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GAUSS MAP OF MINIMAL SURFACES WITH RAMIFICATION

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ABSTRACT. We prove that for any complete minimal surface M immersed in R^n, if in CP^{n-1} there are $q > n(n+1)/2$ hyperplanes H_j in general position such that the Gauss map of M is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^{q}\left(1-\frac{(n-1)}{e_j}\right) > n(n+1)/2,$$

then M must be flat.

1. INTRODUCTION

Let $x: M \to R^n$ be a (smooth, oriented) minimal surface immersed in R^n. Make M into a Riemann surface by decreeing that the 1-form $d\xi_1 + id\xi_2$ is of type $(1, 0)$, where (ξ_1, ξ_2) are any local isothermal coordinates of M. The Gauss map of x is defined to be

$$G: M \to Q_{n-2}(C) \subset CP^{n-1}, \qquad G(z) = [(\partial x/\partial z)]$$

where $[(\cdot)]$ denotes the complex line in C^n through the origin and $(\cdot)$, $z = \xi_1 + i\xi_2$ is the holomorphic coordinate of M, and

$$Q_{n-2}(C) = \{(w_0 : \cdots : w_{n-1};\, w_0^2 + \cdots + w_{n-1}^2 = 0\} \subset CP^{n-1}.$$

By the assumption of minimality of M, G is a holomorphic map of M into CP^{n-1}. It is a natural question to study the "value distribution" properties of the Gauss map G. Fujimoto (see [8]) has shown that the Gauss map of a nonflat minimal surfaces can omit at most $n(n+1)/2$ hyperplanes in general position in CP^{n-1} under the assumption that G is nondegenerate. The "nondegenerate" assumption was removed by the author (see [13]). The purpose of this paper is to study more general "value distribution" properties of the Gauss map. In particular, we study the Gauss map with ramification.

One says that G is *ramified over a hyperplane* $H = \{[w] \in CP^{n-1}: a_0w_0 + \cdots + a_{n-1}w_{n-1} = 0\}$ *with multiplicity at least* e if all the zeros of the function $g_H = (G, A)$ have orders at least e, where $A = (a_0, \ldots, a_{n-1})$. If the image of G omits H, we shall say that G is *ramified over* H *with multiplicity* ∞.

Our main result is the following:

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Theorem 1. *Let M be a complete minimal surface immersed in R^n and assume that the Gauss map G of M is k-nondegenerate (that is $G(M)$ is contained in a k-dimensional linear subspace of CP^{n-1}, but none of lower dimension), $1 \le k \le n-1$. Let $H_i \subset CP^{n-1}$ be q hyperplanes in general position. If G is ramified over H_i with multiplicity at least e_i for each i. Then*

$$\sum_{j=1}^{q}\left(1-\frac{k}{e_j}\right) \le (k+1)\left(n-\frac{k}{2}-1\right)+n.$$

In particular, for any complete minimal surface M immersed in R^n, if in CP^{n-1} there are $q > n(n+1)/2$ hyperplanes in general position such that its Gauss map G is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^{q}\left(1-\frac{(n-1)}{e_j}\right) > n(n+1)/2,$$

then M must be flat.

In the case $m = 3$, $Q_1(C)$ can be identified with CP^1. We have a better result.

Theorem 2. *Let M be a complete minimal surface $(\subset R^3)$. If there are $q (q > 4)$ distinct points $a_1, \dots, a_q \in CP^1$ such that the Gauss map of M is ramified over a_j with multiplicity at least e_j for each j and $\sum_{j=1}^{q}(1-1/e_j) > 4$, then M must be flat.*

In particular, if the Gauss map omits five distinct points, then M must be flat.

2. Facts on holomorphic curves into projective spaces

We shall recall some known results in the theory of holomorphic curves.

(A) **Associated curve.** Let f be a nondegenerated holomorphic map of Δ_R: $\{z : |z| < R\}$ into CP^k, where $0 < R \le \infty$. Take a reduced representation $f = [Z_0 : \cdots : Z_k]$, where $Z = (Z_0, \dots, Z_k) : \Delta_R \to C^{k+1} - \{0\}$. Denote by $Z^{(j)}$ the jth derivative of Z and define

$$\Lambda_j = Z^{(0)} \wedge \cdots \wedge Z^{(j)} : \Delta_R \to \bigwedge^{j+1} C^{k+1}$$

for $0 \le j \le k$. Evidently $\Lambda_{k+1} \equiv 0$.

let P: $\bigwedge^{j+1} C^{k+1} - \{0\} \to CP^{N_j}$ denote the canonical projection, where $N_j = \binom{k+1}{j+1} - 1$. The jth *associated curve* of f is the map $f_j = P(\Lambda_j)$.

It is well known [4] (also see [16]) that the pull-back Ω_j of the Fubini-study metric on CP^{N_j} by f_j is given by

$$\Omega_j = dd^c \log|\Lambda_j|^2 = \frac{i}{2\pi}\frac{|\Lambda_{j-1}|^2|\Lambda_{j+1}|^2}{|\Lambda_j|^4} dz \wedge d\bar{z}, \tag{2.1}$$

for $0 \le j \le k$ and by convention $\Lambda_{-1} \equiv 1$. Note that $\Omega_k \equiv 0$. It follows that

$$\operatorname{Ric}\Omega_j = \Omega_{j-1} + \Omega_{j+1} - 2\Omega_j. \tag{2.2}$$

Take a hyperplane $H\colon (W, A) = 0$, where $A = (a_0, \dots, a_k)$ is a unit vector. Define

$$\varphi_j(H) = \frac{|\Lambda_j \vee A|^2}{|\Lambda_j|^2 |A|^2}.$$

Note that $0 \le \varphi_j(H) \le \varphi_{j+1}(H) \le 1$ for $0 \le j \le k$ and $\varphi_k(H) = 1$.

We need the following well-known lemma (see [4, 16 and 17]).

Lemma 2.1. *Let H be a hyperplane in CP^k, then for any constant $N > 1$, for $0 \le p \le k-1$,*

$$dd^c \log \frac{1}{N - \log \varphi_p(H)} \ge \left\{ \frac{\varphi_{p+1}(H)}{\varphi_p(H)(N - \log \varphi_p(H))^2} - \frac{1}{N} \right\} \Omega_p, \tag{2.3}$$

on $\Delta_R - \{\varphi_p = 0\}$.

(B) **Nochka weights and product to sum estimate.** We consider q hyperplanes H_j $(1 \le j \le q)$ in CP^k which are given by $H_j\colon (W, A_j) = 0$. According to Chen [2], we give the following definition.

Definition 2.2. We say that hyperplanes $H_1, \dots, H_q$ are in n-subgeneral position if, for every $1 \le j_0 < \cdots < j_n \le q$, $A_{j_0}, A_{j_1}, \dots, A_{j_n}$ generate C^{k+1}.

In [11] (see also [2]), Nochka has given the following lemma to prove the Cartan conjecture.

Lemma 2.3. *Let $H_1, \dots, H_q$ be hyperplanes in CP^k located in the n-subgeneral position, where $q > 2n - k + 1$. Then there are some constants $\omega(1), \dots, \omega(q)$ and θ satisfying the following condition:*

(i) $0 < \omega(j)\theta \le 1$ $(1 \le j \le q)$,
(ii) $\theta(\sum_{j=1}^q \omega(j) - k - 1) = q - 2n + k - 1$,
(iii) $1 \le (n+1)/(k+1) \le \theta \le (2n - k + 1)/(k+1)$,
(iv) *if* $R \subset Q$ *and* $0 < \# R \le n + 1$, *then* $\sum_{j \in R} \omega(j) \le d(R)$.

For the proof, see [2] or [11].

Definition 2.4. We call constants $\omega(j)$ $(1 \le j \le q)$ and θ above Nochka weights and a Nochka constant for $H_1, \dots, H_q$ respectively.

Nachka weights are useful because of the following lemma.

Lemma 2.5. *Under the above assumptions. Let $E_1, \dots, E_q$ be a sequence of real numbers with $E_j \ge 1$ for all j. Then for any subset B of the set $\{1, 2, \dots, q\}$ with $0 < \# B \le n + 1$, there exists a subset C of B such that $\{A_j | j \in C\}$ is a base of the linear space spanned by $\{A_j | j \in B\}$ and*

$$\prod_{j \in B} E_j^{\omega(j)} \le \prod_{j \in C} E_j,$$

where $\omega(j)$ are the Nochka weights associated to hyperplanes $H_j\colon (A_j, W) = 0$, $j = 1, 2, \dots, q$.

For the proof, see [2] or [11].

We also have the following product to sum estimate.

Lemma 2.6 (see Chen [2]). *Under the above assumptions. For* $0 \le p \le k-1$, *any constant* $N > 1$, $1/q \le \lambda_p \le 1/(k-p)$, *there exists a positive constant* $c_p > 0$ *only depends on* p *and the given hyperplanes such that*

$$c_p \prod_{j=1}^{q} \left(\frac{\varphi_{p+1}(H_j)^{\omega(j)}}{\varphi_p(H_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right)^{\lambda_p} \le \sum_{j=1}^{q} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N - \log \varphi_p(H_j))^2}, \tag{2.4}$$

on $\Delta_R - \{\varphi_p = 0\}$.

3. Metrics with negative curvature

We retain the notation of the last section. Let $f\colon \Delta_R \to CP^k$ be a nondegenerate holomorphic map. Take a reduced representation $f = [Z_0 : \cdots : Z_k]$ where $Z = (Z_0, \ldots, Z_k)\colon \Delta_R \to C^{k+1} - \{0\}$ is a holomorphic map. Let $H_1, \ldots, H_q$ be hyperplanes in CP^k located in n-subgeneral position. Let $\omega(j)$ be their Nochka weights.

Let f be ramified over H_j with multiplicity at least e_j for each j. Assume that

$$\sum_{j=1}^{q} \left(1 - \frac{k}{e_j}\right) > 2n - k + 1,$$

we shall construct a continuous pseudo-metric on Δ_R such that its Gauss curvature is less than or equal to -1. So that we can use Schwarz lemma to obtain our main inequality.

Let $\Omega_p = \frac{i}{2\pi} h_p(z) dz \wedge d\bar{z}$. Let

$$\sigma_p = c_p \prod_{j=1}^{q} \left[\left(\frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)} \right)^{\omega(j)(1-k/e_j)} \frac{1}{(N - \log \varphi_p(H_j))^2} \right]^{\lambda_p} h_p. \tag{3.1}$$

Where c_p is the constant in the product to sum estimate,

$$\lambda_p = 1 \Big/ \left((k-p) + (k-p)^2 \frac{2q}{N} \right),$$

and $N > 1$.

We take the geometric mean of the σ_p and define

$$\Gamma = \frac{i}{2\pi} c \prod_{p=0}^{k-1} \sigma_p^{\beta_k/\lambda_p} dz \wedge d\bar{z}. \tag{3.2}$$

where $\beta_k = 1/(\sum_{p=0}^{k-1} \lambda_p^{-1})$, and $c = 2(\prod_{p=0}^{k-1} \lambda_p^{\lambda_p^{-1}})^{\beta_k}$.

Let

$$\Gamma = \frac{i}{2\pi} h(z) dz \wedge d\bar{z}. \tag{3.3}$$

We now compute $h(z)$. By (3.1) and (3.2), we have

$$h(z) = c \left[\prod_{j=1}^{q} \frac{k}{\varphi_0(H_j)^{\omega(j)(1-1/e_j)\beta_k}} \prod_{p=0}^{k=1} \frac{h_p^{\beta_k/\lambda_p}}{(N - \log \varphi_p(H_j))} \right]. \tag{3.4}$$

By (2.1),

$$h_p^{1/\lambda_p} = \left(\frac{|\Lambda_{p-1}|^2|\Lambda_{p+1}|^2}{|\Lambda_p|^4}\right)^{(k-p)+(k-p)^2 2q/N},$$

so

$$\prod_{p=0}^{k-1} h_p^{1/\lambda_p} = |\Lambda_0|^{-2(k+1)-(k^2+2k-1)4q/N}|\Lambda_1|^{8q/N}\cdots|\Lambda_{k-1}|^{8q/N}|\Lambda_k|^{2+4q/N}.$$

Notice that $|\Lambda_0| = |Z|$, and $\varphi_0(H_j) = |(Z, A_j)|^2/|Z|^2$, therefore

$$(3.5)\quad h(z) = c\left[\frac{|Z|^{\sum_{j=1}^q \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N}(|\Lambda_1|\cdots|\Lambda_{k-1}|)^{4q/N}|\Lambda_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)}\prod_{p=0}^{k-1}(N-\log\varphi_p(H_j))}\right]^{2\beta_k}.$$

Lemma 3.1. *The function*

$$\frac{|\Lambda_k|}{\prod_{j=1}^q |(Z, A_j)|^{\omega(j)(1-k/e_j)}}$$

is continuous on Δ_R.

Proof. We shall prove that the function

$$P = \left[\frac{|\Lambda_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}}\right]^e$$

is continuous where $e = e_1\cdots e_q$. Lemma 3.1 follows from this. According to the expression of $P(z)$, we only need to consider the points at which (Z, A_j) vanishes. For zero point z_0 of (Z, A_j), since f is ramified over H_j with multiplicity at least e_j for each j, we have

$$(Z, A_j) = (z-z_0)^{\nu_i}Q_j(z)$$

where $Q_j(z_0) \neq 0$, and $\nu_j \geq e_j$ or $\nu_j = 0$. The n-subgeneral position implies that, at each point z, there are at most n of hyperplanes H_j, such that $(Z(z), A_j) = 0$. Thus there exists a constant c_0 (depending only on the given hyperplanes) such that

$$\# B = \#\{j \,|\, |(Z(z), A_j)|/|A_j||Z(z)| \leq c_0\} \leq n.$$

Let $E_j = 1/\varphi_0(H_j)^{\omega(j)(1-k/e_j)}$, then $E_j \leq 1$. If $j \notin B$, then $\varphi_0(H_j) > c_0$, so $E_j \leq c_1$ (depending only on the given hyperplanes).

Applying Lemma 2.5 with E_j above, we obtain

$$\begin{aligned}\frac{|\Lambda_k|^2}{\prod_{j=1}^q \varphi_0(H_j)^{\omega(j)(1-k/e_j)}} &\leq c_2\frac{|\Lambda_k|^2}{\prod_{j\in B}\varphi_0(H_j)^{\omega(j)(1-k/e_j)}}\\ &\leq c_2\frac{|\Lambda_k|^2}{\prod_{j\in C}\varphi_0(H_j)^{(1-k/e_j)}}.\end{aligned}$$

We may assume the index set $C = \{1, 2, \ldots, l\}$ and $l \leq k+1$, therefore

$$\left[\prod_{j\in C}(Z(z), A_j)^{(1-k/e_j)}\right]^e = (z-z_0)^b R(z)$$

where $b = \sum_{j=1}^{l} e\nu_j(1-k/e_j)$ and R is a holomorphic function such that $R(z_0) \neq 0$. Since

$$|\Lambda_k| = \det\begin{vmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_k \\ Z_0' & Z_1' & Z_2' & \cdots & Z_k' \\ \vdots & \vdots & \vdots & & \vdots \\ Z_0^{(k)} & Z_1^{(k)} & Z_2^{(k)} & \cdots & Z_k^{(k)} \end{vmatrix}$$
$$= \det\begin{vmatrix} (Z, A_1) & (Z, A_2) & (Z, A_3) & \cdots \\ (Z, A_1)' & (Z, A_2)' & (Z, A_3)' & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ (Z, A_1)^{(k)} & (Z, A_2)^{(k)} & (Z, A_3)^{(k)} & \cdots \end{vmatrix},$$

we have $\Lambda_k = (z-z_0)^\nu S(z)$, where $\nu = \nu_1 + \nu_2 - 1 + \cdots + \nu_l - k$ and S is a holomorphic function. Hence we obtain

$$P(z) \leq |(z-z_0)^{2p} T(z)|,$$

where

$$p = \frac{ek}{e_1} + \frac{e}{e_2}(k\nu_2 - e_2) + \frac{e}{e_3}(k\nu_3 - 2e_3) + \cdots + \frac{e}{e_l}(k\nu_l - (l-1)e_l) \geq 0,$$

and $T(z)$ is continuous at z_0. Therefore $P(z)$ is bounded around z_0. Therefore $P(z)$ is continuous. Q.E.D.

Lemma 3.2. *If* $\sum_{j=1}^{q}(1-k/e_j) \geq 2n-k+2$, *and*

$$2q/N < \left(\sum_{j=1}^{q} \omega(j)(1-k/e_j) - (k+1)\right)/(k^2+2k),$$

we have

(i) $\operatorname{Ric}\Gamma \geq \Gamma$ *on* $\Delta_R - \bigcup\{\varphi_0(H_j) = 0\}$.

(ii) Γ *is a continuous pseudo-metric on* Δ_R.

Proof. From (3.3) and (3.4) it follows that

$$\begin{aligned} \operatorname{Ric}\Gamma = & -\beta_k \sum_{j=1}^{q} \omega(j)\left(1-\frac{k}{e_j}\right) dd^c \log \varphi_0(H_j) \\ & + \beta_k \sum_{j=1}^{q}\sum_{p=0}^{k-1} dd^c \log(1/(N - \log\varphi_p(H_j)))^2 \\ & + \beta_k \sum_{p=0}^{k-1} (1/\lambda_p) \operatorname{Ric}\Omega_p. \end{aligned}$$

By Lemma 2.1, (2.2), and that $dd^c \log \varphi_0(H_j) = -\Omega_0$, we have

$$\begin{aligned} \operatorname{Ric}\Gamma \geq \beta_k \Bigg(& \sum_{j=1}^{q} \omega(j)\left(1-\frac{k}{e_j}\right)\Omega_0 + 2\sum_{j=1}^{q}\sum_{p=0}^{k-1} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N-\log\varphi_p(H_j))^2}\Omega_p \\ & -\frac{2q}{N}\sum_{p=0}^{k-1}\Omega_p + \sum_{p=0}^{k-1}\left[(k-p) + (k-p)^2\frac{2q}{N}\right]\{\Omega_{p+1} - 2\Omega_p + \Omega_{p-1}\}\Bigg). \end{aligned}$$

Using Lemma 2.6, we obtain

$$\begin{aligned}
\sum_{j=1}^{q} \frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)(N-\log\varphi_p(H_j))^2}\Omega_p
&\le c_p \left[\prod_{j=1}^{q}\left(\frac{\varphi_{p+1}(H_j)}{\varphi_p(H_j)}\right)^{\omega(j)} \frac{1}{(N-\log\varphi_p(H_j))^2}\right]^{\lambda_p} \Omega_p \\
&\ge \frac{i}{2\pi}\sigma_p dz\wedge d\bar z .
\end{aligned}$$

We also notice that $\Omega_k = 0$ so that

$$\sum_{p=0}^{k-1}(k-p)(\Omega_{p+1}-2\Omega_p+\Omega_{p-1}) = -(k+1)\Omega_0$$

and therefore

$$\begin{aligned}
\operatorname{Ric}\Gamma \ge \beta_k \Bigg(&\sum_{j=1}^{q}\omega(j)\left(1-\frac{k}{e_j}\right)\Omega_0 + 2\frac{i}{2\pi}\sum_{p=0}^{k-1}\sigma_p dz\wedge d\bar z - (k+1)\Omega_0 - (k^2+2k)\frac{2q}{N}\Omega_0 \\
&+\sum_{p=1}^{k-2}[(k-p+1)^2-2(k-p)^2+(k-p-1)^2-1]\frac{2q}{N}\Omega_p + \frac{2q}{N}\Omega_{k-1}\Bigg) .
\end{aligned}$$

The following is an elementary inequality:

For all the positive numbers $x_1, \dots, x_n$ and $a_1, \dots, a_n$,

$$a_1x_1+\cdots+a_nx_n \ge (a_1+\cdots+a_n)(x_1^{a_1}\cdots x_n^{a_n})^{1/(a_1+\cdots+a_n)} . \tag{3.6}$$

Letting $a_p = \lambda_p^{-1}$ in (3.6), we have

$$\sum_{p=0}^{k-1}\sigma_p \ge \frac{c}{2\beta_k}\sum_{p=0}^{k-1}\sigma_p^{\beta_k/\lambda_p}$$

and therefore

$$\begin{aligned}
\operatorname{Ric}\Gamma \ge \beta_k \Bigg(&\left(\sum_{j=1}^{q}\omega(j)\left(1-\frac{k}{e_j}\right) - (k+1) - (k^2+2k)\frac{2q}{N}\right)\Omega_0 \\
&+\sum_{p=0}^{k-2}\frac{2q}{N}\Omega_p + \frac{2q}{N}\Omega_{k-1}\Bigg) + \Gamma .
\end{aligned}$$

By Lemma 2.2, we find

$$\begin{aligned}
\theta\left(\sum_{j=1}^{q}\omega(j)\left(1-\frac{k}{e_j}\right)-k-1\right) &= \theta\left(\sum_{j=1}^{q}\omega(j)-k-1\right) - \frac{\sum_{j=1}^{q}\omega(j)\theta k}{e_j} \\
&= q-2n+k-1-\frac{\sum_{j=1}^{q}\omega(j)\theta k}{e_j} \ge q-2n+k-1-\frac{k}{e_j} \\
&= \sum_{j=1}^{q}\left(1-\frac{k}{e_j}\right) - 2n+k-1 > 0
\end{aligned}$$

and $\theta > 0$, so

$$\sum_{j=1}^{q} \omega(j)\left(1 - \frac{k}{e_j}\right) - (k+1) > 0.$$

This implies $\mathrm{Ric}\,\Gamma \geq \Gamma$. Thus (i) is satisfied.

(ii) follows from Lemma 3.1, (3.3) and (3.5). Q.E.D.

We recall the following generalization of the Schwarz lemma.

Lemma 3.3. *Let* $\Gamma = \frac{i}{2\pi} h(z) dz \wedge d\bar{z}$ *be a continuous pseudo-metric on* Δ_R *whose curvature is bounded above by a negative constant. Then, for some positive* c_0, $h(z) \leq c_0(2R/(R^2 - |z|^2))^2$. *For the proof, see* [1, *pp.* 12–14].

The purpose of this section is to obtain the following lemma.

Main Lemma. *Let* $f = [Z_0 : \cdots : Z_k]\colon \Delta_R \to CP^k$ *be a nondegenerate holomorphic map,* $H_1, \ldots, H_q$ *be hyperplanes in* CP^k *in* n*-subgeneral position,* $\omega(j)$ *be their Nochka weights. Let* $H_j\colon (W, A_j) = 0$ *and* $Z = (Z_0, \ldots, Z_k)$. *If* f *is ramified over* H_j *with multiplicity at least* e_j *for each* j, $\sum_{j=1}^{q}(1 - k/e_j) > 2n - k + 1$ *and* $N > 2q(k^2 + 2k)/(\sum_{j=1}^{q} \omega(j)(1 - k/e_j) - (k+1))$, *then there exists a positive constant* c *such that*

$$|Z|^{\sum_{j=1}^{q} \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N} \frac{\prod_{p=0}^{k-1}\prod_{j=1}^{q} |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q+N}}{\prod_{j=1}^{q} |(Z, A_j)|^{\omega(j)(1-k/e_j)}}$$
$$\leq c\left(\frac{2R}{(R^2 - |z|^2)}\right)^{k(k+1)2 + \sum_{p=0}^{k-1}(k-p)^2 2q/N}$$

Proof. Using the above Schwarz lemma for Γ, we obtain

$$h(z) \leq c_0(2R/(R^2 - |z|^2))^2.$$

So by (3.5) we have

(3.7)
$$|Z|^{\sum_{j=1}^{q} \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N} \frac{(|\Lambda_1| \cdots |\Lambda_{k-1})^{4q/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^{q} |(Z, A_j)|^{\omega(j)(1-k/e_j)} \prod_{p=0}^{k-1}(N - \log \varphi_p(H_j))}$$
$$\leq c_0\left(\frac{2R}{R^2 - |z|^2}\right)^{1/\beta_k}.$$

Set $K := \sup_{0<x\leq 1} x^{2/N}(N - \log x)$. Since $\varphi_p(H_j) < 1$ for all p and j we have

$$\frac{1}{(N - \log \varphi_p(H_j))} \geq \frac{1}{K} \varphi_p(H_j)^{2/N} = \frac{1}{K} \frac{|\Lambda_p \vee A_j|^{4/N}}{|\Lambda_p|^{4/N}}.$$

Substituting these into (3.7), we obtain the desired conclusion.

4. Proof of Theorem 1

The proof of Theorem 1 basically follows the argument in [13] using the main lemma (see also the arguments in [6, 7 and 8]). We include our proof here for the convenience of the reader.

We may assume M is simply connected, otherwise we consider its universal covering. By Koebe's uniformization theorem, M is bioholomorphic to C or to the unit disc. For the case $M = C$, Nochka (see [10], also see [16]) proved that if a k-nondegenerate holomorphic map from C to CP^{n-1} is ramified over hyperplanes H_j $(1 \le j \le q)$ with multiplicity at least e_j, where H_j are in general position, then

$$\sum_{j=1}^{q}\left(1-\frac{k}{e_j}\right) \le 2(n-1)-k+1;$$

in this case our Theorem 1 is true. For our purpose it suffices to consider the case $M=\Delta$.

We first prove the first part of Theorem 1.

Assume the first part of Theorem 1 is not true, namely G is ramified over hyperplanes $H_1, \dots, H_q$ in CP^{n-1} in general position with multiplicity e_j and

$$\sum_{j=1}^{q}(1-k/e_j) > (k+1)(n-k/2-1)+n. \tag{4.1}$$

Let $\omega(j)$ be Nochka weights of $\{H_j\}$. Because G is k-nondegenerate, we may assume $G(\Delta) \subset CP^k$, so that $G=[g_0:\cdots:g_k]:\Delta \to CP^k$ is nondegenerate. We consider hyperplanes $H_j \cap CP^k$, obviously these hyperplanes are in $(n-1)$-subgeneral position in CP^k. For the convenience, we still denote these hyperplanes by $\{H_j\}$.

Let $\widetilde{G}=(g_0, \dots, g_k):\Delta \to CP^{k+1}-\{0\}$; then the metric ds^2 on M induced from the standard metric on R^n is given by

$$ds^2 = 2|\widetilde{G}|^2|dz|^2. \tag{4.2}$$

By Lemma 2.2,

$$q-2(n-1)+k-1=\theta\left(\sum_{j=1}^{q}\omega(j)-k-1\right), \qquad 0<\omega(j)\theta \le 1,$$

and

$$\theta \le \frac{2(n-1)-k+1}{k+1} = \frac{2n-k-1}{k+1},$$

so

$$\begin{aligned}
2\left(\sum_{j=1}^{q}\omega(j)\left(1-\frac{k}{e_j}\right)-k-1\right) &= \frac{2\theta\left(\sum_{j=1}^{q}\omega(j)-k-1\right)}{\theta}-2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta e_j}\\
&= \frac{2(q-2n+k+1)}{\theta}-2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta e_j}\\
&\ge \frac{2(q-2n+k+1)}{\theta}-2\sum_{j=1}^{q}\frac{k}{\theta e_j}\\
&= \frac{2\left(\sum_{j=1}^{q}(1-k/e_j)-2n+k+1\right)}{\theta}\\
&\ge \frac{2\left(\sum_{j=1}^{q}(1-k/e_j)-2n+k+1\right)(k+1)}{(2n-k-1)}\\
&> k(k+1) \quad (\text{by } (4.1)).
\end{aligned}$$

Consider numbers

$$\rho = \frac{k(k+1)/2 + \sum_{p=0}^{k-1}(k-p)^2 2q/N}{\sum_{j=1}^{q}\omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N}, \tag{4.3}$$

$$\gamma = \frac{k(k+1)/2 + qk(k+1)/N + 2q/N\sum_{p=0}^{k-1}p(p+1)}{\sum_{j=1}^{q}\omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N}, \tag{4.4}$$

$$\delta = \frac{1}{(1-\gamma)\left(\sum_{j=1}^{q}\omega(j)(1-k/e_j) - (k+1) - (k^2+2k-1)2q/N\right)}. \tag{4.5}$$

Choose some N with

$$\frac{\sum_{j=1}^{q}\omega(j)(1-k/e_j) - (k+1) - k(k+1)/2}{k^2+2k-1+\sum_{p=0}^{k}(k-p)^2} > 2q/N > \frac{\sum_{j=1}^{q}\omega(j)(1-k/e_j) - (k+1) - k(k+1)/2}{1/q + (k^2+2k-1) + k(k+1)/2 + \sum_{p=0}^{k-1}p(p+1)}$$

so that

$$0 < \rho < 1, \quad 2\delta/N > 1. \tag{4.6}$$

Consider the open subset

$$M' = M - \left(\{\widetilde{G}_k = 0\} \bigcup_{1\le j\le q,\ 0\le p\le k-1} \{\widetilde{G}_p \vee A_j = 0\}\right)$$

of M and define the function

$$v = \left(\frac{\prod_{j=1}^{q}|(G, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1}\prod_{j=1}^{q}|\widetilde{G}_p \vee A_j|^{4/N}|\widetilde{G}_k|^{1+2q/N}}\right)^{\delta}$$

on M', where $\widetilde{G}_p = \widetilde{G}^{(0)} \wedge \cdots \wedge \widetilde{G}^{(p)}$. By Lemma 3.1, $\nu(z)$ is strictly positive and continuous on M'.

Let $\pi: \widetilde{M}' \to M'$ be the universal covering of M'. Since $\log v \circ \pi$ is harmonic on $\widetilde{M}'$ by the assumption, we can take a holomorphic function β on $\widetilde{M}'$ such that $|\beta| = v \circ \pi$. Without loss of generality, we may assume that M' contains the origin 0 of C. As in Fujimoto's paper [6, 7, 8], for each point $\tilde{p}$ of $\widetilde{M}'$ we take a continuous curve $\gamma_{\tilde{p}}: [0, 1] \to M'$ with $\gamma_{\tilde{p}}(0) = 0$ and $\gamma_{\tilde{p}}(1) = \pi(\tilde{p})$, which corresponds to the homotopy class of $\tilde{p}$. Let $\tilde{0}$ denote the point corresponding to the constant curve 0. Set

$$w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \beta(z)\,dz.$$

Then F is a single-valued holomorphic function on M' satisfying the condition $F(\tilde{0}) = 0$ and $dF(\tilde{p}) \neq 0$ for every $\tilde{p} \in \widetilde{M}'$. Choose the largest R $(\le \infty)$ such that F maps an open neighborhood U of $\tilde{0}$ biholomorphically onto an open disc Δ_R in C, and consider the map $B = \pi \circ (F|U)^{-1}: \Delta_R \to M'$. By the Liouville theorem, $R = \infty$ is impossible.

By the definition of $w = F(z)$ we have

(4.7) $$|dw/dz| = v(z).$$

For each point $a \in \partial\Delta$ consider the curve

$$L_a\colon w = ta, \qquad 0 \leq t < 1,$$

and the image Γ_a of L_a by B. We shall show that there exists a point a_0 in $\partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M. To this end, we assume the contrary. Then, for each $a \in \partial\Delta_R$, there is a sequence $\{t_\nu\colon \nu = 1, 2, \dots\}$ such that $\lim_{\nu\to\infty} t_\nu = 1$ and $z_0 = \lim_{\nu\to\infty} B(t_\nu a)$ exist in M. Suppose that $z_0 \notin M'$. Let $\delta_0 = 4\delta/N > 1$. Then by Lemma 3.1, we have

$$\liminf_{z\to z_0} |\widetilde{G}_k|^{\delta_0} \prod_{1\leq j\leq q,\ 1\leq p\leq k-1} |\widetilde{G}_p \vee A_j|^{2\delta_0} \cdot v > 0.$$

If $\widetilde{G}_k(z_0) = 0$ or $|\widetilde{G}_p \vee A_j|(z_0) = 0$ for some p and j, we can find a positive constant c such that $v \geq c/|z - z_0|^{\delta_0}$ in a neighborhood of z_0, so that we obtain

$$\begin{aligned} R &= \int_{L_a} |dw| = \int_{L_a} \left|\frac{dw}{dz}\right| |dz| = \int v(z)|dz| \\ &\geq c \int_\Gamma \frac{1}{|z - z_0|^{\delta_0}} |dz| = \infty. \end{aligned}$$

This is a contradiction. Therefore, we have $z_0 \in M'$.

Take a simply connected neighborhood V of z_0 which is relatively compact in M'. Set $C' = \min_{z\in V} v(z) > 0$. Then $B(ta) \in V$ $(t_0 < t < 1)$ for some t_0. In fact, if not, Γ_a goes and returns infinitely often from ∂V to a sufficiently small neighborhood of z_0 and so we get the absurd conclusion

$$R = \int_{L_a} |dw| \geq c' \int_{\Gamma_a} |dz| = \infty.$$

By the same argument, we can easily see that $\lim_{t\to 1} B(ta) = z_0$. Since π maps each connected component of $\pi^{-1}(V)$ bioholomorphically onto V, there exists the limit

$$\tilde{p}_0 = \lim_{t\to 1} (F|U)^{-1}(ta) \in \widetilde{M}'.$$

Thus $(F|U)^{-1}$ has a biholomorphic extension to a neighborhood of a. Since a is arbitrarily chosen, F maps an open neighborhood of $\overline{U}$ biholomorphically onto an open neighborhood of $\overline{\Delta}_R$. This contradicts the property of R. In conclusion, there exists a point $a_0 \in \partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M.

Our goal is to show that Γ_{a_0} has finite length, contradicting the completeness of the given minimal surface M.

By (4.7) we obtain $|dw/dz| = v(z)$. So

(4.8)
$$\begin{aligned} \left|\frac{dw}{dz}\right| &= |v(z)|^{1-\gamma} \left|\frac{dw}{dz}\right|^\gamma \\ &= \left(\frac{\prod_{j=1}^q |(\widetilde{G}, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1}\prod_{j=1}^q |\widetilde{G}_p \vee A_j|^{4/N} |\widetilde{G}_k|^{1+2q/N}}\right)^{1/\left(\sum \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N\right)} \left|\frac{dw}{dz}\right|^\gamma. \end{aligned}$$

Let $Z(w) = \widetilde{G} \circ B(w)$, $Z_0(w) = g_0 \circ B(w), \ldots, Z_k(w) = g_k \circ B(w)$. Then because

$$Z \wedge Z' \wedge \cdots \wedge Z^{(p)} = (\widetilde{G} \wedge \cdots \wedge \widetilde{G}^{(p)}) \left(\frac{dz}{dw}\right)^{p(p+1)/2},$$

it is easy to see that

$$(4.9) \quad \left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^{p} |(Z, A_j)|^{\omega(j)(1-k/e_j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q/N}}\right)^{1/\left(\sum \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N\right)},$$

where $\Lambda_p = Z^{(0)} \wedge \cdots \wedge Z^{(p)}$.

On the other hand, the metric on Δ_R induced from $ds^2 = 2|\widetilde{G}|^2|dz|^2$ through B is given by

$$(4.10) \qquad B^* ds^2 = 2|\widetilde{G}(B(w))|^2 \left|\frac{dz}{dw}\right|^2 |dw|^2 .$$

Combining (4.7) and (4.8) gives

$$B^* ds = 2|Z| \left(\frac{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\Lambda_p \vee A_j|^{4/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^{q} |(Z, A_j)|^{\omega(j)(1-k/e_j)}}\right)^{1/\left(\sum \omega(j)(1-k/e_j)-(k+1)-(k^2+2k-1)2q/N\right)} |dw| .$$

Using the main lemma, we have

$$B^* ds \le c \left(\frac{2R}{R^2 - |w|^2}\right)^{\rho} |dw|,$$

where c is a positive constant. Since $\rho < 1$, it then follows that

$$d(0) \le \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} B^* ds \le c \int_0^R \left(\frac{2R}{R^2 - |w|^2}\right)^{\rho} |dw| < \infty,$$

where $d(0)$ denotes the distance from the origin 0 to the boundary of M. This contradicts the assumption of completeness of M. Hence the proof of the first part of Theorem 1 is complete.

We now prove the second part.

For any complete minimal surface M immersed in R^n, if there are $q > n(n+1)/2$ hyperplanes in general position in CP^{n-1} such that its Gauss map G is ramified over H_j with multiplicity at least e_j for each j and

$$\sum_{j=1}^{q} (1 - n/e_j) > n(n+1)/2,$$

we are going to prove that M is flat. Since M is flat if and only if its Gauss map is a constant map (see [12]), we only need to prove that G is a constant map.

If G is not a constant map, then we may assume that G is k-nondegenerate and $1 \le k \le n-1$. By the first part of the theorem, we have

$$\sum_{j=1}^{q} (1 - k/e_j) \le (k+1)(n - k/2 - 1) + n .$$

Since

$$(k+1)(n - k/2 - 1) + n \le n(n+1)/2,$$

and

$$\sum_{j=1}^{q}(1-(n-1)/e_j) \le \sum_{j=1}^{q}(1-k/e_j),$$

we obtain

$$\sum_{j=1}^{q}(1-(n-1)/e_j) \le n(n+1)/2.$$

This contradicts the assumption. Therefore M is flat. Q.E.D.

5. Proof of Theorem 2

Let $x=(x_1, x_2, x_3): M \to R^3$ be a nonflat minimal surface and $g: M \to CP^1$ the Gauss map. Assume $M = \Delta$ (as the argument above). Set $\varphi_i = \partial x_i/\partial z$ $(i=1, 2, 3)$ and $f = \varphi_1 - \sqrt{-1}\varphi_2$. Then according to [12] or [7], the metric on M induced from R^3 is given by

$$ds^2 = |f|^2(1+|g|^2)^2|dz|^2. \tag{5.1}$$

Take a reduced representation $\tilde{g} = (g_0, g_1)$ of g on M. Then we can rewrite

$$ds^2 = |h|^2|\tilde{g}|^4|dz|^2, \tag{5.2}$$

where $h = f/g_0^2$, and moreover $h \ne 0$. The rest of the steps are the same as the proof of Theorem 1. If M is not flat, then g is not a constant map. Assume that g is ramified over a_j with multiplicity of e_j and $\sum_{j=1}^{q}(1-1/e_j) > 4$, we shall derive a contradiction. Let $P(\alpha_j) = a_j$, $\alpha_j \in C^2$. Consider numbers

$$\rho = \gamma = \frac{1+2q/N}{\sum_{j=1}^{q}(1-1/e_j) - 2 - 2q/N},$$
$$\delta = \frac{1}{(1-\rho)\left(\sum_{j=1}^{q}(1-1/e_j) - 2 - 2q/N\right)}.$$

Choose some N with

$$\frac{\sum_{j=1}^{q}(1-1/e_j) - 3}{3} > 2q/N > \frac{\sum_{j=1}^{q}(1-1/e_j) - 3}{3+1/q}$$

so that $0 < 2\rho < 1$, $\frac{2\delta}{N} > 1$. Consider the open subset $M' = M - (\{\tilde{g}_1 = 0\})$ of M and define the function

$$v = h^{1/(1-\gamma)}\left(\frac{\prod_{j=1}^{q}|(\tilde{g}, \alpha_j)|^{(1-1/e_j-4/N)}}{|\tilde{g}_1|^{1+2q/N}}\right)^{\delta}$$

on M' where $\tilde{g}_1 = \tilde{g} \wedge \tilde{g}'$.

By exactly the same argument as in the proof of Theorem 1, we can find a curve Γ_{a_0} tends to the boundary of M, and we can estimate the pull-back metric, eventually we obtain that Γ_{a_0} has finite length, contradicting the completeness of the given minimal surface M. Q.E.D.

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