \geq -M \psi(c - x)/\|\psi\|.$$

In our next two propositions, we compare Hartley efficiency with super efficiency. Again, super efficient points are Hartley efficient. The two efficiencies coincide when the ordering cone is normal.

Proposition 3.5. *In a normed linear space setting we always have*

$$SE(C, S) \subset HPE(C, S).$$

Proof. Suppose that x is in $SE(C, S)$. Then there is $M > 0$ such that for each c in C and h in X,

$$c \leq_S x + h \quad \text{implies} \quad \|c - x\| \leq M\|h\|.$$

If x is not in $HPE(C, S)$ and $\varepsilon > 0$, the definition of $HPE(C, S)$ fails for $M + \varepsilon$. Then there exist $s_0^+ \in S^+$ with $\|s_0^+\| = 1$ and $x \neq c_0 \in C$ such that $s_0^+(c_0 - x) < 0$; and for all s^+ in S^+, $\|s^+\| = 1$,

$$s_0^+(x - c_0) > (M + \varepsilon)s^+(c_0 - x).$$

This implies that for all s^+ in $S^+ \cap \partial B^*$ (∂B^* is the unit sphere in X^*),

$$\|x - c_0\| \geq (M + \varepsilon)s^+(c_0 - x).$$

Thus for all s^+ in $S^+ \cap B^*$

$$s^+(c_0 - x)/\|c_0 - x\| \leq 1/(M + \varepsilon).$$

Therefore, a standard polarity argument shows

$$(c_0 - x)/\|c_0 - x\| \in (M + \varepsilon)^{-1}(S^+ \cap B^*)^0 = (M + \varepsilon)^{-1}\,\mathrm{cl}(B - S).$$

Hence, $c_0 - x \in \|c_0 - x\|(M + \varepsilon/2)^{-1}(B - S)$, and as $x \in SE(C, S)$,

$$\|c_0 - x\| \leq [M/(M + \varepsilon/2)]\|c_0 - x\| < \|c_0 - x\|.$$

This is a contradiction. □

Proposition 3.6. *Let X be a normed linear space with a normal ordering cone S. Then for any subset C of X, $HPE(C, S) = SE(C, S)$.*

Proof. Let x be in $HPE(C, S)$. Set

$$\|y\|_S := \sup\{|s^+(y)| : s^+ \in S^+, \|s^+\| = 1\} \leq \|y\|.$$

By the normality of S, there is a constant $M_0 > 0$ such that for all y in X, $\|y\| \leq M_0\|y\|_S$, (since S^+ is a strict b-cone, [Peressini 1, p. 76]). Suppose that $c - x \leq_S y$, then for each s^+ in $S^+ \cap \partial B^*$, $s^+(c - x) \leq s^+(y)$. By the definition of $\| \cdot \|_S$, there is, for each $\varepsilon > 0$, an s_1^+ in $S^+ \cap \partial B^*$ such that $\|c - x\|_S \leq |s_1^+(c - x)|(1 + \varepsilon)$. If $s_1^+(c - x) > 0$, then

$$(1 + \varepsilon)^{-1}\|c - x\|_S \leq s_1^+(c - x) \leq s_1^+(y) \leq \|y\|.$$

If $s_1^+(c - x) < 0$, then as x is in $HPE(C, S)$, there exists s_2^+ in $S^+ \cap \partial B^*$ and $M_1 > 0$ such that,

$$s_1^+(x - c) \leq M_1 s_2^+(c - x) \leq M_1 s_2^+(y) \leq M_1\|y\|_S \leq M_1\|y\|.$$

Thus in both cases, we have for all $\varepsilon > 0$, $\|c - x\|_S \leq (1 + \varepsilon)M_2\|y\|$, with $M_2 := \max\{1, M_1\}$. Therefore, $\|c - x\|_S \leq M_2\|y\|$. Finally, we have $\|c - x\| \leq M_0\|c - x\|_S \leq M\|y\|$, where $M = M_0 M_2$. Hence x belongs to $SE(C, S)$. □

When X is $\mathbb{R}^n$ and S is the nonnegative orthant $\mathbb{R}_+^n$, the definition of Hartley efficiency (equivalently super efficiency) coincides with Geoffrion's definition of proper efficiency [Geoffrion 1]. This emphasizes again that super efficient points are indeed the right subclass of efficient points. This is also the point of the following proposition.

Proposition 3.7. *Let X be a normed linear space whose ordering cone S has a weakly compact base. Then for any subset C of X, $PE(C, S) = SE(C, S)$.*

Proof. It suffices to show that any proper efficient point is super efficient. Let x be in $PE(C, S)$. If we define $K := \text{cl}[\text{cone}(C - x)]$ as usual, then $K \cap -S = \{0\}$. We claim that there is an $\varepsilon > 0$ such that

$$(3.8) \qquad K \cap (\varepsilon B - S) \subset (1/\varepsilon)B.$$

Indeed, if (3.8) is not true, then we can $y_n \in ((1/n)B - S) \cap K$ such that $\|y_n\| > n$. Assume that $y_n = b_n/n - s_n$ with b_n in B and s_n in S. Now consider $s_n/\|y_n\|$,

$$\|s_n\|/\|y_n\| \le 1 + 1/(n\|y_n\|),$$

so $\{s_n/\|y_n\|\}$ is bounded. As S is locally weakly compact, it has a weakly compact base Θ [Klee 1], we may assume that $s_n/\|y_n\| = \lambda_n \theta_n$ where $\lambda_n > 0$ and bounded. Now $\theta_n = \lambda_n^{-1}[b_n(n\|y_n\|)^{-1} - y_n\|y_n\|^{-1}]$ converges weakly to some $0 \ne \theta_0$ in Θ. Hence $y_n\|y_n\|^{-1}$ converges weakly to some $0 \ne -s_0$ in $K \cap -S$. This contradicts our assumption that x is in $PE(C, S)$. Therefore (3.8) is verified. And so is $PE(C, S) \subset SE(C, S)$. $\square$

4. Density theorems for super efficiency

We have seen that in general the set of super efficient points is strictly contained in that of the efficient points. In this section we provide the answer to the question of "how many" efficient points are super efficient. First we need another definition.

Definition 4.1. Let C be a subset of a normed linear space X with a closed ordering cone S and let B be the closed unit ball in X. We say that C is *S-lower bounded* if there is some constant $M > 0$ such that

$$(4.1) \qquad C \subset MB + S.$$

Note that if C is S-lower bounded then

$$(4.2) \qquad R_\sigma(C) \subset S.$$

(See §1 for the definition of $R_\sigma(C)$.) Indeed, let $d \in R_\sigma(C)$. By the definition of $R_\sigma(C)$, there exist $t_n \to 0$ and $c_n \in C$ such that $t_n c_n$ tends to d weakly. As C is S-lower bounded there are $b_n \in B$ and $M > 0$ such that $c_n - M b_n \in S$. Now, $t_n c_n - t_n M b_n \in S$ and $t_n M b_n \to 0$ implies that d is in S as S is closed and convex, hence weakly closed. Hence (4.2) holds. As S is pointed we also see that $R_\sigma(C) \cap -S = \{0\}$.

Note that (4.1) holds if either C is bounded topologically or C has an *S-lower bound*, that is, there is m such that $m \le_s c$ for all c in C.

Theorem 4.2. *Let X be a Banach space, S an ordering cone and C a nonempty subset of X. Assume that S has a closed and bounded base Θ. If either of the following is satisfied, then $SE(C, S)$ is norm-dense in the nonempty set $E(C, S)$:*
  (1) *C is weakly compact;*
  (2) *C is weakly closed and S-lower bounded while Θ is weakly compact.*

Proof. Since S has a closed and bounded base, S is closed. Moreover, it is Daniell [Borwein 3]. According to Theorem 2.1 in [Borwein 4], $E(C, S)$ is

then nonempty if either C is weakly compact or C is (weakly) closed and S-lower bounded. Let x_0 be in $E(C, S)$. Then

$$(4.3) \qquad\qquad (C - x_0) \cap -S = \{0\}.$$

Since S has a base Θ, for all $\delta > 0$, one has

$$(4.4) \qquad\qquad (C - x_0) \cap -\delta\Theta = \varnothing.$$

We claim that for any $\delta > 0$, there is an $\varepsilon > 0$ such that

$$(4.5) \qquad\qquad (C - x_0) \cap -S_\varepsilon \subset \delta B.$$

Indeed, if (4.5) were false for some $\delta > 0$, we would be able to find c_n in C, θ_n in Θ, b_n in B and $\lambda_n > 0$ such that $\|c_n - x_0\| > \delta$ and

$$(4.6) \qquad\qquad c_n - x_0 = -\lambda_n(\theta_n + (1/n)b_n).$$

Without loss of generality, we may assume λ_n converges. We consider two cases separately.

(1) When C is weakly compact, $(c_n) \subset C$ is bounded. Extracting subsequences if necessary we may assume that c_n converges weakly to some c in C. If $\lambda_n \to \infty$, then by (4.6) we have

$$-\theta_n = \lambda_n^{-1}(c_n - x_0) + (1/n)b_n \to 0.$$

This is impossible because $0 \notin \mathrm{cl}(\Theta)$. Thus $\lambda_n \to \lambda < \infty$. It is clear that $\lambda > 0$ as $\|c_n - x_0\| > \delta$. Hence

$$\text{w-}\lim \theta_n = \lambda^{-1}(x_0 - c)$$

is in Θ because Θ is weakly closed. Thus

$$(C - x_0) \cap -S \neq \{0\}.$$

This contradicts (4.3). Therefore (4.5) holds.

(2) Suppose now that C is weakly closed and S-lower bounded, and Θ is weakly compact. Then by the weak-compactness of Θ, we may assume that θ_n converges weakly to some θ_0 in Θ. If $\lambda_n \to \infty$, then

$$\lambda_n^{-1}(c_n - x_0) \to -\theta_0 \in -S.$$

Since $\lambda_n^{-1} \to 0$ and c_n are in C, $0 \neq -\theta_0$ is in $R_\sigma(C) \cap -S$. This is a contradiction because C is assumed to be S-lower bounded and hence

$$R_\sigma(C) \cap -S = \{0\}.$$

Thus, $\lambda_n \to \lambda < \infty$. Since Θ is bounded and $\|c_n - x_0\| > \delta$, (4.6) shows $\lambda > 0$. Now c_n converge weakly to $c := x_0 - \lambda\theta_0$. As C is weakly closed, c is in C. But $c - x_0 = -\lambda\theta_0 \in -S\backslash\{0\}$ contradicts the fact that x_0 is in $E(C, S)$. Thus (4.5) holds in this case too.

Note (4.5) implies that for small ε the set $C_\varepsilon := C \cap (x_0 - S_\varepsilon)$ has arbitrarily small diameter. In each case S_ε is Daniell as S has a bounded closed base (Theorem 1.1(5)), and so Theorem 2.1 in [Borwein 4] applies. Thus $E(C_\varepsilon, S_\varepsilon)$ is nonempty. Clearly any point in $E(C_\varepsilon, S_\varepsilon)$ is actually in $E(C, S_\varepsilon)$, and so lies in $SE(C, S)$, by Corollary 2.6. Finally, let $y \in C_\varepsilon \cap E(C, S_\varepsilon)$, then y belonging to $C_\varepsilon = C \cap (x_0 - S_\varepsilon)$ implies that $y - x_0 \in (C - x_0) \cap -S_\varepsilon$. Hence $\|y - x_0\| < \delta$, and we conclude that there is a point in $SE(C, S)$ arbitrarily close

to x_0. As x_0 was arbitrarily chosen from $E(C, S)$, we see that $SE(C, S)$ is dense in $E(C, S)$. □

Observe that the theorem can be viewed as giving density results for Borwein efficiency, Henig efficiency and Hartley efficiency (as super efficient points are in appropriate settings efficient in the sense of those authors). Even these specializations extend many density results in the literature.

5. Chebyshev scalarizations

It is a fundamental principle in vector optimization that optimal elements can be characterized as optimal solutions of certain scalar optimization problems. We have seen that in some important special cases super efficiency can be satisfactorily characterized by scalar optimization problems. In the following we characterize super efficiency by Chebyshev scalarizations, i.e. characterize super efficiency in terms of certain equivalent norms on X.

Definition 5.1. We say that a norm $\| \cdot \|$ is *S-monotone* if

$$0 \leq_S x \leq_S y \text{ implies that } \|x\| \leq \|y\|.$$

In a normed linear space X, when the ordering cone S is normal and has nonempty interior and a set C is contained in the interior of S, then each efficient point in C can be expressed as those points in C with least norm for the equivalent S-monotone norm defined by

$$(5.1) \qquad |||x||| := \inf\{t > 0 | x \in t[-x_0, x_0]_S\},$$

where $x_0 \in E(C, S)$. First, we prove a lemma which shows that the norm defined by (5.1) is indeed a norm equivalent to the original norm $\| \cdot \|$ of the space.

Lemma 5.2. *Let X be an ordered normed space with the norm $\|\cdot\|$. Suppose that the ordering cone S has nonempty interior and is normal. If x_0 is in $\mathrm{int}(S)$, then the norm defined by (5.1) is S-monotone and equivalent to the norm $\| \cdot \|$.*

Proof. The proof that $||| \cdot |||$ defined by (5.1) is a norm can be found in [Jahn 1, p. 27]; that

$$x' \in [0, x]_S \Rightarrow |||x'||| \leq |||x|||$$

is easy and can be seen in [Jahn 1, p. 112]. We show $||| \cdot |||$ is an equivalent norm. Now as $\mathrm{int}(S) \neq \varnothing$, $\mathrm{int}([-x_0, x_0]_S) \neq \varnothing$. We may assume that $\alpha B \subset [-x_0, x_0]_S$, where B is the closed unit ball in X. Let $0 \neq x$ be in X, then $x\|x\|^{-1}$ is in B and hence $\alpha x\|x\|^{-1} \in [-x_0, x_0]_S$. By the definition of $||| \cdot |||$, $\alpha|||x|||/\|x\| \leq 1$. Hence, $|||x||| \leq \alpha^{-1}\|x\|$. The fact that for some $k > 0$ $\|x\| \leq k|||x|||$ follows from the normality of S. Hence, $||| \cdot |||$ is an equivalent norm. □

Proposition 5.3. *Let the ordering cone S be normal with nonempty interior. If the set C is contained in $\mathrm{int}(S)$ and x_0 is in $E(C, S)$, then there exists an equivalent S-monotone norm $||| \cdot |||$ with*

$$(5.2) \qquad 1 = |||x_0||| = \min\{|||x||| \, | x \in C\}.$$

Proof. Let

$$|||x||| := \inf\{t > 0 | x \in t[-x_0, x_0]_S\}.$$

By Lemma 5.2, $|||\cdot|||$ is a monotone norm which is equivalent to $||\cdot||$ with $|||x_0||| \leq 1$. Suppose $|||x||| < 1$ for some x in C. Then x lies in $t[-x_0, x_0]_S$ for some $0 < t < 1$ and so $x \leq_S x_0$, $x \neq x_0$. But $(x_0 - S) \cap C = \{x_0\}$ because $x_0 \in E(C, S)$. Hence (5.2) holds. □

Lemma 5.4. *In an ordered Banach space whose ordering cone S has a bounded base, there is an equivalent norm defined by*

(5.4) $$|||x||| := \inf\{t > 0 \mid x \in t[-a, a]_{S_\varepsilon}\}$$

(for any $0 \neq a$ in S and fixed $\varepsilon > 0$ chosen sufficiently small) satisfies with the property that for some $\delta > 0$,

(5.5) $$0 \leq_s x \leq_s y \Rightarrow |||y||| \geq |||x||| + \delta|||y - x|||.$$

Proof. Let ϕ be in S^{+i} such that $\phi(s) \geq ||s||$ for all s in S, as is possible since S has a bounded base. Let $\Theta^* := \phi^{-1}(1) \cap S$. Then Θ^* is a bounded base for S. Pick $\varepsilon > 0$ small enough that Θ^*_ε is a base. The norm defined by (5.4) is indeed an equivalent S_ε-monotone norm by Lemma 5.2 because $S_\varepsilon(\Theta^*)$ is normal and has a in its interior. Thus, there are $\beta > 0$, $\beta_1 > 0$ such that

$$\beta_1||x|| \geq |||x||| \geq \beta||x||.$$

Suppose $0 \leq_s \leq_s y$ with $y \neq x$. Let $s_0 := y - x$. Then s_0 is in $S/\{0\}$. Set $x^* := x/\phi(s_0)$, $y^* := y/\phi(s_0)$, $s_0^* := s_0/\phi(s_0)$. Then $s_0^* \in \Theta^*$. Now

$$s_0^* + \varepsilon B \subset \varepsilon B + \Theta^* \subset S_\varepsilon.$$

Thus, $s_0^* - \varepsilon y^*/||y^*|| \in S_\varepsilon$ and

$$(||y^*|| - \varepsilon)y^*/||y^*|| = x^* + s_0^* - \varepsilon y^*/||y^*|| \geq_{S_\varepsilon} x^* \geq_s 0.$$

In particular, $||y^*|| \geq \varepsilon$ since $y^* \in S_\varepsilon$ and S_ε is pointed while y is nonzero. As $|||\cdot|||$ is S_ε-monotone, $(||y^*|| - \varepsilon)|||y^*|||/||y^*|| \geq |||x^*|||$ and

$$|||y^*||| \geq |||x^*||| + \varepsilon|||y^*|||/||y^*|| \geq |||x^*||| + \varepsilon\beta.$$

Therefore,

$$|||y||| \geq |||x||| + \varepsilon\beta\phi(y - x) \geq |||x||| + \varepsilon\beta||y - x||$$
$$\geq |||x||| + \varepsilon\beta\beta_1^{-1}|||y - x||| = |||x||| + \delta|||y - x|||$$

where $\delta := \varepsilon\beta\beta_1^{-1}$. If $y = x$ this also holds and (5.5) is satisfied. □

We observe that we could have used any equivalent $S_\varepsilon(\Theta^*)$-monotone norm.

Let us say that a norm satisfying (5.5) is *strongly S-monotone*. Clearly any such norm is *strictly S-monotone* in the sense that

$$0 \leq_s x \leq_s y \text{ and } x \neq y \text{ implies that } ||x|| < ||y||.$$

Now we are able to give a characterization of x_0 being super efficient in terms of Chebyshev scalar optimization.

Theorem 5.5. *In an ordered Banach space, if the closed ordering cone S has a bounded base Θ and C is contained in $S\backslash\{0\}$, then the following are equivalent:*
  (i) $x_0 \in SE(C, S)$;
  (ii) *There is an equivalent strongly S-monotone norm $|||\cdot|||$ such that*

(5.6) $$|||x_0||| = \min\{|||x||| \mid x \in C\}.$$

Proof. Let $x_0 \in SE(C, S)$. Consider Θ as built in Lemma 5.4. Select a sufficiently small $\varepsilon > 0$ so that

(i) the norm defined in Lemma 5.4 with $a := x_0$ is strongly S-monotone and

(ii) $x_0 \in E(C, S_\varepsilon(\Theta))$; since x_0 lies in $\text{int} S_\varepsilon$. As in the proof of Proposition 5.3 we see that (5.6) holds.

Conversely, suppose there exists an equivalent strongly S-monotone norm such that (5.6) is satisfied. Suppose that y is in X with $c - x_0 \leq_s y$ for some c in C. Then $0 \leq_s c \leq_s x_0 + y$. This implies that

$$|||c||| + \delta|||x_0 + y - c||| \leq |||x_0 + y|||.$$

Hence, using (5.6),

$$|||x_0||| + \delta|||x_0 + y - c||| \leq |||x_0||| + |||y|||,$$

which, in turn, implies that

$$\delta|||x_0 - c||| - \delta|||y||| \leq |||y|||.$$

Therefore, $|||x_0 - c||| \leq (1 + \delta^{-1})|||y|||$, i.e. $x_0 \in SE(C, S)$. $\square$

Note that when the cone S has a weakly compact base, the base Θ constructed above is also weakly compact (any other closed bounded base is). In addition, super efficient and proper efficient points coincide (by Proposition 3.7) and so we have the following corollary which extends a result in [Jahn 2].

Corollary 5.6. *In an ordered Banach space, if the closed ordering cone S has a weakly compact base Θ and C is contained in $S\backslash\{0\}$, then the following are equivalent:*

(i) $x_0 \in PE(C, S)$;

(ii) *There is an equivalent strongly S-monotone norm $||| \cdot |||$ such that*

$$(5.6) \qquad |||x_0||| = \min\{|||x||| \,|\, x \in C\}.$$

Conclusion

It should be clear that super efficiency is a very flexible kind of proper efficiency and that there is a lot more that can be said about this concept. Some other results about super efficiency will be summarized in forthcoming papers. Simple expressions of super efficiency in vector lattices and scalarization results of super efficiency provide promises of practical application of the theory. Density results of super efficiency can be applied to establish some tangency formulas in nonsmooth analysis as demonstrated in [Zhuang 1]. We hope that super efficiency will enter the vector optimization literature and find its application in optimization practice.

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